Master Document

Week 3, Day 4 — Counting and Probability

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised May, 2006 and July, 2009

Week 3, Day 4 — Counting and Probability

<u>Materials Needed</u>	<u>Allocated Time</u>
Activity #1 — Ducks and Dinosaurs	5 minutes

- no materials needed

Activity #2 —On the Trail to Science Camp . . . . 40 minutes

- A set of large signs naming and illustrating each of the five exploration activities available at Science Camp: Plants, Weather, Birds, Rocks, and Bugs.
- A transparency of a flagpole plus five small transparencies with pictures of the five exploration activities.
- Ten sets of five post-it notes with pictures of each of the five exploration activities.
- Five "Yes/No"placards with strings to be worn around the neck.

Activity #3 — Review of Counting Methods . . . . 20 minutes

- no materials needed

Activity #4 — Probability Explorations . . . . 40 minutes

- no materials needed

Activity #5 — Lottery . . . . 20 minutes

- no materials needed

. . . . TOTAL WORKSHOP TIME: 125* minutes

* In addition, 10 minutes are allocated for a break in this 2 ¼ hour workshop.

Activity #1. Ducks and Dinosaurs.
(Allocated time = 5 minutes)

A. Discuss with participants their strategies for this game. (You may wish to use TSP #12 from Day 3.) Try to avoid having one person or group "spill the beans" at the beginning. Ask how many think that Player 1 has a winning strategy, and focus on them first — providing responses to all suggested first moves for Player 1. At some point, someone will recognize and observe that the options at the outset of the game are the same as those of 3-2-1 Nim, at which point confirm that this is actually 3-2-1 Nim. A three-dimensional version of this problem appears on TSP #1; of course, you should not tell the participants this right away, but let them come to this conclusion after understanding the rules presented on TSP #1 . Refer participants to similar problems on Pages 6-7 of yesterday's Resource Book.

Activity #2. On the Trail to Science Camp
(Allocated time = 40 minutes)

A. Today's focus will be counting and probability, and we'll begin with a brief review of some of the counting principles that were discussed last summer. Before the workshop begins, put up the five large signs illustrating the five exploration activities offered at Science Camp in a horizontal row in a prearranged area of the room so that they will be ready to use later in the activity. The signs should be hung high enough so that they can be read above the head of a person standing in that area.

Announce to the group that we are going to Science Camp! Referring to the large signs, describe the five areas that are available for exploration at Science Camp: Plants, Weather, Birds, Rocks, and Bugs. Explain that the camp often communicates information by hanging flags on the camp's tall vertical flagpole. For example, if the next activity were to explore Bugs, then the flag for Bugs might be displayed on the flagpole.

Note that, by tradition, at the start of each day at Science Camp, the flagpole displays all five flags representing the five exploration activities. Using the transparency of the flagpole and the transparency pictures of the five exploration activities, demonstrate a few possible ways that the five flags could be hung from the flagpole - one beneath the other.

Tell the participants that the camp is open each day during April, May, and June

and that the camp director would like to fly the flags in a different way each morning that the camp is open. Ask the participants, in how many different ways could the five flags be hung on the flagpole? Are there enough different ways that a different one could be used each day of the months April, May, and June? Elicit from them the fact that since there are five choices for the flag that will be at the top of the flagpole, then four choices for the flag that will be next to the top flag, etc., by the multiplication rule of counting there are 5! = 120 ways that the five flags can be hung from the flagpole. So, yes, the flags can be hung in a different way each of the 91 mornings.

Next ask the participants in how many ways could three of the five flags be hung vertically on the flagpole? Using the transparency, demonstrate a few examples. Elicit the response that, since there are five choices for flag at the top, four choices for the second flag, and three choices for the third flag, by the multiplication rule of counting, there are 5 x 4 x 3 = 60 ways to display three of the five flags on the flagpole.

B. Designate a section of the board as the camp bulletin board. Tell the participants that they have the opportunity to select three of the five exploration activities to do while they are at camp. Ask volunteers to demonstrate the possible ways to do this by selecting post-it notes for three of the five activities and placing the three post-it notes together on the bulletin board. They should be placed so that all three activities can be seen but it is clear that these post-it notes form a group of three. After several participants have each put up a group of three post-it notes, ask if any possible ways to select three of the five activities have been left out. Call on volunteers to continue displaying possible groupings until all ten are displayed. Be sure that all agree that there are ten possible ways to select a group of three of the five activities. Remind the participants that using a systematic method for listing all of these groups would have been helpful in finding all of the possibilities.

C. Give "Yes/No" placards to five participants. Select one of the groupings of three activities shown on the bulletin board and ask the five participants to demonstrate how this grouping could be represented by standing in appropriate places in the area of the room with the horizontal arrangement of the five large signs (that can be read above theirheads) and displaying the appropriate "Yes" and "No" placards . Then select a different grouping of three activities and ask the five participants with the "Yes/No" placards to represent this new group. Do this a few times until all participants understand this method of representing the groupings.

Note that each group of three activities can be represented by three "Yes"and two "No"placards. Therefore, the number of groupings of three activities is the same as the number of words that can be made using three Ys and two Ns. Remind the participants

of the BOOBOO problem from last year and help them recall that we called the number of ways to select three activities from five possible choices: 5 choose 3. Further, for those participants who wish to, help them recall that we could have calculated the ten possible ways to select three of the five activities using 5!/(3!x 2!).

Also, remind the participants that there is another way to calculate the choose numbers and that the next activity should help them remember this alternative.

Activity #3. Review of Counting Methods.
(Allocated time = 20 minutes)

A. Distribute HO #1 (=TSP #2) and ask people to systematically list and count the number of paths that spell out the word "TRAIL", and the number of trails that end at each "L". Ask them to tell you what they can say about the mathematics in this problem — anticipating that they will mention systematic listing, Pascal's triangle, choose numbers, $2^4 = 16$, tree diagrams, etc.

B. Then hand out HO #2 (the first page of the Resource Book) and review the multiplication rule for counting, systematic listing, the choose numbers, and Pascal's triangle. (See TSP #3, #4, and #5.) Discuss Pascal's triangle (on TSP #5) before the pizza problem (on TSP #4), so that you can use the pizza problem to summarize these ideas. (Pizza problem: How many ways are there of choosing three toppings from among six available toppings? Four toppings? Any number of toppings?) Note that the answers to all the questions in the pizza problem are contained in the sixth row of Pascal's triangle.

[Time for a 5-10 minute break]

Activity #4. Probability Explorations.
(Allocated time = 40 minutes)

A. Use HO #3 (= TSP #6) to conduct a probability experiment. After each person completes the chart, someone at each table should add up the results for the table. Then ask each table to report the results, and record them on TSP #7. Add the columns to get the totals for each L; the total will be out of 200-300+ trials.

Naturally, participants shouldn't "work together" by writing down the same results.

Each participant should be doing his/her very own experiments and getting independent results. Watch to make sure that the following scenario doesn't happen as a result of a misunderstanding of the instructions — the participants at a table simultaneously flip coins, and they all write down the same answers!

B. Before analyzing these results, we will look at our data and see if they jive with what we would expect. Ask "How many of you expected that the number of times we landed at each L would be about equal? After all, there are five possibilities, and so each one should have happened about 1/5 of the time." Elicit from the participants the idea that not all of the five possibilities were "equally likely." Lead them to explain that there are 6 ways to get to L-3 and only 1 way to get to L-1 or L-5, and that any way to get through "TRAIL" is as likely as any other way, so that you should get to L-3 six times more often than to L-1 or L-5. Examine the data to see whether that happens, approximately. Ask whether there should be any relation between the totals in the L-2 and L-4 columns; elicit the response that since the number of ways to get to L-2 and L-4 are the same, the totals in the two columns should be approximately the same.

C. Let's see how these ideas are expressed in terms of the mathematical theory of probability. Review TSP #8 and TSP #9 which express the probabilities we discussed above in terms of "favorable outcomes divided by total possible outcomes" or of "desired outcomes over possible outcomes". (Note that the question of why the 16 outcomes of the TRAIL activity are equally likely is addressed below.) Now let's compare the results of our experiment with the results predicted by probability. Enter the probabilities at the bottom of TSP #9 onto the T-row of TSP #7; "T" stands for "theoretical". Of course, we can't expect them to be equal, but if we continued with the experiment, we would expect them to get very close. Ask participants to convert the fractions obtained experimentally to decimals, and enter the results on the E-row of TSP #7; "E" stands for "experimental". Most likely, participants will be surprised by the results — all calculations should result in numbers which differ by at most .05.

Introduce the terms "theoretical probability" and "experimental probability" in the context of this activity — and note that the experimental results are very close to what would be predicted theoretically. Indeed, that's what usually happens. What if they don't agree? Then either the theoretical model is inadequate, or the experiment doesn't measure what the theory is describing — so the two methods, theoretical and experimental, are often used hand-in-hand, as checks on one another. Moreover, there are cases where it is hard to arrive at conclusions theoretically; in that case, the only way to get an answer is using experimental *probability.*

D. Let's now look a little more closely at some of the assumptions and implications of the mathematical theory of probability. Hand out the Resource Books and review TSP #<u>10</u> and TSP #<u>11</u>; the material on these pages can be found on page 3 of the Resource Book. (We hand out the Resource Books at this point so that the participants will have the basic information about probability handy while trying the problems on the next handout.)

E. Distribute HO #<u>4</u> (= TSP #<u>12</u>) involving a variety of probability questions. After the participants have an opportunity to work on these questions, review them all. Particular attention should be made to the errors that are made frequently. For example, many will overlook that there are 36 equally likely possibilities that result from tossing two dice. Although it is rare that someone will suggest that the numbers from 2 to 12 are equally likely outcomes for the sum of two dice, there may be some who will think that there are 21 equally likely outcomes — deleting one of each pair {m,n}. Use TSP #<u>13</u> to reinforce this; note that this display is particularly helpful to visual learners. Note that question #i is intentionally ambiguous, and that question #g is intended to evoke a misapplication of the addition rule, which will result in the conclusion that it's impossible to throw a 5! The same error is often made (see question #l) when predictions of 50% of rain on each of Saturday and Sunday are combined to a prediction of 100% of rain on the weekend. Remind participants that there are many more such probability problems in the Resource Book. Question #m revisits a problem on yesterday's homework. What's the answer to this question if the limitation on the size of the piles is removed? Probably one. The point is that it is very unlikely that three piles chosen at random will be "balanced", so that if you let your opponent choose the size of the piles, you will almost certainly win if you are Player 1, since your opponent will essentially be choosing at random, unless s/he is familiar with the binary analysis of NIM. (To avoid "accidents", however, your opponent should be required to make all piles larger than, say, 5.)

Activity #5 — Lottery
(Allocated time = 20 min)

A. A Simple Lottery. This lottery consists of two numbers picked from 9 numbers (1-9), and a player wins if they pick both of the numbers correctly. Analyze this game on TSP #<u>14</u> where participants should discover that the chances of their winning if they play once is 1/36, and the chances of their winning if they get six opportunities to guess the two numbers 6/36 or 1/6.

B. Lottery. Distribute HO #5 (= TSP #15). After participants have had an opportunity to work on this problem, review their suggestions and the following solution: Four numbers are picked for the lottery from 1 to 25.

(a) How many ways can four numbers be chosen from the numbers 1 to 25? (12,650 which is 25 choose 4)

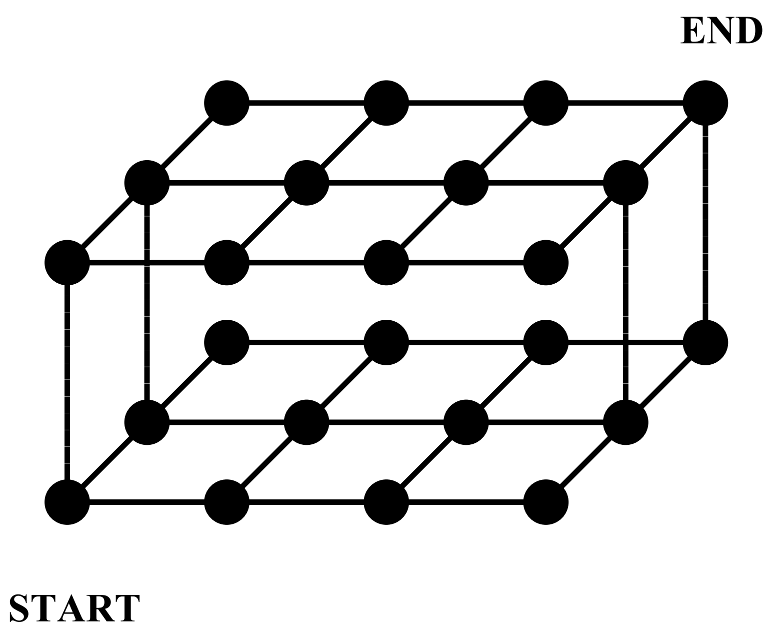
(b) How many different ways are there of matching all four numbers? (1)

(c) How many different ways are there of matching exactly 3 out of the 4 numbers? (this is (4 choose 3) × (21 choose 1) = 84)

(d) If you buy 10 tickets every day, how often should you expect to win? (85×10/12650 = .067 or one out of 15 days, enough to get you hooked)

A Three-dimensional game

A marker is placed at the vertex marked "START". Two players take turns moving the marker from one vertex to another in any one of three directions — to the right (straight), to the back (slanted up and to the right), and up (even though vertical lines are not always drawn). The marker may just move in one of the directions per turn, although it may move any number of spaces in that direction. The winner is the player who moves the marker to "END". Which player should win this game, Player 1 or Player 2?



Take a Hike

How many different ways are there to spell out the word “TRAIL” in the diagram below?

How many trails end at each of the five L’s?

Provide systematic lists to justify your answers.

L				
I	L			
A	I	L		
R	A	I	L	
T	R	A	I	L

A Review of Counting Principles

A. Multiplication rule for counting.

If there are A ways for one thing to take place, B ways for a second thing to take place, C ways for a third thing to take place, etc., and these are all independent, then the total number of possibilities is $A \times B \times C \times ...$

In our example, where we are counting the number of ways of spelling TRAIL, there are two possible moves at each step, East and North, and there are altogether four steps, so the total number of possibilities is $2 \times 2 \times 2 \times 2 = 16$.

B. Systematic listing.

We can make a systematic list (or a tree) of all possible paths, treating each path as a “word” involving E (for East) and N (for North). Here the list is presented alphabetically:

EEEE	ENEE	NEEE	NNEE
EEEN	ENEN	NEEN	NNEN
EENE	ENNE	NENE	NNNE
EENN	ENNN	NENN	NNNN

C. The Choose Numbers

Each path that ends at the middle L in the diagram involves two steps North and two steps East. It therefore corresponds to a word involving E’s and N’s which has exactly two E’s and two N’s. The number of such words is “4 choose 2” since it corresponds to the number of ways of choosing two slots for E’s among four slots corresponding to the four steps; its value is $(4\times3)/(2\times1) = 6$.

D. The Pizza Problem

You wish to buy a pizza with three different toppings, and there are six toppings available. In how many different ways can you do this?

How about with four toppings? Five toppings?

Any number of toppings?

E. Pascal's Triangle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5   10  10  5   1
1   6   15  20  15  6   1
```

The entries in the 6th row of Pascal's triangle, for example, add up to $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$, since they represent all the paths that you can take from the 1 at the top down to the 6th row, with each move being to the left or to the right.

Each particular entry in the 6th row represents the paths that end at that entry; for example, 20 represents all the paths from the 1 at the top down to the 20 — that is, all paths in which 3 out of the 6 moves are to the left; this is "6 choose 3", whose value is $(6 \times 5 \times 4)/(3 \times 2 \times 1) = 20$.

Hand-out #2. Random Trails

L-1				
I	L-2			
A	I	L-3		
R	A	I	L-4	
T	R	A	I	L-5

1. Toss a coin four times, and record the results in the left column.

2. Convert H(eads) to E(ast) and T(ails) to N(orth) and record the results in the second column.

3. Walk the trail indicated by the sequence in the second column, and mark in the third column the L at which the trail ends.

4. Repeat this 8 times.

	H's and T's	E's and N's	Ending at...
Sample	H-H-T-H	E-E-N-E	L-4
1			
2			
3			
4			
5			
6			
7			
8			

Record below the number of times each L appears in the right column:

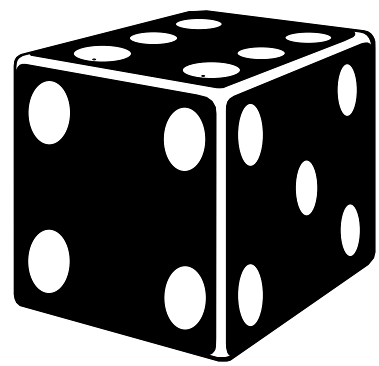
	L-1	L-2	L-3	L-4	L-5
Number of times					

TABLE	L-1	L-2	L-3	L-4	L-5	SUM
1						
2						
3						
4						
5						
6						
7						
8						
TOTAL						
E						
T						

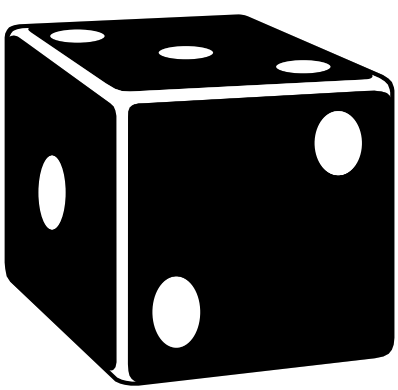
Introduction to Probability

Events are called "equally likely" if the probability of each of them occurring is the same as the probability of any other one occurring.

For example, when you toss a die, there are six possible outcomes, all of which are equally likely. Or, when you toss a coin, there are two possible outcomes, both equally likely. Or, when you take a walk on the TRAIL diagram, there are sixteen possible outcomes, all equally likely.



Suppose there are K outcomes of an experiment, all equally likely, and A of those outcomes constitute an "event", then the probability of that event is A/K.



Example: The probability of getting a 1, 2, or 3 when a die is rolled is ½ because three of the six outcomes constitute this event.

Our Experiment

In our experiment, since there are sixteen equally likely outcomes, the probability of ending at

L-1 is 1/16 = .0625 (NNNN)

L-2 is 4/16 = .25 (ENNN, NENN, NNEN, NNNE)

L-3 is 6/16 = .375 (EENN, ENEN, ENNE, NEEN, NENE, NNEE)

L-4 is 4/16 = .25 (EEEN, EENE, ENEE, NEEE)

L-5 is 1/16 = .0625 (EEEE)

Addition Rule for Probabilities.

If events are mutually exclusive (not more than one can occur at the same time), then the probability of one of them happening is the sum of their individual probabilities.

> Example: Since the events of throwing a 3 and a 5 are mutually exclusive, the probability of throwing a 3 or a 5 is the sum of the two probabilities.

In our example, the probability of ending up at L-1, L-2, L-3, L-4, and L-5 are mutually exclusive, so the probability of ending up at one of them is the sum of the five probabilities. Since one of these five events must happen, the total of the five probabilities must be 1.

Another example. From the individual probabilities, we see that the probability of ending up at either L-2 or L-4 is exactly 1/2.

Multiplication Rule for Probabilities.

If the probability of event A is p, the probability of event B is q, the probability of event C is r, etc., and these are independent experiments, then the probability of event

A and B and C and ...

is $p \times q \times r \times \ldots$.

> Example: If you throw a die and toss a coin, then the probability that you will throw a 5 and toss an H is the product $(1/6) \times (1/2) = 1/12$.

In our example, the probability that you will take the NENE trail is $(1/2) \times (1/2) \times (1/2) \times (1/2) = 1/16$. The same is true for each trail, which is why the 16 trails are all equally likely, as claimed earlier.

Complementary Events

The complementary event to A consists of all outcomes not in A. For example, the complement of (rolling a 1 or 3) is (rolling a 2, 4, 5, or 6). The probability of an event plus the probability of its complement is always 1.

Probability examples

What is the probability of ...

a. ... rolling a 3 with a die?

b. ... rolling a 3 and a 4 with two dice?

c. ... rolling two dice whose sum is 3?

d. ... rolling two dice whose sum is 7?

e. ... rolling two dice whose sum is 1?

f. ... rolling two dice whose sum is not 7?

g. ... rolling two dice whose sum is either even or 1, 2, or 3?

h. ... drawing a king from a standard deck of cards?

i. ... drawing another king immediately afterwards?

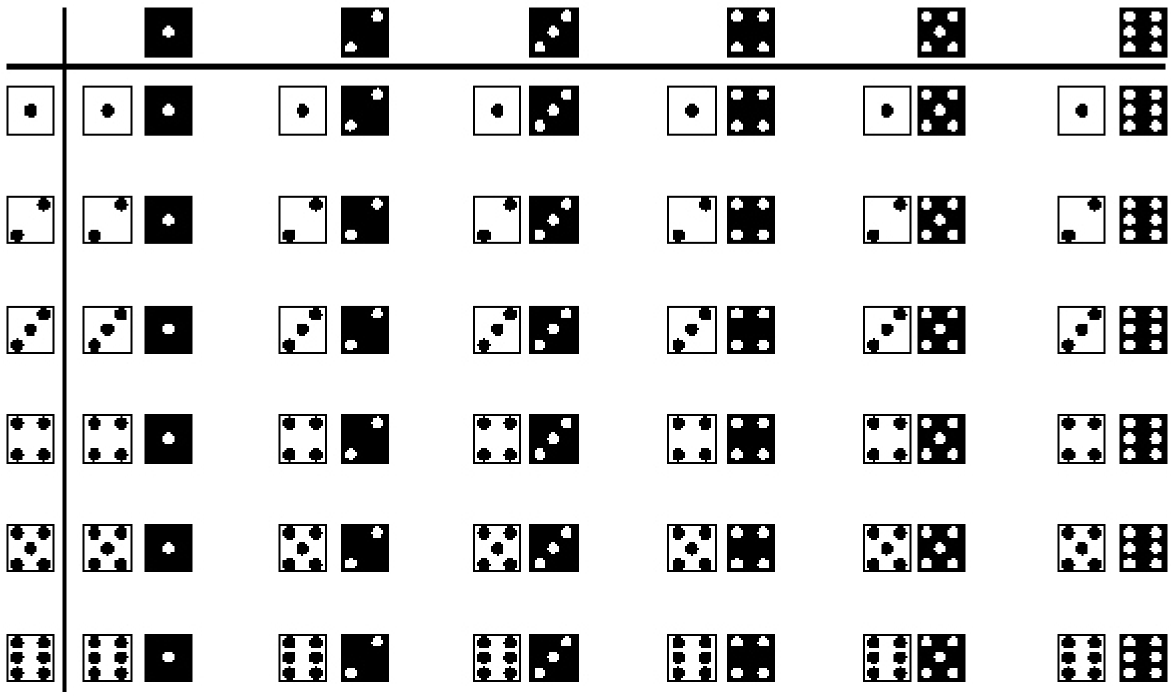
j. ... drawing the three of hearts?

k. ... tossing four coins and getting exactly three heads?

l. ... rain sometime this weekend if the probability of rain is 50% on Saturday and 50% on Sunday?

m. ... winning at Three-Pile NIM if your opponent picks the sizes of the three piles at random (all piles at most 5) and you are Player 1?

THE OUTCOMES FOR TOSSING TWO DICE



2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

A Simple Lottery

1 2 3 4 5 6 7 8 9

I have chosen two of these numbers.

What is the probability that you can guess both of my numbers on a single try?

What is the probability that, given six tries, you can guess my numbers?

Hand-out #5 — Lottery

Four numbers are picked each day from 1 to 25, and you win if your ticket matches 3 or 4 of the numbers.

a. How many ways can 4 numbers be selected from the numbers 1 to 25?

b. How many different ways are there of matching all 4 of the numbers?

c. How many different ways are there of matching exactly 3 of the 4 numbers?

d. If you buy 10 tickets every day, how often should you expect to win?

Hand-out #1 — How Many Trails?

How many different ways are there to spell out the word "TRAIL" in the diagram below?

How many trails end at each of the five L's?

Provide systematic lists to justify your answers.

```
L

I   L

A   I   L

R   A   I   L

T   R   A   I   L
```

Hand-out #2 — A Review of Counting Principles

A. Multiplication rule for counting.

If there are A ways for one thing to take place, B ways for a second thing to take place, C ways for a third thing to take place, etc., and these are all independent, then the total number of possibilities is $A \times B \times C \times \ldots$

In our example, where we are counting the number of ways of spelling TRAIL, there are two possible moves at each step, East and North, and there are altogether four steps, so the total number of possibilities is $2\times2\times2\times2 = 16$.

B. Systematic listing.

We can make a systematic list (or a tree) of all possible paths, treating each path as a "word" involving E (for East) and N (for North). Here the list is presented alphabetically:

EEEE	ENEE	NEEE	NNEE
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EENN	ENNN	NENN	NNNN

C. The Choose numbers.

Each path that ends at the middle L in the diagram involves two steps North and two steps East. It therefore corresponds to a word involving E's and N's which has exactly two E's and two N's. The number of such words is "4 choose 2" since it corresponds to the number of ways of choosing two slots for E's among four slots corresponding to the four steps; its value is $(4\times3)/(2\times1) = 6$.

D. The Pizza Problem

You wish to buy a pizza with three different toppings, and there are six toppings available. In how many different ways can you do this? How about with four toppings? Five toppings? Any number of toppings?

E. Pascal's Triangle

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          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
1   6  15  20  15   6   1
```

The entries in the 6th row of Pascal's triangle, for example, add up to $2\times2\times2\times2\times2\times2 = 64$, since they represent all the paths that you can take from the 1 at the top down to the 6th row, with each move being to the left or to the right.

Each particular entry in the 6th row represents the paths that end at that entry; for example, 20 represents all the paths from the 1 at the top down to the 20 — that is, all paths in which 3 out of the 6 moves are to the left; this is "6 choose 3", whose value is $(6\times5\times4)/(3\times2\times1) = 20$.

Hand-out #3 — Random Trails

L-1				
I	L-2			
A	I	L-3		
R	A	I	L-4	
T	R	A	I	L-5

1. Toss a coin four times, and record the results in the left column.

2. Convert H(eads) to E(ast) and T(ails) to N(orth) and record the results in the second column.

3. Walk the trail indicated by the sequence in the second column, and mark in the third column the L at which the trail ends.

4. Repeat this 8 times.

	H's and T's	E's and N's	Ending at...
Sample	H-H-T-H	E-E-N-E	L-4
1			
2			
3			
4			
5			
6			
7			
8			

Record below the number of times each L appears in the right column:

	L-1	**L-2**	**L-3**	**L-4**	**L-5**
Number of times					

Hand-out #4 — Probability Examples

What is the probability of ...

a. ... rolling a 3 with a die?

b. ... rolling a 3 and a 4 with two dice?

c. ... rolling two dice whose sum is 3?

d. ... rolling two dice whose sum is 7?

e. ... rolling two dice whose sum is 1?

f. ... rolling two dice whose sum is not 7?

g. ... rolling two dice whose sum is either even or 1, 2, or 3?

h. ... drawing a king from a standard deck of cards?

i. ... drawing another king immediately afterwards?

j. ... drawing the three of hearts?

k. ... tossing four coins and getting exactly three heads?

l. ... rain sometime this weekend if the probability of rain is 50% on Saturday and 50% on Sunday?

m. ... winning at Three-Pile NIM if your opponent picks the sizes of the three piles at random (all piles at most 5) and you are Player 1?

Hand-out #5 — Lottery

Four numbers are picked each day from 1 to 25, and you win if your ticket matches 3 or 4 of the numbers.

a. How many ways can 4 numbers be selected from the numbers 1 to 25?

b. How many different ways are there of matching all 4 numbers?

c. How many different ways are there of matching exactly 3 of the 4 numbers?

d. If you buy 10 tickets every day, how often should you expect to win?

Week 3, Section 4 — Counting and Probability — Exercises

Practice Problems

1. Suppose you toss a coin 3 times. What is the probability of each of the following happening:
   a) you get 3 heads
   b) you get 3 tails
   c) you get heads, tails, heads
   d) you get tails, tails, heads
   e) you get at least one tails
   f) you get tails on the second toss

2. Alice remembers that Bob's birthday was in July, and that the date was a 2 digit number, and that the second digit was odd, but she can't remember anything else about it. If she just guesses based on that information, what is the probability that she will guess the correct day?

3. Bob was *so* impressed that Alice remembered his birthday that he quickly had to try to remember hers. All he could remember was that it was also in July, came after his (which turned out to be the 17th) and had a different first digit. What is the probability that he correctly guesses hers?

Study Group Problems

4. a) Alice and Bob get married. Now they have to remember an anniversary! They can both remember that they got married on the 1st of some month, but can never remember which month. What is the probability that in any given year, Bob will guess the wrong month but Alice will get it correct?

   b) Bob and Alice have three kids, all born in the same month. What is the probability of this, assuming it was purely coincidental? (Hint: the answer is not 1/1728)

5. In Arizona, **LOTTO** is played by picking 6 of the numbers from 1 to 42.
   a) How many ways can you pick 6 numbers from 42? (a calculator can help here!)
   b) What is the probability that you will win this game with the 6 numbers *you* pick?
   c) What is the probability that the 6 winning numbers drawn on Saturday will be *exactly* the same as the 6 winning numbers drawn on the previous Wednesday?
   d) What surprising conclusion about your own chances of winning can you draw from comparing your answers in parts b) and c)?

6. What is the probability of ...
   the sum of two dice being a multiple of four?
   tossing four coins and getting exactly two heads?
   drawing an "S" from a box with the letters of MISSISSIPPI on separate cards?
   getting three consecutive jacks in dealing from a deck of cards?
   rolling two dice and getting a sum of 13?
   drawing two consecutive hearts in dealing from a deck of cards?

7. What is the probability of ...
   the sum of two dice equaling 7 or 11?
   tossing four coins and getting exactly one head?
   drawing a "B" from a box with the letters of PROBABILITY on separate cards?
   getting 2-3-4 consecutively in dealing from a deck of cards?
   getting 2-3-4 not necessarily consecutively in dealing three cards from the deck?
   rolling one die and getting a seven?
   drawing no hearts in dealing two cards from a deck?

8. Suppose it is known that the probability of rain is 50% on Saturday and 60% on Sunday:
   a. What is the probability of no rain this weekend?
   b. What is the probability of rain sometime this weekend?
   c. What is the probability of rain both days this weekend?

9. Would you rather be Player 1 or Player 2 if ...
   a. Three dice are tossed. Player 1 wins $1 if all three dice are different, and Player 2 wins $1 if two or more dice are the same.
   b. Two dice are tossed. Player 1 wins $1 if the product is less than 11, and Player 2 wins if the product is more than 11.
   c. Six coins are tossed. Player 1 wins $1 if there are either 1, 3, or 5 heads, and Player 2 wins $1 otherwise.

10. Three dice are tossed. What is the probability that the sum will be 8 or less?

11. For each of the first 10 figures (top three rows and first one in the bottom row) on the next page, find the probability that a dart which lands randomly in the figure will actually be in the shaded region.

Extension Problems:

12. For each of the last 2 figures (bottom row) on the next page, find the probability that a dart which lands randomly in the figure will actually be in the shaded region.

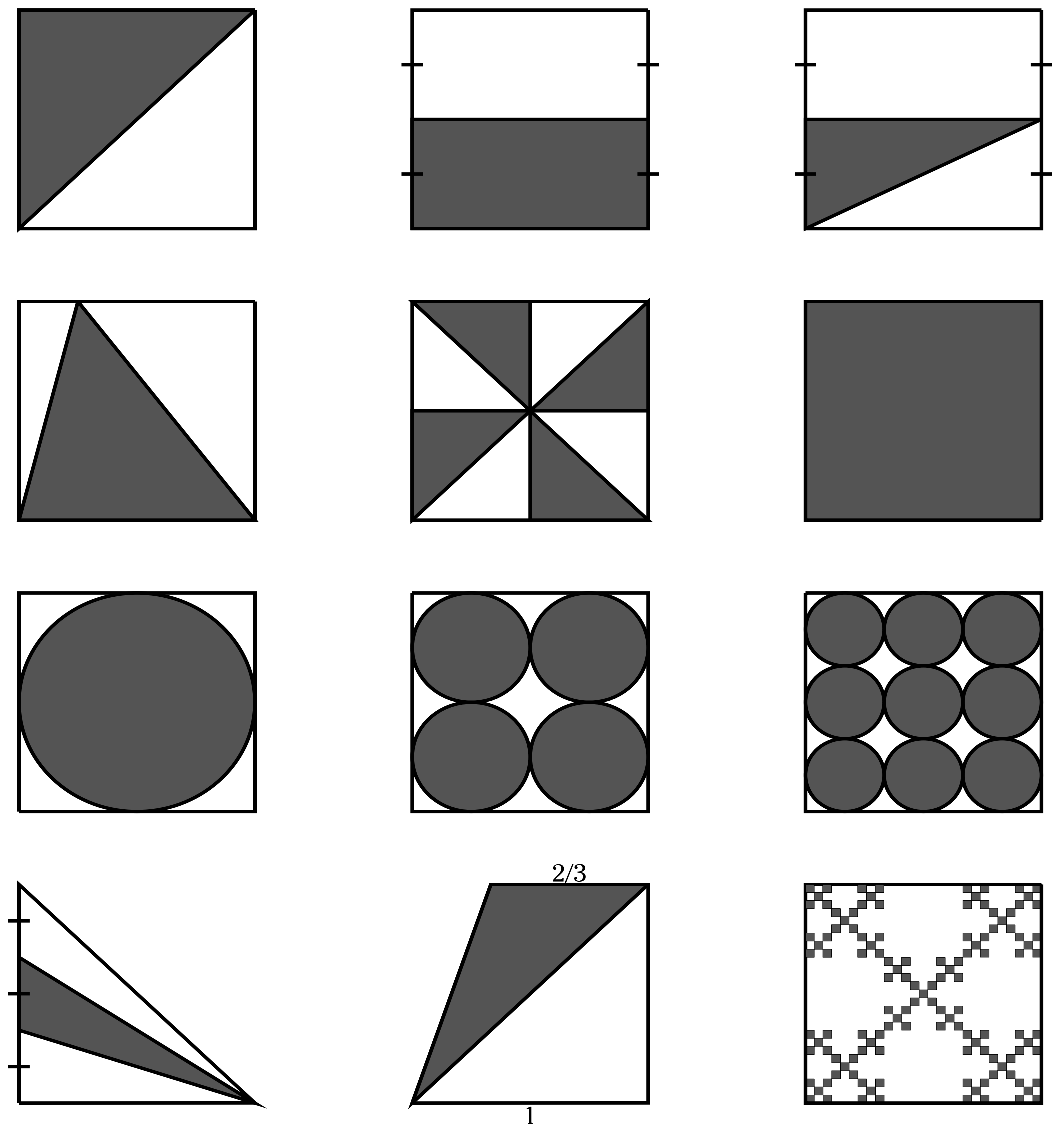
13. Five numbers are picked each day from 1 to 20, and you win if your ticket matches 3, 4, or 5 of the numbers. You buy fifteen tickets every day. How often should you expect to win a prize? (Before making the calculation, guess how the answer to this problem compares to the answer of the problem below.)

14. Five number are picked each day from 1 to 25, and you win if your ticket matches 3, 4, or 5 of the numbers. You buy fifteen tickets every day. How often should you expect to win a prize? (Before making the calculation, guess how the answer to this problem compares to the answer of the problem above.)

15. How many tickets would you have to buy each day to win as frequently in problem #14 as in problem #13?

GEOMETRIC PROBABILITIES

For each of the figures (dartboards) below, find the probability that a dart which lands randomly in the figure will land in the shaded region.



Resource Book

Week 3, Section 4: Counting and Probability

A Review of Counting Principles

A. Multiplication rule for counting.

If there are A ways for one thing to take place, B ways for a second thing to take place, C ways for a third thing to take place, etc., and these are all independent, then the total number of possibilities is $A \times B \times C \times ...$

In our example, where we are counting the number of ways of spelling TRAIL, there are two possible moves at each step, East and North, and there are altogether four steps, so the total number of possibilities is $2\times2\times2\times2 = 16$.

B. Systematic listing.

We can make a systematic list (or a tree) of all possible paths, treating each path as a "word" involving E (for East) and N (for North). Here the list is presented alphabetically:

EEEE	ENEE	NEEE	NNEE
EEEN	ENEN	NEEN	NNEN
EENE	ENNE	NENE	NNNE
EENN	ENNN	NENN	NNNN

C. The Choose numbers.

Each path that ends at the middle L in the diagram involves two steps North and two steps East. It therefore corresponds to a word involving E's and N's which has exactly two E's and two N's. The number of such words is "4 choose 2" since it corresponds to the number of ways of choosing two slots for E's among four slots corresponding to the four steps; its value is $(4\times3)/(2\times1) = 6$.

D. The Pizza Problem

You wish to buy a pizza with three different toppings, and there are six toppings available. In how many different ways can you do this? How about with four toppings? Five toppings? Any number of toppings?

E. Pascal's Triangle

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5   10  10  5   1
1   6   15  20  15  6   1
```

The entries in the 6th row of Pascal's triangle, for example, add up to $2\times2\times2\times2\times2\times2 = 64$, since they represent all the paths that you can take from the 1 at the top down to the 6th row, with each move being to the left or to the right.

Each particular entry in the 6th row represents the paths that end at that entry; for example, 20 represents all the paths from the 1 at the top down to the 20 — that is, all paths in which 3 out of the 6 moves are to the left; this is "6 choose 3", whose value is $(6\times5\times4)/(3\times2\times1) = 20$.

Resource Book

Week 3, Section 4: Counting and Probability

How many trails?

How many different ways are there to spell out the word "TRAIL" in the diagram below? How many trails end at each of the five L's? Provide systematic lists to justify your answers.

L				
I	L			
A	I	L		
R	A	I	L	
T	R	A	I	L

Introduction to Probability

Events are called "equally likely" if the probability of each of them occurring is the same as the probability of any other one occurring.

> For example, when you toss a die, there are six possible outcomes, all of which are equally likely. Or, when you toss a coin, there are two possible outcomes, both equally likely. Or, when you take a walk on the TRAIL diagram, there are sixteen possible outcomes, all equally likely.

Suppose there are K outcomes of an experiment, all equally likely, and A of those outcomes constitute an "event", then the probability of that event is A/K.

> Example: The probability of getting a 1, 2, or 3 when a die is rolled is ½ because three of the six outcomes constitute this event.

Resource Book

Week 3, Section 4: Counting and Probability

Random trails.

L-1				
I	L-2			
A	I	L-3		
R	A	I	L-4	
T	R	A	I	L-5

1. Toss a coin four times, and record the results in the left column.
2. Convert H(eads) to E(ast) and T(ails) to N(orth) and record the results in the second column.
3. Walk the trail indicated by the sequence in the second column, and mark in the third column the L at which the trail ends.
4. Repeat this 8 times.

	H's and T's	E's and N's	Ending at...
Sample	H-H-T-H	E-E-N-E	L-4
1			
2			
3			
4			
5			
6			
7			
8			

Record below the number of times each L appears in the right column:

	L1	**L2**	**L3**	**L4**	**L5**
Number of times					

Resource Book

Week 3, Section 4: Counting and Probability

In finding a TRAIL, there are sixteen outcomes, each equally likely, so the probability of ending

at L-1 is 1/16 = .0625	(NNNN)
at L-2 is 4/16 = .25	(ENNN, NENN, NNEN, NNNE)
at L-3 is 6/16 = .375	(EENN, ENEN, ENNE, NEEN, NENE, NNEE)
at L-4 is 4/16 = .25	(EEEN, EENE, ENEE, NEEE)
at L-5 is 1/16 = .0625	(EEEE)

Addition Rule for Probabilities.

If events are mutually exclusive (not more than one can occur at the same time), then the probability of one of them happening is the sum of their individual probabilities.

> Example: Since the events of throwing a 3 and a 5 are mutually exclusive, the probability of throwing a 3 or a 5 is the sum of the two probabilities.

In our example, the probability of ending up at L-1, L-2, L-3, L-4, and L-5 are mutually exclusive, so the probability of ending up at one of them is the sum of the five probabilities. Since one of these five events must happen, the total of the five probabilities must be 1.

> Another example. From the individual probabilities, we see that the probability of ending up at either L-2 or L-4 is exactly 1/2.

Why are the sixteen possibilities listed in the problem above equally likely? This illustrates the multiplication rule for probabilities (paralleling the multiplication rule for counting).

Resource Book

Week 3, Section 4: Counting and Probability

Multiplication Rule for Probabilities.

If the probability of event A is p, the probability of event B is q, the probability of event C is r, etc., and these are independent experiments, then the probability of event

A and B and C and ...

is $p \times q \times r \times \ldots$.

> Example: If you throw a die and toss a coin, then the probability that you will throw a 5 and toss an H is the product $(1/6) \times (1/2) = 1/12$.

In our example, the probability that you will take the NENE trail is $(1/2) \times (1/2) \times (1/2) \times (1/2) = 1/16$. The same is true for each trail, which is why the 16 trails are all equally likely, as claimed earlier.

Complementary Events

The complementary event to A consists of all outcomes not in A. For example, the complement of (rolling a 1 or 3) is (rolling a 2, 4, 5, or 6). The probability of an event plus the probability of its complement is always 1.

Resource Book

Week 3, Section 4: Counting and Probability

Probability examples

What is the probability of ...

a. ... rolling a 3 with a die?

b. ... rolling a 3 and a 4 with two dice?

c. ... rolling two dice whose sum is 3?

d. ... rolling two dice whose sum is 7?

e. ... rolling two dice whose sum is 1?

f. ... rolling two dice whose sum is not 7?

g. ... rolling two dice whose sum is either even or 1, 2, or 3?

h. ... drawing a king from a standard deck of cards?

i. ... drawing another king immediately afterwards?

j. ... drawing the three of hearts?

k. ... tossing four coins and getting exactly three heads?

l. ... rain sometime this weekend if the probability of rain is 50% on Saturday and 50% on Sunday?

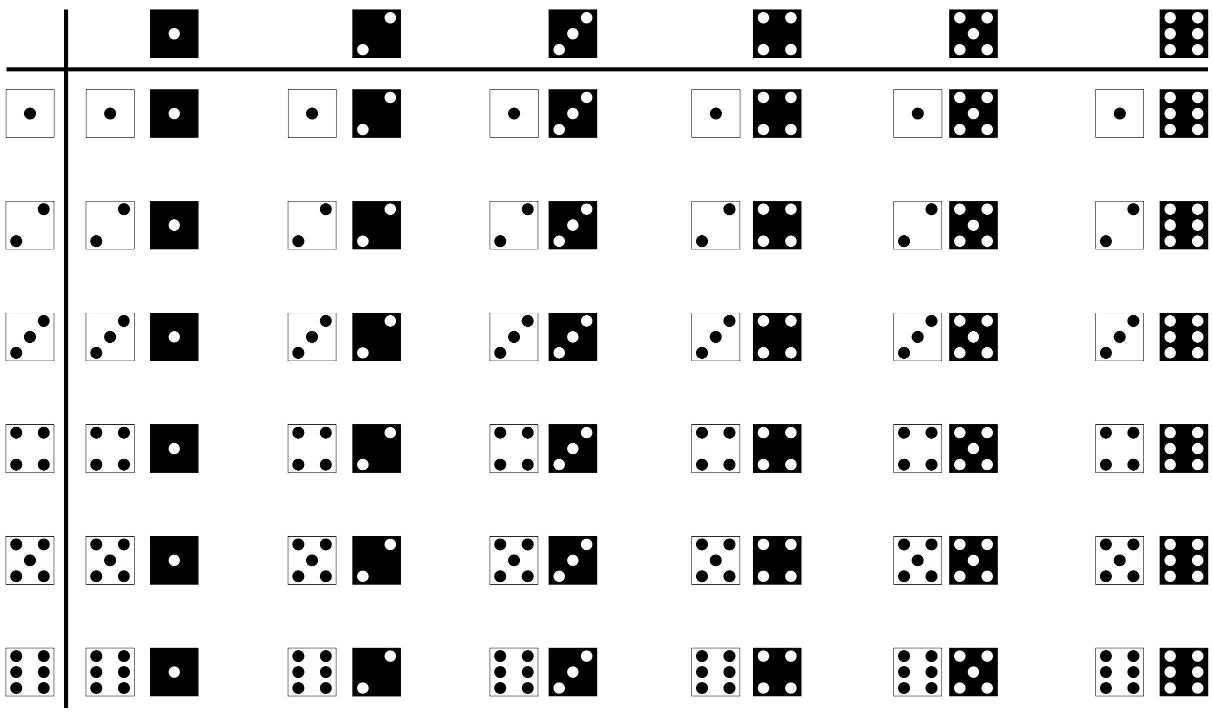
m. ... winning at Three-Pile NIM if your opponent picks the sizes of the three piles at random (all piles at most 5) and you are Player 1?

Lottery

Four numbers are picked each day from 1 to 25, and you win if your ticket matches 3 or 4 of the numbers.

a. How many different ways are there of matching all four numbers?
b. How many different ways are there of matching exactly 3 of the numbers?
c. If you buy 10 tickets every day, how often should you expect to win?

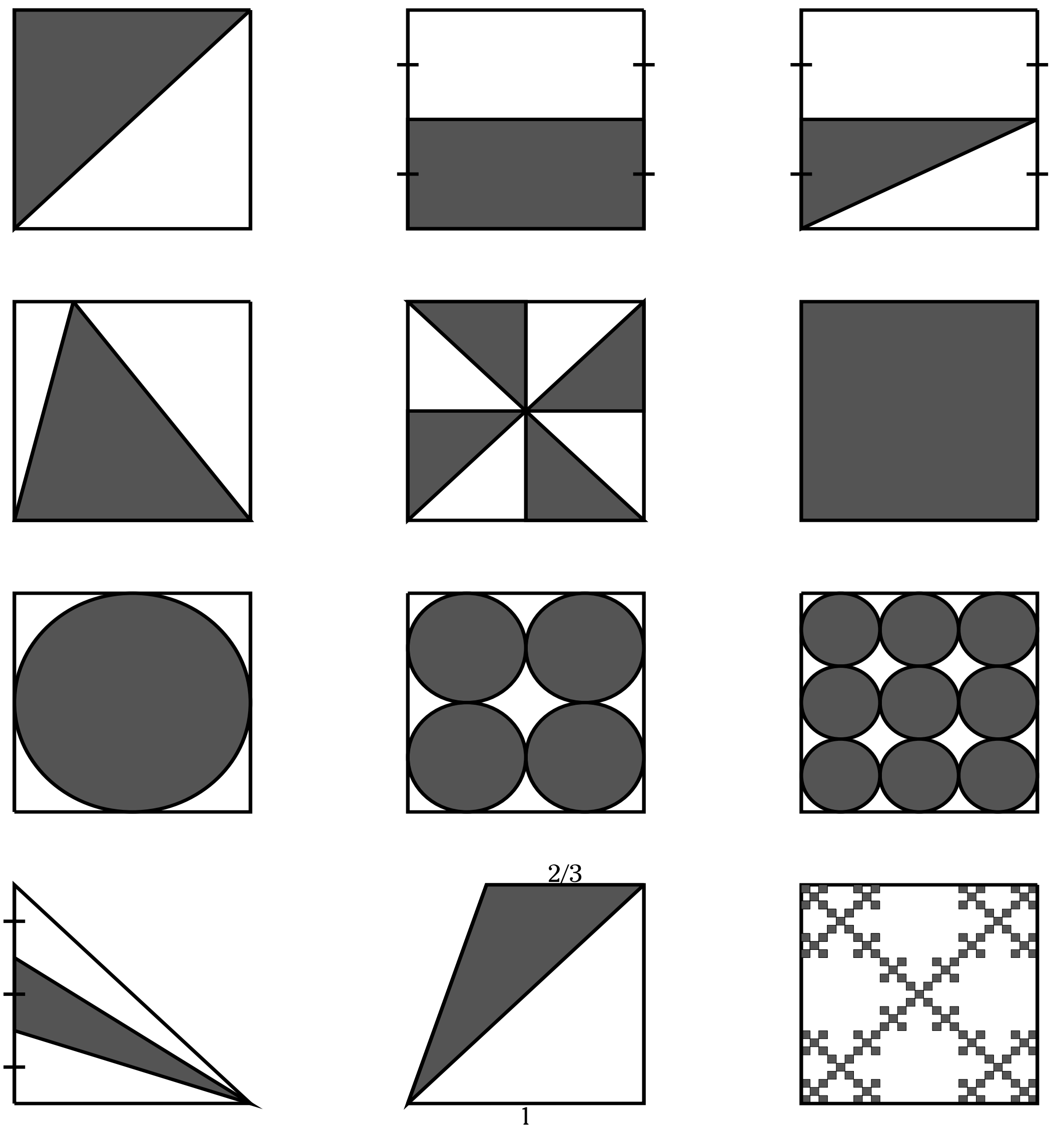
THE 36 OUTCOMES FOR TOSSING TWO DICE



2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

GEOMETRIC PROBABILITIES

For each of the figures (dartboards) below, find the probability that a dart which lands randomly in the figure will land in the shaded region.



WITPO (What Is the Probability Of)

BY DONNA BIRD ERICKSEN, MARTHA L. FRANK, and RYAN KELLEY

Games can be used in the classroom for many educational purposes. For example, they can be used to introduce and develop a concept. This purpose was illustrated by Noone (1988) with the game "Chuck-a-luck: Learning Probability Concepts with Games of Chance." Games can also be incorporated at the end of a unit to review material. Students enjoy studying probability because it often involves games. The following probability game, WITPO, was developed by Ryan Kelley, a preservice secondary school teacher.

WITPO (What Is the Probability Of) motivates students to practice basic probability skills and algorithms. Some examples include computing compound probability and using Pascal's triangle to solve binomial probability problems. Students can play this game after completing an introductory unit on probability.

WITPO was classroom tested with a class of first-year-algebra students who had just completed a short enrichment unit on probability. The students played the game in groups of three or four. They had very little difficulty understanding the rules and were enthusiastic about playing the game. Students especially enjoyed the chance element of the game.

To make game playing a positive experience for students, teachers should consider the following model for implementation of classroom games.

Looking ahead

Before introducing a game, some advance preparation is useful.

1. Teachers should familiarize themselves with the rules of the game.

2. Teachers should decide whether the game really is at the appropriate level for their students. If not, consideration should be given as to how the game can be modified. For example, with WITPO, remove cards that do not fit the content studied. These cards should be replaced with cards that better reflect individual purposes.

3. Grouping is another important concern. Teachers need to decide if the game should be in a small-group or whole-group setting. If small groups are used, they can be randomly selected or organized based on either ability or compatibility.

4. In any game-playing situation, cheating can occur. It is necessary to decide in advance how such infractions will be handled.

The game requires a die, a playing board (reproduced in this article), four playing pieces, and a deck of 3 in. × 5 in. question cards. Suggested questions for these cards are included at the end of this article. Answers, as well as questions, should be printed on the cards to allow the teacher or a student quickly to check the reply of the player answering. Teachers are encouraged to modify or supplement these questions to reflect their own instructional objectives. To avoid duplication of questions, a WITPO deck should contain at least forty questions.

WITPO is designed to be played in small groups of from two to four players. The rules of the game are easy for students to follow. The playing time is approximately forty-five minutes for each game. This time can be shortened if a time limit is set for answering each question.

Rules

1. Roll the die to determine who goes first. The player rolling the highest number plays first. Play moves counterclockwise.

2. The student who is to play second reads a question from the cards to the first player.

- If the question is answered correctly, then player 1 rolls the die and moves his or her marker the resulting number of spaces

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forward on the playing board. Player 1 then tries to get bonus spaces by rolling the die a second time. If the number on the die matches the number in the playing board where the player's game piece rests, then player 1 moves the marker ahead that many spaces.

- If the question is answered incorrectly, the marker stays in the same space.

3. Player 3 next reads a question to player 2. Play continues in this manner until one player reaches the finish.

4. Place the used cards (questions) on the bottom of the stack.

5. Lose a Turn: Leave the marker on the "Lose a Turn" space and the player loses his or her next turn.

6. If a player gives a wrong answer, he or she should be shown the correct answer. Players should be encouraged to discuss the right answer and how it is obtained.

Teachers can adapt this game to a whole-class activity by copying the playing board on an overhead transparency. Students should be divided into two large teams. The teacher directs questions to a student on one of the teams. If a question is answered incorrectly, the same question is repeated to a member of the second team. Play continues in this manner until one team correctly responds to the question. If a student answers a question correctly, the teacher allows that student to roll a die. The teacher or a student then moves the team's playing piece around the board on the overhead.

Looking around

Playing the game showed that the questions were at an appropriate level of difficulty for the students. The game generated some good discussions of probability concepts. Even when it wasn't their turn to answer, other students in the playing group were attentive and would also try to work out the correct answer as a way of checking the player. When students got wrong answers, the group almost always discussed how the correct result was obtained.

Looking back

After the game is finished, teachers need to reflect a bit about questions the students asked while they were playing.

1. If these questions dealt with content, perhaps a review of the unit is needed.

2. If they dealt with understanding of the rules, modification may be needed to make rules easier to follow.

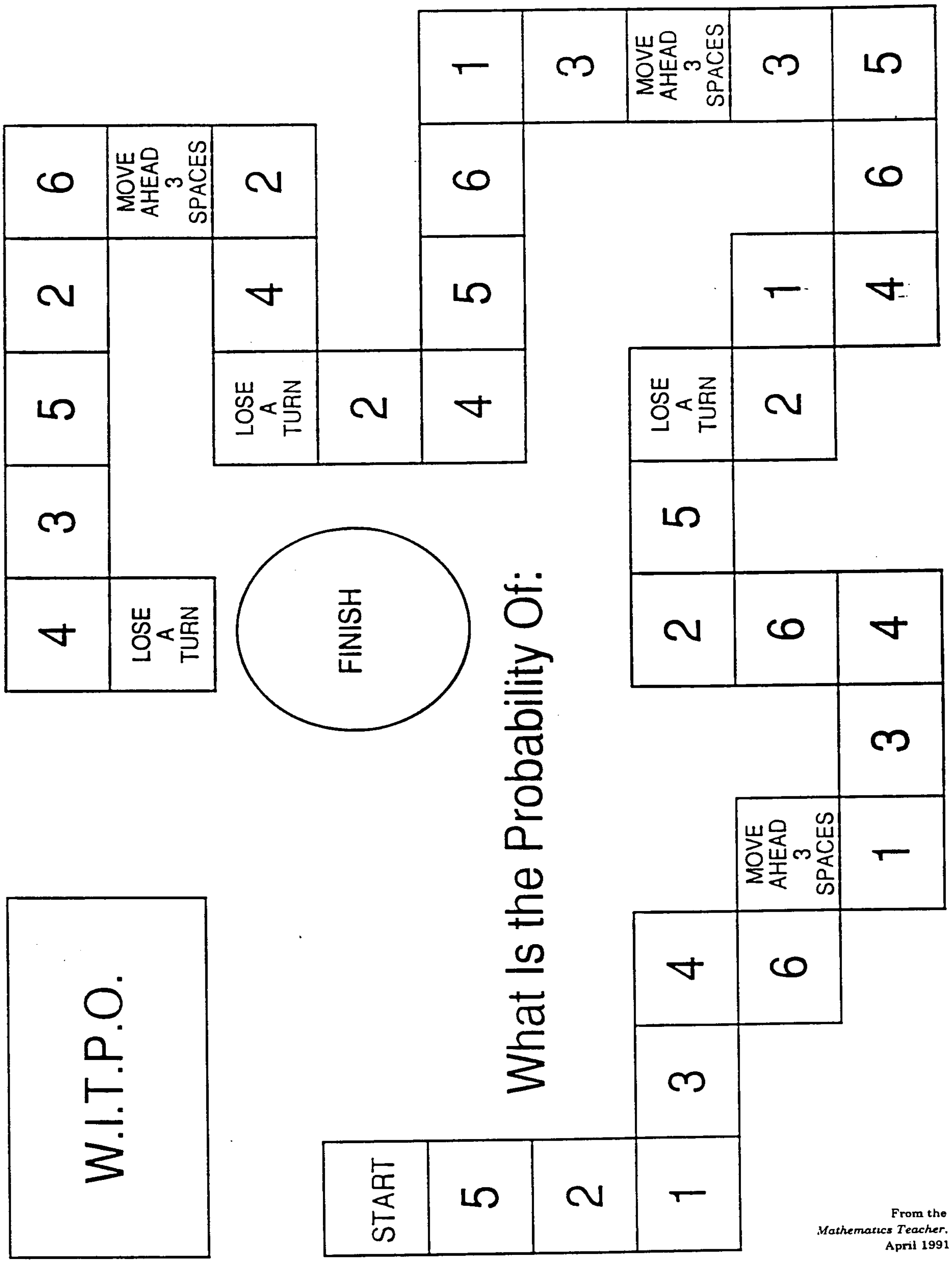
Conclusions

Game playing can be a supplement to any unit of instruction. Students' positive attitudes toward games can carry over into day-to-day instruction. Teachers can encourage students to adapt the game to fit other mathematical topics, develop new questions, and maybe even invent new games.

REFERENCE

Noone, E. T., Jr. "Chuck-a-luck: Learning Probability Concepts with Games of Chance." *Mathematics Teacher* 81 (February 1988): 121–23.

Week 3, Section 4: Counting and Probability



From the *Mathematics Teacher*, April 1991

Week 3, Section 4: Counting and Probability

WITPO Questions and Answers

Q: What is the probability of: Drawing a king from a standard deck of cards in one draw? A: 4/52 = 1/13.	Q: What is the probability of: Rolling a "3" with a die? A: 1/6
Q: What is the probability of: Drawing a red or green marble out of a bag containing 16 red marbles, 10 green marbles, 14 blue marbles, and 10 yellow marbles? A: 16/50 + 10/50 = 26/50 = 13/25.	Q: What is the probability of: The sum of two dice equaling 3? A: 2/36 = 1/18.
Q: What is the probability of: The sum of two standard dice being equal to 1? A: 0/36 = 0.	Q: What is the probability of: The sum of the faces of two dice *not* equaling 12? A: 35/36
Q: What is the probability of: Tossing four coins and getting exactly three heads? A: 4/16 = 1/4.	Q: What is the probability of: Tossing a coin three times and getting exactly three heads? A: 1/2 × 1/2 × 1/2 = 1/8.
Q: What is the probability of: On the first try, working through a maze with 10 "forks in the road," where you must decide to go left or right? A: 1/1024	Q: What is the probability of: Picking out a green marble in one draw from a jar containing 3 red, 2 blue, 5 yellow, and 6 green marbles? A: 6/16 = 3/8.
Q: What is the probability of: In only three draws, drawing three of a kind from a standard deck of cards without replacement? A: 52/52 × 3/51 × 2/50 = 6/2550 = 1/425.	Q: What is the probability of: Correctly answering exactly two of three true-false questions (assuming you are guessing)? A: 3/8

From the *Mathematics Teacher*, April 1991

Week 3, Section 4: Counting and Probability

WITPO Questions and Answers—*Continued*

Q: What is the probability of: Tossing four coins and getting exactly four tails? A: 1/16	Q: What is the probability of: Drawing an "E" from a box with the letters S, N, E, E, Z, and E in it? (Each letter is on a separate card within the box.) A: 3/6 = 1/2.
Q: What is the probability of: Drawing a face card from a standard deck of cards? A: 12/52 = 3/13.	Q: What is the probability of: Rolling a multiple of three with a die in one roll? A: 2/6 = 1/3.
Q: What is the probability of: The sum of two dice being a multiple of four? A: 9/36 = 1/4.	Q: What is the probability of: The sum of two dice equaling 7 or 11? A: 6/36 + 2/36 = 8/36 = 2/9.
Q: What is the probability of: Drawing a king of hearts from a standard deck of cards in one draw? A: 1/52	Q: What is the probability of: Tossing four coins and getting exactly two heads? A: 6/16 = 3/8.
Q: What is the probability of: Drawing three jacks on three consecutive draws without replacement? A: 4/52 × 3/51 × 2/50 = 24/132 600 = 1/5 525.	Q: What is the probability of: *Guessing* correctly on four true-false questions in a row? A: 1/2 × 1/2 × 1/2 × 1/2 = 1/16.
Q: What is the probability of: Drawing two jacks on two consecutive draws without replacement? A: 4/52 × 3/51 = 12/2652 = 1/221.	Q: What is the probability of: Drawing an "S" from a box with the letters M, I, S, S, I, S, S, I, P, P, and I in it? (Each letter is on a separate card within the box.) A: 4/11

From the *Mathematics Teacher*, April 1991

Week 3, Section 4: Counting and Probability

WITPO Questions and Answers—*Continued*

Q: What is the probability of: The sum of two dice equaling 2, 3, or 12? A: 4/36 = 1/9.	Q: What is the probability of: The sum of two dice being 13? A: 0/36 = 0.
Q: What is the probability of: Getting a head on the first toss and a tail on the second toss when tossing a coin twice? A: 1/2 × 1/2 = 1/4.	Q: What is the probability of: Rolling a seven with a die in one roll? A: 0/6 = 0.
Q: What is the probability of: The sum of two dice *not* equaling 7? A: 30/36 = 5/6.	Q: What is the probability of: Drawing any pair from a standard deck of cards in two draws without replacement? A: 52/52 × 3/51 = 3/51 = 1/17.
Q: What is the probability of: Drawing a spade from a standard deck of cards? A: 13/52 = 1/4.	Q: What is the probability of: Drawing a "7" or an "8" from a standard deck of cards in one draw? A: 8/52 = 2/13.
Q: What is the probability of: Drawing 4 hearts in 4 draws from a standard deck of cards without replacement? A: 13/52 × 12/51 × 11/50 × 10/49 = 17 160/6 497 400 = 143/54 145.	Q: What is the probability of: Pulling a red marble out of a bag containing 16 red marbles, 10 green marbles, 14 blue marbles, and 10 yellow marbles? A: 16/50 = 8/25.
Q: What is the probability of: Rolling a die and its not coming up a "1"? A: 5/6	Q: What is the probability of: The sum of two dice equaling 7? A: 6/16 = 3/8.

From the *Mathematics Teacher*, April 1991

Week 3, Section 4: Counting and Probability

WITPO Questions and Answers—*Continued*

Q: What is the probability of: Drawing the ace of spades from a standard deck of cards in one draw?
A: 1/52

Q: What is the probability of: Rolling a "6" in one roll with a die?
A: 1/6

Q: What is the probability of: The sum of two dice equaling 2?
A: 1/36

Q: What is the probability of: Drawing a red card (hearts or diamonds) from a standard deck of cards?
A: 26/52 = 1/2.

Q: What is the probability of: A family with three children having all girls? Assume that $P(\text{girl}) = P(\text{boy})$.
A: $1/2 \times 1/2 \times 1/2 = 1/8$.

Q: What is the probability of: Tossing a coin five times and getting five tails?
A: $1/2 \times 1/2 \times 1/2 \times 1/2 \times 1/2 = 1/32$.

Q: What is the probability of: Rolling an odd number with a die in one roll?
A: 3/6 = 1/2.

Q: What is the probability of: Tossing a coin twice and getting one head and one tail in any order?
A: 2/4 = 1/2.

