# Master Document

## Week 3, Day 3 — Games and Strategies

# LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

## Instructor's Notes

## Revised May, 2006 and July, 2009

## Week 3, Day 3 — Games and Strategies

| **Materials Needed** | **Allocated Time** |
|---|---|
| Activity #1 — Review of Homework Problems | 30 minutes |
| • ten markers (e.g., pennies) to place on the overhead projector | |
| Activity #2 — Tree Games | 20 minutes |
| • no materials needed | |
| Activity #3 — Strategies for Three-Pile Nim | 35 minutes |
| • linker cubes (4 sets of 14 per table, if available) | |
| Activity #4 — Solving Three-Pile Nim | 20 minutes |
| • no materials needed | |
| Activity #5 — Ducks and Dinosaurs | 20 minutes |
| • no materials needed | |
| TOTAL WORKSHOP TIME: | 125* minutes |

* In addition, 10 minutes are allocated for a break in this 2 ¼ hour workshop.

**Activity #1. Review of Homework Problems on Games and Strategies.**
**(Allocated time = 25 minutes for part A and 5 minutes for part B.)**

**Introduction: To help set the stage for Activity #4, ask a participant (or two) to demonstrate how they teach or review writing numbers in base 2. As part of the discussion, be sure that everyone has written the numbers 5, 6, 7, 9 as well as the numbers you plan to use in Activity #4 in base 2.**

**A. Since participants often have more difficulty with games and strategies than other concepts, this session begins with a detailed review of the homework problems. The following points constitute a structured review of a number of the homework problems. Into the discussion of these points, the instructor should incorporate responses to questions raised at the homework review session, comments on the similarities and differences between the various games, and a review of TSP #1 (yesterday's TSP #17, which introduces the notions of winning position, winning move, and winning strategy).**

**1. As an introductory activity, have the participants play a game where they take turns placing circular plates on a table-top, or pennies (or 2x2 squares made from linker cubes) on the overhead projector. The first player who can't make a move without overlapping loses. Participants will recognize the symmetry and first suggest that Player 2 should mimic each of Player 1's moves, placing a plate diagonally opposite to Player 1's move. You might call this Player 2's winning strategy. But then you might ask if there is any move that Player 2 can't mimic, and someone will notice that Player 1 can first place a plate in the center of the table, and thereafter mimic Player 2's moves — in other words, Player 1 can "steal" what we called Player 2's winning strategy.**

**2. Did "stealing the winning strategy" arise in two-pile Nim? Elicit the response that if the two piles are equal, Player 2 has a winning strategy, but that if the piles are unequal, Player 1 can make them equal and thereafter follow (i.e., steal) Player 2's strategy.**

**3. Turning to the Kayll's game with seven markers, although the situation is symmetric, there is a marker at the center — so that Player 1 can create two equal piles by taking the center marker, and thereafter mimic each of Player 2's moves — another example where Player 1 steals Player 2's strategy.**

**Ask whether this reasoning would work if the number of markers were some number other than seven. Expect them to recognize that Player 1 has a winning strategy whenever the number of markers is odd. Place ten markers on the overhead projector, and ask for their suggestions about this game. Many might conclude that Player 2 wins this game, but some should realize that there is still a center and suggest that Player 1 take two markers from the center. Thus no matter how long the row of markers in Kayll's game, Player 1 has a winning strategy.**

**4. Ask them to review the Daisy Chain game in their groups. Several should realize that after Player 1 chooses 1 or 2 markers, Player 2 should choose 2 or 1 markers on the opposite side. Elicit the response that after Player 1's first move, we have Kayll's game, so that Player 2 has the winning strategy.**

**5. A very useful technique for analyzing games is to look at all game positions as "winning positions" or "losing positions". A player cannot move from one winning position to another winning position — thus if your opponent has achieved a winning position, you have to end up with a losing position. Thus, for example, the winning strategy in Two-Pile Nim is easily derived inductively by considering that 0-0 is a winning position, so 0-anything else cannot be a winning position, since you can get to 0-0 from such a position. This leaves 1-1 as the next "smallest" position to consider, and it is a winning position, so 1-anything else is a losing position. This leaves 2-2 as the next "smallest position to consider, and it is a winning position, so 2-anything is a losing position. Etc.**

**6. Note that it often takes a while to catch on to the idea of a particular game, and that that typically involves first playing the game a number of times. Note that the game operates on two levels — the play of the game, and the reflection about the game. Games are often praised as teaching children problem-solving skills, because children love to play games. But the real problem-solving comes not from the play of the game, but from the reflection about the game. So you can help your students become better problem solvers if the perspective you convey to them is "play the game, reflect on the game, play the game some more, reflect some more, etc." This is the perspective of the article in the Resource Book on the game called "Chomp", where the teacher describes his students' explorations of this game which got more complicated as the size of the initial board increased. Also in the Resource Book is the game of**

**“Sprouts”.**

**7. Someone might ask what happens in Joe’s Game (yesterday’s TSP #14) if the path ends at an unmatched vertex. Respond that in that case, Player 2 did not start with a maximum matching and that this situation is discussed in the final pages of the Resource Book for Day 2.**

**B. Optional Activities. If participants had few problems with yesterday’s games, and there is time left over, you might try one of the following.**

**1. Dominoes on Grids. This is a variation of the activity in A.1 above. Player 1 and Player 2 alternate placing dominoes on a 4x4 grid. Dominoes cannot overlap. The person placing the last domino wins. Which player has a winning strategy? The 5x5 grid is more difficult.**

**2. Dots and Boxes. You can mention that “dots and boxes” was introduced as problem #12 on yesterday’s homework’s extension problems. In the 1x2 “dots and boxes” game, if Player 1 starts with the middle edge, then a tie results, but if Player 1 starts with any other edge, then Player 2 wins both boxes. (This can be demonstrated using TSP #<u>13</u>.) Thus Player 1’s best strategy leads to a tie. With larger versions of “dots and boxes”, as with most other games, the games become complicated very quickly. For example, it is already difficult to develop a strategy for “2x2 dots and boxes”. It is known that Player 1 wins in “2x2 dots and boxes” and that Player 2 wins in “3x3 dots and boxes”, but the strategies are quite complex. (The two-volume set “Winning Ways” has an extensive treatment of “dots and boxes”.)**

**Activity #2. Tree games.**
**(Allocated time = 20 minutes)**

**A. Any tree can be considered as a game. Draw the tree so that the root (at the top) is labeled “start”, and explain that the players take turns moving down the tree, each player selecting a vertex which is below the previous vertex selected. The player who selects the final vertex is the winner of the game. Play the game on TSP #<u>2</u> until they agree that Player 1 has the winning strategy. Add a vertex at the bottom of one of the leafs on the right, and elicit the conclusion that now Player 2 has the winning strategy. Develop a tree game which**

**represents 2-1 Nim (the tree on TSP #2) and then show and discuss TSP #3 which has a tree game which represents 2-2 Nim. Note that all games like Nim, where the last player to move wins (or loses) — including chess — are actually tree games. Note also that tree games are not necessarily useful in playing games (as with chess), but do provide another way of thinking about them.**

**Activity #3. Strategies for Three-pile Nim**
**(Allocated time = 35 minutes)**

**A. Remind them that the basic premise of Nim is that you have piles of objects, that each person removes an arbitrary number of objects from one pile, and that the person who removes the last objects wins. (Nim is sometimes called the "Marienbad game" because it was featured in a 1960s movie called "Last Year at Marienbad".) Remind them of the winning strategy in two-pile Nim — to move to a position where the two stacks were the same size, the winning positions. Distribute HO #1 (= TSP #5). Have participants play for a while, and hope that they will recognize that they can make use of the information they gained about two-pile Nim in playing three-pile Nim. It may be necessary to elicit this somewhere in the middle, and to suggest that they should make a list of all possible moves by Player 1 and the appropriate responses by Player 2. In any case, they should be able to conclude that Player 2 has a winning strategy in the 1-2-3 game. Unlike the previous examples, where broad strategies sufficed, here we need to look at all six possible moves by Player 1 and convince ourselves that Player 2 can get a winning position in each case. This can be done at the overhead projector using linker cubes.**

**B. Let's take the game up a level. Distribute HO #2 (= TSP #6) which asks them to determine who wins in the 2-2-3 game, the 1-3-3 game, the 1-2-4 game, the 2-3-4 game, and the 1-4-5 game. Note that for the 1-4-5 game, Player 2 can get to a winning position for each of the 10 first moves of Player 1. (Linker cubes can be used here as well.)**

**TIME FOR A FIVE-MINUTE BREAK!**
**(Allocated time = 5-10 minutes)**

**Activity #4. Solving Three-Pile Nim**
**(Allocated time = 20 minutes)**

**A. Three-pile Nim actually has a general solution, which is described in terms of the binary representation of numbers; this topic is discussed on TSP #7 and TSP #8. Participants should be given the Resource Books so that they can follow and take notes on this discussion. A winning strategy for Nim is described on TSP #9, TSP #9, and TSP #11, where the winning strategy is demonstrated for the 5-7-9 game. After these transparencies are reviewed, pick some large numbers at random, e.g., 17-34-29 (at least one larger than 32), and go through the steps of Player 1's winning strategy with the participants.**

**Activity #5. Ducks and Dinosaurs**
**(Allocated time = 15 minutes)**

**Let's now look at a different game — Ducks and Dinosaurs — on HO #3 (which is TSP #12). Have participants play the game, try to determine the winning positions, and which player has a winning strategy. Tell participants that if their group finds a solution, they should not reveal it; if they don't find a solution, they should play the game some more later. It will be discussed tomorrow. HO #4 can be distributed for the participants to try later on.**

# Terminology

**Winning position.** A position in a game is called a winning position if, no matter what moves the next player makes, the other player can always win the game.

**Winning move.** A move is called a winning move if making it results in a winning position.

**Winning strategy.** A winning strategy is one which results in a victory for the player with the strategy. For example, a winning strategy might involve moving to a winning position, and then continuing to move to winning positions in response to the other player's moves.

**Player-1-winning-game.** A Player-1-winning-game is one where Player 1 has a winning strategy.

**Player-2-winning-game.** A Player-2-winning-game is one where Player 2 has a winning strategy.

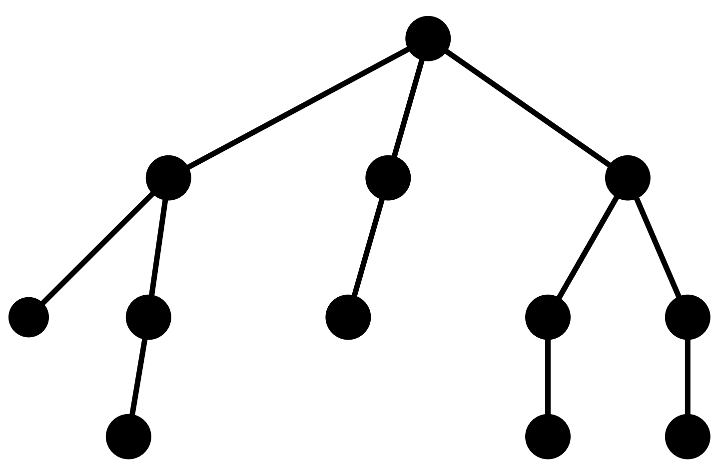
# Tree Game

**The players take turns moving down the tree, beginning at the "Start" vertex, with each player selecting a vertex which is below and connected to the previous vertex selected.**

**The player who selects the final vertex is the winner of the game.**

**Which player has a winning strategy?**

**START**



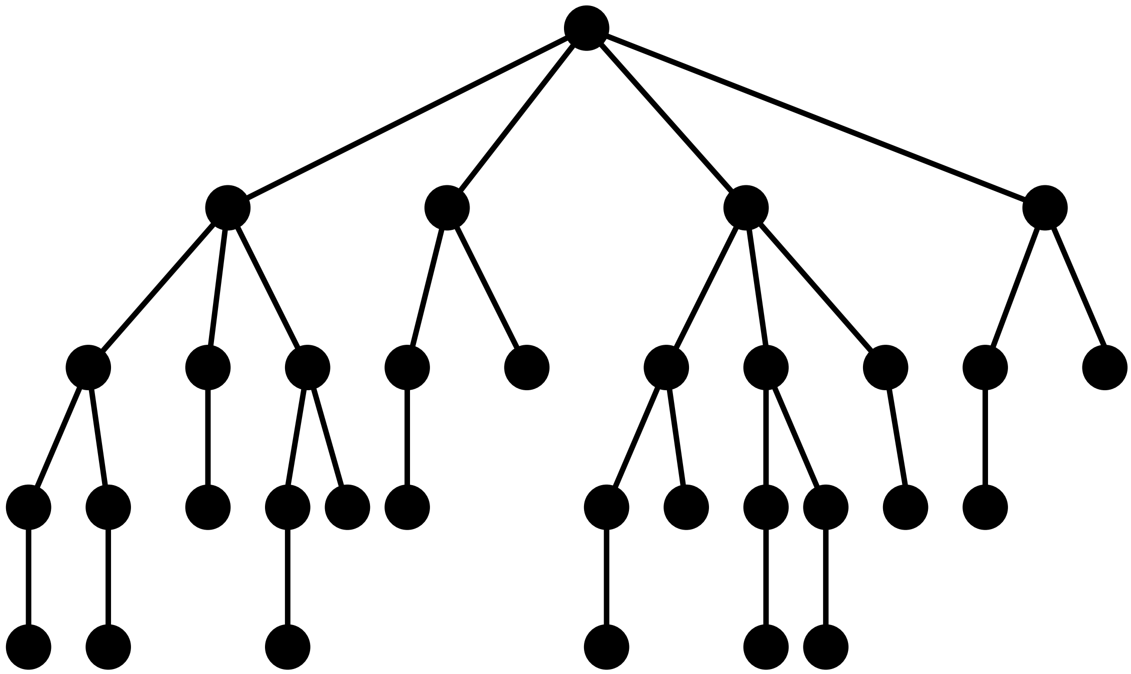
**1st move**

**2nd move**

**3rd move**

# Tree Game for 2-2 NIM

## START



The “start” vertex represents the position 2-2, and the vertices below it represent the positions 2-1, 2-0, 1-2, and 0-2 respectively. Note that the trees below the 2-0 vertex and the 0-2 vertex are essentially the same, as is the case for the 2-1 vertex and the 1-2 vertex; this is because the games starting from those two positions are essentially the same.

You can see from the tree that taking two from one pile is a bad first move for Player 1 since s/he ends up at vertex 2-0 or 0-2, and then Player 2 can win by going to the right to a bottom vertex. (Note that all bottom vertices represent the game 0-0.)

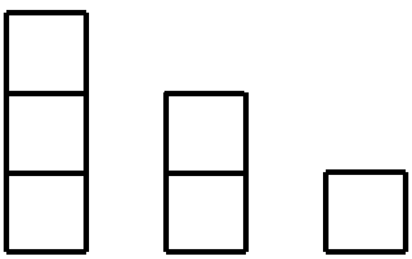
So Player 1 should go to vertex 2-1 or 1-2 (by taking one from one of the piles). Then Player 2 has to be careful to choose the best of the three moves available — taking one from the other pile.

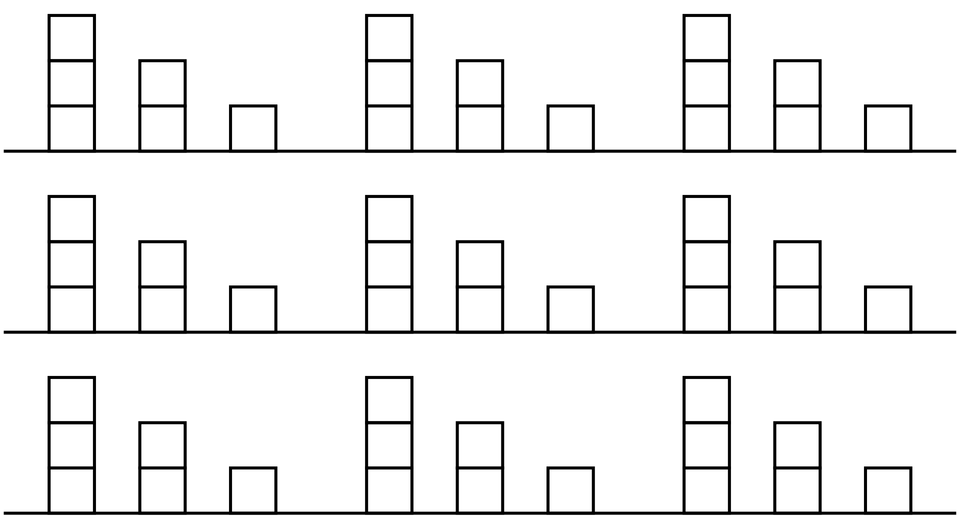
**Hand-out #1.**

**"Three pile Nim"**

Start with three piles — one with three cubes, one with two cubes, and one with one cube. Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last cube wins. Which player has a winning strategy?

(Below is a diagram of the beginning of this NIM game. At the bottom of the page are nine more smaller diagrams on which you can practice playing the game — although you might find it better to use linker cubes.)





**Hand-out #2.**

More on “Three-Pile Nim”.

In “Three-Pile Nim”, there are three piles of cubes, and Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last cube wins. Which player has a winning strategy in each of the following variations of “Three-Pile Nim”?

A. 2-2-3

B. 1-3-3

C. 1-2-4

D. 2-3-4

E. 1-4-5

Can you make any generalizations to other variations of three pile Nim?

# Binary representation of numbers

In the usual decimal representation of numbers, where we use digits from 0 to 9 (one less than ten), the number 4358 is understood to denote

4 thousands, 3 hundreds, 5 tens, and 8 units

or

$4 \times 10^3$, $3 \times 10^2$, $5 \times 10$, and $8 \times 1$

In the binary representation of numbers, where we use digits from 0 to 1 (one less than two), the number 1101 is understood to denote

1 eight, 1 four, 0 two, and 1 unit

or

$1 \text{ x } 2^3$, $1 \text{ x } 2^2$, $0 \text{ x } 2^1$, and $1 \text{ x } 2^0$

In decimal notion, this number would be

$$8 + 4 + 1 \text{ or } 13.$$

## How do we find the binary representation of a given decimal number?

For example, to find the binary representation of 27, we first find the highest power of 2 less than 27 — that is 16.

Subtract 16 from 27 to get 11.

The highest power of 2 less than 11 is 8; the remainder is 3 which is 2+1.

In other words, when we write 27 as a sum of distinct powers of 2, we get

$$27 = 16 + 8 + 2 + 1$$

In binary form, this is

$$(1\times 2^4) + (1\times 2^3) + (0\times 2^2) + (1\times 2^1) + (1\times 2^0)$$

$$= 11011$$

# A Winning Strategy for Nim

In the game of Nim, there are three piles of objects, and the two players take turns removing any number of objects from a single pile. The person who removes the last object wins the game.

A particular version of this game is determined by the number of objects in each pile, so that, for example, 5-7-9 Nim is the version where the three piles have five, seven, and nine objects.

It is possible to determine which player has a winning strategy in any version of Nim by converting the three numbers of objects in the piles into binary numbers.

For example, five, seven, and nine in binary form are

| | | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| 5 | = | | | | |
| 7 | = | | | | |
| 9 | = | | | | |

## A Winning Strategy for Nim (continued)

The basic rule is that if the number of 1's in each column is even, then Player 2 has a winning strategy.

In this case, we see that three columns, the 8's, the 2's, and the 1's column, have an odd number of 1's. If Player 1 can remove objects from one pile so that the binary forms of all three resulting numbers do have an even number of 1's in all columns, then Player 1 will have a winning strategy.

This is always possible. In our example, Player 1 should remove objects from the 9 pile, leaving a 0 in the 8's column, a 0 in the 4's column, a 1 in the 2's column, and a 0 in the 1's column. In other words, the result should be to reduce the third pile

from "1 0 0 1" to "0 0 1 0",

which equals 2; that is, Player 1 should remove 7 objects from the third pile, leaving piles of 5-7-2.

# A Winning Strategy for Nim (continued)

Suppose now that Player 2 removes one object from the 7 pile. Now the piles are 5-6-2. What is the winning move for Player 1?

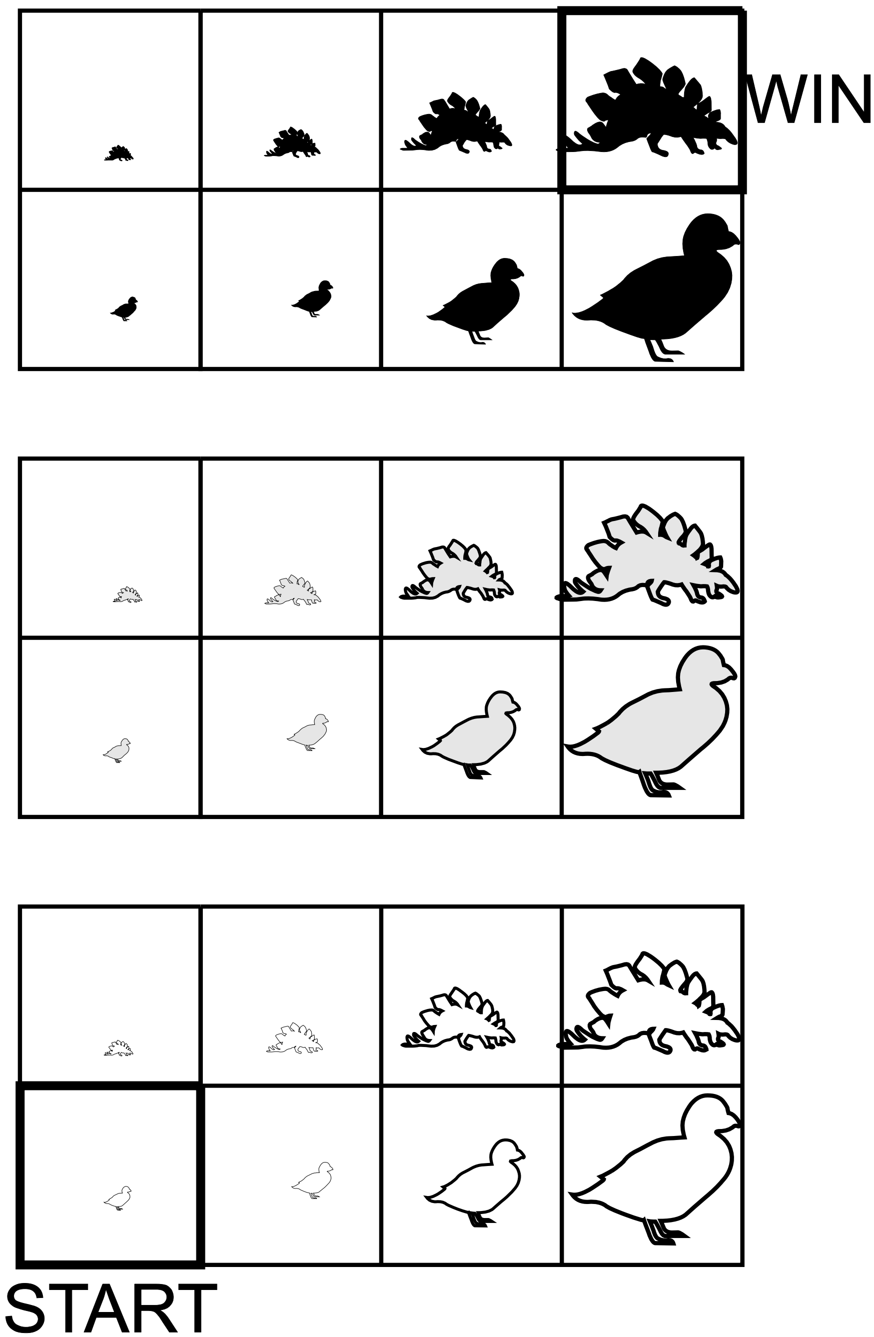
| | | 8 | 4 | 2 | 1 |
|---|---|---|---|---|---|
| 5 | = | | | | |
| 6 | = | | | | |
| 2 | = | | | | |

In that case, Player 1's response would be to remove ________ from the __ pile.

As in two-person Nim, the winning positions are those in which the piles are balanced. With two piles, the winning positions are those where the two piles have the same number of objects; with three piles, the winning positions are those where each 8 is balanced by an 8, each 4 is balanced by a 4, each 2 is balanced by a 2, and each 1 is balanced by a 1.

# Ducks and Dinosaurs

By. Robert Hochberg

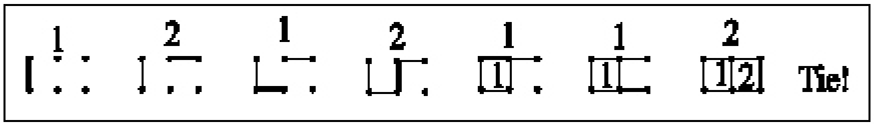



Children place a marker at START and take turns moving it.

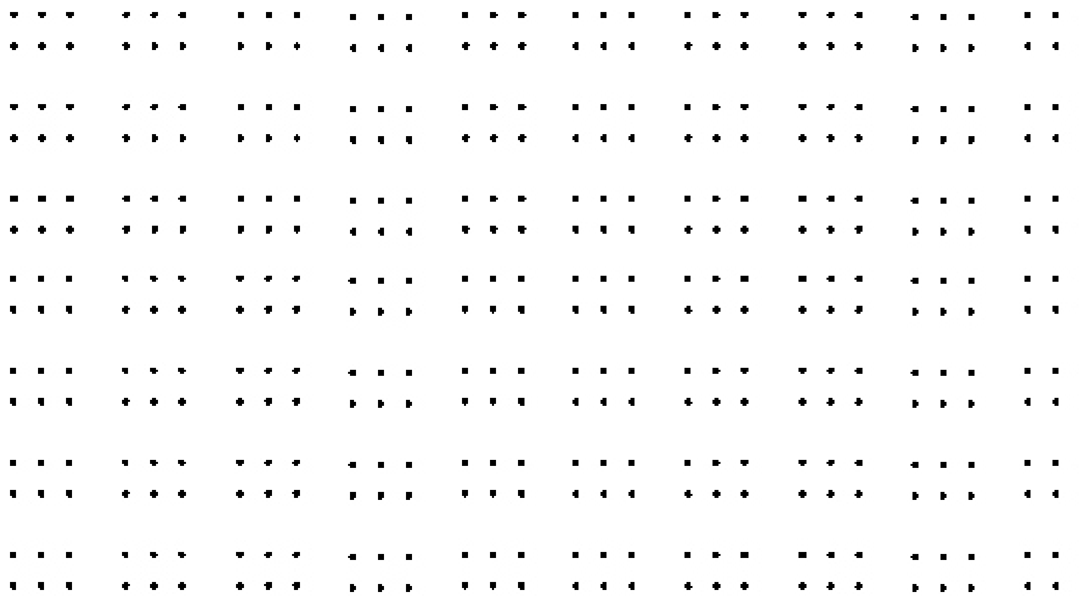
Players may move the marker to a bigger, badder or bolder animal on each move, but may change only one attribute per turn.

The winner is the player who moves the marker to WIN.

## SOME DOTS AND BOXES GRIDS




A sample game



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In this game, players take turns adding a vertical or horizontal edge to connect two adjacent vertices. Here are the rules:

1. Players alternate drawing edges.
2. When a player completes a box he puts his initial in the box, and then must make another move.
3. The game ends when all 7 edges have been drawn and both boxes initialed.

The winner is the player with more initialed boxes..
One box each is, of course, a tie.

## Hand-out #1.

### "Three pile Nim"

Start with three piles — one with three cubes, one with two cubes, and one with one cube. Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last cube wins. Which player has a winning strategy?

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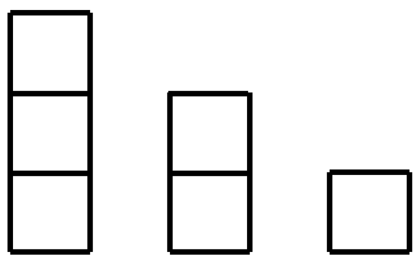
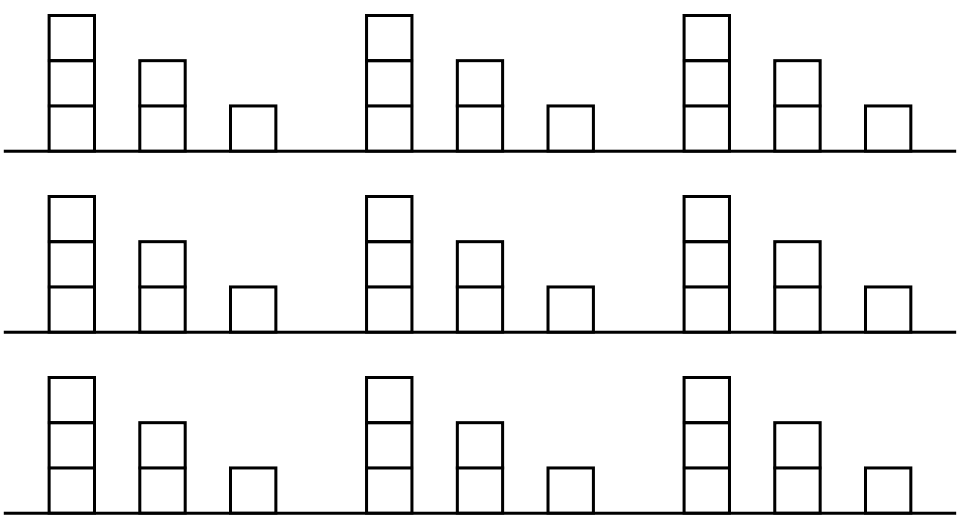




**Hand-out #2.**

More on “Three pile Nim”.

In “Three pile Nim”, there are three piles of cubes, and Player 1 and Player 2 take turns removing any number of cubes from a single pile. Whoever removes the last cube wins.

Which player has a winning strategy in each of the following variations of “Three pile Nim”?

A. 2-2-3

B. 1-3-3

C. 1-2-4

D. 2-3-4

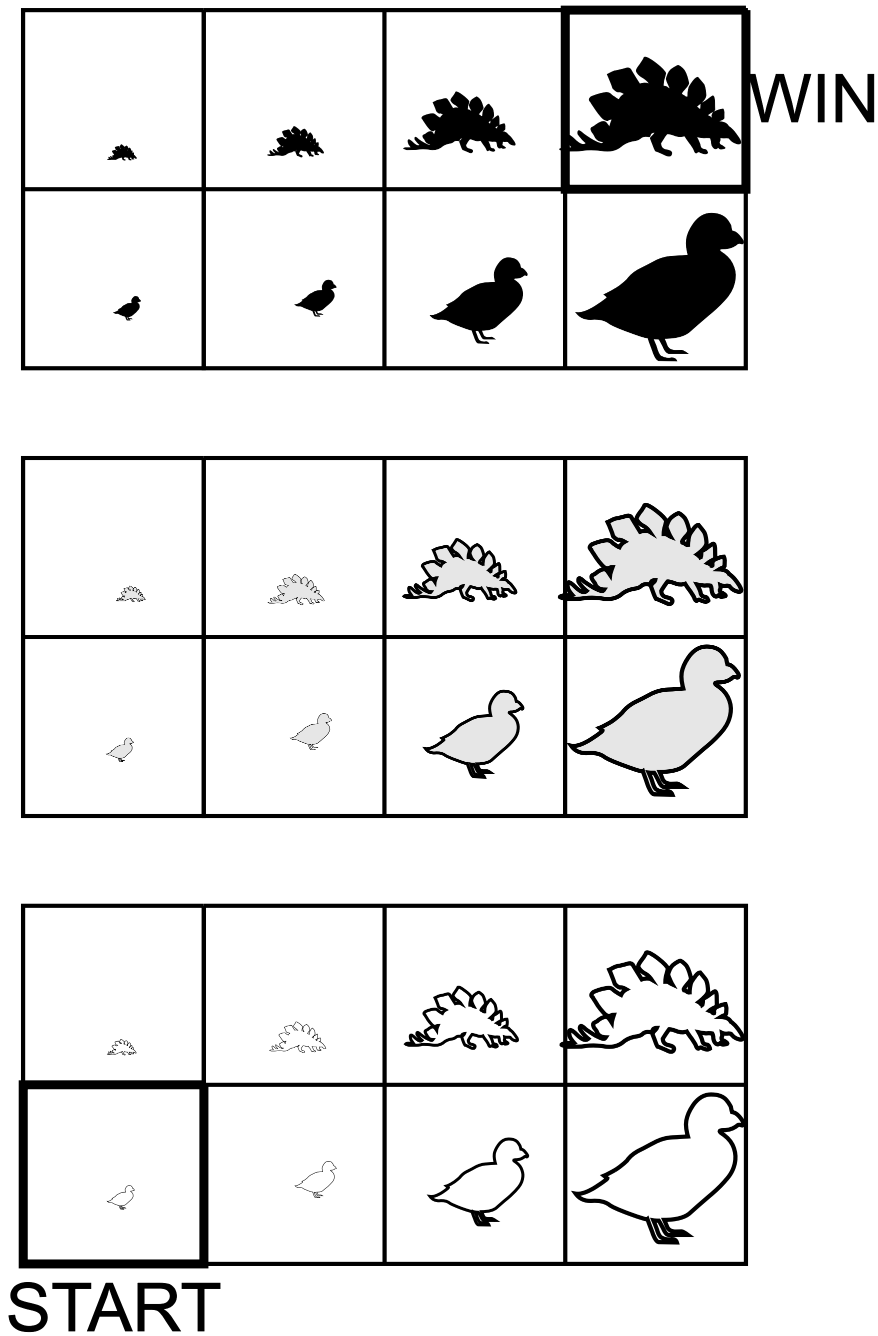
E. 1-4-5

Can you make any generalizations to other variations of three pile Nim?

**Hand-out #3.**

# Ducks and Dinosaurs

By. Robert Hochberg



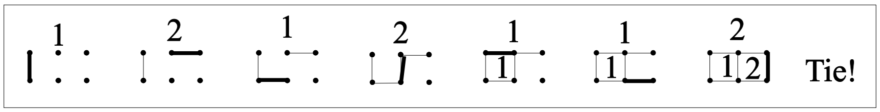

Children place a marker at START and take turns moving it.

Players may move the marker to a bigger, badder or bolder animal on each move, but may change only one attribute per turn.

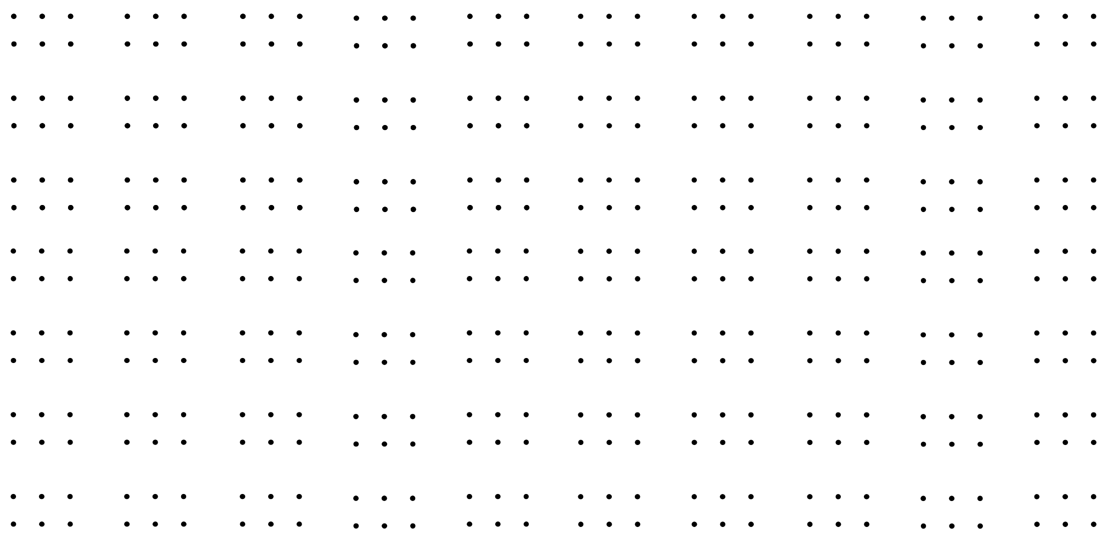
The winner is the player who moves the marker to WIN.

# Hand-out #4.

## SOME DOTS AND BOXES GRIDS




A sample game



In this game, players take turns adding a vertical or horizontal edge to connect two adjacent vertices. Here are the rules:

1. Players alternate drawing edges.
2. When a player completes a box he puts his initial in the box, and then must must make another move.
3. The game ends when all 7 edges have been drawn and both boxes initialed.

The winner is the player with more initialed boxes..

One box each is, of course, a tie.

**Week 3, Session 3 — Games and Strategies — Exercises**

**Practice Problem**

1. Determine which player has a winning strategy in the 2-3-5 Nim game, where there are three piles with two, three, and five cubes, each player in turn removes any number of objects from one pile, and whoever removes the last object wins.

**Study Group Problems**

2. Determine which player has a winning strategy in each of the Nim games 3-4-5, 2-4-6, and 3-4-6, by describing the winning strategy explicitly; this should be done without converting all the numbers to binary form.

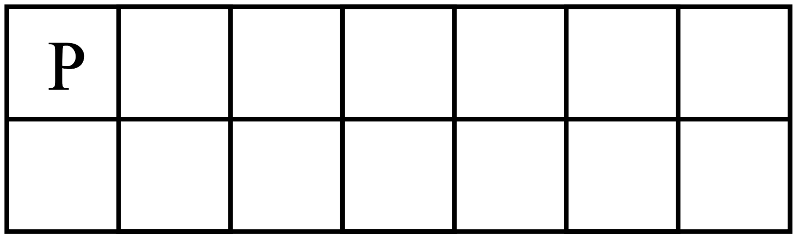
3. Show that Player 2 has a winning strategy in the 3-5-6 Nim game by giving Player 2's response to each possible first move by Player 1; this should be done without converting all the numbers to binary form..

4. Compile a list of all winning positions (games where player 2 wins) in 3-pile Nim with largest pile at most 5. You can simplify your task if you first explain why any position where two of the piles are the same size is not a winning position. What fraction of the total number of Nim positions with largest pile at most 5 do these winning positions comprise?

5. Use binary representations to determine who wins in the 7-10-13 and 8-10-11 Nim games. If Player 1 has the winning strategy, give Player 1's first move. If Player 2 has the winning strategy, give Player 2's correct responses if Player 1 removes one cube from the first pile, or one cube from the second pile, or one cube from the third pile.

6. Who wins the 4-pile Nim game where the piles have the following sizes? What are the winning moves?
   a. 2, 2, 2 and 2?
   b. 2, 2, 5 and 5?
   c. 1, 2, 3 and 4?
   d. 3, 3, 4 and 5?

7. In the following simplified version of "Chomp" (see Resource Book), each player in turn chomps the grid by picking one square and eating it and all squares which are either below it or to its right. For example, if you chomped the center square in the top row of the diagram below, eight squares would be eaten; on the other hand, if you chomped the center square in the bottom row, only four squares would be eaten. The object of the game is to get one's opponent to eat the poison square P. Who has the winning strategy in this game?

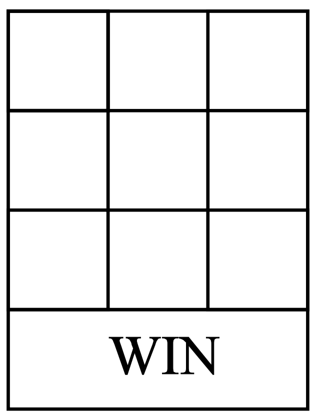



8. Who wins a game of six pile Nim if the piles have sizes 1, 1, 2, 2, 3 and 3?

9. In the game board below, Player 1 and Player 2 alternate moving a marker from one cell to another according to the following rules:

    Player 1 starts by placing the marker in any square in the top row.
    On his or her turn, each player moves the marker one square right, left, or down, but may not move it to the square it just came from.

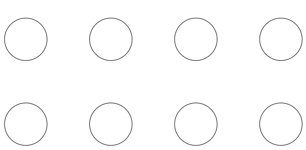
    Whoever moves the marker into the bottom rectangle wins. Which player has a winning strategy?

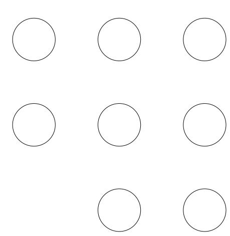


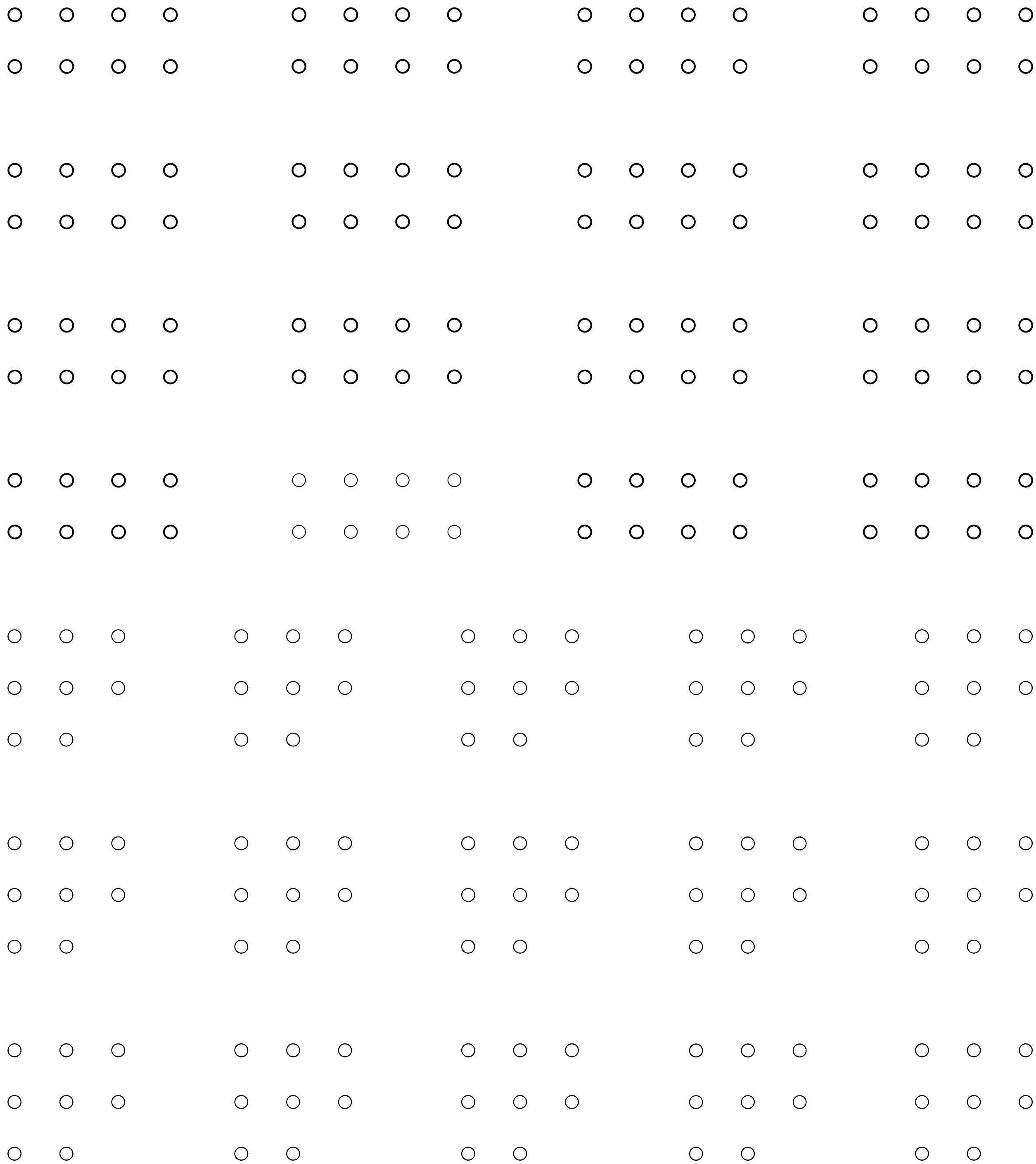

**Extension problems:**

10. Find a winning strategy for dots and boxes played on the game board below (on the left).

11. Find a winning strategy for dots and boxes played on the game board below (on the right).







# Resource Book

## Week 3, Section 3: Games and Strategies

## Binary Representations of Numbers

In the usual decimal representation of numbers, where we use digits from 0 to 9 (one less than ten), the number 4358 is understood to denote

4 thousands, 3 hundreds, 5 tens, and 8 units

or

$4{\times}10^3$, $3{\times}10^2$, $5{\times}10$, and $8{\times}1$

In the binary representation of numbers, where we use digits from 0 to 1 (one less than two), the number 1101 is understood to denote

1 eight, 1 four, 0 two, and 1 unit

or

$2 \text{ x } 2^3$, $1 \text{ x } 2^2$, $0 \text{ x } 2^1$, and $1 \text{ x } 2^0$

In decimal notion, this number is 8 + 4 + 1 or 13.

**How do we find the binary representation of a given decimal number?**

For example, to find the binary representation of 27, we first find the highest power of 2 less than 27 — that is 16. Subtract 16 from 27 to get 11. The highest power of 2 less than 11 is 8; the remainder is 3 which is 2+1. In other words, when we write 27 as a sum of distinct powers of 2, we get

$$27 = 16 + 8 + 2 + 1$$

In binary form, this is

$$(1 \text{ x } 2^4) + (1 \text{ x } 2^3) + (0 \text{ x } 2^2) + (1 \text{ x } 2^1) + (1 \text{ x } 2^0)$$

or

11011

# Resource Book

## Week 3, Section 3: Games and Strategies

## The Game of Nim

In the game of Nim, there are three piles of objects, and the two players take turns removing any number of objects from a single pile. The person who removes the last object wins the game.

A particular version of this game is determined by the number of objects in each pile, so that, for example, 5-7-9 Nim is the version where the three piles have five, seven, and nine objects.

It is possible to determine which player has a winning strategy in any version of Nim by converting the three numbers of objects in the piles into binary numbers. For example, five, seven, and nine in binary form are

| | 8 | 4 | 2 | 1 |
|---|---|---|---|---|
| 5 = | | 1 | 0 | 1 |
| 7 = | | 1 | 1 | 1 |
| 9 = | 1 | 0 | 0 | 1 |

The basic rule is that if the number of 1's in each column is even, then Player 2 has a winning strategy.

In this case, we see that three columns, the 8's, the 2's, and the 1's column, have an odd number of 1's. If Player 1 can remove objects from one pile so that the binary forms of all three resulting numbers have an even number of 1's in all columns, then Player 1 will have a winning strategy.

This is always possible. In our example, Player 1 should remove objects from the 9 pile, leaving a 0 in the 8's column, a 0 in the 4's column, a 1 in the 2's column, and a 0 in the 1's column. In other words, the result should be to reduce the third pile from "1 0 0 1" to "0 0 1 0", which equals 2; that is, Player 1 should remove 7 objects from the third pile, leaving piles of 5-7-2.

Suppose now that Player 2 removes one object from the 7 pile. Now the piles are

| | | | |
|---|---|---|---|
| 5 = | 1 | 0 | 1 |
| 6 = | 1 | 1 | 0 |
| 2 = | 0 | 1 | 0 |

In that case, Player 1's response would be to remove ________ from the ____ pile.

As in two-person Nim, the winning positions are those in which the piles are balanced. With two piles, the winning positions are those where the two piles have the same number of objects; with three piles, the winning positions are those where each 8 is balanced by an 8, each 4 is balanced by a 4, each 2 is balanced by a 2, and each 1 is balanced by a 1.

# Resource Book

## Week 3, Section 3:  Games and Strategies

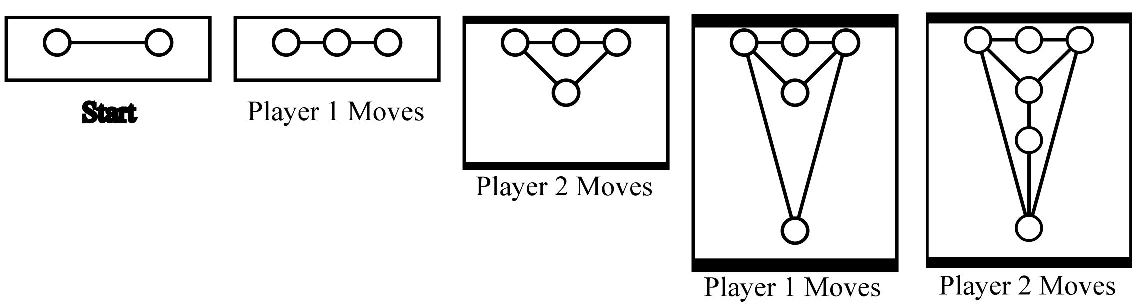
## The Game of Sprouts

Two players take turns building a graph until one of them (the winner) completes the graph.  The rules are:

** start the game with two or three vertices;
** each player adds an edge joining two vertices, and then adds a new vertex at the center of that edge;
** no more than three edges can occur at a vertex; and
** edges may not cross.

Here is a sample game:

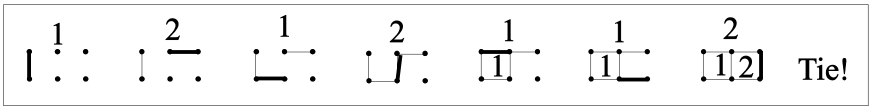
In this sample game, Player 2 wins because no matter how you draw an edge connecting the only two vertices with degree less than three, it would cross an existing edge, so the graph is complete.



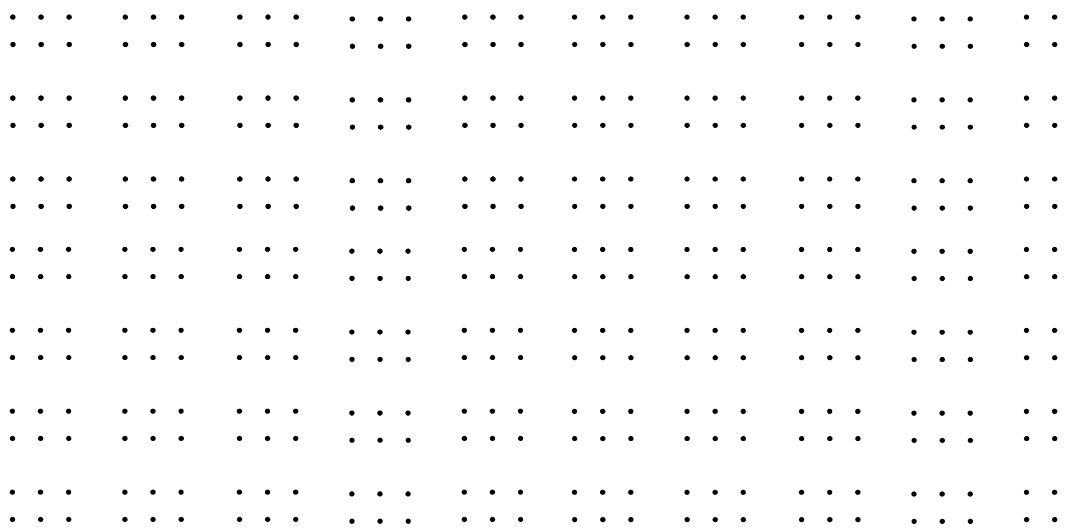
# Resource Book

## Week 3, Section 3:  Games and Strategies

SOME DOTS AND BOXES GRIDS




A sample game



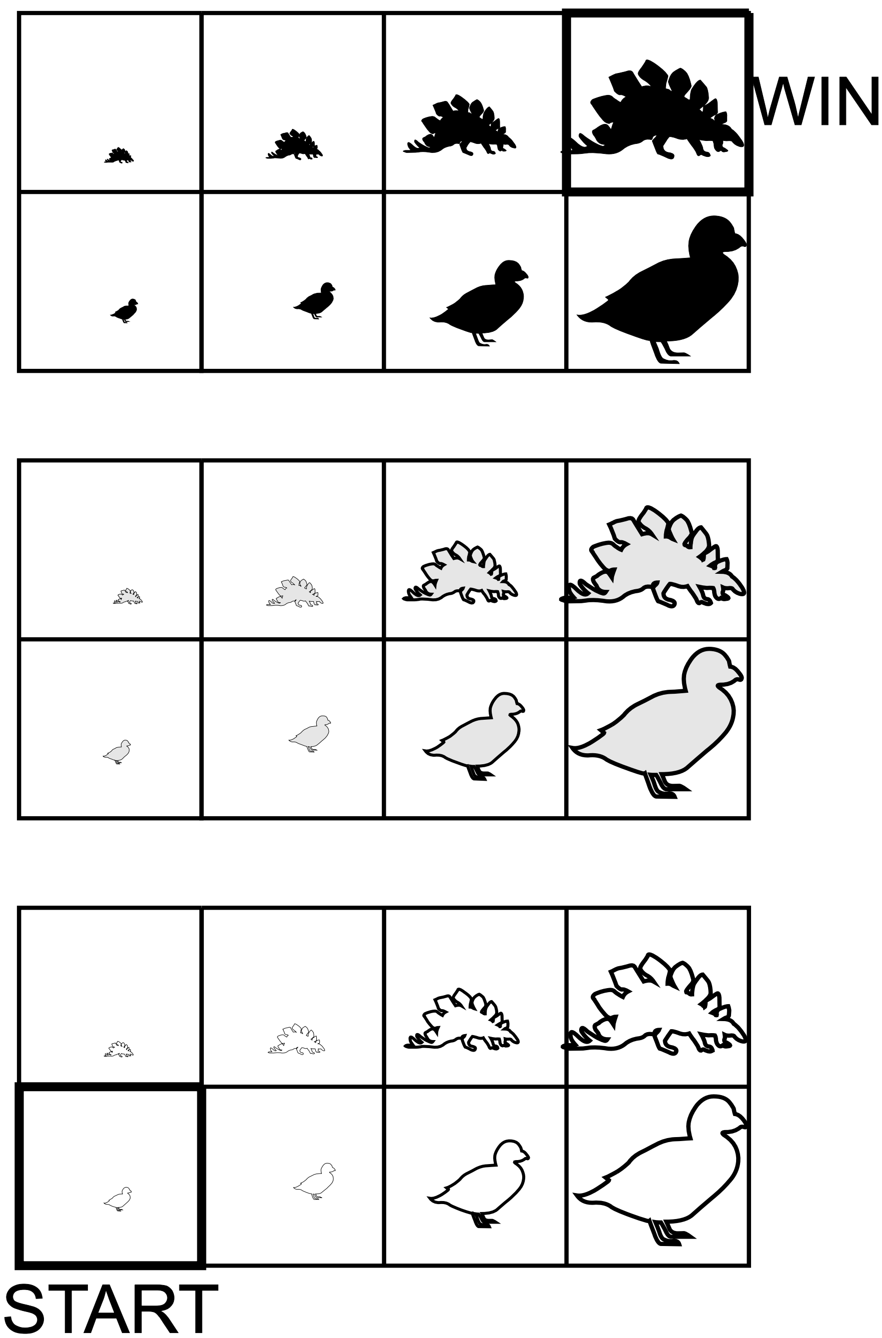
In this game, players take turns adding a vertical or horizontal edge to connect two adjacent vertices.  Here are the rules:

1. Players alternate drawing edges.
2. When a player completes a box he puts his initial in the box, and then must must make another move.
3. The game ends when all 7 edges have been drawn and both boxes initialed.

The winner is the player with more initialed boxes..
One box each is, of course, a tie.

# Ducks and Dinosaurs

By. Robert Hochberg

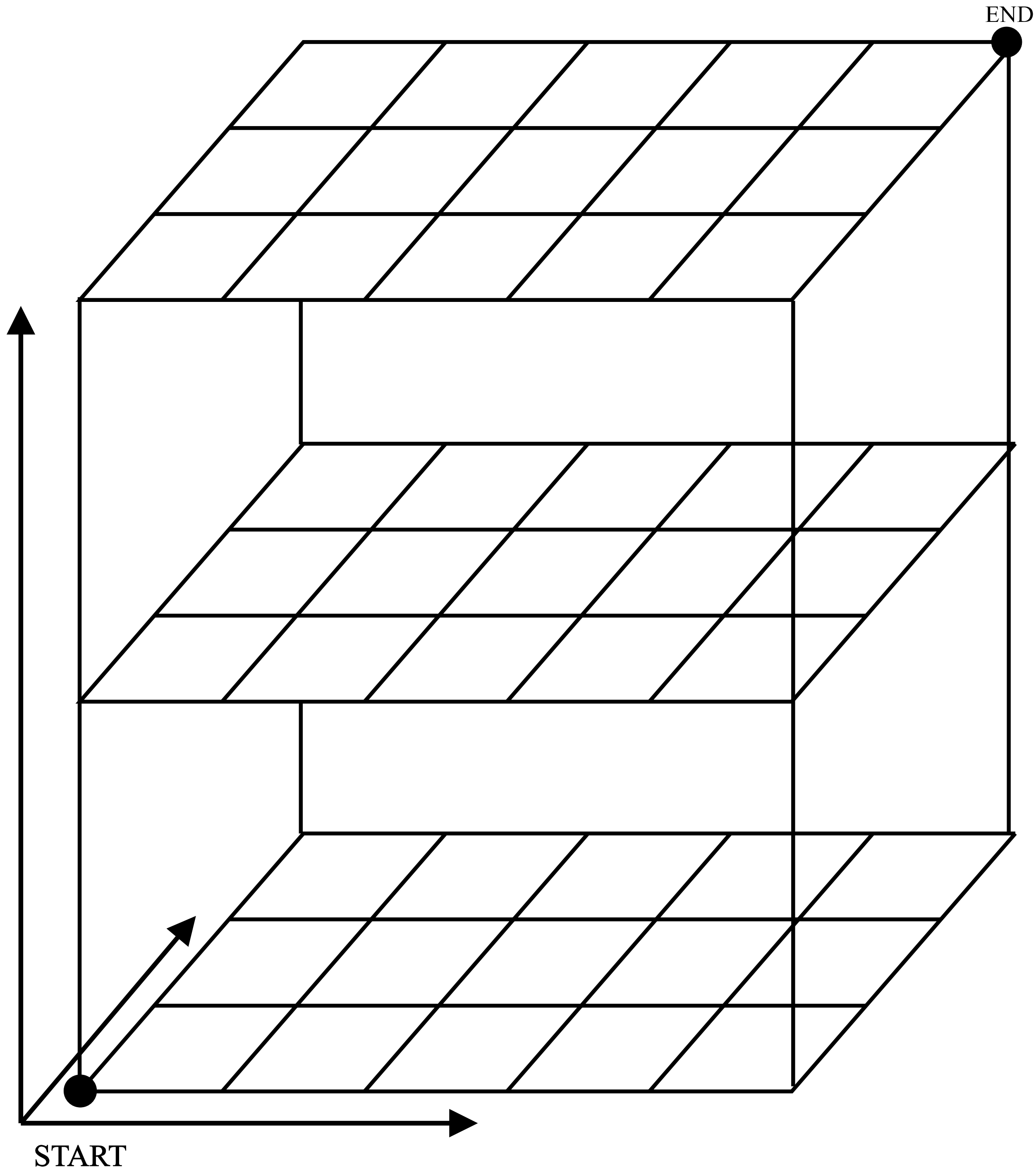



Children place a marker at START and take turns moving it.

Players may move the marker to a bigger, badder or bolder animal on each move, but may change only one attribute per turn.

The winner is the player who moves the marker to WIN.

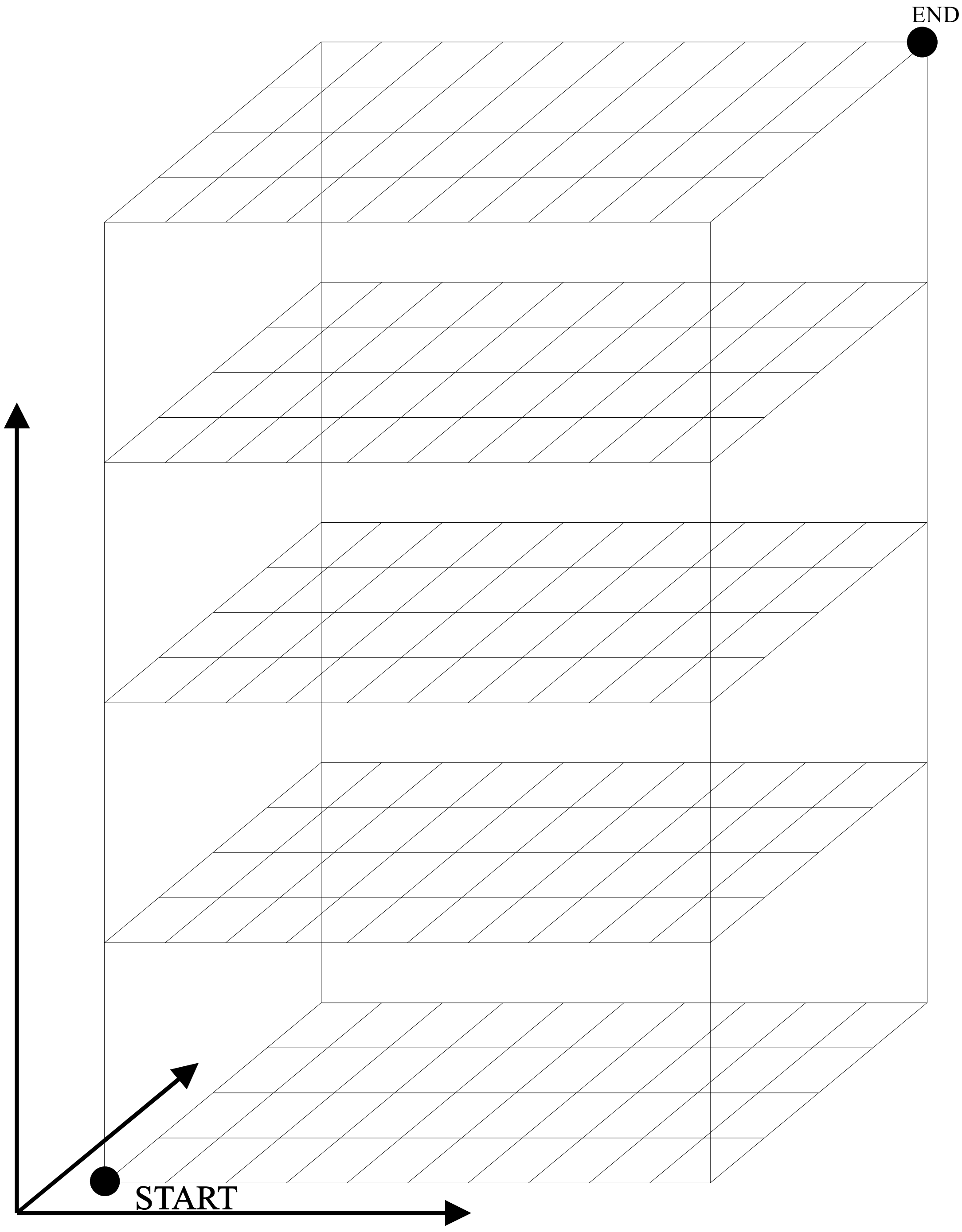
## Week 3, Section 3: Games and Strategies




A marker is placed at "Start". Players alternate taking turns moving the marker in the directions indicated, though the marker may move in just one of the three directions per turn.
The winner is the player who moves the marker to "End." Note that players may move "up" from one grid to another, even though there is no line drawn.

# Resource Book

## Week 3, Section 3: Games and Strategies

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
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## Fair Games

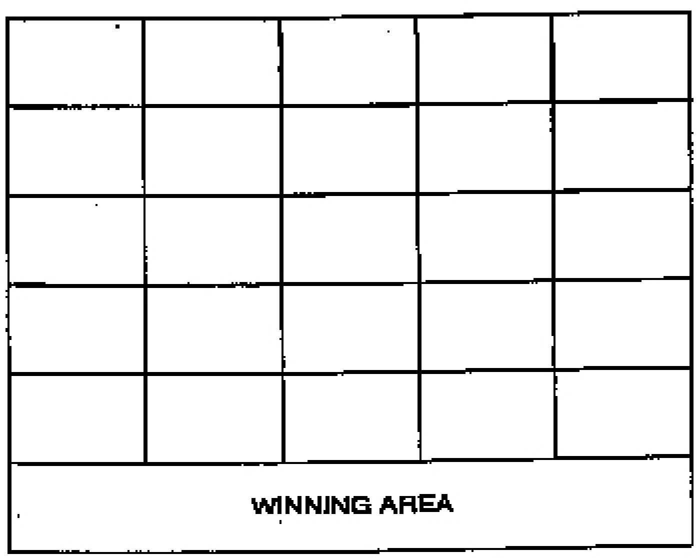
Playing games is a frequent source of entertainment. We enjoy the stimulation, the challenge to fin winning strategies, and the competition. The following activities will allow you to explore three differer games involving two players.

### Game 1

Use a marker and the game board below. The rules for play are as follows:

1. Player A places the marker in any one of the empty cells in the top row.
2. Player B moves the same marker one cell to the right, one cell to the left, or one cell straight down. player is *not* allowed to move up or diagonally.
3. Players alternate turns.
4. A player is *not* allowed to move to the previously occupied cell.
5. The first player who moves the marker into the winning area wins the game.

Play the game several times with another student.




Can you find a winning strategy? Does one appear to gain any advantage by being first or seconc Analyze the game together with another student. Try to find a winning strategy. Find a way so that you ca win every time.

# Chomp—an Introduction to Definitions, Conjectures, and Theorems

By ROBERT J. KEELEY, Denver Christian Middle School, Denver, CO 80210

The concept of mathematics as an organized system of carefully defined ideas and logical extensions is a difficult one for students to grasp. The game of Chomp provides an opportunity to explore this concept by building a system while playing a game and giving students a chance to look at some open questions.

The game requires two players. Begin by making a rectangular array of $x$'s. The upper-left element is considered poison (see fig. 1). Each player in turn takes a bite of this "cookie" with his or her "Chomp monster." The Chomp monster's mouth is shaped so that the bite is always in the form of a right angle at the lower right. A typical bite is also shown in figure 1. The object of the game is to force one's opponent into eating the poison. A sample game is shown in figure 2.

After explaining the game and letting students play it for a while, I noticed that some students began developing a set of strategies that were quite successful. In discussing the strategies, we became aware that some precise definitions and arguments were needed to go along with our ideas. We then started on a set of definitions and conjectures. When a particular strategy always seemed to work for a small group of students, it was presented to the entire class, which gave us the perfect opportunity to discuss the concept of proof. The students were motivated to learn about this concept and had fun looking for new theorems. An important aspect of this activity is that students are involved in stating and clarifying their observations. Your

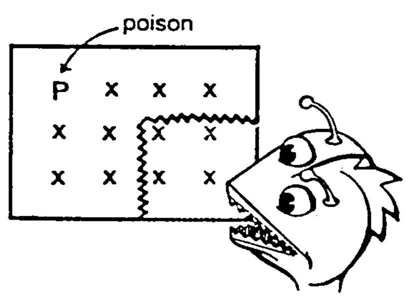



Fig. 1. A 3 × 4 Chomp board showing a typical monster chomp

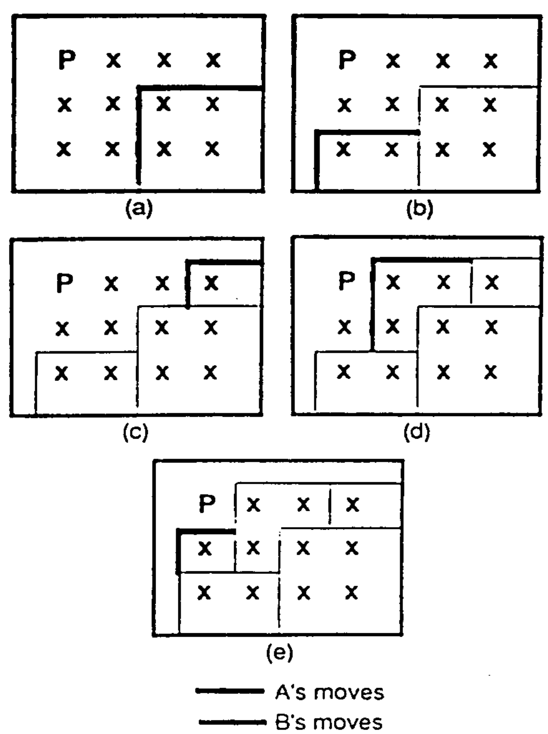



Fig. 2



students may come up with different wordings and other observations.

The following set of theorems, definitions, and corollaries is a compilation of the work we did for about a week during the first fifteen minutes of our eighth-grade mathematics class. Every day we tried to extend our work a little farther, and I encouraged students to come to class with new conjectures. Although many of the ideas were the students', they needed help in formalizing the conjectures and in proving them. All theorems assume that players make the best possible move at all times. The wording represents our final result. The original suggestions needed polishing and discussion and have been rewritten for this article. I found that some students caught on rather quickly to the use of our definitions, whereas others did not use very precise language. My intention was not to teach the students to speak or write mathematics formally but rather to let them see and be a part of developing a system.

DEFINITION. *Each x on a Chomp board is called a cell.*

THEOREM 1. *Eating either cell horizontally or vertically next to the poison cell leads to a loss.*

*Proof.* Let $C$ be an $n \times m$ Chomp board. If player A takes either of the cells next to the poison, exactly one row or column of cells remains. Player B takes all but the poison and wins.

DEFINITION. *A stairstep is a configuration of cells in which exactly two rows or columns remain and the row or column with the poison has one more cell than the row or column without it.* (See fig. 3.) *The length of the stairstep is the number of cells in the longest row or column.*

```
P x x x
x x x
```

Fig. 3. Stairstep of length 4

THEOREM 2. *Making a stairstep leads to a win.*

*Proof.* Let $S$ be a Chomp board on which player A has created a stairstep of length 2. Player B must take either all three of the remaining cells, including the poison, which is a loss by definition, or one of the cells next to the poison, which leads to a loss by theorem 1.

Let $T$ be a Chomp board on which player A has created a stairstep of length $m > 2$. Player B has two possible moves:

1. Take $h < m$ cells from the poison row or column and $h - 1$ cells from the other row or column. Player A then takes one cell from the nonpoison row or column to make another stairstep. This process continues until a stairstep of length 1 or 2 is made and a win is guaranteed by theorem 1.

2. Take $g < m$ cells from the nonpoison row or column. Player A then takes $g$ cells from the poison row or column to make another stairstep.

DEFINITION. *An equal edge is a configuration in which the only cells remaining lie in a poison row and column of the same length.* (See fig. 4.)

```
P x x x
x
x
x
```

Fig. 4. Equal edge of length 4

THEOREM 3. *Making an equal edge guarantees a win.*

*Proof.* Let $C$ be a Chomp board with an equal edge of length $n$. Player B has two possible moves:

1. Take all the remaining cells and lose by definition.

2. Take $c < n$ cells from one row or column. Player A can take $c$ cells from the row or column opposite the ones that player B took. This move creates another equal edge. This process continues until player B takes all the remaining cells or leaves a stairstep of length 2.

DEFINITION. *A square game is a Chomp board of dimension n × n.*

THEOREM 4. *A square game is a win for the first player.*

*Proof.* Let *C* be an *n* × *n* Chomp board. If player A takes all but the poison row and column, he or she makes an equal edge and guarantees a win by theorem 3.

At this point we started looking at some specific game settings, starting with the easiest nontrivial board—a 2 × 3 board.

THEOREM 5. *In a 2 × 3 Chomp game, taking one cell from the nonpoison row guarantees a win.*

*Proof.* Let *C* be a 2 × 3 Chomp board. Player A takes one cell from the nonpoison row to make a stairstep and wins by theorem 2.

COROLLARY 2. *In a 2 × n Chomp board, taking one cell from the nonpoison row guarantees a win.*

*Proof.* See theorem 5.

The next theorem takes the game one step farther—to a 3 × 4 board. The number of possible responses to the winning move increases dramatically, and a new system of notation is necessary to communicate the moves. This system is demonstrated in figure 5. All points from which bites "enter" or "exit" the Chomp board have been marked with a letter. The move shown can thus be called B–E.

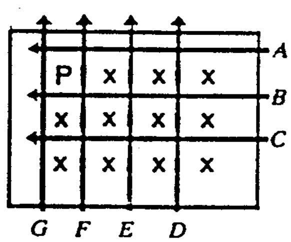



Fig. 5. A 3 × 4 Chomp board marked with a system to describe chomps

THEOREM 6. *In a 3 × 4 Chomp game, taking a 2 × 2 piece on the first move guarantees a win.*

*Proof.* Let *C* be a 3 × 4 Chomp board on which player A has taken a 2 × 2 piece (B–E) on the first move. We shall look at all options that player B has and indicate player A's responses that lead to a win.

(*a*) Player B takes A–D. Player A takes C–G to make a stairstep and win by theorem 2 (fig. 6(a)).

(*b*) Player B takes A–E. Player A takes C–F to make a stairstep and win by theorem 2 (fig. 6(b)).

(*c*) Player B takes A–F. Player A wins by theorem 1 (fig. 6(c)).

(*d*) Player B takes A–G. Player A wins

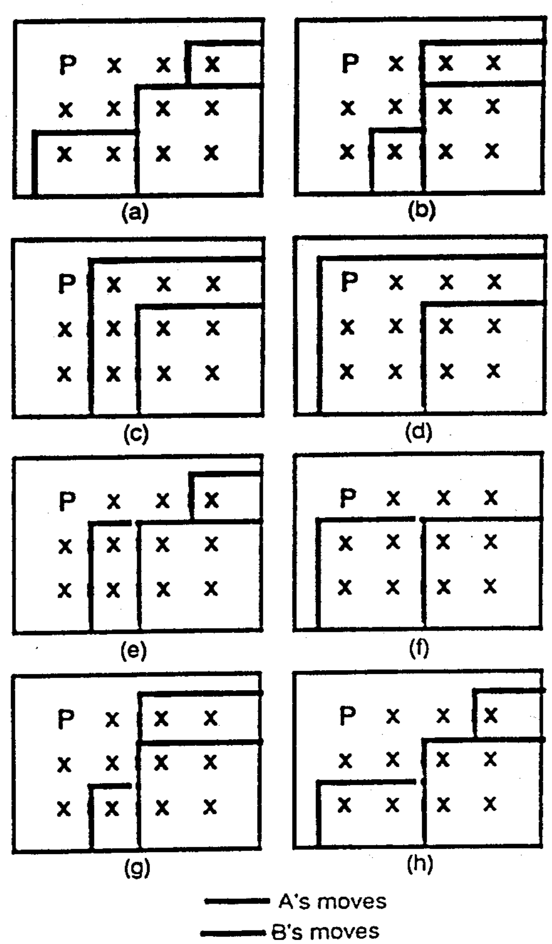



Fig. 6. Theorem 6

by definition (fig. 6(d)).

(*e*) Player B takes B–F. Player A takes A–D to make an equal edge and win by theorem 3 (fig. 6(e)).

(*f*) Player B takes B–G. Player A wins by theorem 1 (fig. 6(f)).

(*g*) Player B takes C–F. Player A takes A–E to make a stairstep and win by theorem 2 (fig. 6(g)).

(*h*) Player B takes C–G. Player A takes A–D to make a stairstep and win by theorem 2 (fig. 6(h)).

We also looked at a number of conjectures that dealt with larger games. We discovered, of course, that as the games got bigger, it was more difficult to come up with a first move that guaranteed a win. We are left with some open questions: Does a winning first move exist for a $4 \times 5$ game? For an $n \times n + 1$ game? Can we use these theorems to program a computer to play Chomp?

These questions led to some lively classes and some good conjectures. Most important, the students were using terms like *conjecture*, *theorem*, and *proof* with some comfort and authority and had experience in creating some original mathematics.