Master Document

Week 3, Day 1 — Paths and Matchings

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Revised May, 2006

Instructor's Notes

Week 3, Day 1 — Paths and Matchings

Materials Needed	**Allocated Time**

Activity #1 — Octagonal Review . . . . 35 minutes

- A large octagonal graph (see next page) constructed on the floor using centers of paper plates and masking tape. Each vertex is connected to five other vertices, excepting only the vertices which are two vertices away. Each exterior edge is about 3' long, so that opposite sides are about 7.2' apart. The four longest edges, connecting diagonally opposite vertices, should be laid last, so that they can be easily removed. (This graph will also be used for the workshop on the following day, so it should not be removed.)
- 24 hanging placards numbered 1 to 8, three for each number.
- 28 cardboard arrows, each 4" wide and 30" long.

Activity #2 — The Art Museum Curator's Problem . . . . 50 minutes

- The octagonal graph, hanging placards, and arrows used for Activity #1.
- Four short (1') pieces of string.

Activity #3 — Vertices of Odd Degree . . . . 10 minutes

- no materials needed

Activity #4 — Matching in Graphs . . . . 30 minutes

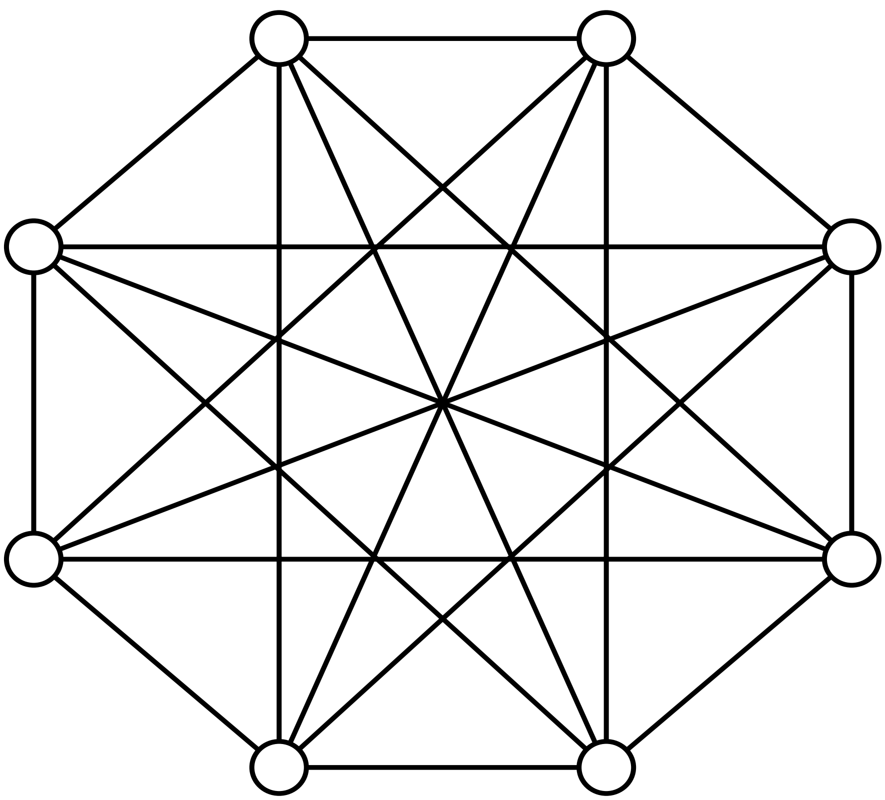
- no materials needed

. . . . TOTAL WORKSHOP TIME: 125* minutes

* In addition, 10 minutes are allocated for a break in this 2 ¼ hour workshop.
* Note that it is important that at least 25 minutes be spent on Activity #4.

Octagonal Graph

The following map should be laid out on the floor — each exterior edge should be about 3' long so that opposite edges are about 7.2' apart.



Activity 1: Octagonal review of Euler circuits, Euler paths, triangular numbers, the handshake problem, and the Chinese postman problem.

(Allocated time = 35 minutes, 20 for parts A/B and 15 for parts C/D)

A. Start with a graph on the floor (constructed using masking tape, with numbered plates — rather, their centers — as vertices) consisting of an octagon in which every vertex is connected to five other vertices (excluding only those which are two vertices away). (Each edge of the octagon is 3', so that opposite sides are about 7.2' apart.) Remind them of the definitions of paths and circuits in graphs by walking along the edges, and then remind them of the definitions of Euler paths and circuits. Ask whether this graph has an Euler path or Euler circuit, and elicit the response that the graph has neither an Euler path nor an Euler circuit, and the reason for that. Discuss Euler's theorem and the justification for it. Delete one of the edges that connects opposite vertices (these edges were laid last so that they could be removed easily); has the situation changed? No. Delete another edge and repeat the question, and then delete a third edge and again repeat the question. Now participants will volunteer that there must be an Euler path, and that it must start and end at the two vertices of odd degree.

It is important to remember that most participants are rusty on most of what was learned last summer — so we are using the octagon graph as an opportunity to review everything as a prelude to moving ahead. It is also very important to convey to participants that we are not expecting them to remember everything that they did last summer — that we will be reviewing key concepts and will be building on that review. For those who forgot all about Euler paths and circuits, this activity has brought the concepts to the fore wonderfully — so that they were able to proceed as if they had remembered them all along.

B. How do we find an Euler path? Let's use a human graph. Have two people stand at each vertex of degree four, one behind the other, and three people at the two odd vertices — with tags indicating their vertices. (The tag of the first person at one odd vertex should also be labeled "start", and the tag of the third person at the other odd vertex should also be labeled "end"). Each person should be holding a cardboard arrow which is 3' long. "Start"points to someone, then that person points to someone along an unused edge, etc., until the last person points to "end" and there are no more vertices left; once a person has pointed to someone, he/she should lay down an arrow pointing to that person along the edge of the graph, and should then join a line that is forming nearby, with each person joining hands with the person who pointed to her/him and then with the person to whom s/he pointed. The result is that they have formed a human path. (Note that it is possible that some vertices will be left over, in

which case backtracking will be needed until the unused edges form a connected subgraph.) Verify by looking at their tags that each of the edges on the graph appears on the path as a pair of joined hands. How many edges are there are on the graph? Try to get the participants to suggest that since there are 18 people (2 at each of six vertices and 3 at each of two vertices) forming a path, there are 17 edges, corresponding to the 17 pairs of hands being held; they might also be able to suggest that since we started with a graph each of whose eight vertices have degree 5, the total of the degrees is 40 which makes for 20 edges, from which we removed 3, leaving 17. (We will use this reasoning again in Activity #3.)

Participants are pleasantly surprised by the outcome. Although on one level they understand that the result of the pointing is a path, they are able to understand the result differently as a result of this activity — by recreating it as a human chain, the path becomes a real path. An important part of the process is the verification that each of the 17 edges on the graph actually appeared on the chain as a pair of joined hands.

C. Holding hands reminds us of the "handshake problem". If there are eight people in a room, and everyone shakes hands with every other person exactly once, how many handshakes have taken place? Elicit several different solutions — answers and explanations — to this problem — for example, eight choose two, 7+6+5+4+3+2+1, and (eight times seven) divided by two. Elicit that the numbers obtained in this manner are the triangular numbers. Use this opportunity to briefly review some of the counting techniques from last summer. Demonstrate the triangular numbers by drawing dots in a triangular pattern on a transparency. Ask the participants how many total edges are in a complete graph with five vertices. Connect this result o finding 5 choose 2 and to the BOOBOO problem YYNNN.

Relate the handshake problem to the complete graph on eight vertices — each handshake corresponds to an edge on the graph — and have eight people stand on the vertices of the octagon imagining that they are on the vertices of a complete graph. Now the question — tell them not to answer until we've actually tried it — Is it possible for all 28 handshakes to take place in sequence, with no handshakes repeated, where the recipient of each handshake initiates the next handshake. (Alternative descriptions: a. In each handshake there is one person involved in the previous handshake and one person involved in the next handshake. b. Each handshake involves someone in the previous handshake, and no one can be in three handshakes in a row.) Choose a first person, and have her/him, and each subsequent person, stride across the graph to shake hands with someone, laying down an arrow as s/he does so, and then return to her/his vertex . After about half a dozen handshakes, stop the activity and elicit that we are

creating a path in the graph, and that in order to complete the 28 handshakes as specified, we have to create an Euler circuit. Why is that impossible here? Because that would be an Euler circuit on the complete graph with eight vertices, where each vertex has odd degree, and an Euler circuit is only possible in a graph which has no vertices of odd degree. But it should be possible for an odd number of people to do this, because in the complete graph on 5 or 7 vertices every vertex has even degree, so there is an Euler circuit. Let's try it with five people in a smaller circle — everyone keep track of whom you've shaken with — pick a person to initiate the first handshake (and be involved in the last one) — have the audience count handshakes — we need to get to 5 choose 2, or 10. Try it again with 7 people, and have the audience count to 21! It may be helpful for the initiator of each handshake to lay down an arrow representing that handshake. (If they discover on-line the idea of going around the outside, then skipping one, then skipping two, ask why this wouldn't work with 9 people — it's not a prime! — although it would work with 11 people.)

D. Have the participants return to their tables and, using TSP #1, TSP #2, and TSP #3, summarize the above activity. Hand out Hand-out #1 so that participants can review and take notes on the summary while you are presenting it.

Activity 2. The Art Museum Curator's Problem.

(Allocated time = 50 minutes, 25 minutes for part A and 25 minutes for parts B-D)

A. Show TSP #4 and discuss the Art Museum Curator's Problem; As curator of a large art museum, it is your job to frequently check along all corridors to make sure that no painting has been disturbed. In order to plan a way to do this, you make a graph whose edges correspond to the corridors of your museum, and you look for an Euler path or circuit. (Show the graph on TSP #5 and explain how it corresponds to the map of the museum.) Unfortunately, graph theorist that you are, you quickly discover that this graph has no Euler path (why?). Undaunted, and confident in your ability to solve problems, you decide on the following: You will decompose the edges of the graph into as few paths as possible, with no overlaps. (The reason for no overlaps is that guards on overlapping rounds generally stop to gab.) Then you will pace one path and hire some assistants to pace the other paths. The question is: how can you discover the smallest number of assistants you will need to hire?

Before working on this problem, let's try some simpler problems. First use TSP #6 to demonstrate the idea of decomposing a graph into trails. The shaded and unshaded edges of the graph form two non-overlapping trails which include each edge exactly once. Demonstrate a second way to decompose this graph into two non-overlapping trails. You may want to prepare some simple models of these trails to help

with this demonstration. Then distribute Hand-out #2 (= TSP #7) and ask participants to try to determine how many trails are needed for each of the four graphs; you might express this by saying that you want to create a bunch of paths that together cover all of the edges. Introduce the notion of a trail as a path which doesn't repeat any edges, and explain why this notion is important here; the distinction between trail and path is difficult for many participants, and will have to be repeated a number of times.

> *While they are working on this activity, put back the three edges that had been removed from the graph on the floor and Eulerize the original octagonal graph by adding four curved edges, using masking tape, outside of the octagon, joining adjacent vertices.*

Review their answers, using different color transparency markers on a blank overlay of TSP #7 . If there is time, ask whether it is possible to arrange it so that all of the trails have equal length — in the second graph it is possible to have two trails each of length six, and in the third graph it is possible to have three trails each of length five, but in the fourth graph the number of edges (22) is not divisible by 4.

[Time for a 5-10 minute break]

B. Ask them if they can tell how many trails are needed without actually drawing all the trails. Elicit that each trail starts and ends at odd degree vertices, and that no odd degree vertex is used for more than one trail, so that the number of trails is half the number of odd vertices. But how do you know that this many trails will always cover all of the edges? Back to the graph on the floor.

C. The octagonal graph we discussed earlier had eight vertices of degree 5, so we had reason to believe that we could decompose the graph into four trails. How can we actually find four trails? Let's remind ourselves of the Chinese Postman Problem. "Suppose your delivery route requires you to walk on every edge of this graph. What is the smallest number of edges you have to repeat?" To solve this problem we Eulerize the graph — we have accomplished that here by adding four edges with tape alternating around the perimeter of the octagon. Note that the graph now has an Euler circuit because all vertices have even degree. Let's find an Euler circuit, using three people at each vertex — note that 24 people are needed for this activity! Repeat the activity carried out earlier, with the arrows used to mark the edges that have been traversed and participants forming a path, except that the four extra edges should be represented by short strings rather than handshakes. At the end, the last person connects to the first person, so that everyone is now arranged into one big circuit. The instructor then shows how removing te strings leaves the four trails that we were trying

to find.

D. After participants return to their seats, review these conclusions on TSP #8, TSP #9, and TSP #10; participants should receive their Resource Books at this point so that they can make notes on the summary. Return to TSP #5 and ask participants to determine the number of paths need here; a problem on the homework will ask them to actually find the paths. Then mention some additional examples (such as fire escape routes, UPS deliveries, paper routes) of situations in which decomposing a graph into trails would be helpful.

Activity #3. The number of vertices of odd degree in a graph.
(Allocated time = 10 minutes)

It is important that at least 25 minutes be available to complete Activity #4, since many of the homework problems deal with matchings. Activity #3 should be shortened (by eliminating the hand-out and doing this as a group activity) or even deleted altogether if the workshop is behind schedule at this point.

A. By the way, what happens if there are an odd number of vertices of odd degree? Have participants do Hand-out #3 (= TSP #11) in which they are asked to add up the degrees of all vertices of several graphs and compare that with the number of edges in the graph. Elicit the response that the total of all the degrees is twice the number of edges since when you add the total of all the degrees you count each edge twice. So the sum of all the degrees of the vertices must be even. Can it happen that there are an odd number of vertices of odd degree? Show TSP #12 to explain why this cannot happen.

Activity #4. Matching in graphs.
(Allocated time = 30 minutes)

A. Make it clear to the participants that you are switching gears here. The topic to be discussed next (matchings) is different from the topic of trails. Put up "The 5 carpoolers problem" on TSP #13 and tell the following story:

The 5 carpoolers. *There were five friends (who owned large cars) who wished to car pool to work during the week. Because of the needs of other members of their families, they could only volunteer their cars on certain days of the week. Listed are the five friends and the days that they can have the car. Can you schedule a car* *pooling assignment where each person drives once a week?*

Ask them to find an assignment of days to drivers, and then, more importantly, ask how they could model this problem with a graph. They should be able to come up, with your guidance, with the idea of creating a graph whose vertices are the days and the drivers, as on TSP #14. Work through the idea of a matching with the participants, putting up TSP #15 to show the terminology. Tomorrow, we will reflect back on this and the next example and introduce the idea of a bipartite graph; although they have not yet seen this concept, we can still draw the graphs as bipartite — with drivers on one side and days on the other.

B. Distribute the "Children and Pets" problem (Hand-out #4 = TSP #16), which lists each child's preferred pets, and asks whether it is possible to assign each child a pet from among those that s/he prefers. The graph associated with this problem (TSP #?), which should be reviewed after participants have completed the hand-out, does indeed have a perfect matching. One thing they should discover by doing this problem is the tactic of first making assignments to pets and children of low degree. Note that in much larger matching problems it can be very difficult to find a best matching. If you have time, mention some other examples of matching problems such as assigning medical students to hospitals for residency.

Terminology

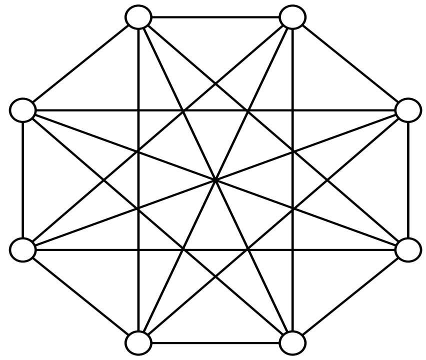
Euler path. An Euler path in a graph is a path which includes every edge of the graph exactly once.

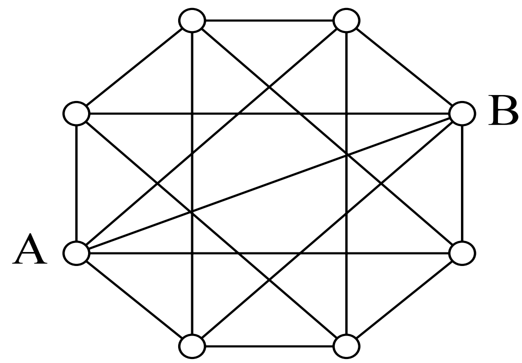
Euler circuit. An Euler circuit in a graph is an Euler path which ends where it begins.

If a graph has no vertices of odd degree, then it has an Euler circuit.

If a graph has two vertices of odd degree, then it has an Euler path, which begins at one of the two vertices of odd degree and ends at the other.

If a graph has more than two vertices of odd degree, then it has neither an Euler circuit nor an Euler path.



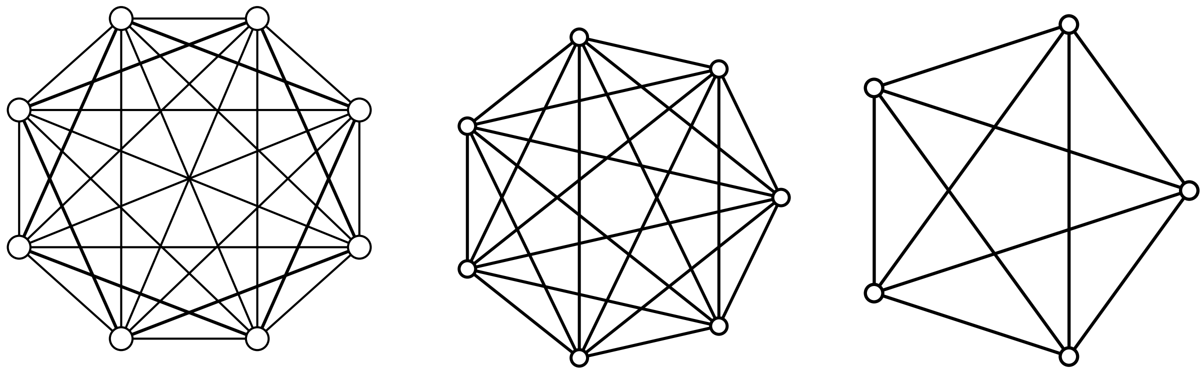


Activity1: There is no Euler path or circuit in the first graph.

Activity 2: There is an Euler path in the second graph, starting with A and ending with B.

If we start with three people standing at A and B and two at each other vertex, this Euler path can be represented as a chain of 18 people holding hands; the 17 pairs of hands corresponds to the 17 edges of the graph.

Activity 3: Handshake Problem (revisited):



If you have 8 people (or any other number), can you have them complete all their handshakes in such a way that in the recipient of each handshake initiates the next one?

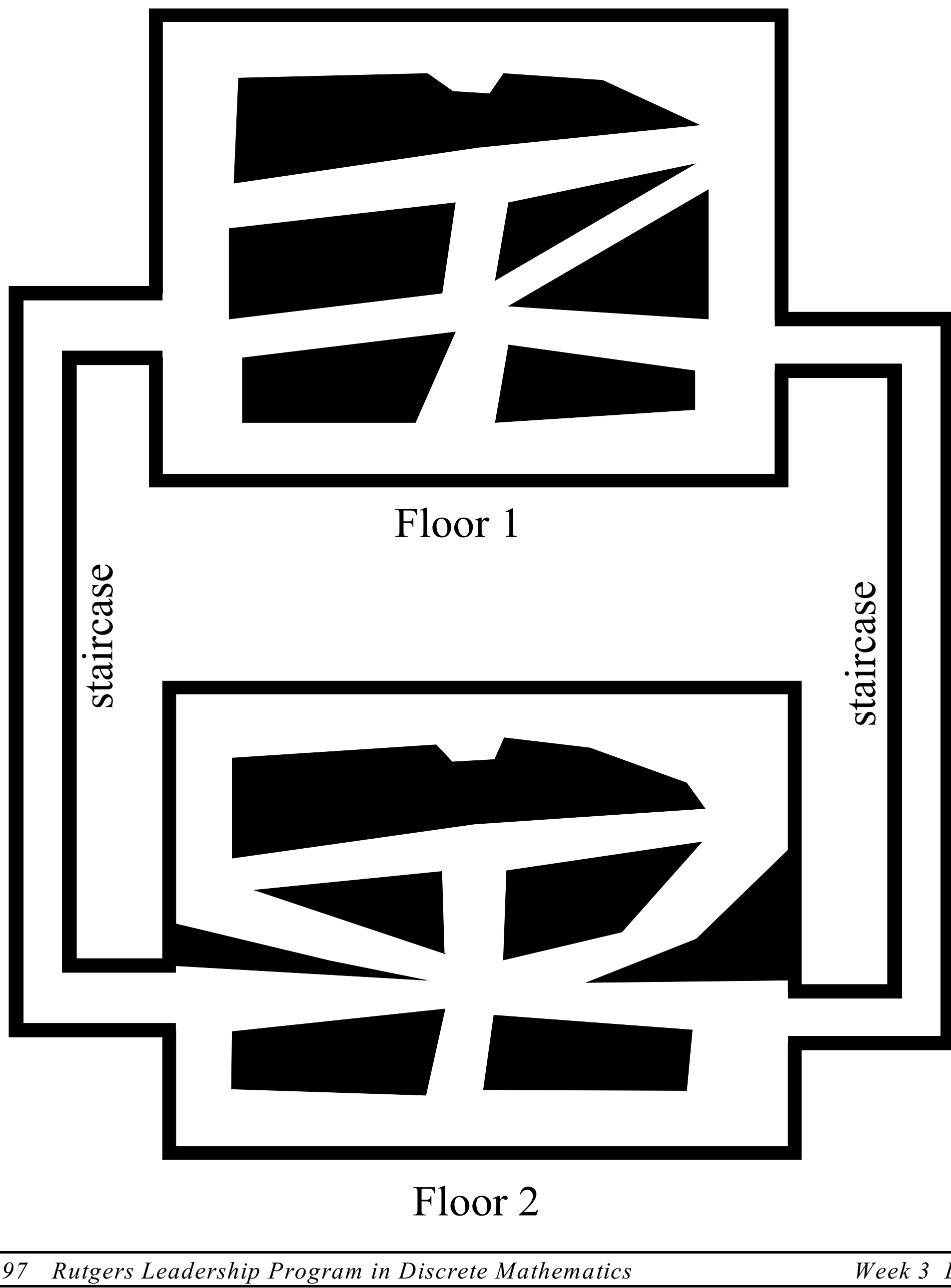
This is the same as asking for an Euler circuit in a complete graph, because each handshake corresponds to walking down one edge of the graph (the previous handshaker striding across the graph to shake hands with the next handshaker).

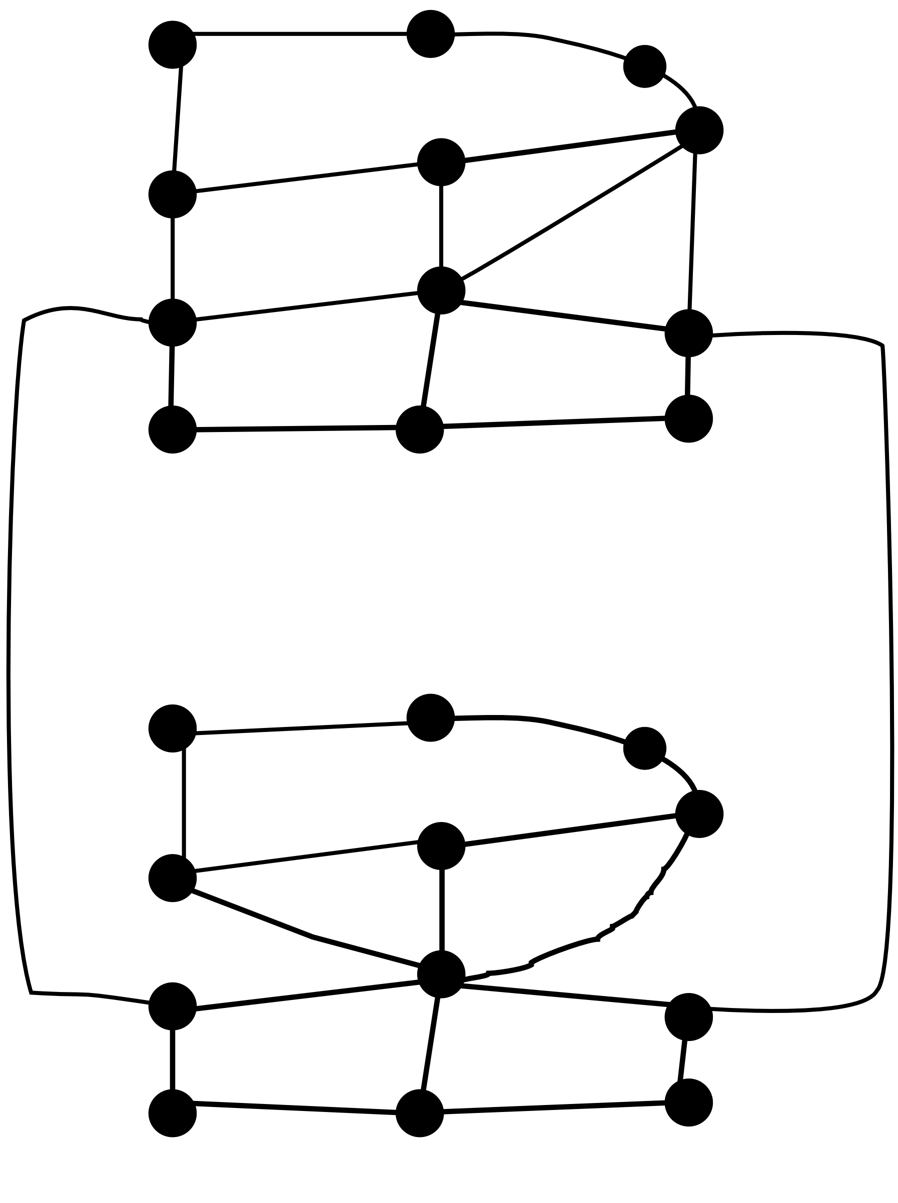
This is impossible if there are an even number of people, because in that complete graph each vertex has an odd number of edges (an odd number of people to shake hands with).

If there are an odd number of people, then it is possible, and we demonstrated it for 5 people and 7 people.

The Art Museum Curator

You want to assign to each museum guard an inspection route so that every corridor in the museum is passed through exactly once by exactly one guard. What is the smallest number of museum guards that you need?

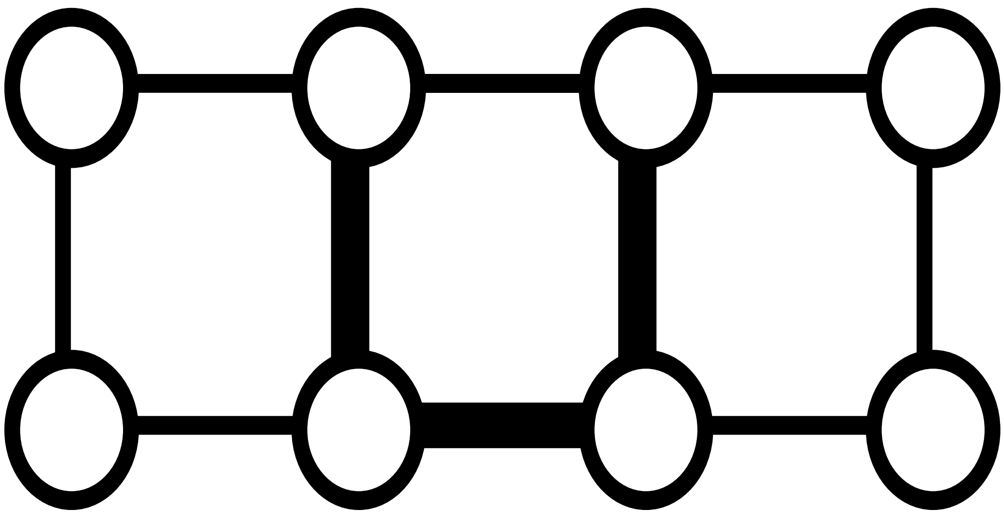




Trails

A trail is a path which has no repeated edges.

In the graph shown below, the shaded and unshaded edges form two non-overlapping trails which cover all of the edges of the graph.



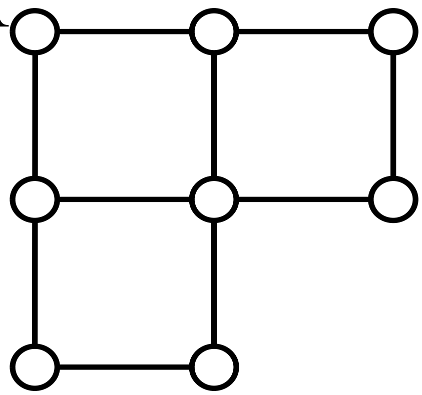
Can you find a different way to cover all of the edges of this graph with two non-overlapping trails?

Hand-out #2

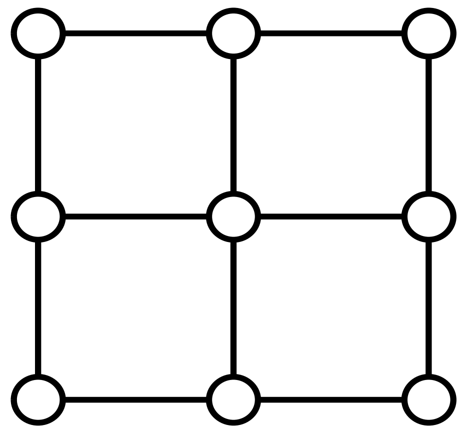
A trail is a path which has no repeated edges. How many non-overlapping trails does it take to cover all the edges of each of these graphs?

How many non-overlapping trails does it take to cover all the edges of each of these graphs?

A



B



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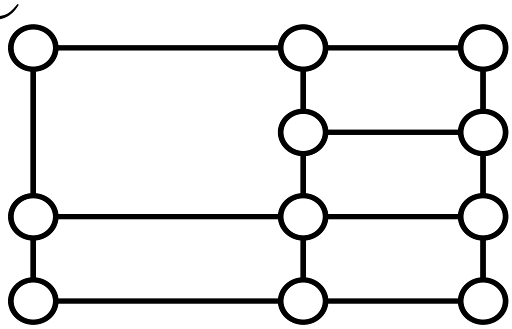

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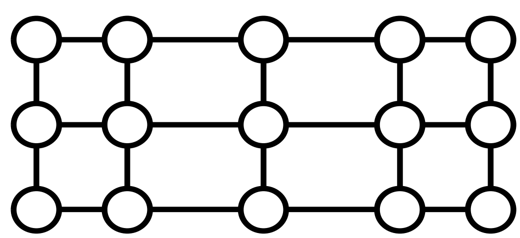
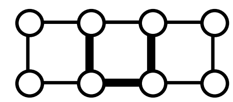


Terminology

Trail. A trail is a path which has no repeated edges.

Using this terminology, an Euler path is a trail which includes all edges of the graph, and an Euler cycle is an Euler path which ends where it begins.

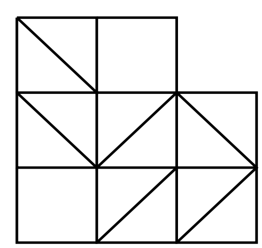
Euler number of a graph. The "Euler number" of a graph is the least number of trails which will include each edge exactly once.



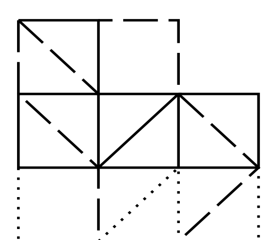
A graph with an Euler path or circuit has Euler number 1. The graph at the right has Euler number 2, since it has no Euler path or circuit, and since the bold and non-bold edges form two trails which include each edge exactly once.

Decomposition of a graph. In a decomposition of a graph, the edges of the graph are divided into components so that each edge is assigned to exactly one component.

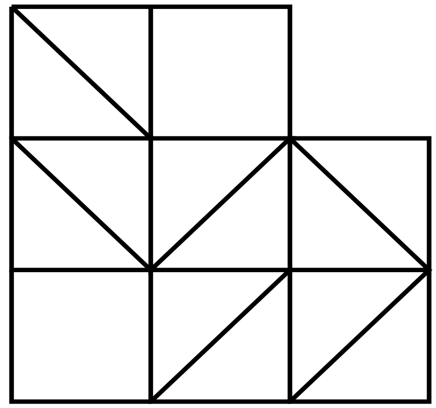
The Euler number of a graph is the smallest number of trails which together form a decomposition of the graph.



For example, the graph at the right can be decomposed into three trails — the solid trail, the dashed trail, and the dotted trail — which don't overlap yet which include all edges on the graph. Can this graph be decomposed into two trails?



Euler Number of a Graph



In a decomposition of a graph into trails, each vertex of odd degree must be an endpoint of at least one trail. Therefore, if a graph is decomposed into trails, the number of trails must be at least half the number of odd vertices.

Since the graph above has six vertices of odd degree, it cannot be decomposed into fewer than three trails, so the Euler number of the graph is 3.

In fact, if a graph has 2n vertices of odd degree, then it can always be decomposed into n trails, so its Euler number is n. That is, the Euler number of a graph is equal to half of the number of vertices of odd degree.

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Here's how you find the *n* trails:

- First, Eulerize the graph (make all degrees even by adding extra edges).

- Second, find an Euler circuit.

- Third, drop the extra edges from the Euler circuit.

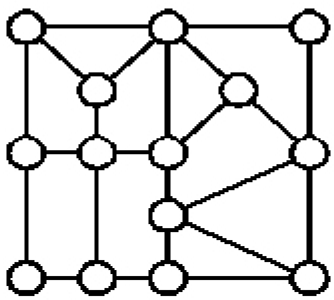
The remaining parts of the path provide a decomposition of the graph into *n* trails.

SUMMARY: The smallest number of trails into which any graph can be decomposed is half its number of odd vertices.

Sums of Degrees

For each of the following graphs, add up the degrees of all its vertices, and compare the total with the number of edges of the graph. Can you explain the pattern?

A



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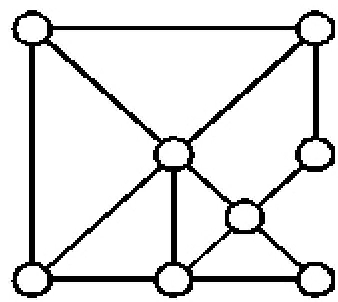

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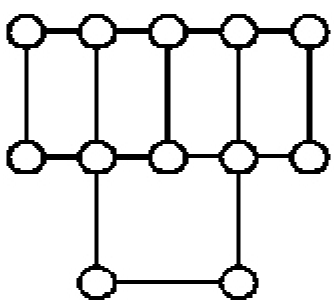

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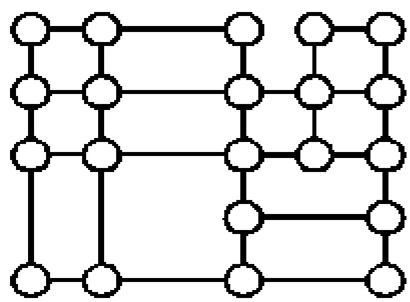


G is a graph with *n* vertices, numbered so that the odd vertices are listed first:

Degree of vertex 1	=	odd	
Degree of vertex 2	=	odd	
Degree of vertex 3	=	odd	add all odd degrees to get A
Degree of vertex 4	=	odd	
. . .	=	. . .	
. . .	=	odd	
. . .	=	even	
. . .	=	even	
. . .	=	. . .	add all even degrees to get B
. . .	=	even	
Degree of vertex *n*	=	even	

Sum of all degrees A + B must be even, since it is twice the number of edges.

Since B is even, A must be even also. But an odd number of vertices of odd degree would make A odd.

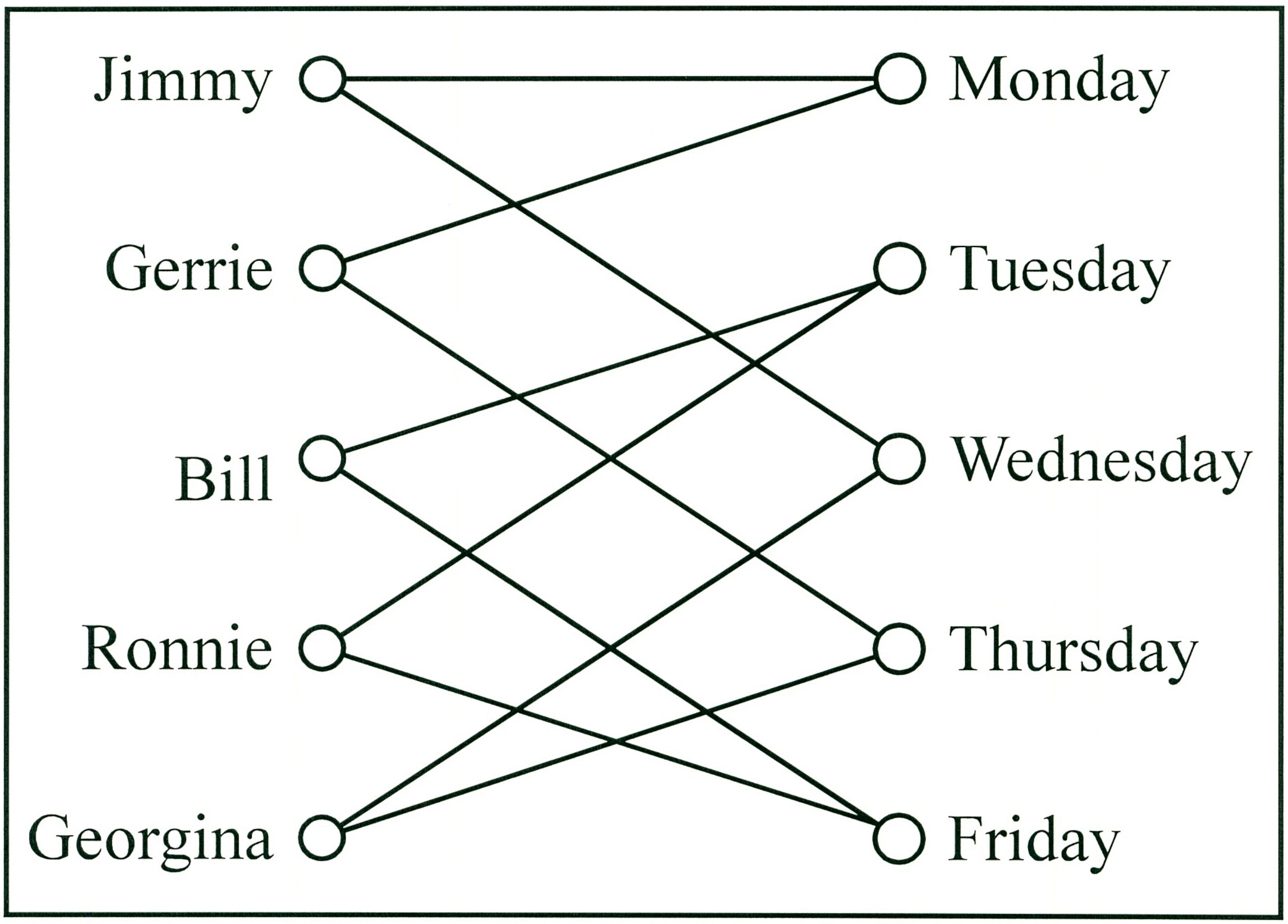
So the number of vertices of odd degree must be even.

Carpoolers

Jimmy	**Monday Wednesday**
Gerrie	**Monday Thursday**
Bill	**Tuesday Friday**
Ronnie	**Tuesday Friday**
Georgina	**Wednesday Thursday**

Can you assign a day for each driver so that no one need drive more than one day per week?

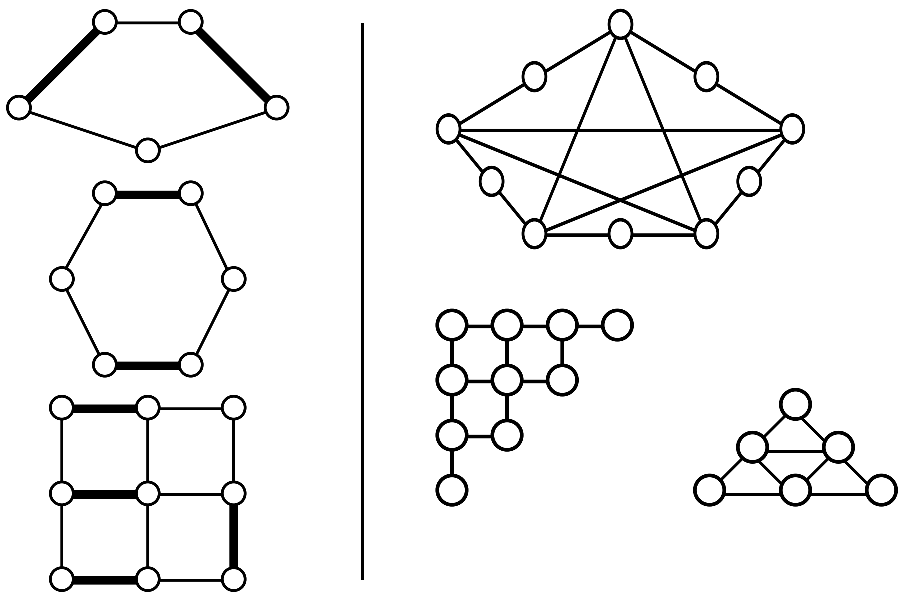
The Carpoolers' compatibility graph



Can you find a perfect matching in this graph?

Matchings in graphs

Matching. A matching M is a set of edges of the graph, no two of which share a vertex.



If one of the edges connects the vertex v to the vertex w, then we say that the matching M matches v to w.

Maximum matching. A maximum matching is one which matches as many vertices as possible.

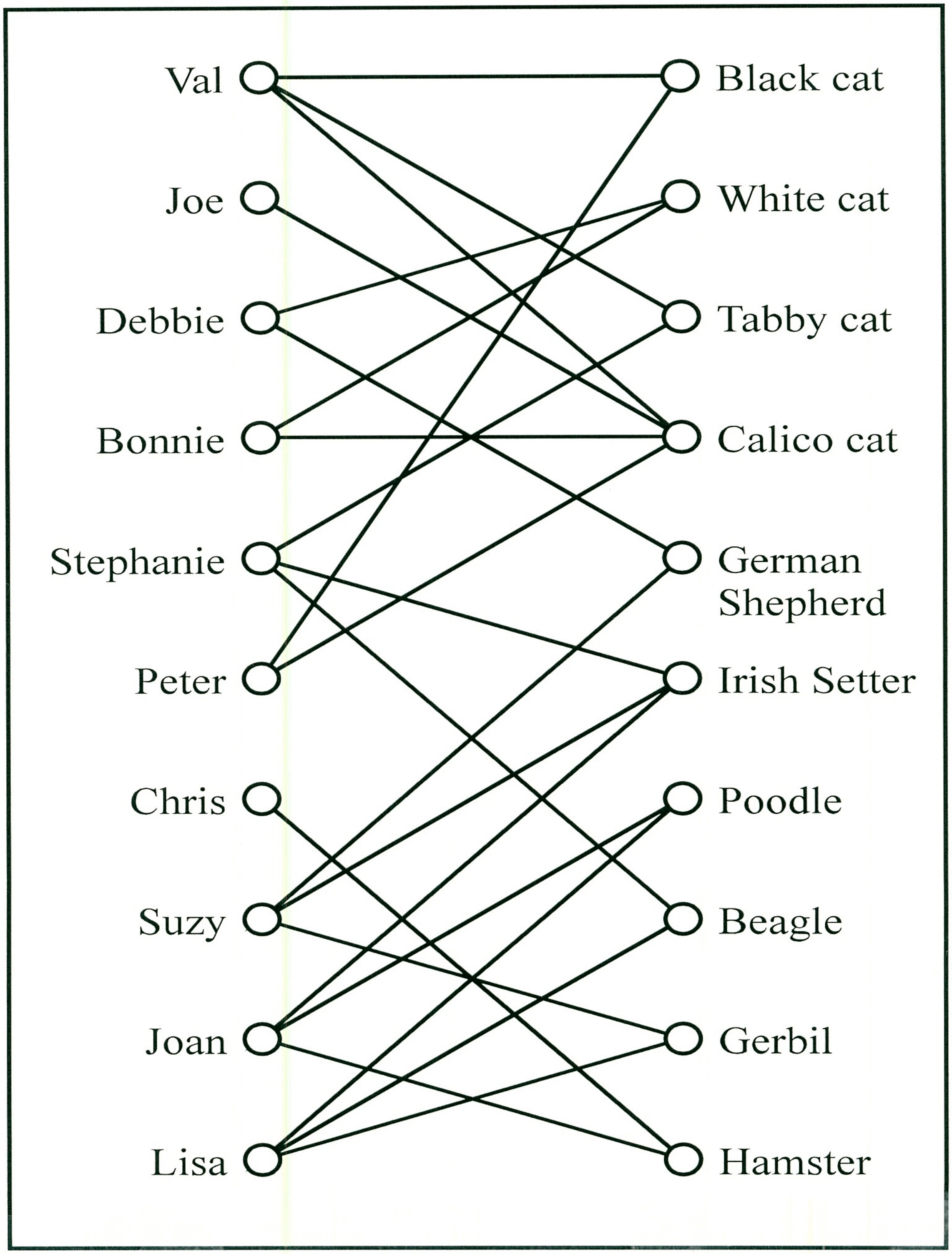
Perfect matching. A perfect matching is one which matches every vertex; this can happen only if the number of vertices is even.

Children and Pets

Val	Black cat, Tabby cat, Calico cat
Joe	Calico cat
Debbie	White Cat, German Shepherd
Bonnie	White cat, Calico cat
Stephanie	Tabby cat, Irish setter, Beagle
Peter	Black cat, Calico cat
Chris	Hamster
Sue	German shepherd, Irish setter, Gerbil
Joan	Irish setter, Poodle, Hamster
Lisa	Poodle, Beagle, Gerbil

Can you match pets to children so that each child gets one of the pets that s/he prefers?

A graph showing child/pet compatibilities



Can you find a perfect matching in this graph?

Handout #1 Terminology

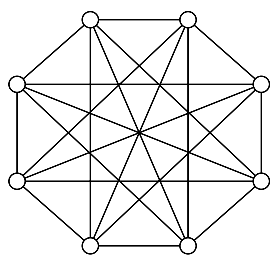
Euler path. An Euler path in a graph is a path which includes every edge of the graph exactly once.

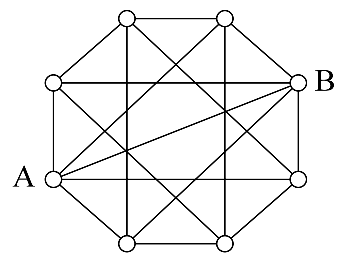
Euler circuit. An Euler circuit in a graph is an Euler path which ends where it begins.

If a graph has no vertices of odd degree, then it has an Euler circuit.

If a graph has two vertices of odd degree, then it has an Euler path, which begins at one of the two vertices of odd degree and ends at the other.

If a graph has more than two vertices of odd degree, then it has neither an Euler circuit nor an Euler path.



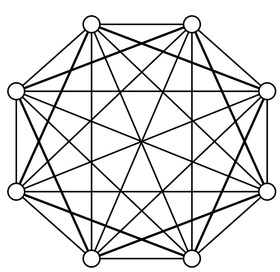


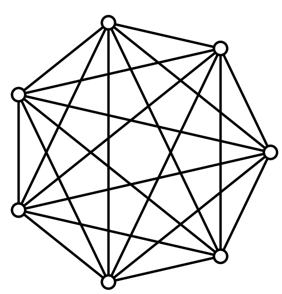
Activity 1: There is no Euler path or circuit in the graph above and to the left.

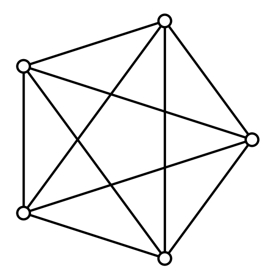
Activity 2: There is an Euler path in the second graph, starting with A and ending with B.

If we start with three people standing at A and B and two at each other vertex, this Euler path can be represented as a chain of 18 people holding hands; the 17 pairs of hands corresponds to the 17 edges of the graph.

Activity 3: Handshake Problem (revisited):







If you have 8 people (or any other number), can you have them complete all their handshakes in such a way that the recipient of each handshake initiates the next one? This is the same as asking for an Euler circuit in a complete graph, because each handshake corresponds to walking down one edge of the graph (the previous handshaker striding across the graph to shake hands with the next handshaker).

This is impossible if there are an even number of people, because in that complete graph each vertex has an odd number of edges (an odd number of people to shake hands with).

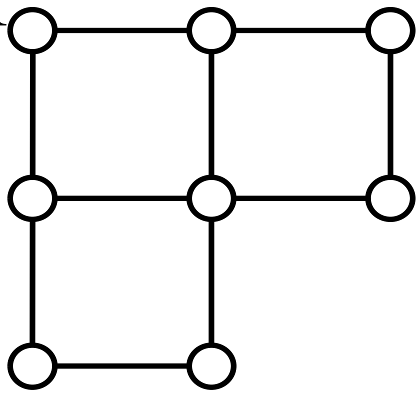
If there are an odd number of people, then it is possible, and we demonstrated it for 5 people and 7 people.

Hand-out #2

A trail is a path which has no repeated edges. How many non-overlapping trails does it take to cover all the edges of each of these graphs?

How many non-overlapping trails does it take to cover all the edges of each of these graphs?

A



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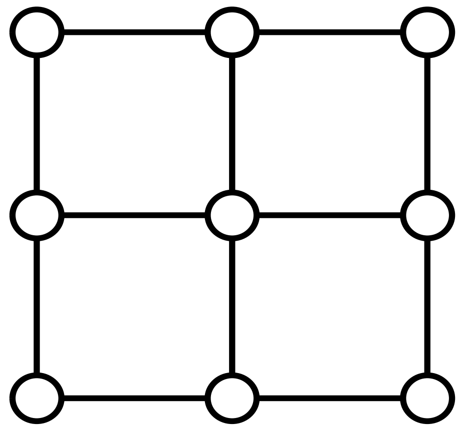

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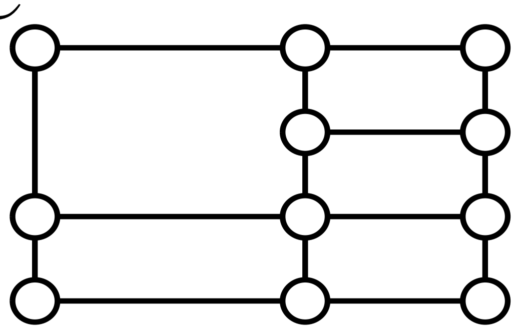

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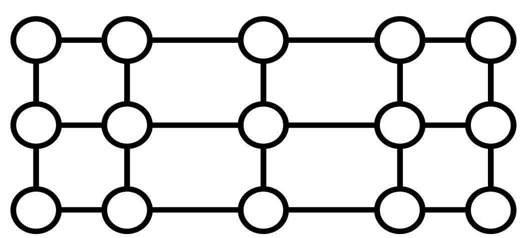
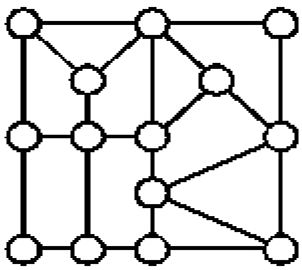


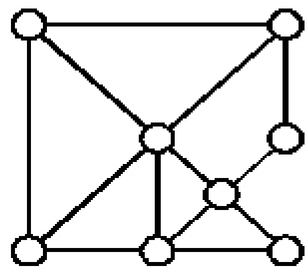
Hand-out #3

For each of the following graphs, add up the degrees of all its vertices, and compare the total with the number of edges of the graph. Can you explain the pattern?

A



B



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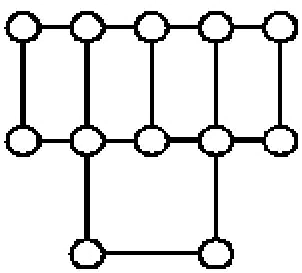

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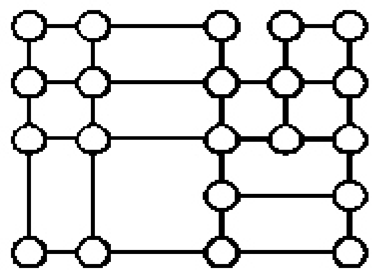


Hand-out #4

Children and Pets

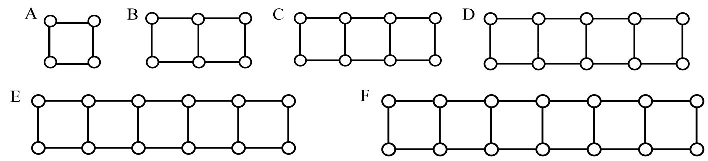
Val	Black cat, Tabby cat, Calico cat
Joe	Calico cat
Debbie	White Cat, German Shepherd
Bonnie	White cat, Calico cat
Stephanie	Tabby cat, Irish setter, Beagle
Peter	Black cat, Calico cat
Chris	Hamster
Sue	German shepherd, Irish setter, Gerbil
Joan	Irish setter, Poodle, Hamster
Lisa	Poodle, Beagle, Gerbil

Can you match pets to children so that each child gets one of the pets that s/he prefers?

Week 3, Session 1 — Paths and Matchings — Exercises

Practice Problems:

1. For each of the graphs to the right, find a decomposition involving the smallest number of trails.



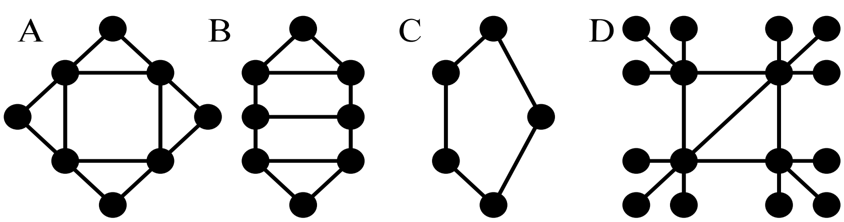
2. Find a perfect matching in each of the graphs above. Can you find more than one perfect matchings in these graphs?

Study Group Problems:

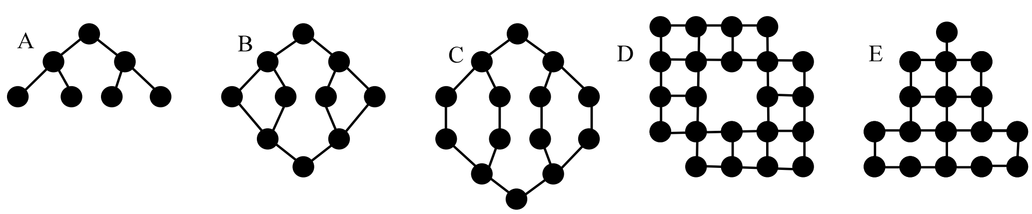
3. How many trails are needed in the Art Museum Curator's Problem (see page EX 3)? Can you decompose the edges into five trails whose lengths are 8, 8,8, 4, and 7?

4. Sequence the handshakes among seven people (called A,B,C,D,E,F,G) so that each new handshake is initiated by the person who received the previous handshake. How can you use an Euler circuit on the complete graph with seven vertices?

5. What is the largest number of edges you can have in a matching in each of the graphs below?



6. What is the largest number of edges you can have in a matching in each the graphs below?



7. The computer center has four files, F, G, H, and I, to be stored. Each of the possible storage locations, L, M, N, and O, can hold at most one of the files. F can be stored in M or O; G can be stored in L, M, or N; H can be stored in M, N, or O; and I can be stored in M or N. Decide where the files could be stored.

8. Circle one letter in each of the following ten words so that no letter is circled in more than one word: AGE, FICHE, BADGE, BAGGAGE, DEJA, EDGE, GAB, HEAD, EDIFICE, JIBE. Describe how a graph could be used to solve this problem. Change one word so that it becomes impossible to circle the letters as required.

9. a. Graph C in problem 1 has 10 edges and 4 vertices of odd degree. Can you decompose it into two trails each of which has five edges?

   b. Graph E in problem 1 has 16 edges and 8 vertices of odd degree. Can you decompose it into four trails each of which has four edges?

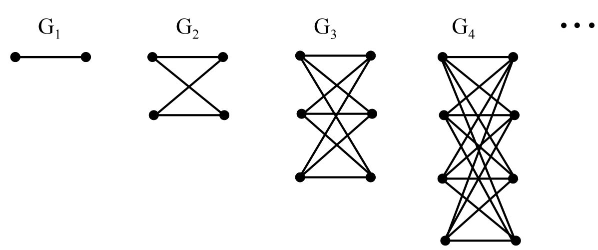
   c. The Art Museum Curator's Problem has 35 edges and 10 vertices of odd degree (see graph at right of page EX 3). Can you decompose it into 5 trails each of which has seven edges? If not, how close to equal can you get the trails (so that each of the guards routes are close to the same number of edges)?

10. Do the "School Dance" problem on page EX 4. If you can't find a perfect matching, find a maximum matching.

Extension problems:

11. For each of the graphs in problem 1, count the number of different perfect matchings you can find. What sequence do you get? Can you say why that sequence arises in this situation?

12. For the graphs below, find the number of different perfect matchings in each graph. Can you see how and why the pattern continues?

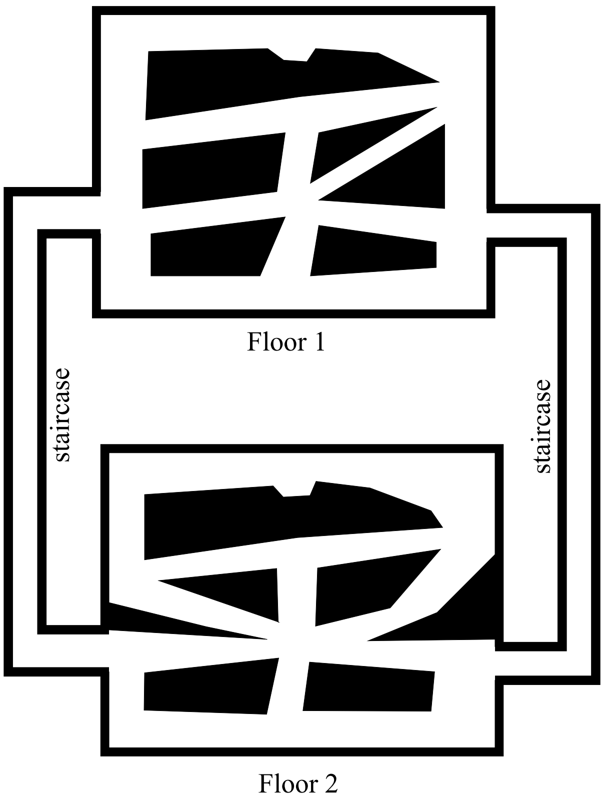


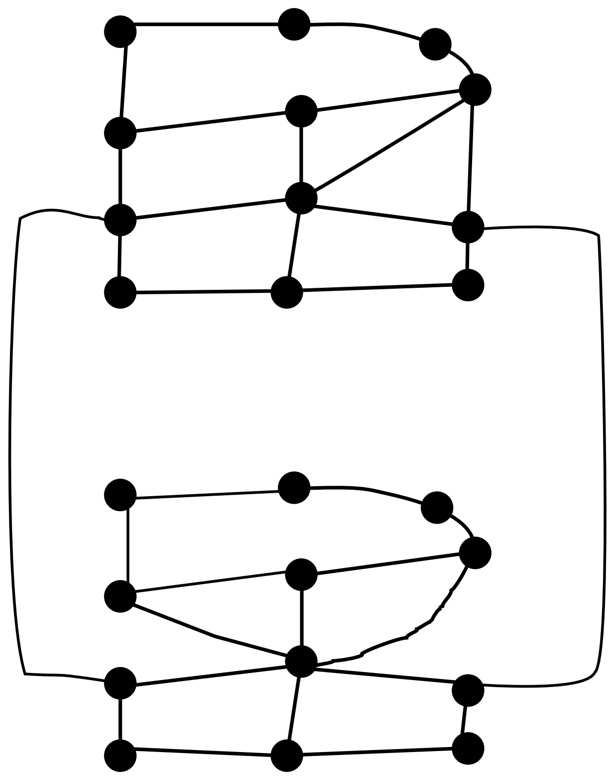
13. How would you find a maximum matching in a graph which has a Hamilton circuit?

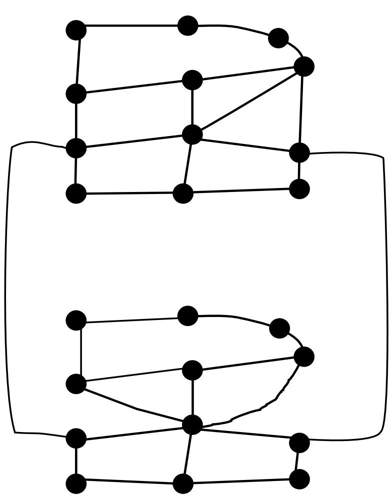
14. Try problem 4 again with 9 or 11 people instead of 7. Can you find a way to describe this process in general?

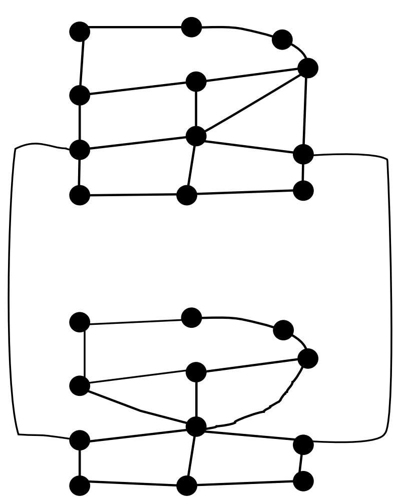
The Art Museum Curator’s Problem

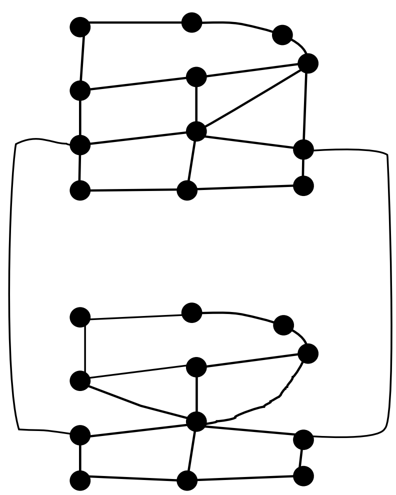
You want to assign to each museum guard an inspection route so that every corridor in the museum is passed through exactly once by exactly one guard. What is the smallest number of museum guards that you need? Can you make all of their trails contain the same number of edges?











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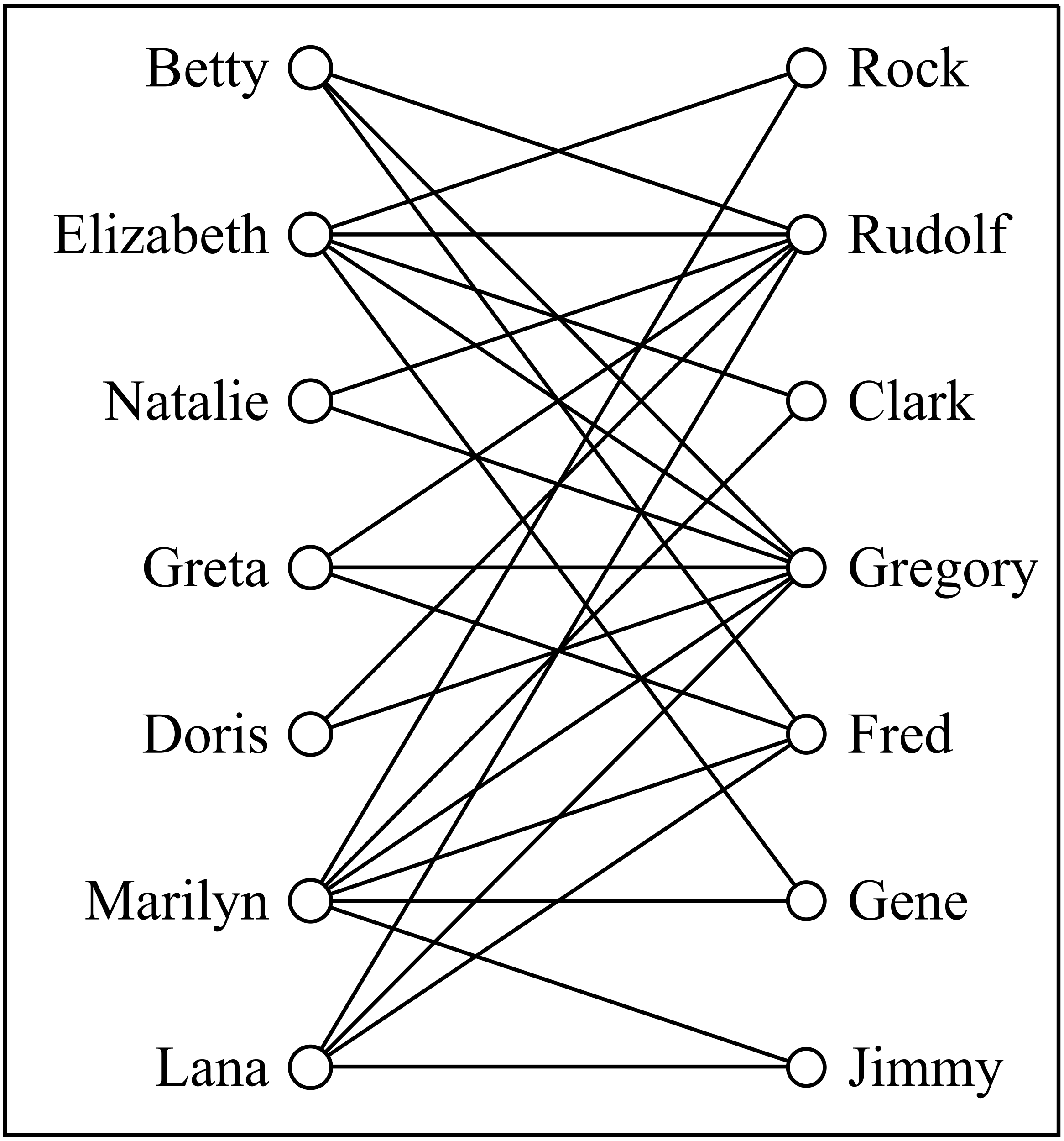
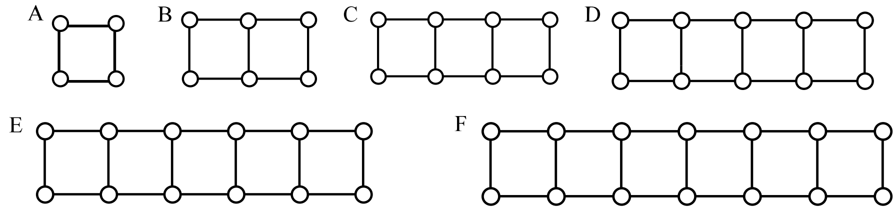
The School Dance compatibility graph



Can you find a perfect matching in this graph?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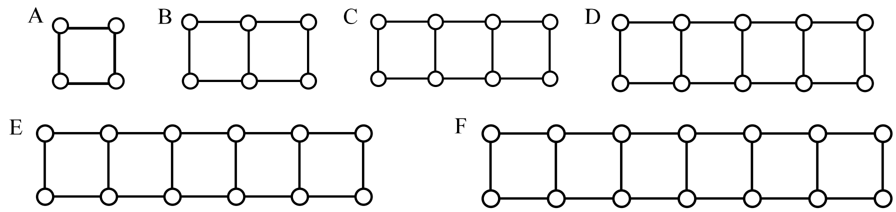
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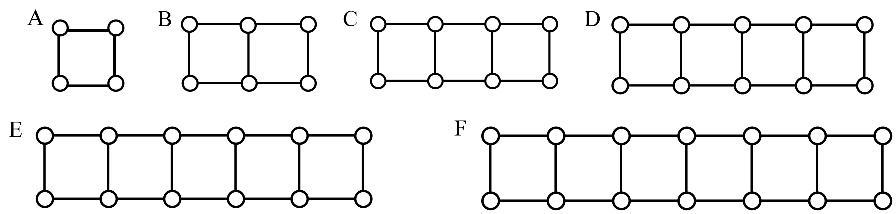
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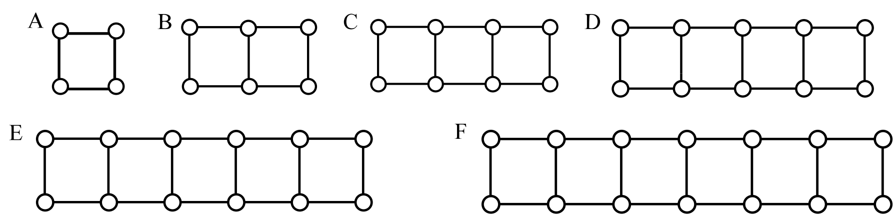
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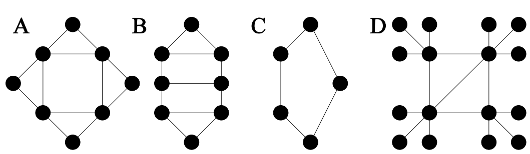
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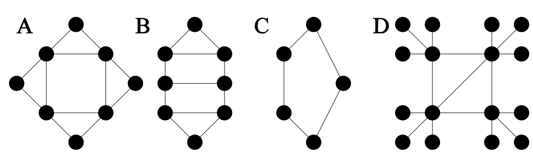
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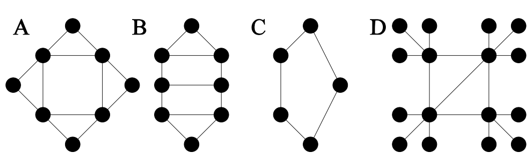
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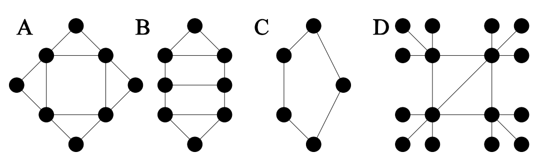
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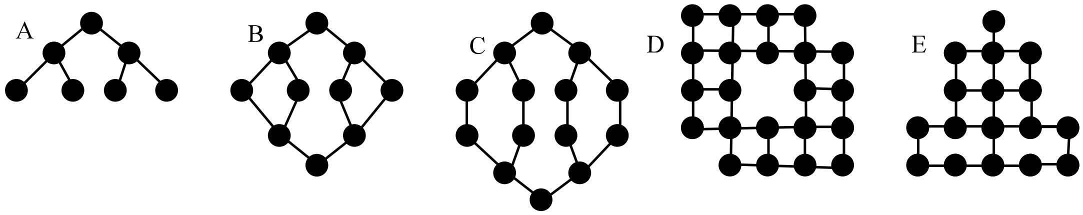

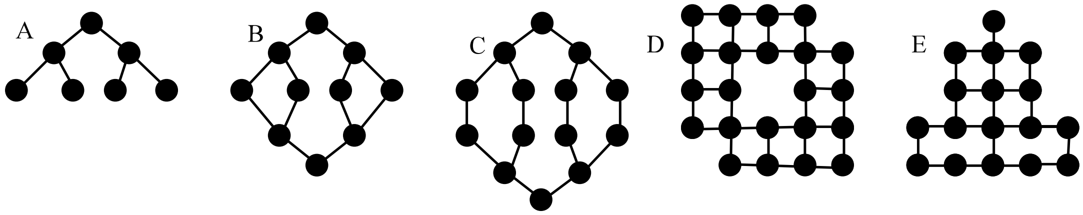

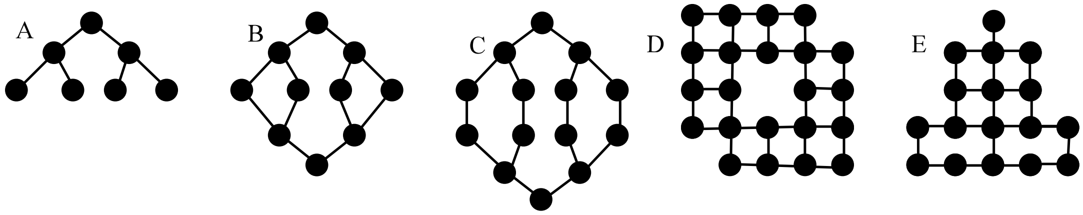

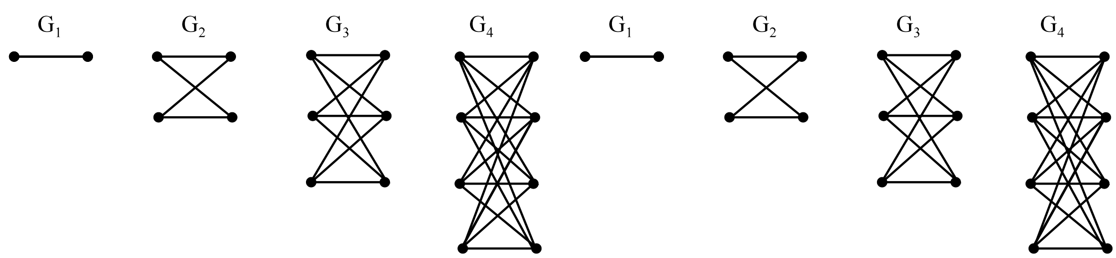

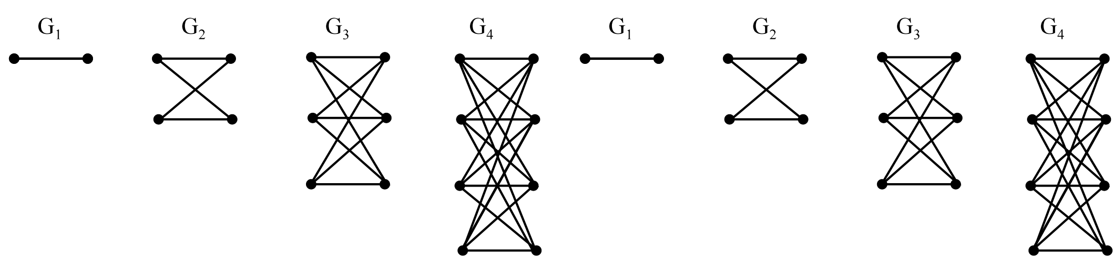

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Resource Book

Week 3, Section 1: Paths and Matchings

Mathematical background

Euler path. An Euler path in a graph is a path which includes every edge of the graph exactly once.

Euler circuit. An Euler circuit in a graph is an Euler path which ends where it begins.

If a graph has no vertices of odd degree, then it has an Euler circuit.

If a graph has two vertices of odd degree, then it has an Euler path, which begins at one of the two vertices of odd degree and ends at the other.

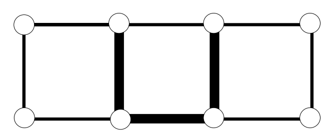
If a graph has more than two vertices of odd degree, then it has neither an Euler circuit nor an Euler path.

Trail. A trail is a path which has no repeated edges.

Using this terminology, an Euler path is a trail which includes all edges of the graph, and an Euler circuit is an Euler path which ends where it begins.

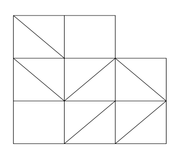
Euler number of a graph. The "Euler number" of a graph is the least number of trails which will include each edge exactly once.

A graph with an Euler path or circuit has Euler number 1.

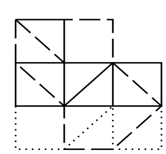


The graph at the right has Euler number 2, since it has no Euler path or circuit, and since the shaded and unshaded edges form two trails which include each edge exactly once.

Decomposition of a graph. In a decomposition of a graph, the edges of the graph are divided into components so that each edge is assigned to exactly one component.



The Euler number of a graph is the smallest number of trails which together form a decomposition of the graph.



For example, the graph at the right can be decomposed into three trails — the solid trail, the dashed trail, and the dotted trail — which don't overlap yet which include all edges on the graph.

Resource Book

Week 3, Section 1: Paths and Matchings

Mathematical background (continued)

Can the graph on the previous page be decomposed into two trails?

In a decomposition of a graph into trails, each vertex of odd degree must be an endpoint of at least one trail. Therefore, if a graph is decomposed into trails, the number of trails must be at least half the number of odd vertices.

Since the graph above has six vertices of odd degree, it cannot be decomposed into fewer than three trails, so the Euler number of the graph is 3.

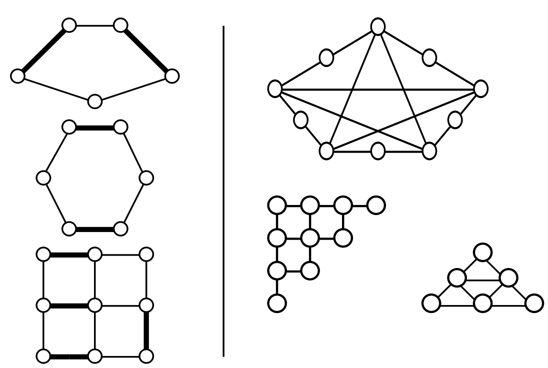
In fact, if a graph has 2n vertices of odd degree, then it can always be decomposed into n trails, so its Euler number is n. That is, the Euler number of a graph is equal to half of the number of vertices of odd degree.

Here's how you find the *n* trails. First, Eulerize the graph. Second, find an Euler path. Third, drop the extra edges from the Euler path. The remaining parts of the path provide a decomposition of the graph into *n* trails.

Theorem: The smallest number of trails into which any graph can be decomposed is half its number of odd vertices.

Vertices of odd degree. In any graph, the number of vertices of odd degree must be even. This is because the sum of the degrees of all the vertices in the graph equals twice the number of edges (since each edge is counted twice in the total, once for each endpoint) and if there were an odd number of odd degrees, this total would be odd rather than even.

Matching. A matching M is a set of edges of the graph no two of which share a vertex. If one of the edges connects the vertex v to the vertex w, then we say that the matching M matches v to w.



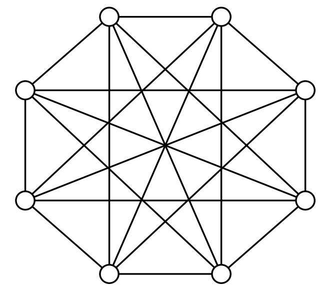
Maximum matching. A maximum matching is one which matches as many vertices as possible.

Perfect matching. A perfect matching is one which matches every vertex; this can happen only if the number of vertices is even.

Resource Book

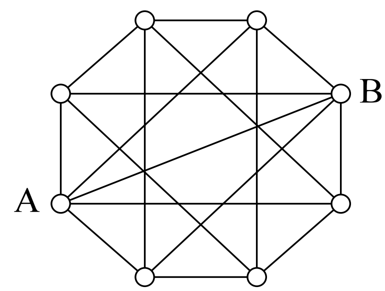
Week 3, Section 1: Paths and Matchings

Activity 1: There is no Euler path or circuit in the first graph.

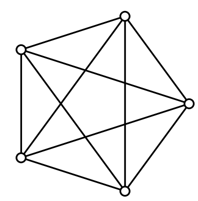


Activity 2: There is an Euler path in the second graph, starting with A and ending with B.

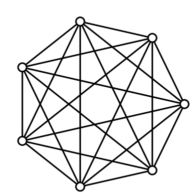
If we start with three people standing at A and B and two at each other vertex, this Euler path can be represented as a chain of 18 people holding hands; the 17 pairs of hands corresponds to the 17 edges of the graph.



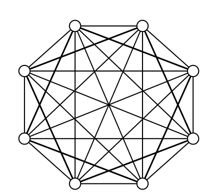
Activity 3: Handshake Problem (revisited):
If you have 8 people (or any other number), can you have them complete all their handshakes in such a way that the recipient of each handshake is the initiator of the next handshake?



This is the same as asking for an Euler circuit in a complete graph, because each handshake corresponds to walking down one edge of the graph (the previous handshaker striding across the graph to shake hands with the next handshaker).



This is impossible if there are an even number of people, because in that complete graph each vertex has an odd number of edges (an odd number of people to shake hands with).



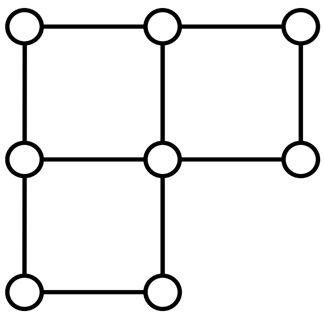
If there are an odd number of people, then it is possible, and we demonstrated it for 5 people and 7 people.

Resource Book

Week 3, Section 1: Paths and Matchings

How many non-overlapping trails does it take to cover all the edges of each of these graphs?

A



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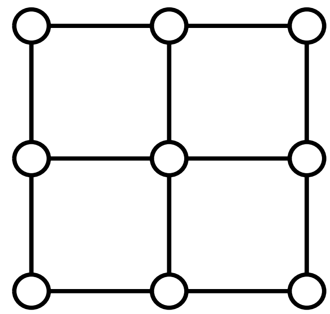

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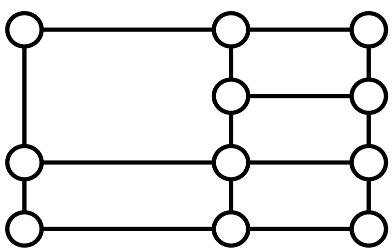

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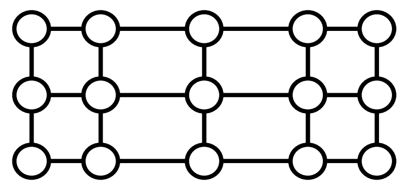

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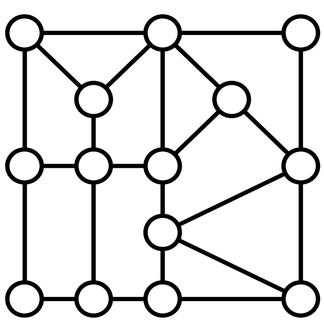
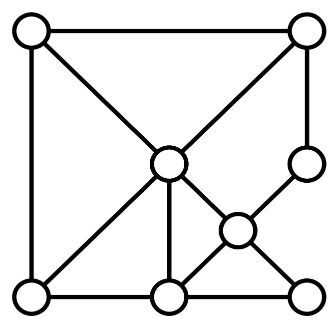


F



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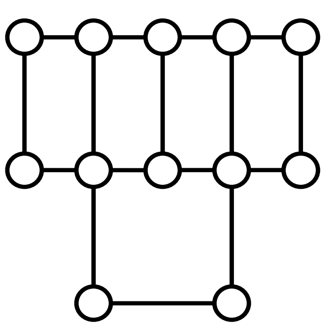
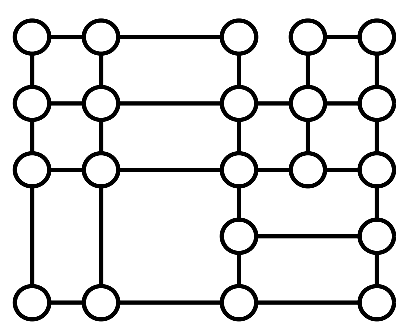


H

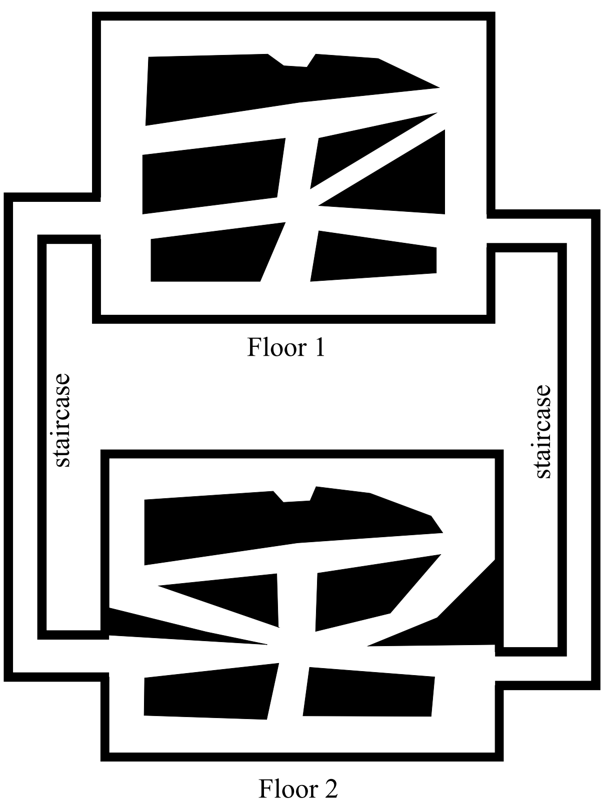


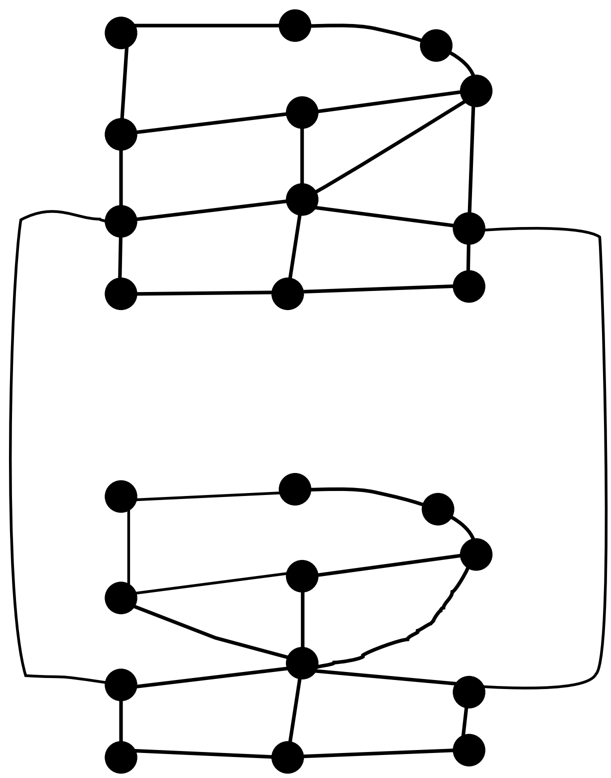
Resource Book

Week 3, Section 1: Paths and Matchings

The Art Museum Curator's Problem

You want to assign to each museum guard an inspection route so that every corridor in the museum is passed through exactly once by exactly one guard. What is the smallest number of museum guards that you need?



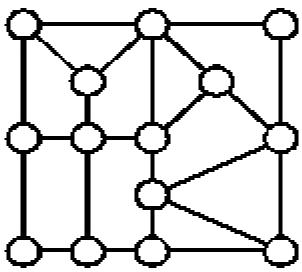


Resource Book

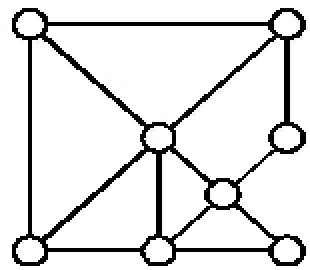
Week 3, Section 1: Paths and Matchings

For each of the following graphs, add up the degrees of all its vertices, and compare the total with the number of edges of the graph. Can you explain the pattern?

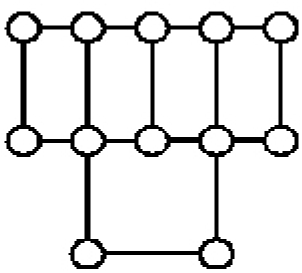
A



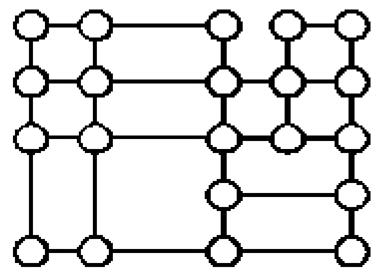
B



C



D



Total of degrees of all vertices =
Twice the number of edges (which must be even) =
(Total of degrees of all even vertices) + (Total of degrees of all odd vertices) =
(Even + Even + Even + ... + Even) + (Odd + Odd) + ··· + (Odd + Odd)
Even + Even + + Even
Adds up to an even number — **Each two vertices of odd degree contribute an even number to the sum.**

Any left over vertex of odd degree would contribute an odd number, making the total odd.

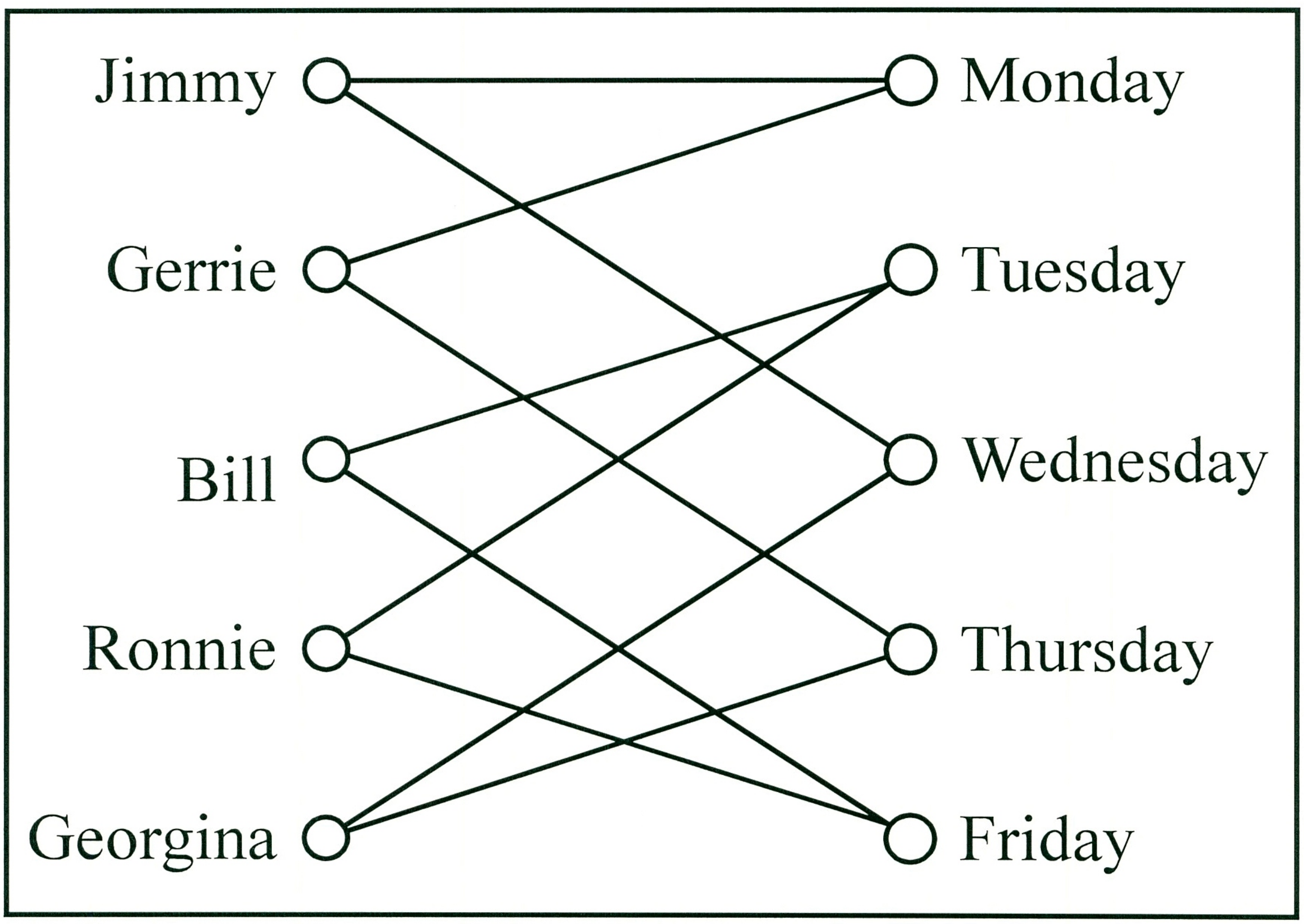
The total of the degrees of all the vertices in a graph must be even, since it is twice the number of edges. If the graph had an odd number of vertices of odd degree, then the total of the degrees of all the vertices would also be odd, which it can't be, since no number is both odd and even. That means that it is impossible to have a graph which has an odd number of vertices of odd degree!

Carpoolers

Jimmy	**Monday** **Wednesday**
Gerrie	**Monday** **Thursday**
Bill	**Tuesday** **Friday**
Ronnie	**Tuesday** **Friday**
Georgina	**Wednesday** **Thursday**

Can you assign a day for each driver so that no one need drive more than one day per week?

The Carpoolers' compatibility graph



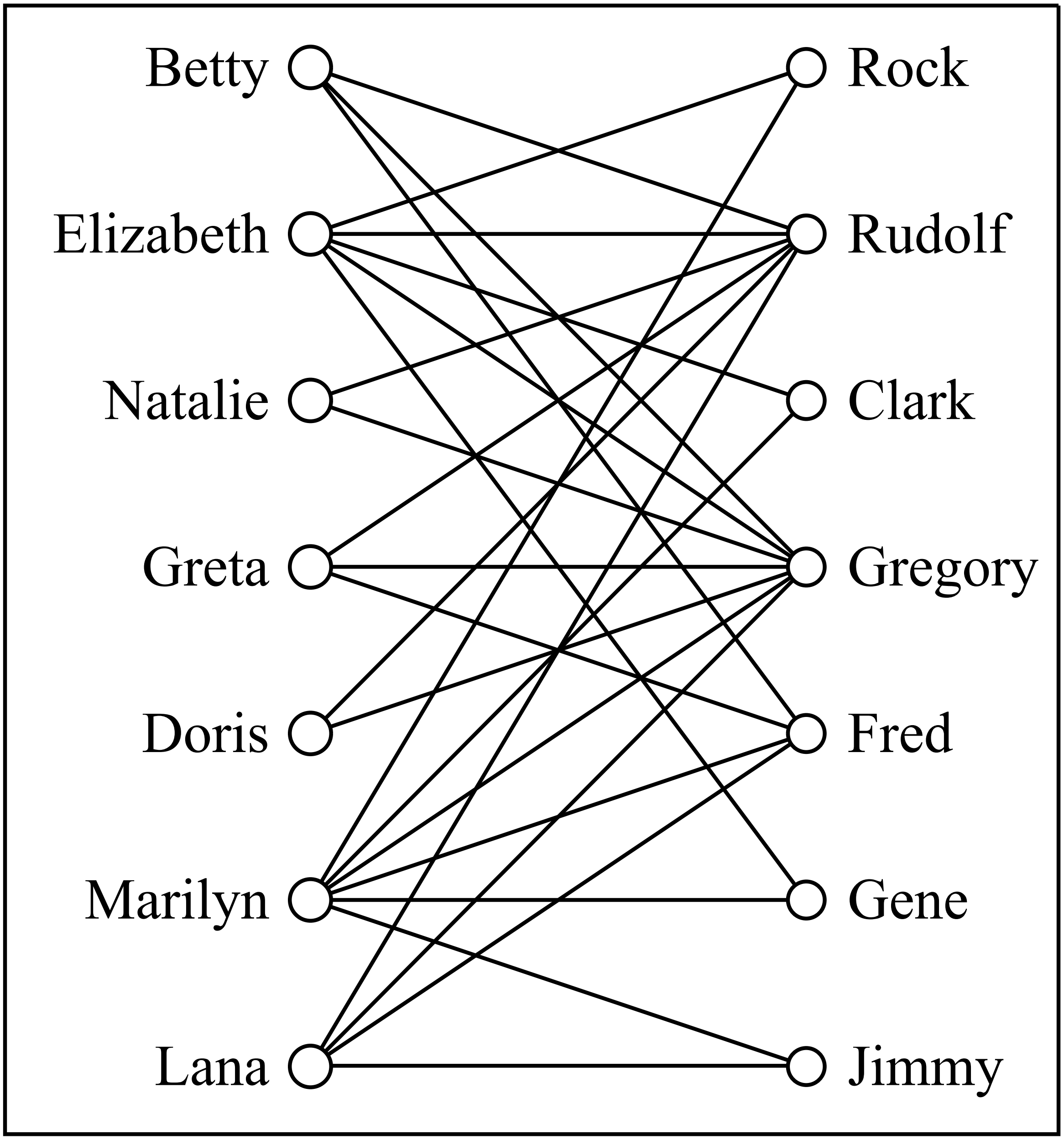
Can you find a perfect matching in this graph?

THE SCHOOL DANCE

Betty	**Rudolf** **Gregory** **Fred**
Elizabeth	**Rock** **Rudolf** **Clark** **Gregory** **Gene**
Natalie	**Rudolf** **Gregory**
Greta	**Rudolf** **Gregory** **Fred**
Doris	**Rudolf** **Gregory**
Marilyn	**Rock** **Clark** **Gregory** **Fred** **Gene** **Jimmy**
Lana	**Rudolf** **Gregory** **Fred** **Jimmy**

Can you match up the women with men that they are willing to dance with?

The School Dance compatibility graph



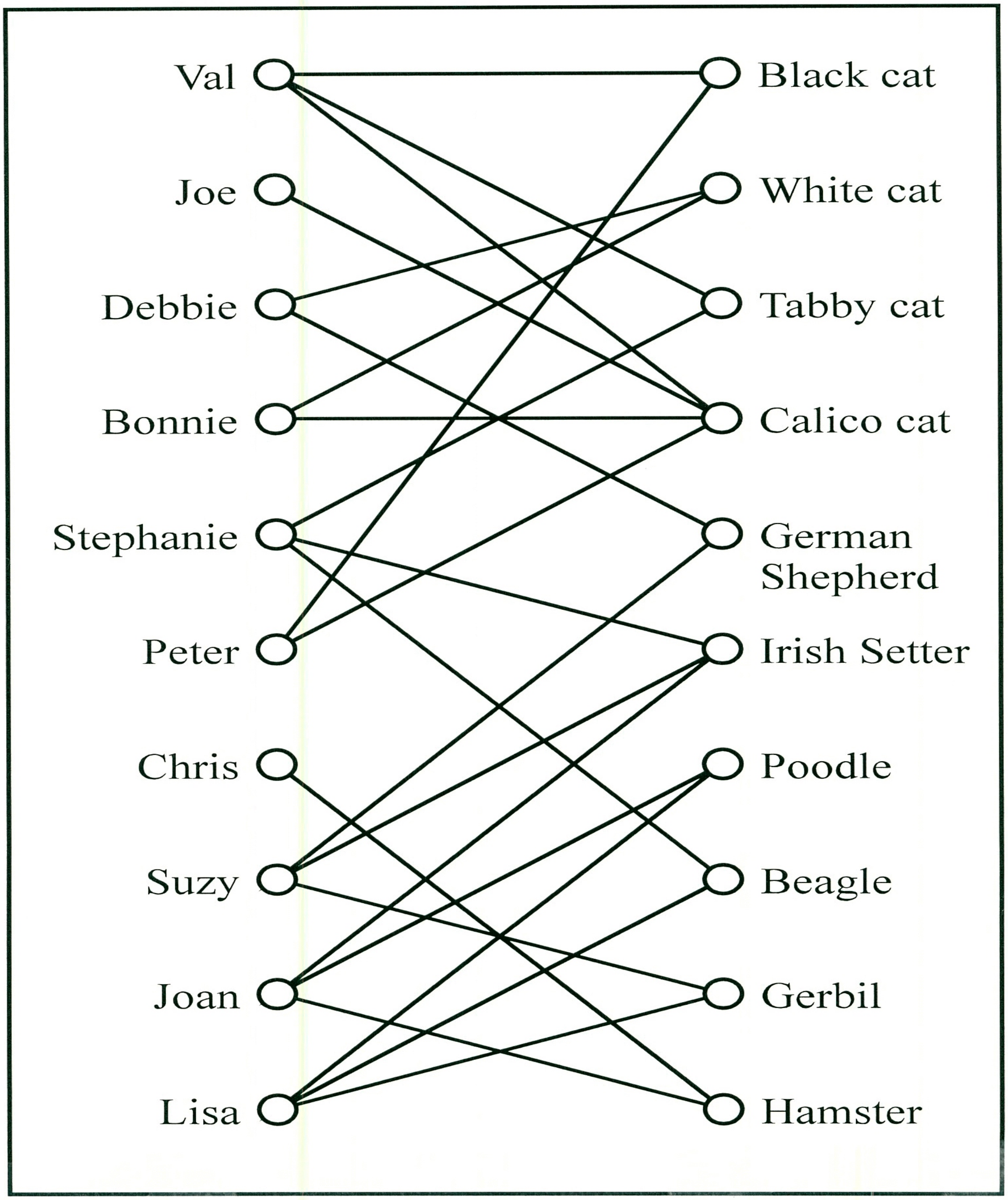
Can you find a perfect matching in this graph?

Children and Pets

Val	Black cat, Tabby cat, Calico cat
Joe	Calico cat
Debbie	White Cat, German Shepherd
Bonnie	White cat, Calico cat
Stephanie	Tabby cat, Irish setter, Beagle
Peter	Black cat, Calico cat
Chris	Hamster
Sue	German shepherd, Irish setter, Gerbil
Joan	Irish setter, Poodle, Hamster
Lisa	Poodle, Beagle, Gerbil

Can you match pets to children so that each child gets a pet that s/he prefers?

A graph showing child/pet compatibilities



Can you find a perfect matching in this graph?