Master Document

Workshop 9 — Patterns in Geometry

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised July 18, 1999

Workshop 9 — Patterns in Geometry

<u>Materials Needed</u> **<u>Allocated Time</u>**

Activity #1 — Review .... 10 minutes

Activity #2 — Introduction to Geometric Patterns in Nature .... 10 minutes
- Queen Anne's Lace or fern to place on overhead projector

Activity #3 — Geometric Iteration .... 35 minutes
- Large equilateral triangle paper for each participant

Activity #4 — The Sierpinski Triangle .... 70 minutes
- 27 triangular sheets of paper (9″x 9″x 9″) for building a stage 4 Sierpinski triangle
- 729 triangular pattern blocks (or 1″x 1″x 1″ triangles cut from green card stock)
- For each two person team: a ruler, a die, a penny, an erasable transparency marker, and a specially prepared transparency.

.... TOTAL WORKSHOP TIME: 125* minutes

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Activity #1 — Review
(Allocated time = 10 minutes.)

Before beginning the day's activities, take a few minutes to review homework from the previous day. Be sure to review the outcome of the "spirolateral" activity of the previous day. Also, if you didn't get a chance to do this at the end of yesterday's class, take a few minutes to demonstrate spirolaterals on the triangular dot paper. Do examples of both 60 degree and 120 degree spirolaterals, as described on the previous day's instructor's notes. (You might do the sequences (4,2), (4,2,6), (4,2,6,2), and (4,2,6,2,1) for each angle; interesting patterns emerge for all of the 60 degree figures and particularly for the final 120 degree figure.)

Some points in particular that you may wish to make are that the 3n+1 problem (Odds and Evens) is still open, in the sense that no one has proven that every starting point eventually results in the 1-4-2-1 cycle. Computers have verified that this is the case for thousands of small numbers, but it is still unsolved in general.

For some of the recursive procedures that they saw on yesterday's homework, it was impossible to tell from most positions where you had come from on the previous step. For example, with the "write the number in words" procedure, if you are at '6', you might have come from '11' or '12' or '80.' There's just no way to tell. Another interesting thing about this "write the number in words" procedure is that every number in the English language ends up at '4.' But this is not the case in other languages. For example, in French you have the 4-6-3-5-4 cycle (quatre-six-trois-cinq-quatre), and in Hungarian you have 5-2-5 (öt-kettő-öt).

Activity #2 — Introduction to Geometric Patterns in Nature
(Allocated time = 10 minutes)

A. On the overhead projector, display an example of a fractal-like structure from nature, for example, some Queen Anne's Lace or a fern. Broccoli and cauliflower also have some self-similarity, but they don't show up so well on an overhead projector. Show how small pieces of the figure look like reduced copies of the whole — not exactly like the whole, but roughly.

This is called "self-similarity" because a portion of the whole is similar to the whole itself. We have observed a natural phenomenon — and like we've done with so many other observations of nature, we can abstract what we've observed to make a *mathematical model. That is the motivation behind the following activities.*

B. Show the color fractal slides (TSP# 1 and TSP #2) of the computer-generated fern and green tree. Point out features of self-similarity, how parts look like reduced (and in the case of the fern, stretched or compressed in a skew direction) copies of the whole.

What we will be doing in this and the next workshops is trying to imitate what we saw in nature via mathematical constructions. We will take as our motivation this self-similarity and try to reproduce it mathematically. But since our models will be greatly simplified (compared to the complexities of nature) our figures will not look exactly like those found in nature. Conversely, we shouldn't feel bad if nature does not exhibit the mathematical exactness found in our fractals. This is not nature's shortcoming, but her beauty.

To summarize these ruminations, define "fractal" as found on TSP #3. Note that the square on the bottom of that TSP is self-similar (there are reduced copies at every scale) but we don't wish to call it a fractal because it isn't infinitely complex. The other figure (the sequence of circles) is ever-diminishing and endlessly repeating, but it also isn't infinitely complex, so we don't want to call it a fractal.

Explain that today's workshop will involve exploring these ideas.

Activity #3 — Geometric Iteration

(Allocated time = 35 minutes, 15 for part A and 20 minutes for B)

Note that activities A and B introduce ideas (self-similarity and geometric iteration) which the participants may wish to discuss at length. Put off detailed discussion because these ideas will be discussed in much more interesting contexts in the subsequent activities.

A. Take an equilateral triangle of paper and show TSP #4. Note that this activity involves a starting point and an iterating rule. Distribute equilateral triangle paper to each of the participants so they can model this also. Fold the top corner down to the center of the bottom, crease, and unfold. (TSP #5 gives an idea of what we're aiming for here.) Your iterating rule here is "*every time you see an empty triangle, fold the top corner down to the middle of the opposite side, and then crease and unfold.*" Put up TSP #5 when they have completed a few iterations. Ask them how long they think this sequence can continue. Help them understand that this procedure can continue indefinitely, even though you can't draw it. Show TSP #6 with the "end stage" of this

MUST imagine end stage – since can't actually do it sequence of figures. Point out the self-similarity of the figure, how if you enlarge the region contained inside the 20th triangle in the figure, it looks *exactly* like the entire figure itself, just smaller! Thus, at the top we can find infinitely many reduced copies of the whole at every scale, no matter how small. This kind of self-similarity is rather limited, however, (it could be called "self-similar at the top"), so that even though this figure is ever-diminishing, we would not consider this to be a fractal, because it isn't infinitely complex.

Saying that this is not a fractal because it isn't infinitely complex is a rather subjective comment. You should emphasize that there is no universal agreement among mathematicians (or artists) as to what constitutes a fractal. In fact, it is rather subjective.

As a review of the choose numbers, ask how many trapezoids there are in each figure, and elicit the answer that if the triangle has n horizontal lines, there are "n choose 2" trapezoids; point out, if no one mentions it, that this is the same as a homework problem which asks for the number of rectangles in a 1xn grid.

B. Distribute HO #1 (= TSP #7) and model the activity with the participants. Explain that what is shown is the first stage in the generation of a fractal tree, according to the rule shown on the page. Ask them what they would do next. If it seems they have it, let them try the next stage by themselves, and regroup to review it as a class. If that went well, let the participants finish, drawing as many stages as they can.

Distribute HO #2 (= TSP #8) and have the participants to items 2 to 4, which include some numerical activities related to the figure just generated. Create a chart that gives for each length of a branch the number of branches of that length and their total length.

When this is successfully completed, take some time to discuss the self-similarity of the figure, how you can find reduced copies of the whole. Note that the figure is considered to be infinitely complex since there are more and more copies of the original at reduced sizes. Tomorrow, you may wish to come back to this figure and review it in terms of reduce, replicate and rebuild.

[Time for a 5 -10 minute break]

Activity #4 — The Sierpinski Triangle

(Allocated time = 65 minutes; 15 minutes for parts A and B; 20 minutes for part C; 5 minutes for part D and 30 minutes for parts E, F, G, and H.)

A. Building on old themes, put up TSP #9 (with the triangulated triangles) and, to make a quick connection with earlier work, remind them how the square numbers and triangular numbers are contained in the figures. Have them see that there is an *explicit* rule for generating the *n*th figure: Figure *n* is generated by breaking the equilateral triangle into *n* pieces along each side and drawing the appropriate lines.

But there is also an *iterative* procedure for generating these figures, namely adding a new bottom row which has one more triangle than the existing bottom row. This rule works at every stage. Try to elicit such a rule from them that will work at every stage, forcing them to be precise. Take the time to take some of their rules and follow them literally on the overhead, exposing deficiencies in their precision and encouraging them to revise them. This can take a while, but it is worth it. Very soon they will be seeing (and finding) simpler rules for generating much more complex figures.

B. Repeat all the discussion of the previous activity with the figure on TSP #10; at last the famous Sierpinski triangle. Ask them to suggest rules to generate the figure (remove the middle quarter of each shaded triangle?), and again, be critical. Discuss for a few minutes the beautiful properties of Sierpinski's triangle.

Stage 7 in the generation of the Sierpinski Triangle is shown on TSP #11. Tell how in any part of the figure that includes some shaded portion you can find a reduced copy of the entire figure. Also mention how each stage in the development of Sierpinski's Triangle looks like three reduced copies of the previous figure.

A nice way to emphasize the self-similarity of this figure is to take a piece of paper with a hole punched out and place it over the figure. When the hole reveals a filled-in part of the triangle, you can see a solid triangle through the key-hole, i.e., a triangle which is solid in stage 7. You can then ask what would happen to that solid triangle in subsequent stages, noting that in that triangle, Sierpinski's Triangle will ultimately appear. That is, if what we were looking originally were actually the end stage, i.e., the real Sierpinski Triangle, then instead of that little solid triangle we would have actually seen a full Sierpinski Triangle. In other words, the Sierpinski Triangle contains a reduced copy of itself no matter where you look!

C. Now put up the "Tree" picture by Scott Kim (TSP #12). Spend a few minutes discussing with them a precise way to generate this figure, using transparencies which contain the initial stages of this tree. (That is, prepare ahead of time four extra copies of the transparency and cut out each of the first four stages of the tree, so that participants can see more clearly the rule for generating the tree.)

Initially, they will be nervous and uncomfortable since the figure seems hopelessly complex. But as you guide them into discovering more and various self-similarities, and show how portions here and there contain the whole figure, just shrunk, rotated, perhaps flipped and translated, they will be pleasantly surprised that they can comprehend this picture in its entirety.

Although you shouldn't dwell on this topic, if there is extra time you might encourage them to point out some properties of this figure (for examples, there are lots of spirals, some copies of the figure are flipped, most are rotated, etc.) Ask them to tell various ways in which they can see parts which look like the whole. Perhaps they can even draw their ideas on a blank transparency overlaid on the Scott Kim picture. This helps in the very important task of visualization.

D. Distribute Handout #3, the hexagonal grid paper (TSP #13). Instruct them to create about ten rows of Pascal's triangle in the grid in pencil -- starting at the top of the page. This done, have participants use markers (or colored pencils) to color every cell containing an even number red and every cell containing an odd number blue.

fill in Pascal #6

At this point, they may recognize a bit of Sierpinski's triangle developing. If they don't, don't give it away.

Invite them to continue Pascal's triangle, together with the coloring, to the bottom of the page. However, since the entries rapidly become unwieldy, you can suggest they try to find an easier way. All they need to use is the rule for adding odd and even numbers together — this way they can drop the numbers altogether and simply add colors — red + red = red; blue + blue = red; red + blue = blue; and blue + red = blue.

E. On the overhead projector, put up a single triangular pattern block and call it stage 0 in the process of making the Sierpinski triangle. Next take 2 more triangles and, together with the first triangle, form stage 1 in the process of making Sierpinski's triangle. To form stage 2 of the process, take 2 more configurations like the one you

just made (9 triangles altogether) and arrange them in a triangle. This should be sufficient for them to get the idea of how to build a Sierpinski triangle from scratch.

F. Give each of 27 people a 9″x 9″x 9″ triangular sheet of paper. Ask them to come up one at a time and place their triangles on the floor so that when they are finished, the 27 triangles will form a stage 4 Sierpinski triangle. You will need to guide them through the process.

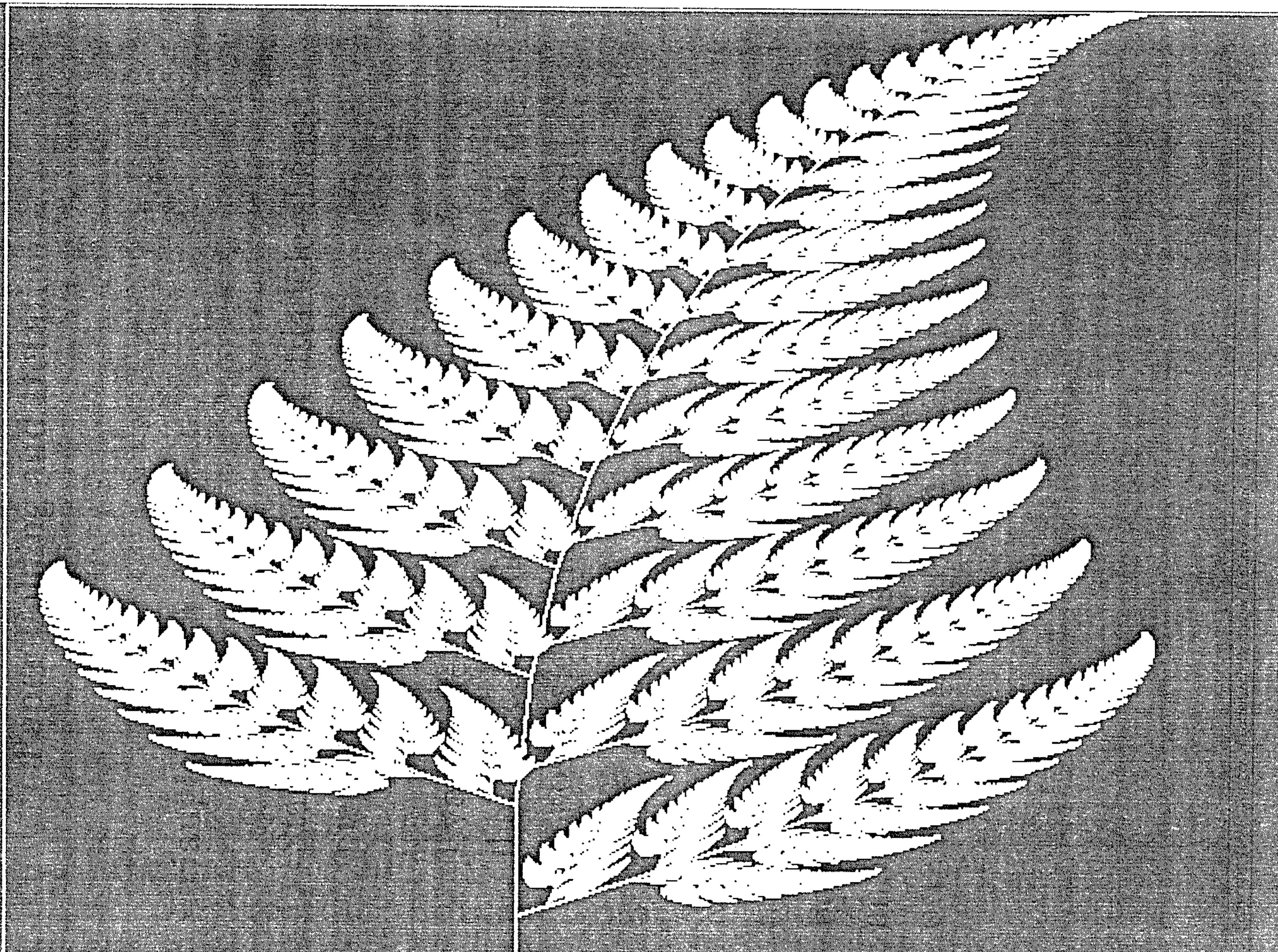
G. Give each person 27 triangular pattern blocks (or 1″x 1″x 1″ triangles cut from green card stock) and ask them to place the pattern blocks inside their paper triangle to create a stage 4 Sierpinski triangle. If there are enough triangular pattern blocks to go around (you'll need 729), then the result will be a stage 7 Sierpinski triangle.

If there won't be enough time for Activity #4H, then suggest that they complete this Activity #4G later.

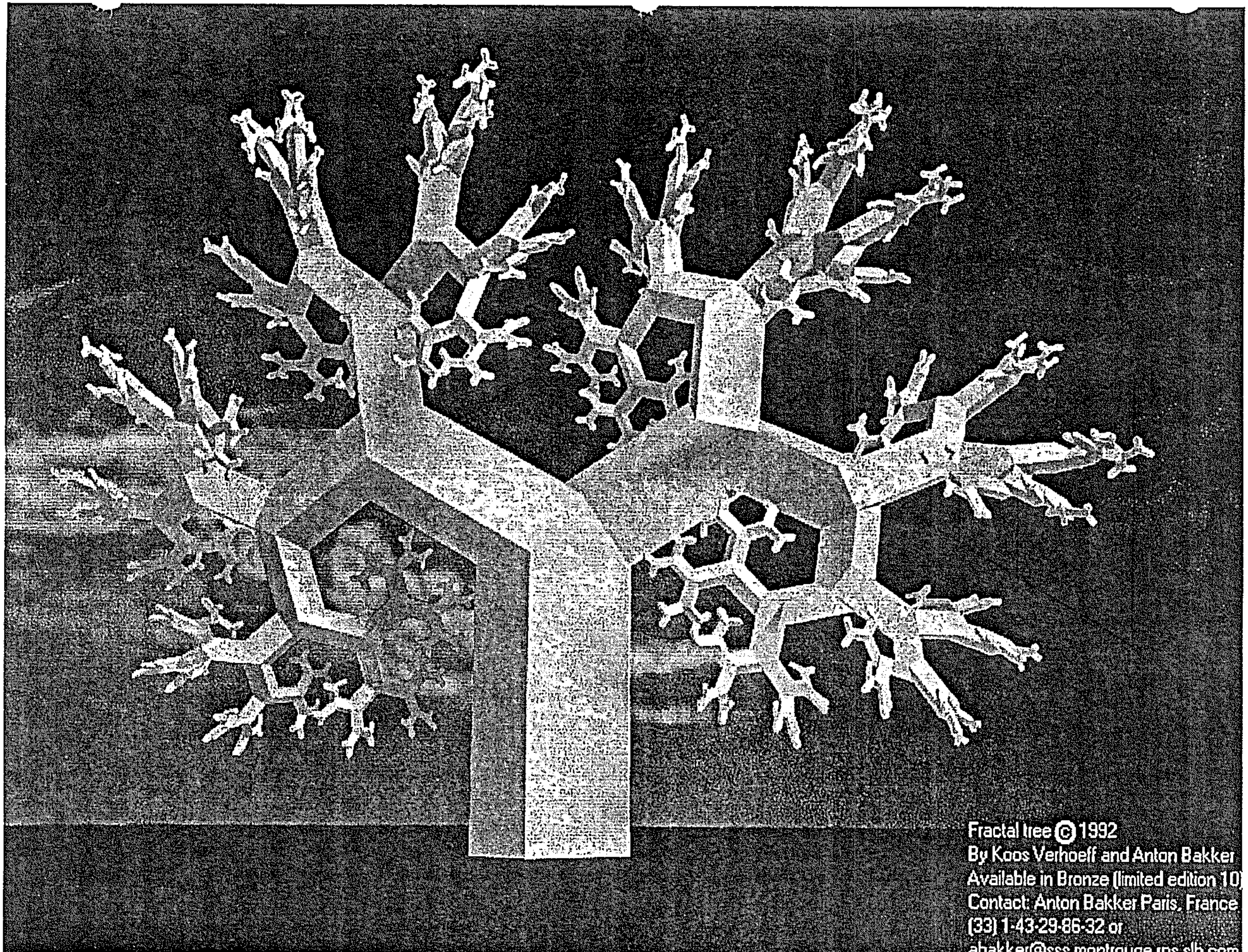
Mention that the participants may want to cut out the Pascal triangles they made earlier, now transformed into stage 4 Sierpinski triangles, and put 27 of them together on the wall to create a stage 7 Sierpinski triangle. That they can do later if they wish.

H. The Chaos Game. Provide each pair of participants with a copy of TSP #14, which has on it the playing board for the Chaos Game. Each team will also need a ruler, a die, and an erasable transparency marker. Show the instructions on TSP #15 and, after explaining them, ask each pair to repeat the procedure ten times (or more if they work quickly). Note that they should not draw the lines, only the dots! Once all the teams have completed this, ask them whether any patterns have emerged. Probably nothing will be noticeable. Collect all the transparencies, superimpose them (by lining up every "T" on top of every other "T", every "R" on top of every other "R", and every "L" on top of every other "L"), and place them on the overhead projector. Lo and behold, what appears should bear a close resemblance to the Sierpinski triangle. If they are skeptical, rotate some of the triangles and reverse some of the others, and then superimpose them. Ask for explanations of why this happens.

These activities are all appropriate across the K-8 curriculum. Teachers at every grade level can find something instructive for their students. Tomorrow, the participants will rephrase many of today's concepts in terms of "reduce, replicate and rebuild." In this activity they are replicating and rebuilding, but not reducing.



Fractal Fern (Dimension = 1.[illegible])


Fractal tree © 1992
By Koos Verhoeff and Anton Bakker
Available in Bronze (limited edition 10)
Contact: Anton Bakker Paris, France
(33) 1-43-29-86-32 or
abakker@sss.montrouge.rps.slb.com

Substitute this page with the blue fern transparency page.

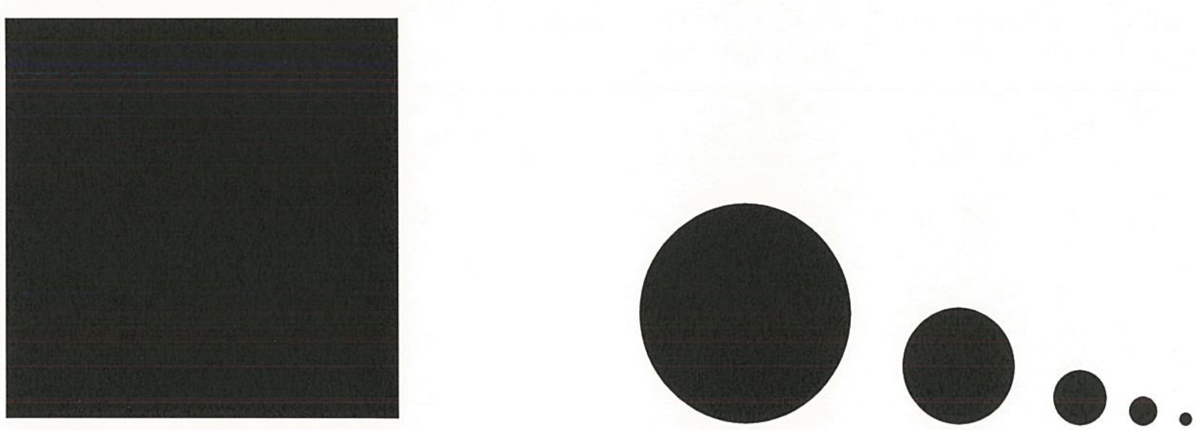
Substitute this page with the green tree transparency page

A Fractal Is...

- **Endlessly Repeating** — meaning that you can find an endless number of copies of the whole figure within the figure, at a reduced scale, and possibly rotated or stretched. This is also called *self-similarity*.

- **Ever Diminishing** — meaning that you can find copies of the whole figure at every scale, no matter how small.

- **Infinitely Complex** — meaning that the figure should have detail and delicateness as well as self-similarity. You can find copies of the whole figure throughout the figure.



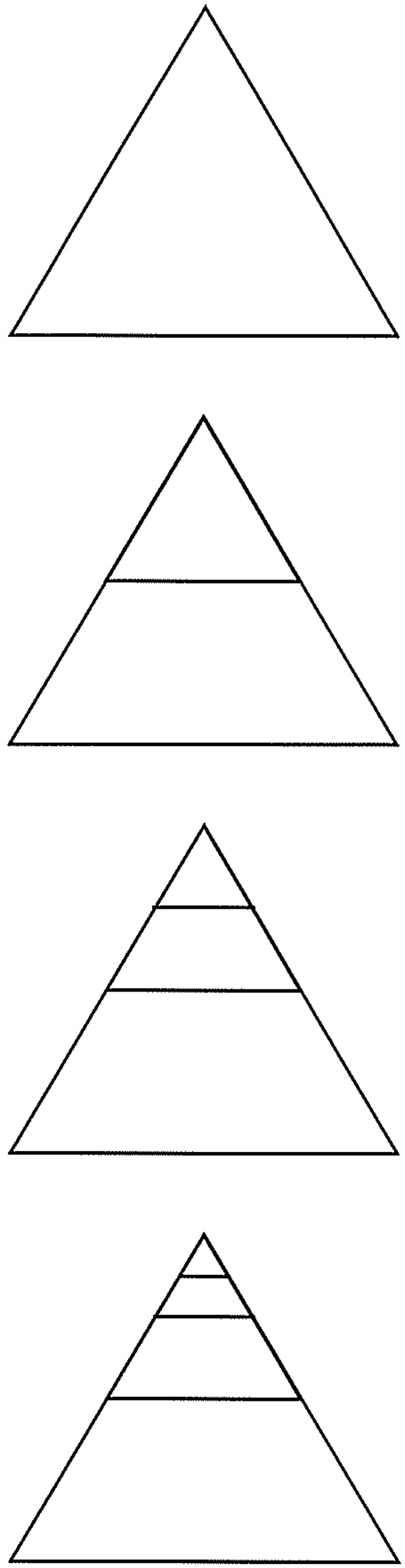
Paper Folding and Iteration

Triangle Pattern

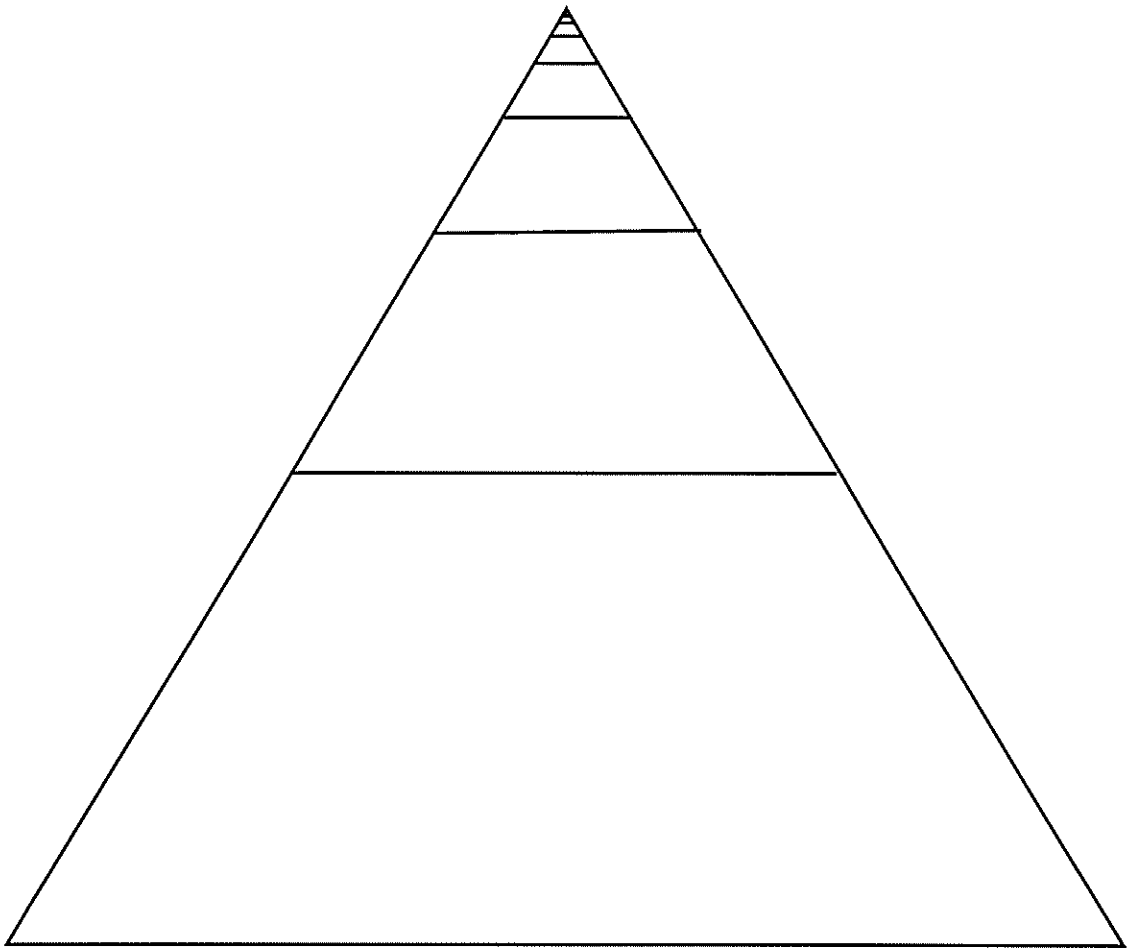
1. Start with an equilateral triangle-shaped piece of paper.

2. Every time you see an empty triangle, fold the top corner down to the middle of the opposite side, and then crease and unfold.

Clarification: At each stage, fold the top of the smallest triangle down to the center of its bottom side, make a crease along the fold line, and then unfold so that you now have the original triangle, but with one more crease.



The End Stage



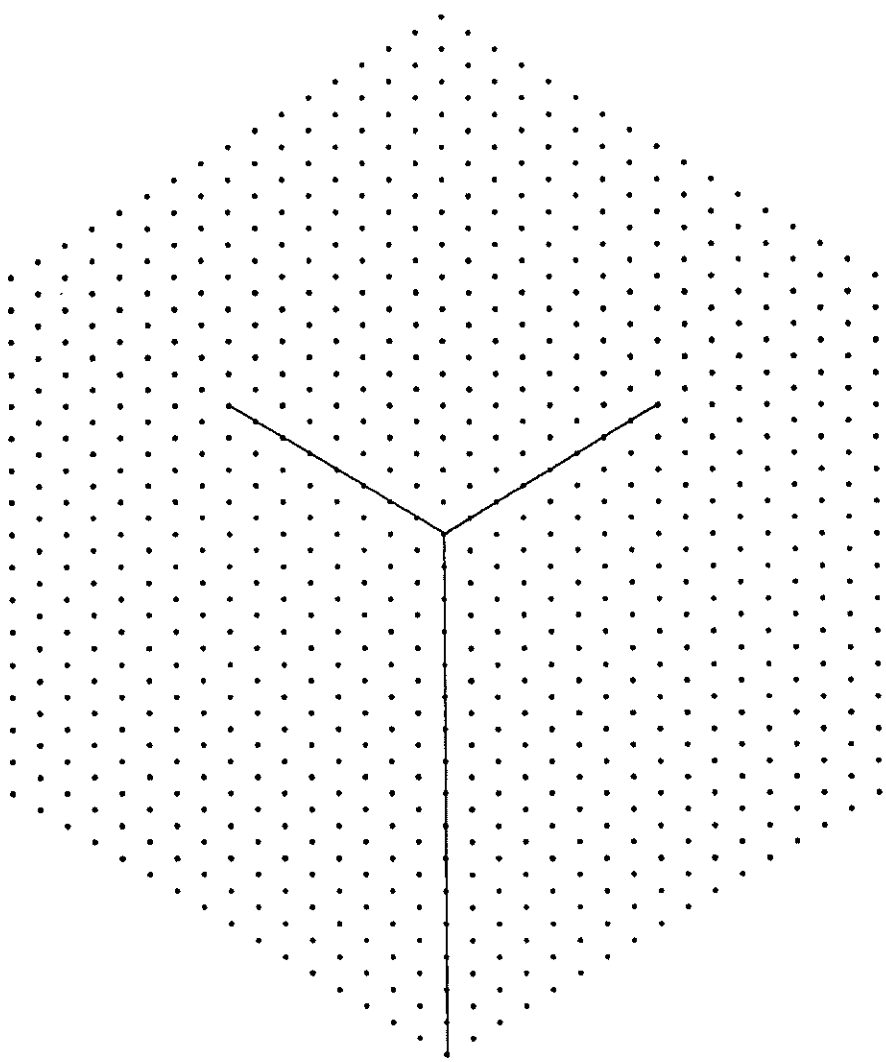
Handout #1 Generating Trees

1.5 TREES **1.5A**

As trees grow, they branch out. From big branches grow smaller ones. From these grow smaller ones still, and so on. Use this dot paper to draw a mathematical tree with some of the same properties as the live ones.

Construction From the endpoint of each branch, draw two new branches half as long growing off at 60o in opposite direction.

1. Stage 1 of the tree has already been drawn. Draw the four new branches for stage 2 by connecting endpoints to the appropriate dots on the grid. Draw the eight new branches for stage 3. Repeat again for stage 4. Endpoints should still be on the dots of the grid. Continue the growing process until the branches become too small to draw.



Handout #2: Generating Trees (cont.)

1.5B

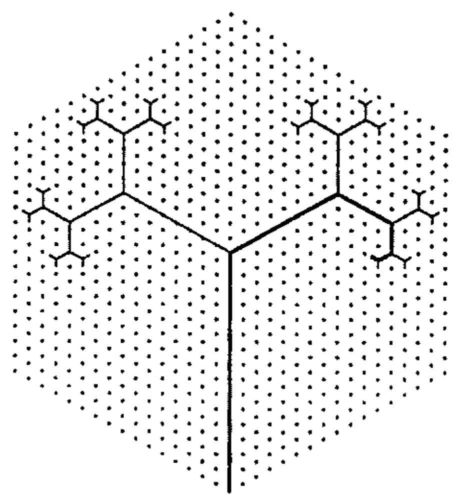
Suppose the tree starts with an initial vertical segment of 1 unit as the trunk. Imagine further that the tree continues growing branches, over and over by the process given, until fully grown. Visualize this completed tree.

2. How many branches have lengths of 1/4? of 1/16? What is the sum of the lengths of all branches 1/4 long? 1/16 long?

3. What is the total length of all branches of the completed tree?

4. Are there parts of the completed tree that look like the entire tree? Using the tree just drawn as a model of a fully grown tree, draw a hexagon around a part that would be an exact image of the tree itself. Draw another using a hexagon of a different size.

One interesting shape found on the completed tree is a *spiral*. Start at the base of the tree and turn right at each and every junction point. Note how these particular branches trace out a spiral.

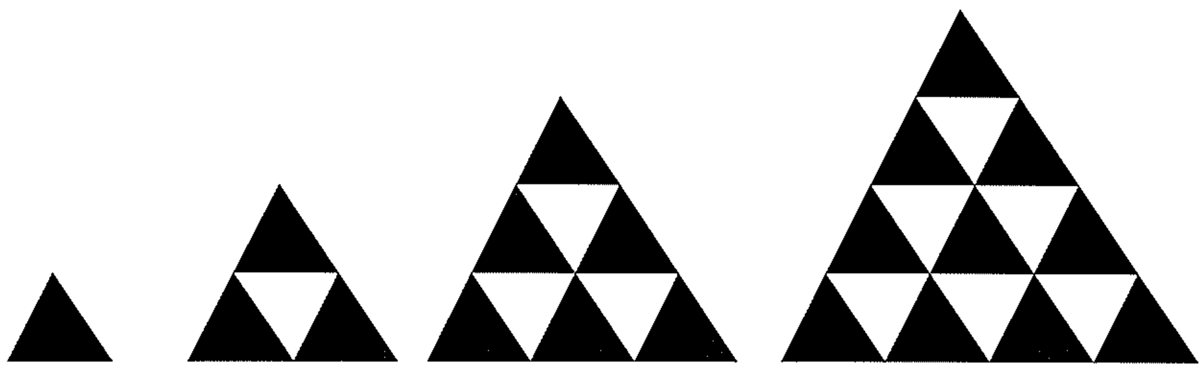


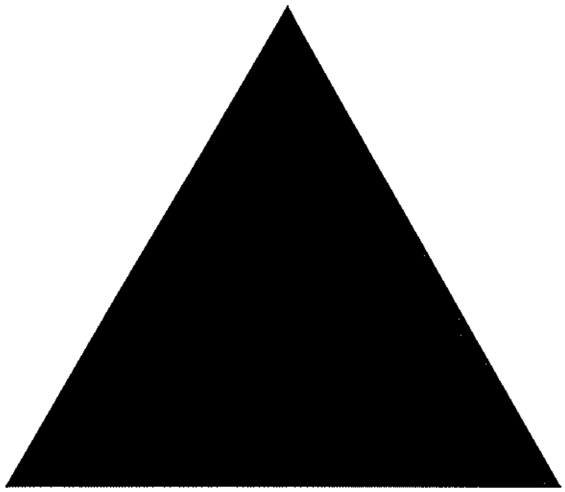
5. Find another spiral that is a reflection of the one just described. What is the length of this spiral?

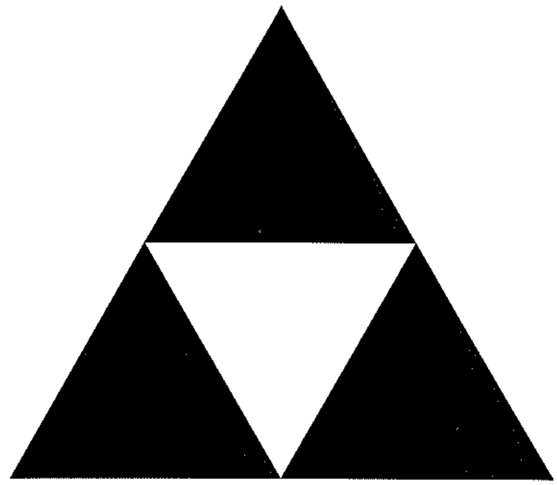
6. Find four spirals with half the length of the one just described. How many spirals in the tree have one-quarter the length of the original?

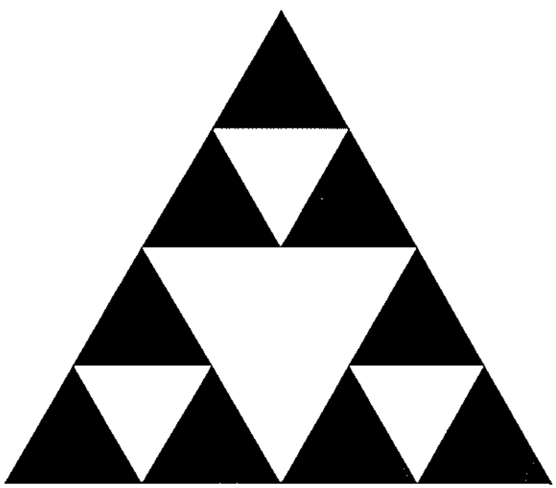
7. Consider all such spirals of all sizes that can be found on the tree. They all do not have the same length. Are they all exact replicas of each other except for size?

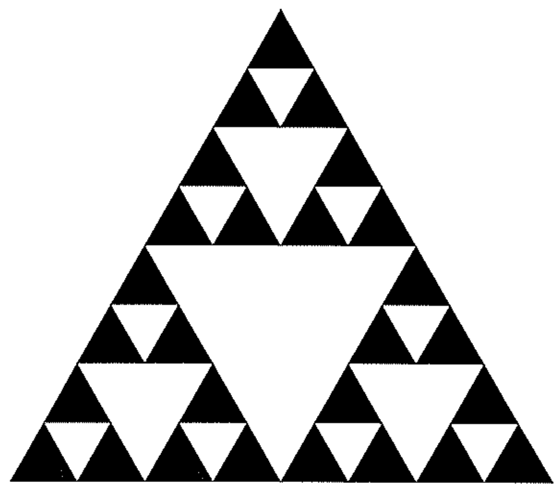
Can you give an iterative rule for generating these figures?

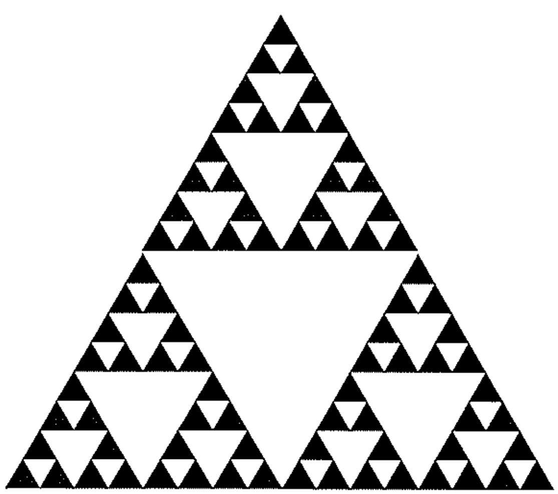






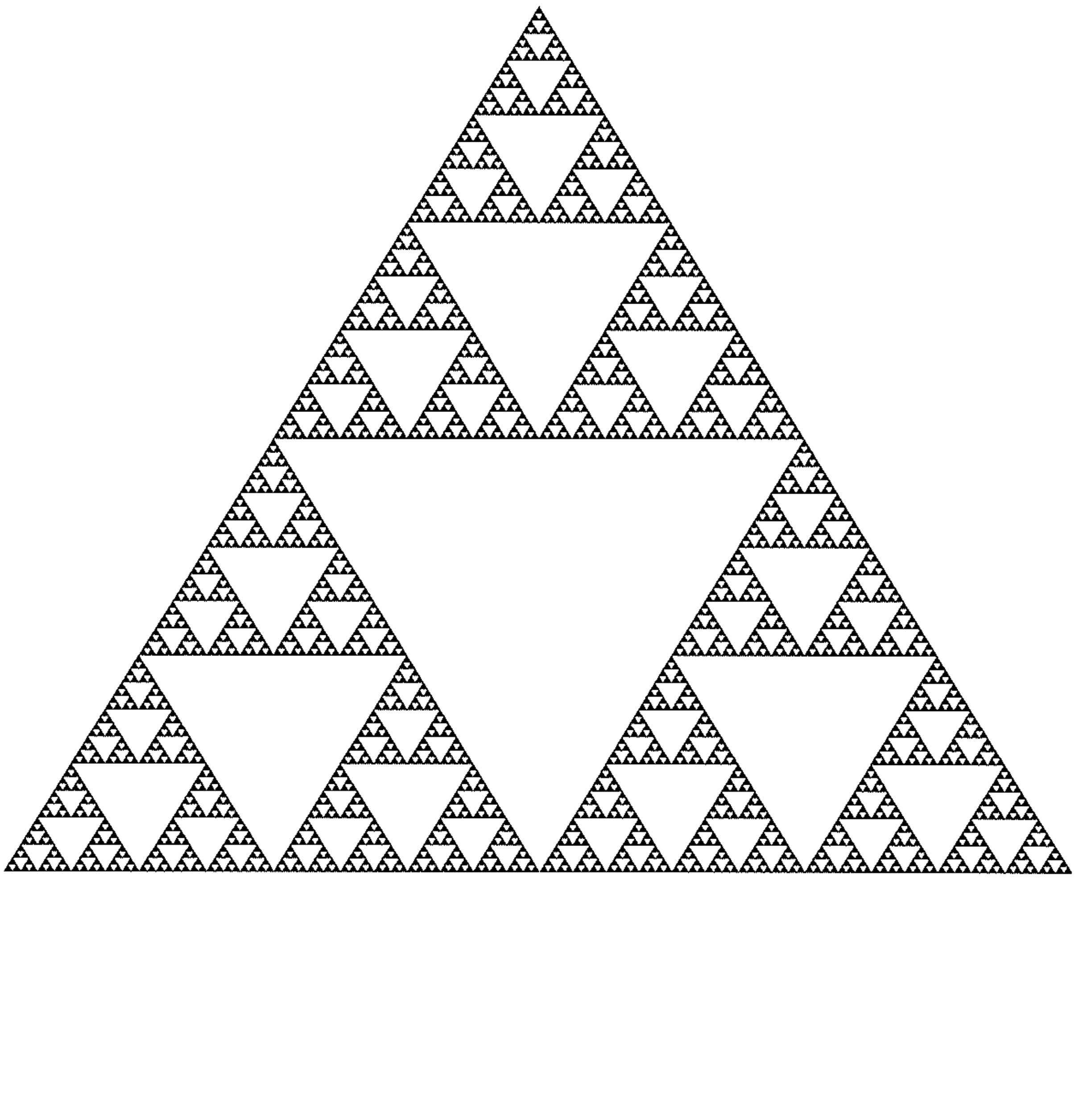






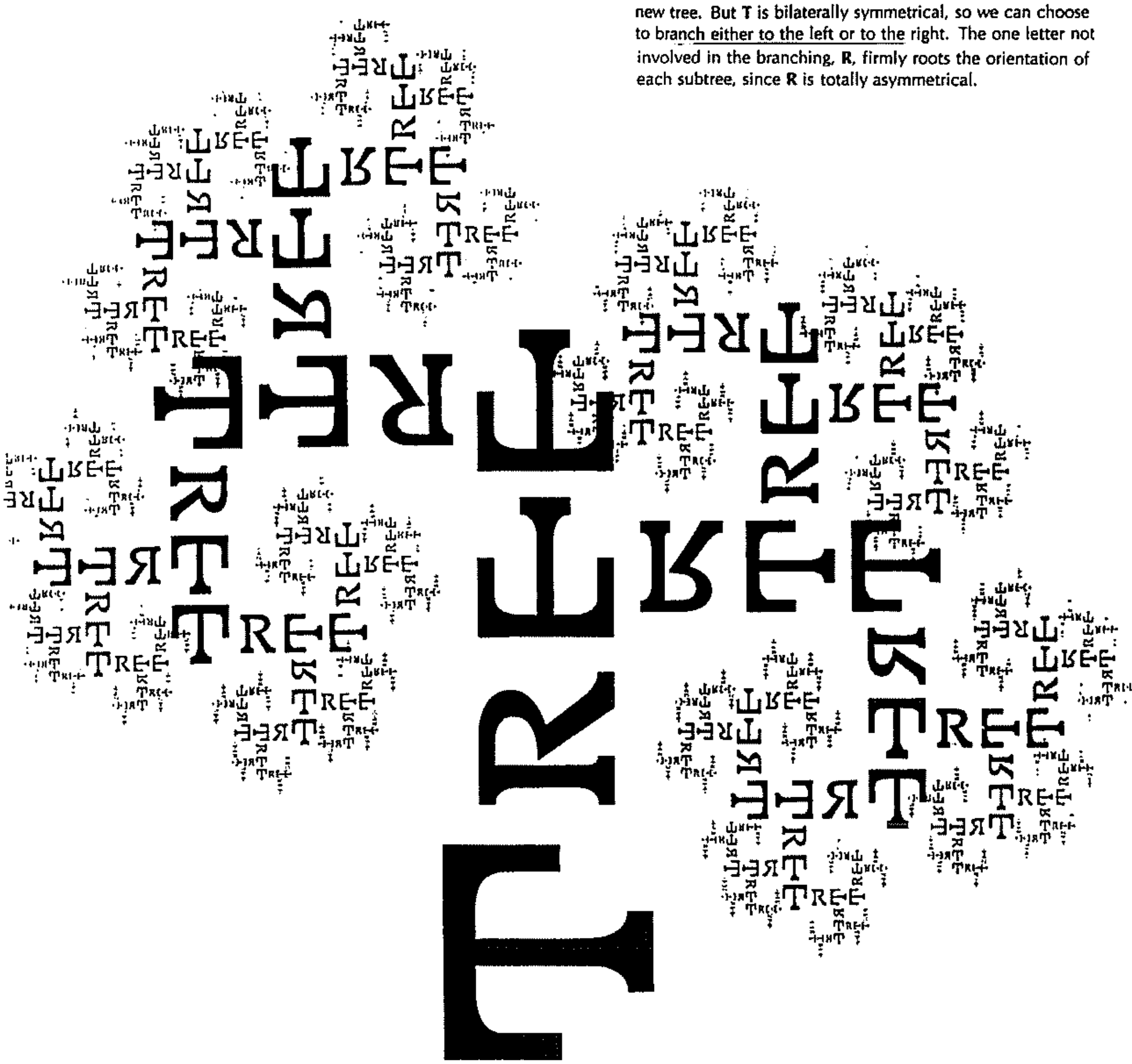
These are the first 5 stages in the construction of Sierpinski's Triangle. Can you describe the iterative process here?

Stage 7 of Sierpinski's Triangle

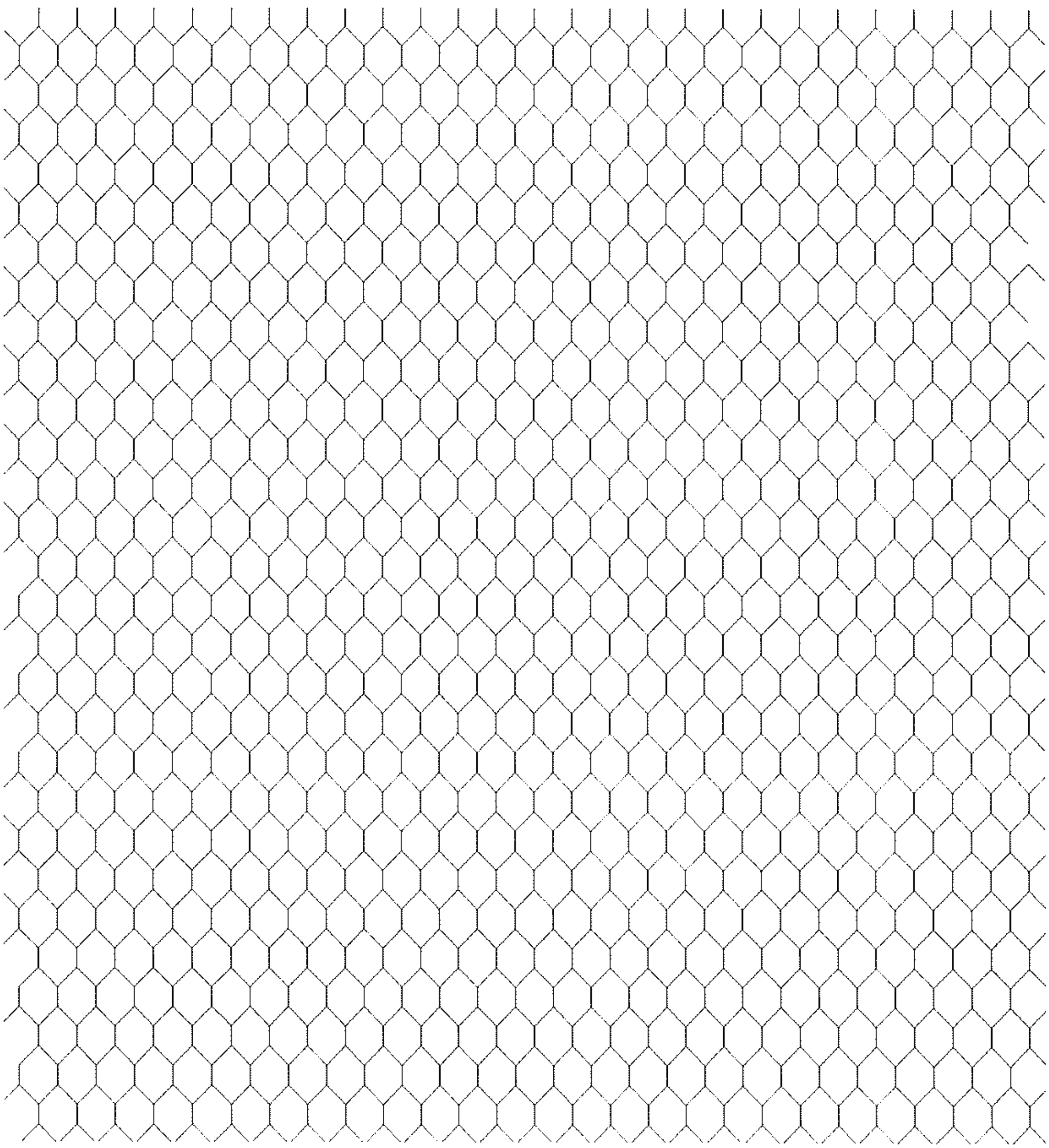


Can you give an iterative rule for generating this figure?

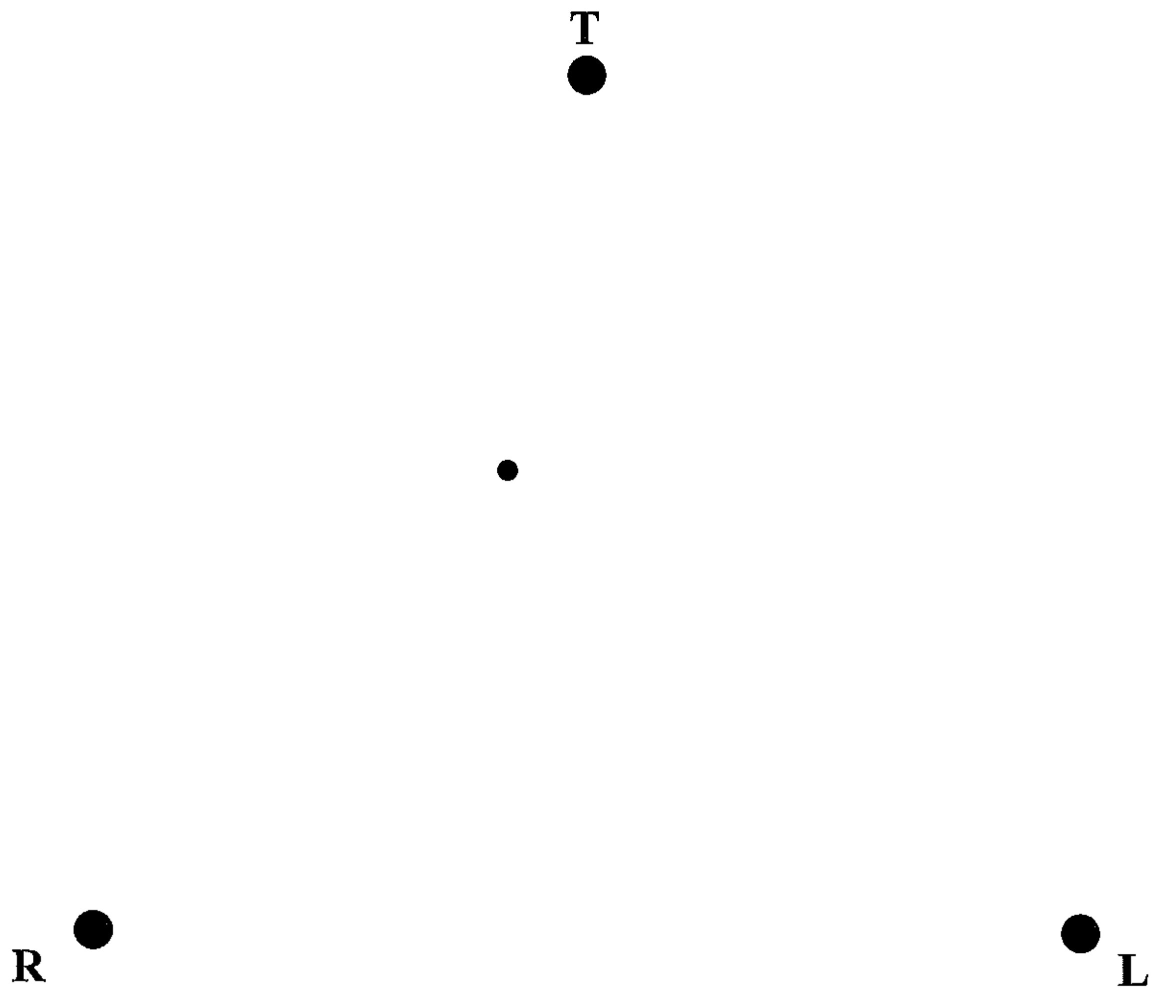
TREE (Recursion). A tree made of trees. Where do the branches end and the leaves begin? As in all recursive structures, the parts here are the same as the whole. **E** (with a long crossbar) turned sideways becomes **T**—the seed of a new tree. But **T** is bilaterally symmetrical, so we can choose to branch either to the left or to the right. The one letter not involved in the branching, **R**, firmly roots the orientation of each subtree, since **R** is totally asymmetrical.



SOME HEXAGONAL GRID PAPER



Game Board for The Chaos Game



Rules for "The Chaos Game"

1. Start with the point located in the interior of triangle labeled "TRL."

2. Toss the die.

3. A. If the result is "1" or "2", measure halfway from the current point to the vertex marked "T" and create a new point.

   B. If the result is "3" or "4", measure halfway from the current point to the vertex marked "R" and create a new point.

   C. If the result is "5" or "6", measure halfway from the current point to the vertex marked "L" and create a new point.

4. Cover the previous point with a penny so that you will remember which point is now the current point.

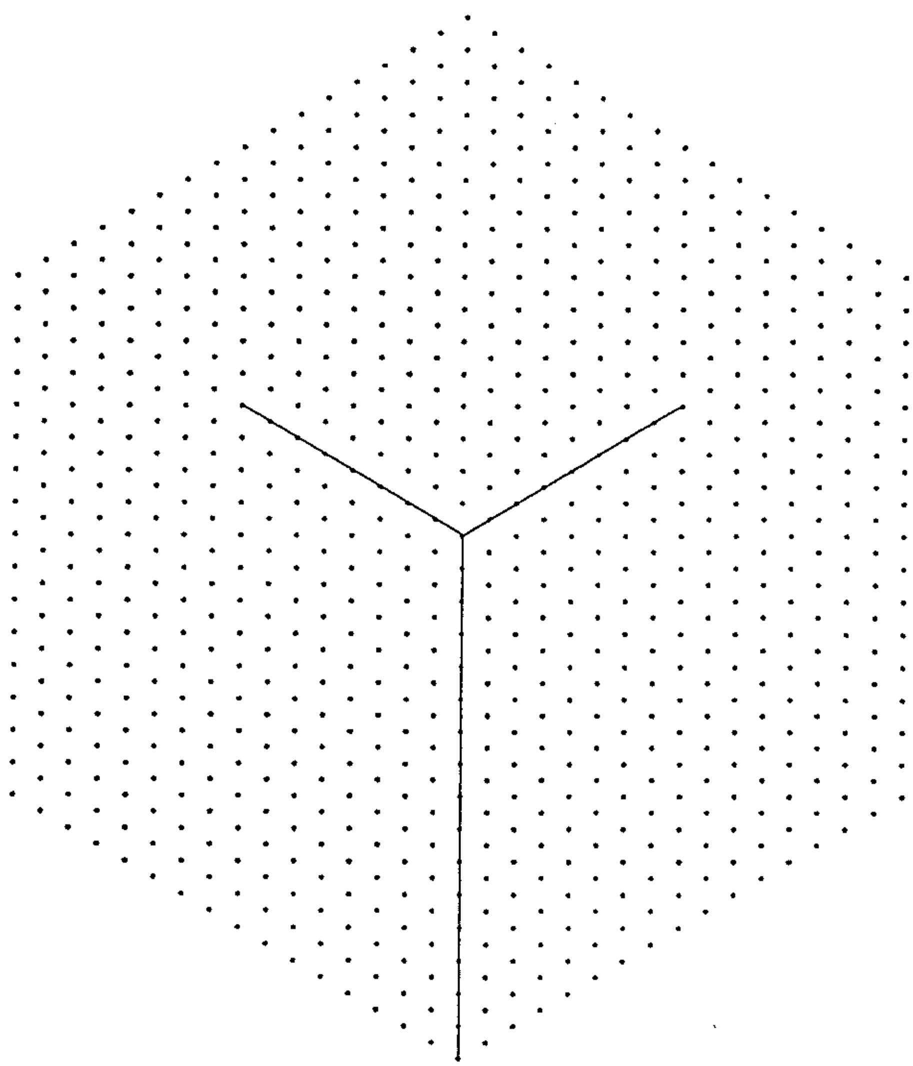
5. Repeat steps 2 to 4 another ten times.

1.5 TREES

As trees grow, they branch out. From big branches grow smaller ones. From these grow smaller ones still, and so on. Use this dot paper to draw a mathematical tree with some of the same properties as the live ones.

Construction From the endpoint of each branch, draw two new branches half as long growing off at 60° in opposite direction.

1. Stage 1 of the tree has already been drawn. Draw the four new branches for stage 2 by connecting endpoints to the appropriate dots on the grid. Draw the eight new branches for stage 3. Repeat again for stage 4. Endpoints should still be on the dots of the grid. Continue the growing process until the branches become too small to draw.

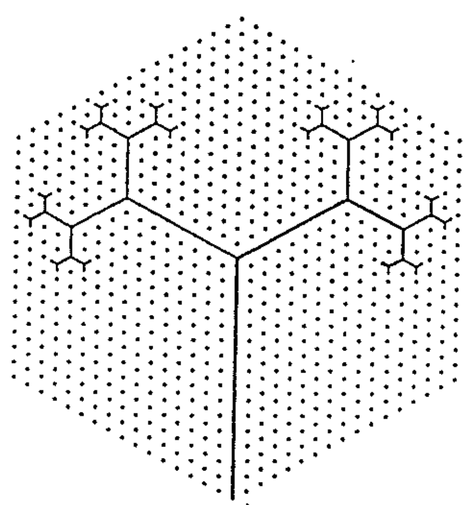


Suppose the tree starts with an initial vertical segment of 1 unit as the trunk. Imagine further that the tree continues growing branches, over and over by the process given, until fully grown. Visualize this completed tree.

2. How many branches have lengths of 1/4? of 1/16? What is the sum of the lengths of all branches 1/4 long? 1/16 long?

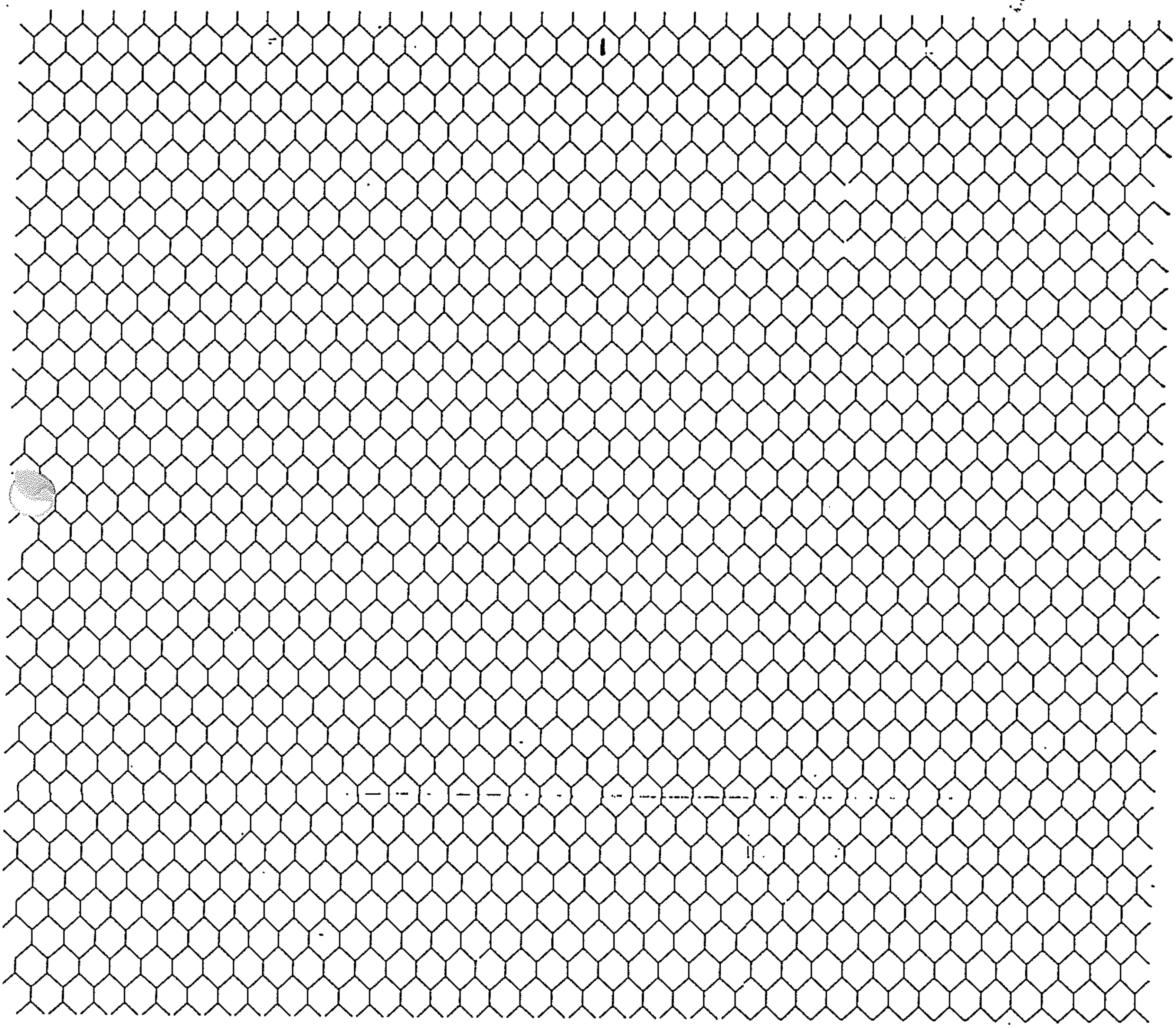
3. What is the total length of all branches of the completed tree?

4. Are there parts of the completed tree that look like the entire tree? Using the tree just drawn as a model of a fully grown tree, draw a hexagon around a part that would be an exact image of the tree itself. Draw another using a hexagon of a different size.



Handout #3 — Odds and Evens in Pascal's Triangle

SOME HEXAGONAL GRID PAPER



Session 9 — Patterns in Geometry — Exercises

Practice Problems:

1. How many shaded triangular regions are in stage 5 of the developing Sierpinski triangle? Use the idea of recursion to describe how to find the number at stage 6 from the number at stage 5.

2. How much of the area of the original triangle remains at stage 5? Use the idea of recursion to describe how to find the shaded area at stage 6 from the shaded area at stage 5.

3. Use recursion to describe how to find the perimeter around all the shaded triangular regions at stage 6 from the corresponding perimeter at stage 5.

Study Group Problems:

4. List the number of shaded triangular regions in the first 8 stages of the developing Sierpinski triangle. Take first, second, and third differences. What patterns can you find in these differences? Look carefully!

5. Complete parts 1-3 on the worksheet on exercise page <u>3</u>, taken from *Fractals for the Classroom.* At what stage will there be enough small shaded triangular regions so that one could be given to every person living in New Jersey? in the United States? in the world? (The population of NJ is about 20 million, the population of the US is about 280 million, and the population of the world is about 5.5 billion.)

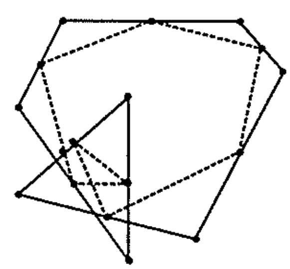
6. Start with an equilateral triangle with 4-inch sides, and generate the first five stages of the Sierpinski triangle (as on exercise page <u>3</u>). Make a chart which gives the perimeter around all the shaded triangular parts at each stage. From this pattern, can you tell what this total perimeter will be at each successive stage of the construction? At what stage will the perimeter around all the shaded triangular parts first be greater than the distance around the earth at the equator, which is about 25,000 miles?

7. Complete parts 4-6 on the worksheet on exercise page <u>3</u>. At what stage will the shaded area become less than one square millimeter, assuming that the area of the initial triangle is one square meter?

8. Complete the dot-paper drawing titled *Triangle Variation,* found on exercise page <u>4</u>, and complete questions 1-6, found on exercise page <u>5</u>.

9. Consider the four figures on exercise page <u>5</u>. Stage 0 has three corners, stage 1 has 10 corners, and if you count carefully, you'll find that stage 2 has 52 corners.
   a. How many corners do stages 3 and 4 have?
   b. What is the recursive rule for generating this "number of corners" sequence?

10. On a piece of paper draw 8 dots in any chaotic fashion, number them 1 to 8 and connect the dots in order — 1 to 2, 2 to 3, etc..., 7 to 8 and 8 to 1, to create an octagon. (It will work best if you start with a large figure.) Then iterate the following procedure: *Draw a dot at the midpoint of each edge of the most recent octagon, and connect the dots on adjacent edges to create a new octagon.* One iteration is shown to the right, where the dotted figure is the most recent octagon. What do you notice happening as you iterate this procedure? Try this with more complex examples and see if it still works.



11. Read and then complete the page titled *Cellular Automata* on exercise page 6.

Extension Problems:

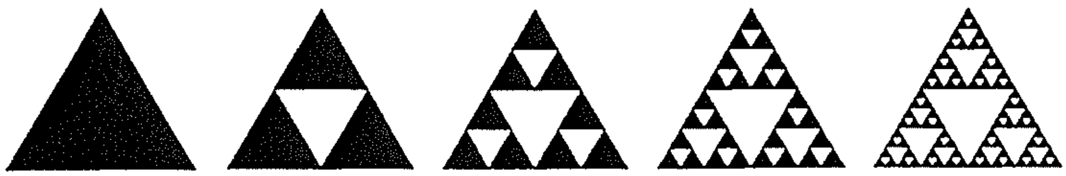
12. Use dot paper to draw the figure referred to in question 2 of *Triangle Variation* from exercise page 4 introduced on Day 10. Describe the number patterns generated by counting triangles and by counting holes at each stage. Dot paper is provided on exercise page 7.

13. Think of the *Triangle Variation* activity from exercise page 4 yet again. Give a geometric and numerical analysis of the stages of a similar fractal generation where only the three corners were used repeatedly in the building process. How will the results compare to the Sierpinski triangle? More dot paper is provided on exercise page 8.

1.2 NUMBER PATTERNS WITH VARIATIONS 1.2A

This activity explores some of the number patterns found in the Sierpinski triangle.

DIRECTIONS---The first four stages of the construction of the Sierpinski triangle are shown below. In subsequent stages, the subdivision continues into smaller and smaller triangles. Use these figures to explore number patterns that emerge as the Sierpinski triangle is developed through successive iterations.



Stage 0 Stage 1 Stage 2 Stage 3 Stage 4

NUMBER OF TRIANGLES

1. Count the number of shaded triangles at each stage 0 through 4.

STAGE	0	1	2	3	4	5	...	n
NUMBER	1							

2. Extend the pattern to predict the number of triangles at stage 5.
   What constant multiplier can be used to go from one stage to the next?

3. Generalize to find the number of triangles for level n.
   As n becomes large without bound, what happens to the number of triangles?

AREA OF TRIANGLES

4. Let the area at stage 0 be 1. Find the total shaded areas at stages 1 through 4.

STAGE	0	1	2	3	4	5	...	n
AREA	1							

5. Extend the pattern to predict the total area at stage 5.
   What constant multiplier can be used to go from one stage to the next?

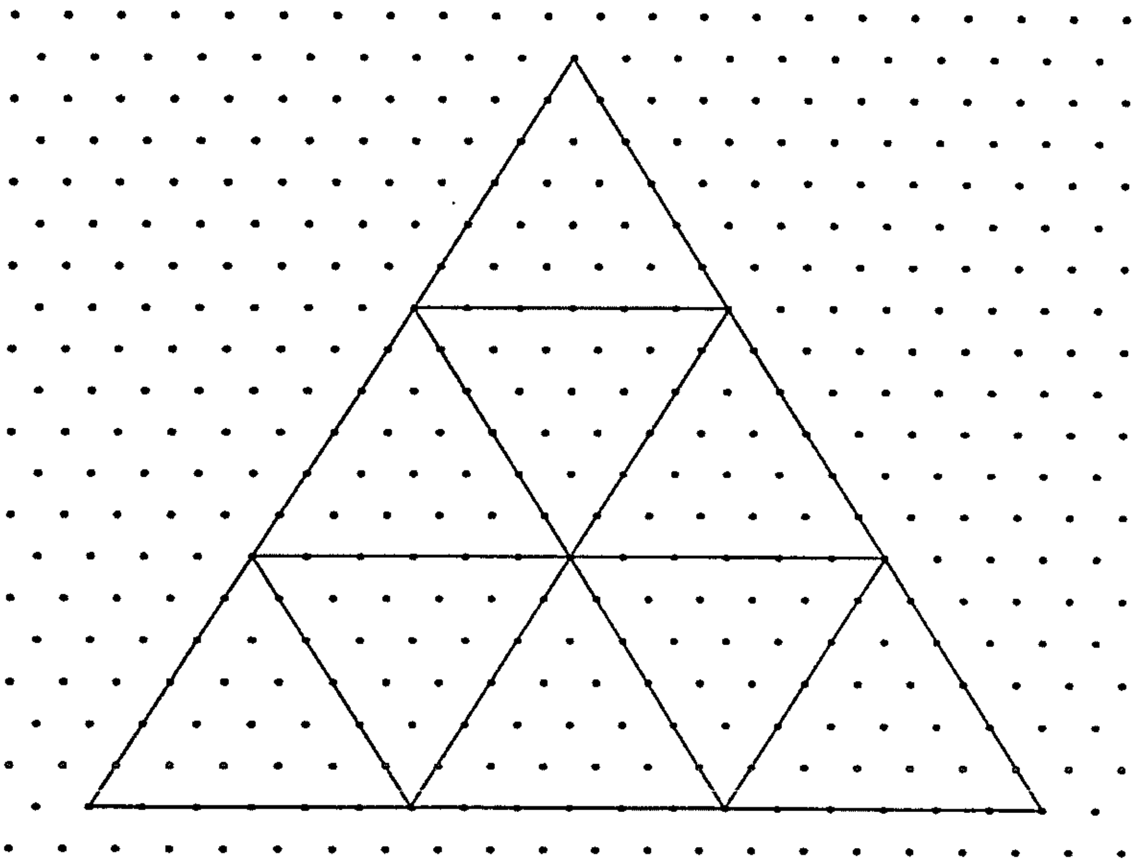
6. Generalize to find the total area at stage n.
   As n becomes large without bound, what happens to the shaded area?

TRIANGLE VARIATION 1.1B

When repeated over and over, this construct variation generates another fractal.

Construction — Connect trisection points on the sides as shown, ~~keeping~~ shading only the six border subtriangles.

In this variation, the sides of the triangle are divided into thirds. Repeat the process through a second iteration using exactly the same procedure in each of the six border subtriangles shown in this first stage. Count dots carefully. Each vertex of each of the 36 congruent subtriangles that emerge at the second stage are on dots of the grid paper. Shade in these triangles.



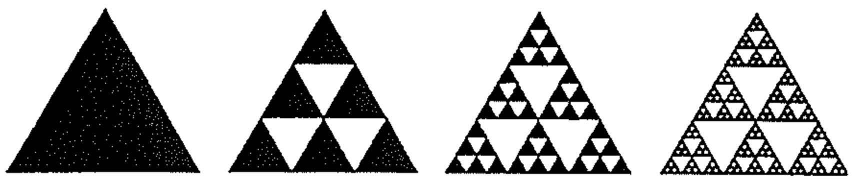
1. Imagine repeating the process over and over. At each stage, each triangle is transformed into six new subtriangles with sides one-third as long. Describe what you would see of the original triangle if the process were continued on without end.

2. Change the algorithm from ~~keeping~~ shading the six border subtriangles to ~~keeping~~ shading the three inner ones. What kind of figure would emerge after two iterations?

NUMBER PATTERNS FROM TRISECTION 1.2B

Varying the construction algorithm varies not only the figures but the number patterns produced as well.

DIRECTIONS The first three stages of the triangle construction using the trisection algorithm are shown below. In subsequent levels, smaller and smaller subtriangles are formed. Use these figures to explore the number patterns that emerge as more and more iterations are performed on the figure.



Stage 0 Stage 1 Stage 2 Stage 3

NUMBER OF TRIANGLES

1. Count the number of shaded triangles at each stage 0 through 3.

STAGE	0	1	2	3	4	5	...	n
NUMBER	1							

2. Extend the pattern to predict the number of shaded triangles at stages 4 and 5. What constant multiplier can be used to go from one stage to the next? As n becomes large without bound, what happens to the number of shaded triangles?

3. Compare this number pattern to that for the number of shaded triangles for stages in the Sierpinski triangle. In which case are the numbers increasing more rapidly?

AREA OF TRIANGLES

4. Let the area at stage 0 be 1. Find the total shaded area at stages 1 through 3.

STAGE	0	1	2	3	4	5	...	n
AREA	1							

5. Extend the pattern to predict the total area at stages 4 and 5. What constant multiplier can be used to go from one stage to the next? As n becomes large without bound, what happens to the shaded area?

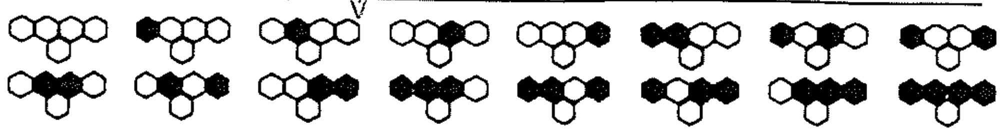
6. Compare this number pattern to that for the areas for stages in the Sierpinski triangle. In which case are the areas decreasing more rapidly?

CELLULAR AUTOMATA

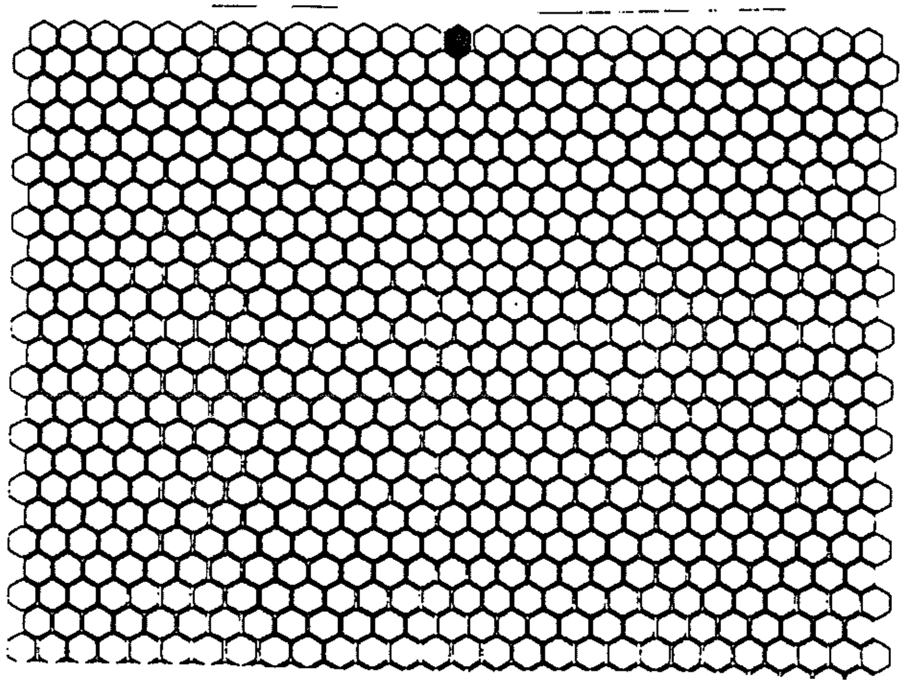
A coloring look-up table supplies a visual definition of all the rules needed to color any particular cell based upon the colors of the cells immediately above it.

In the exercise below, the top row has been colored---all white except for one black cell. Once a row has been colored, you can determine the color of each cell in the next row by looking at the colors of the 4 cells immediately above it, looking for that pattern in the look-up table that you defined in part 1, and coloring the cell accordingly. Different ways of defining the look-up table will yield different final patterns.

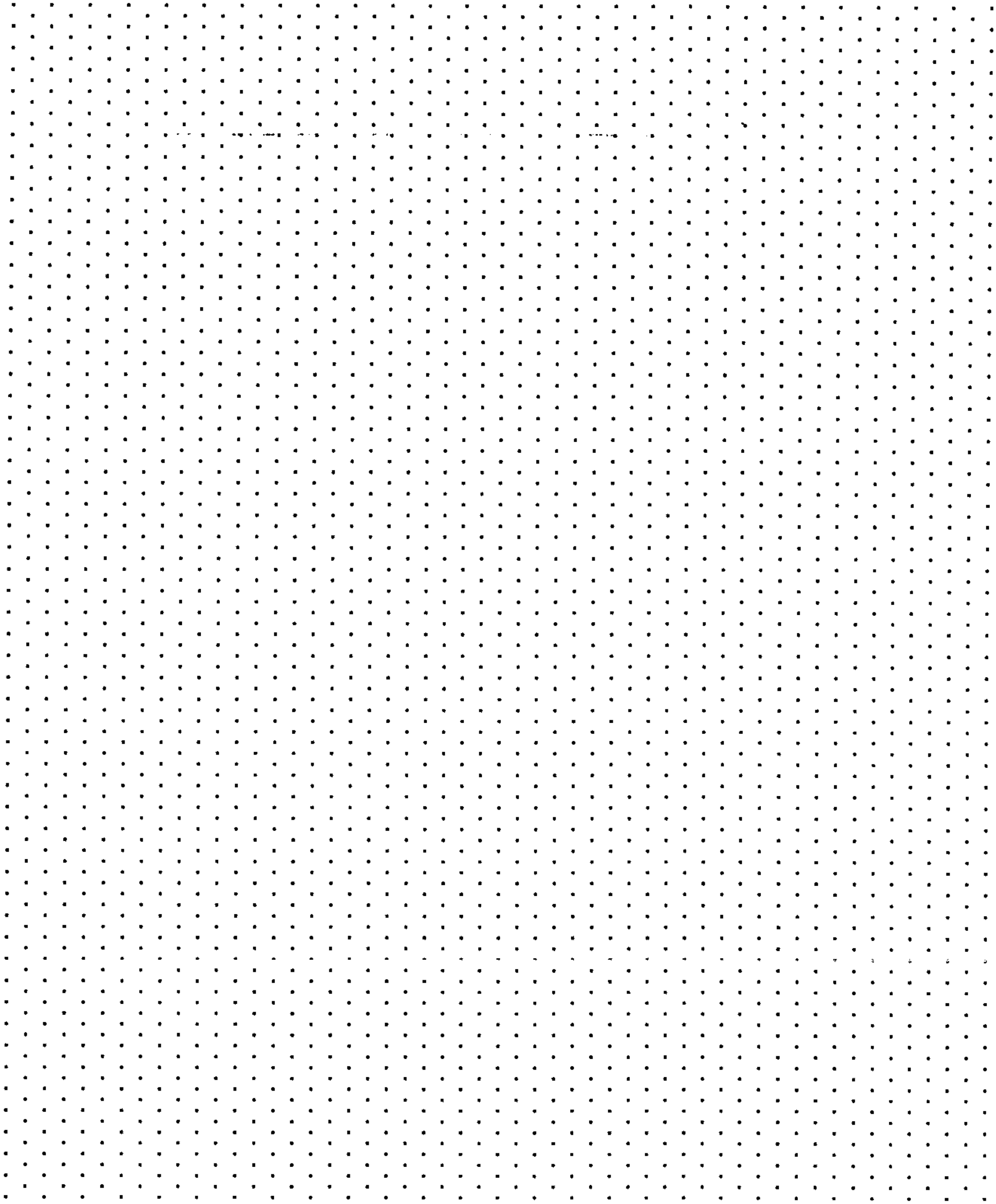
1. Complete the coloring look-up table by making your own choices for each of the 16 bottom cells in this table.

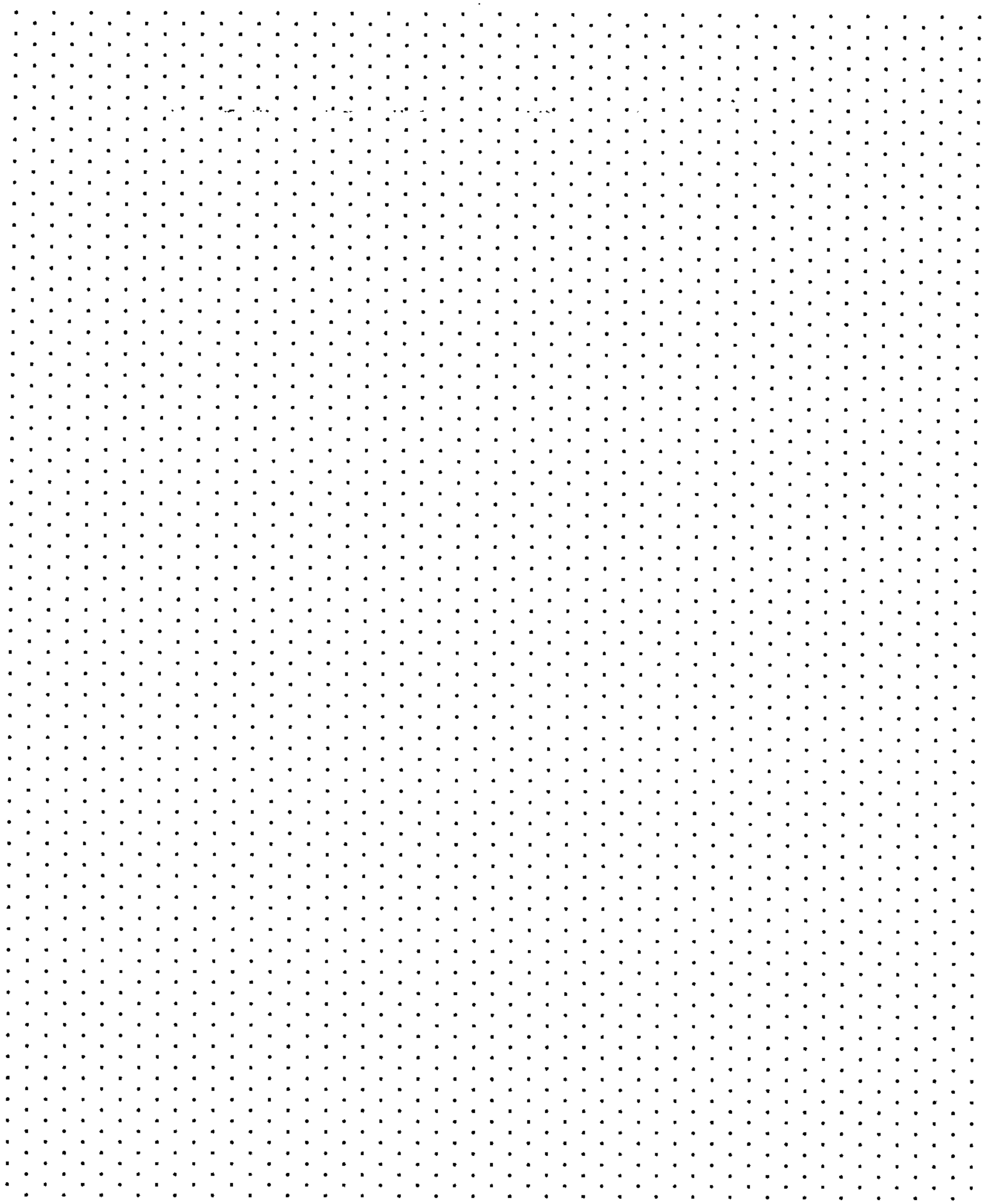


2. Apply the rules from your completed coloring look-up table to the cellular array given below. Do any geometric patterns or structures emerge?



3. Why are there 16 patterns to be accounted for in question 1? How many different ways could you have completed this coloring look-up table?





Resource Book

Workshop 9: Patterns in Geometry

Table of Contents

Resource Book

Workshop 9: Patterns in Geometry

Mathematical Background

- Congruency: Geometric figures are congruent if they have the same shape and the same size.

- Similarity: Geometric figures are similar if they have the same shape but not necessarily the same size.

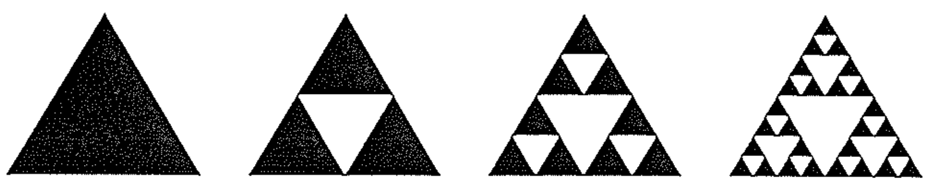
- Self-similarity: A geometric figure is self-similar if it contains reduced, repeated images of itself at all scales. For example, the Sierpinski triangle is self-similar.

- Fractals: Fractals emerge from certain iterative, geometric procedures. The single most characteristic property of most fractals is *self-similarity.*

  Visual characteristics included in the structure of a fractal are that it is
  Endlessly repeating,
  Ever diminishing, and
  Infinitely complex.

  The complexity of a fractal is reported as its *dimension*, the precise definition of which is rather complicated. Fractal dimensions are not the familiar Euclidean dimensions of 1, 2 and 3.

  This series of figures shows the initial stages in the generation of the *Sierpinski Triangle.* The actual fractal itself is the final state. Its fractal dimension is about 1.58.



Stage 0 Stage 1 Stage 2 Stage 3

Resource Book

Workshop 9: Patterns in Geometry

Workshop Outline

1. Geometric Patterns in Nature
   a. We saw some examples of fractal-like structures in nature.
      i. Queen Anne's Lace was circular in structure, and we saw that it was composed of many smaller circles, which each looked like small Queen Anne's Laces.
      ii. A fern branch had many small branches which all looked like smaller versions of the whole branch
      iii. A computer-generated tree consisted of a trunk with two branches, each of which had two branches, and each of those had two branches, and so-on. The result was rather like a real tree.
   b. We called this notion of "parts looking like reduced copies of the whole" *self-similarity.* Our ideas from here on were attempts to model mathematically what we saw happening in nature.
   c. We defined a fractal to be an object or figure which was endlessly repeating, ever diminishing and infinitely complex.

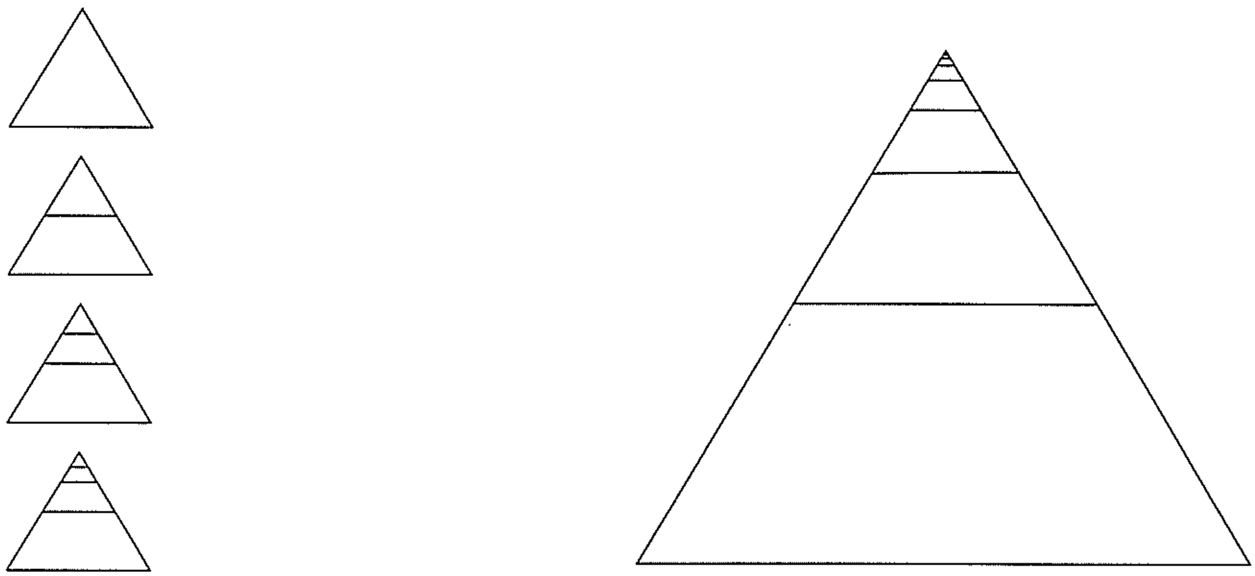
2. Geometric Iteration
   a. We folded a triangular piece of paper, always folding the top of the newest, smallest triangle down to the center of the opposite side, creating a crease, and then unfolding. This resulted in a figure which was self-similar only at the top; moreover, it could be considered self-similar only if we consider the "end stage" of the figure. The end stage, which can't be drawn or folded, is what we imagine if we had done infinitely many stages.
   b. We created fractal trees by starting with a 16-unit-long trunk and adding two branches, 60 degrees apart in opposite directions, half as long as the trunk. Then we iterated this procedure, adding to each branch, two branches, 60 degrees apart in opposite directions, half as long. The result looked rather like a tree.

3. The Sierpinski Triangle, and friends
   a. We saw a sequence of triangulated triangles with the up-pointing triangles shaded. Participants were asked to find a rule for generating this sequence of figures.
   b. Then we generated the Sierpinski Triangle
      i. We considered a sequence of figures generated by starting with a solid equilateral triangle, and at each stage removed the middle quarter from each solid triangle we saw
      ii. The figure that results from continuing this process indefinitely is called the Sierpinski Triangle. This is a fractal, and has self-similarity at *every* point.
      iii. As you construct later and later stages of the Sierpinski Triangle, the area decreases to zero, but the perimeter increases without bound.
   c. The TREE picture by Scott Kim (http://www.scottkim.com) was made up of many rotated and flipped copies of the word "TREE," and the resulting figure looked like a tree. This fractal exhibited a great deal of self-similarity.
   d. We colored the odds and evens of Pascal's triangle, and discovered the Sierpinski Triangle!
   e. We modeled the Sierpinski Triangle with pattern blocks, and with large paper triangles, and then we put pattern blocks inside the triangles to get a stage 7 version of Sierpinski Triangle.
   f. Finally, we played the Chaos Game, and out came ... The Sierpinski Triangle!

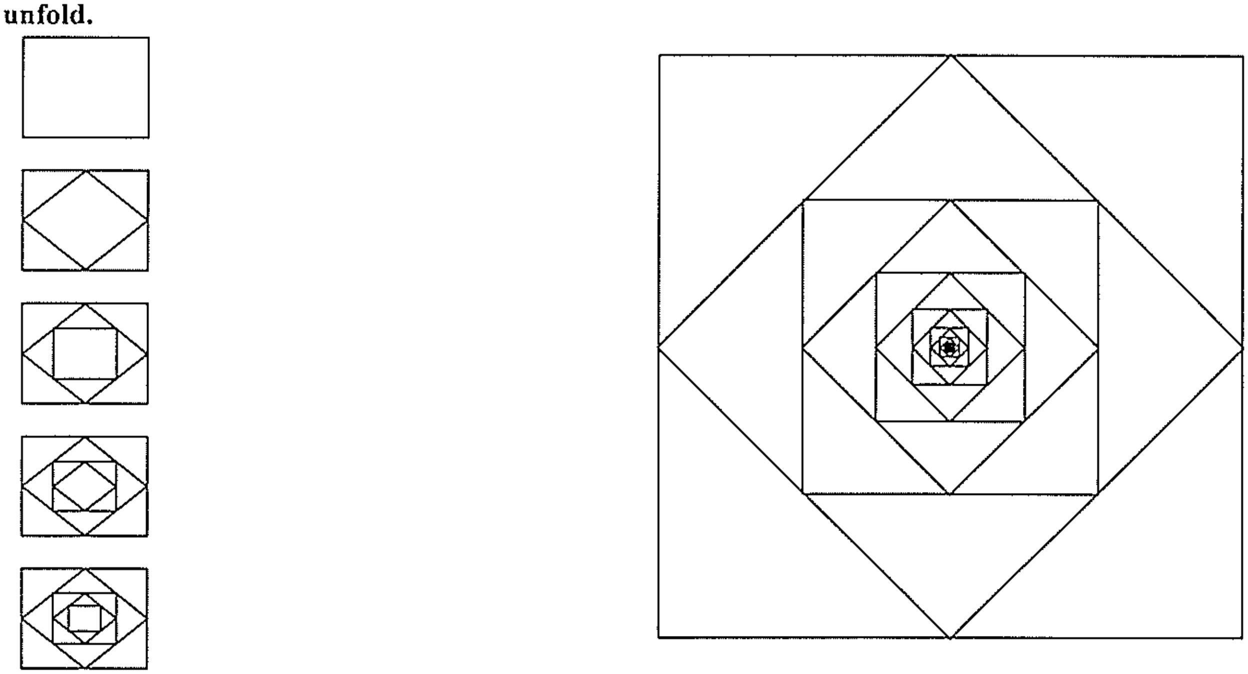
Workshop 9: Patterns in Geometry

Some iterative paper-folding procedures together with their "end stages."

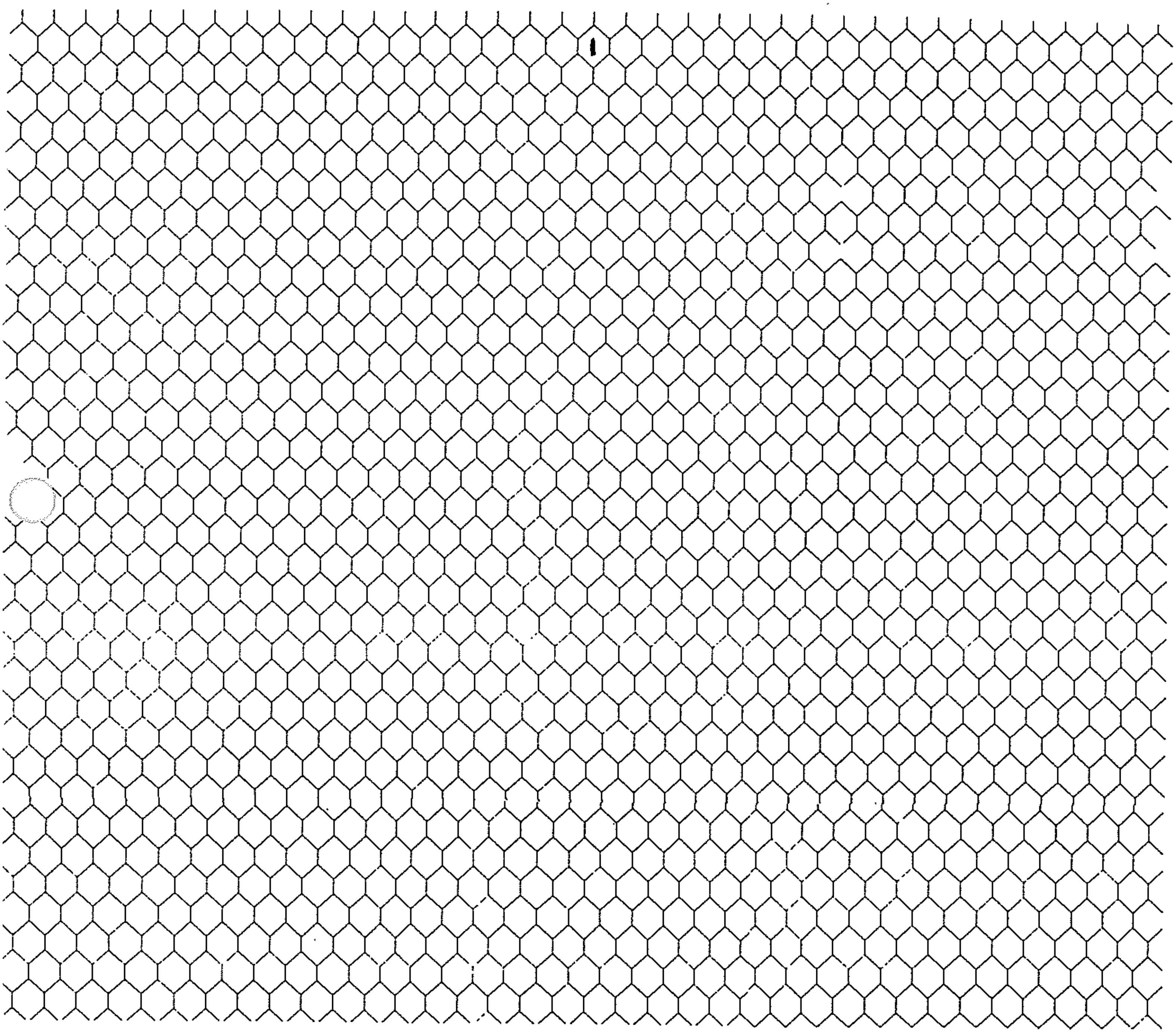
Triangle Pattern: Start with an equilateral triangle-shaped piece of paper. When you see an empty triangle, fold the top corner down to the middle of the opposite side, crease, and unfold.



Square Pattern: Start with a square sheet of paper with the center identified by light folds. At each stage, consider the smallest empty square and fold its corners down to the center of that square. Then make crease lines along the folds, but only within that smallest square, and then unfold.



SOME HEXAGONAL GRID PAPER



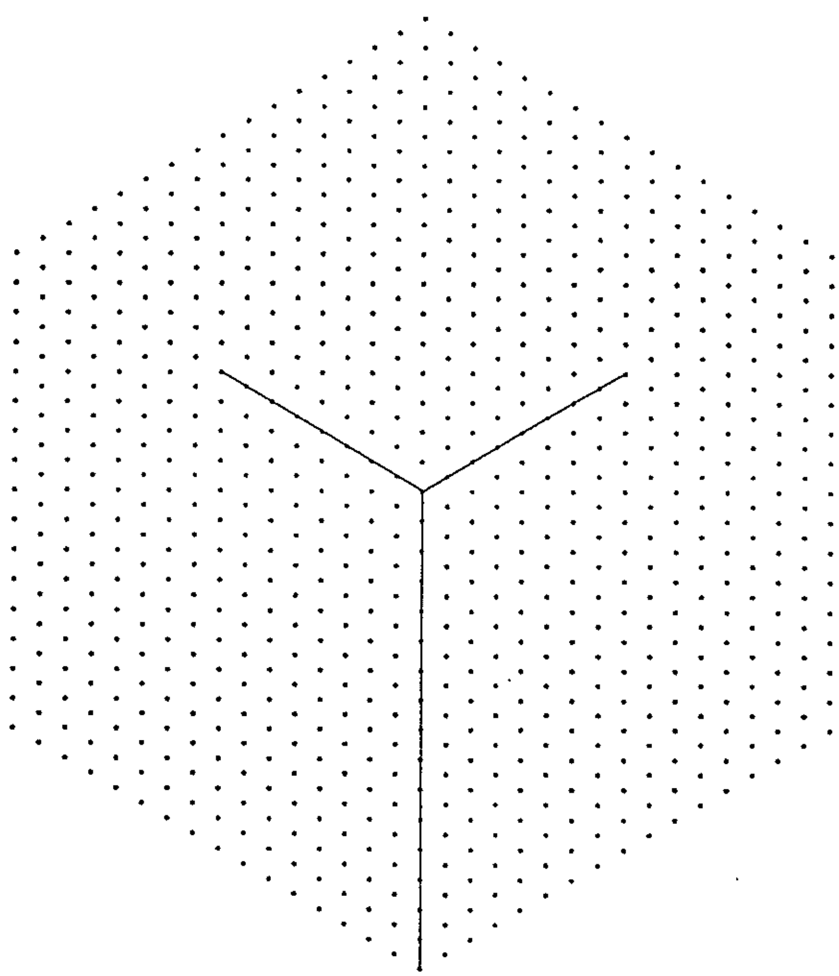
Section 9: Patterns in Geometry

1.5 TREES **1.5A**

As trees grow, they branch out. From big branches grow smaller ones. From these grow smaller ones still, and so on. Use this dot paper to draw a mathematical tree with some of the same properties as the live ones.

Construction From the endpoint of each branch, draw two new branches half as long growing off at 60° in opposite direction.

1. Stage 1 of the tree has already been drawn. Draw the four new branches for stage 2 by connecting endpoints to the appropriate dots on the grid. Draw the eight new branches for stage 3. Repeat again for stage 4. Endpoints should still be on the dots of the grid. Continue the growing process until the branches become too small to draw.



1.5B

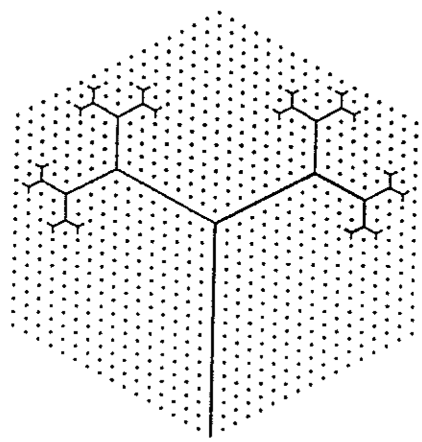
Suppose the tree starts with an initial vertical segment of 1 unit as the trunk. Imagine further that the tree continues growing branches, over and over by the process given, until fully grown. Visualize this completed tree.

2. How many branches have lengths of 1/4? of 1/16? What is the sum of the lengths of all branches 1/4 long? 1/16 long?

3. What is the total length of all branches of the completed tree?

4. Are there parts of the completed tree that look like the entire tree? Using the tree just drawn as a model of a fully grown tree, draw a hexagon around a part that would be an exact image of the tree itself. Draw another using a hexagon of a different size.

One interesting shape found on the completed tree is a *spiral*. Start at the base of the tree and turn right at each and every junction point. Note how these particular branches trace out a spiral.



5. Find another spiral that is a reflection of the one just described. What is the length of this spiral?

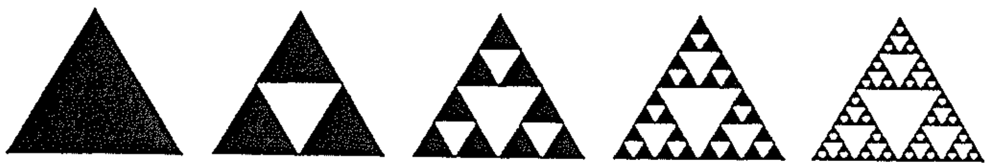
6. Find four spirals with half the length of the one just described. How many spirals in the tree have one-quarter the length of the original?

7. Consider all such spirals of all sizes that can be found on the tree. They all do not have the same length. Are they all exact replicas of each other except for size?

1.2 NUMBER PATTERNS WITH VARIATIONS 1.2A

This activity explores some of the number patterns found in the Sierpinski triangle.

DIRECTIONS The first four stages of the construction of the Sierpinski triangle are shown below. In subsequent stages, the subdivision continues into smaller and smaller triangles. Use these figures to explore number patterns that emerge as the Sierpinski triangle is developed through successive iterations.



NUMBER OF TRIANGLES

1. Count the number of shaded triangles at each stage 0 through 4.

STAGE	0	1	2	3	4	5	...	n
NUMBER	1							

2. Extend the pattern to predict the number of triangles at stage 5.
   What constant multiplier can be used to go from one stage to the next?

3. Generalize to find the number of triangles for level n.
   As n becomes large without bound, what happens to the number of triangles?

AREA OF TRIANGLES

4. Let the area at stage 0 be 1. Find the total shaded areas at stages 1 through 4.

STAGE	0	1	2	3	4	5	...	n
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5. Extend the pattern to predict the total area at stage 5.
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NUMBER PATTERNS FROM TRISECTION 1.2B

Varying the construction algorithm varies not only the figures but the number patterns produced as well.

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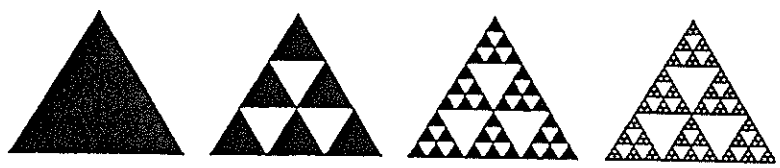

Stage 0 Stage 1 Stage 2 Stage 3

NUMBER OF TRIANGLES

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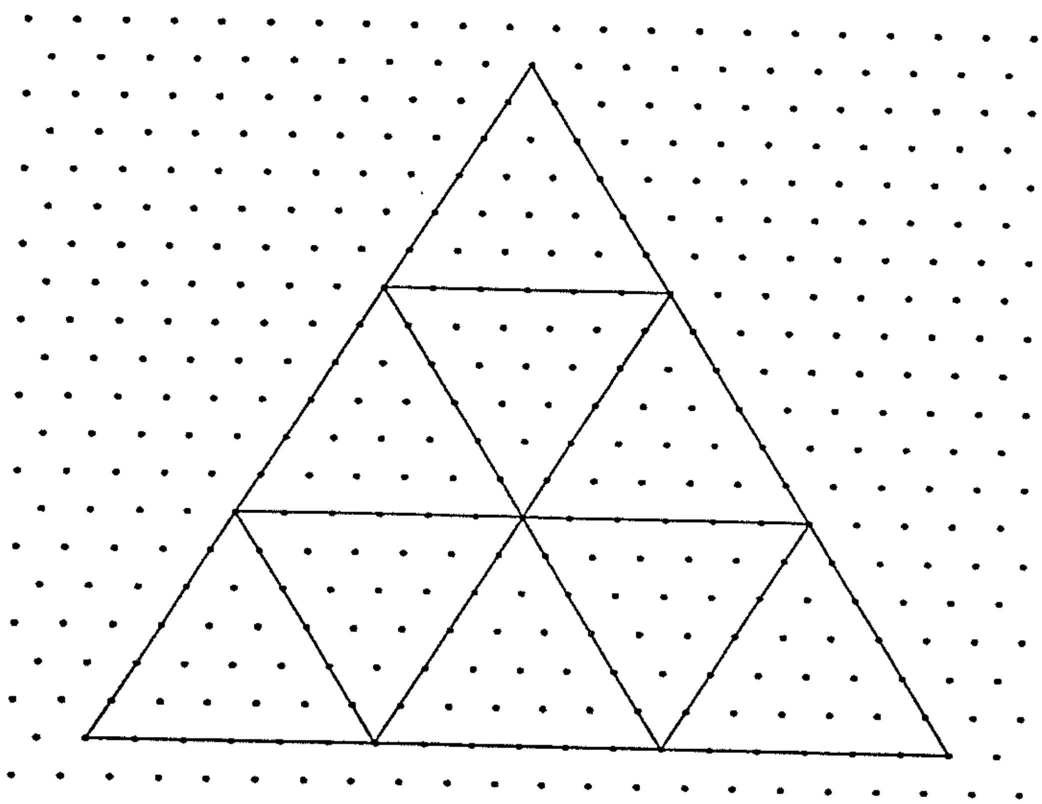
TRIANGLE VARIATION

1.1B

When repeated over and over, this construct variation generates another fractal.

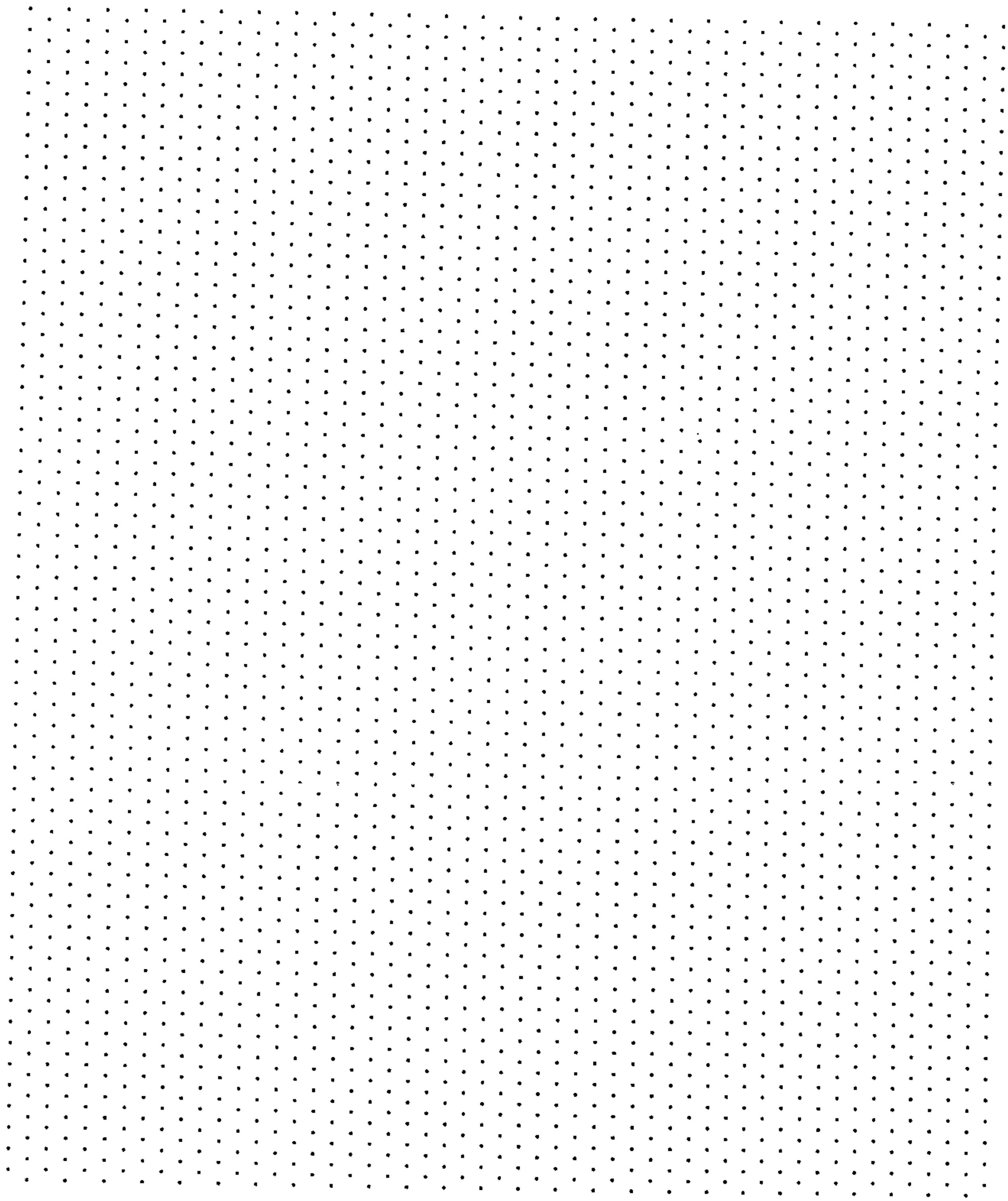
Construction Connect trisection points on the sides as shown, keeping only the six border subtriangles.

In this variation, the sides of the triangle are divided into thirds. Repeat the process through a second iteration using exactly the same procedure in each of the six border subtriangles shown in this first stage. Count dots carefully. Each vertex of each of the 36 congruent subtriangles that emerge at the second stage are on dots of the grid paper. Shade in these triangles.



1. Imagine repeating the process over and over. At each stage, each triangle is transformed into six new subtriangles with sides one-third as long. Describe what you would see of the original triangle if the process were continued on without end.
2. Change the algorithm from keeping the six border subtriangles to keeping the three inner ones. What kind of figure would emerge after two iterations?

Section 9: Patterns in Geometry



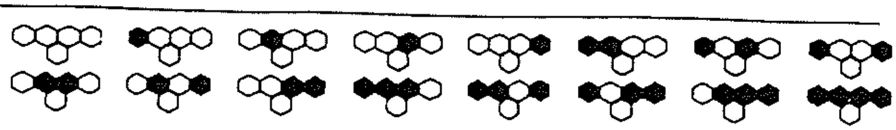
Section 9: Patterns in Geometry

CELLULAR AUTOMATA

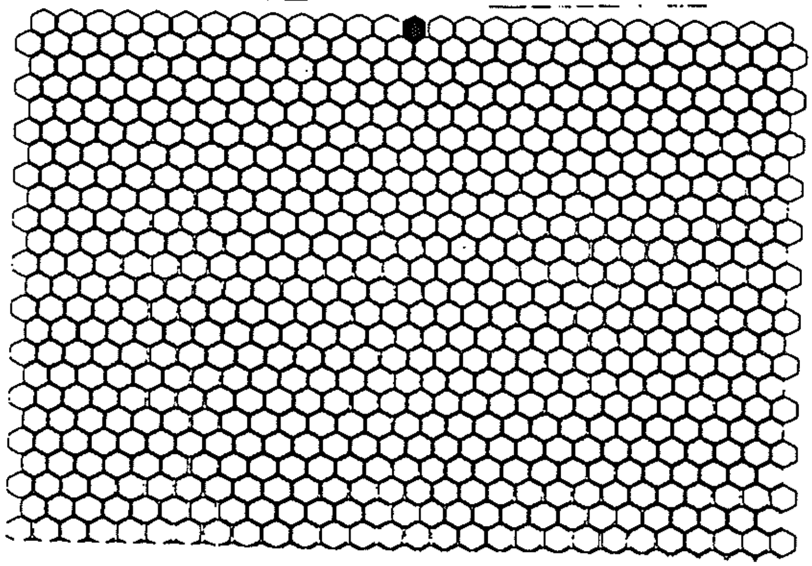
A coloring look-up table supplies a visual definition of all the rules needed to color any particular cell based upon the colors of the cells immediately above it.

In the exercise below, the top row has been colored---all white except for one black cell. Once a row has been colored, you can determine the color of each cell in the next row by looking at the colors of the 4 cells immediately above it, looking for that pattern in the look-up table that you defined in part 1, and coloring the cell accordingly. Different ways of defining the look-up table will yield different final patterns.

1. Complete the coloring look-up table by making your own choices for each of the 16 bottom cells in this table.

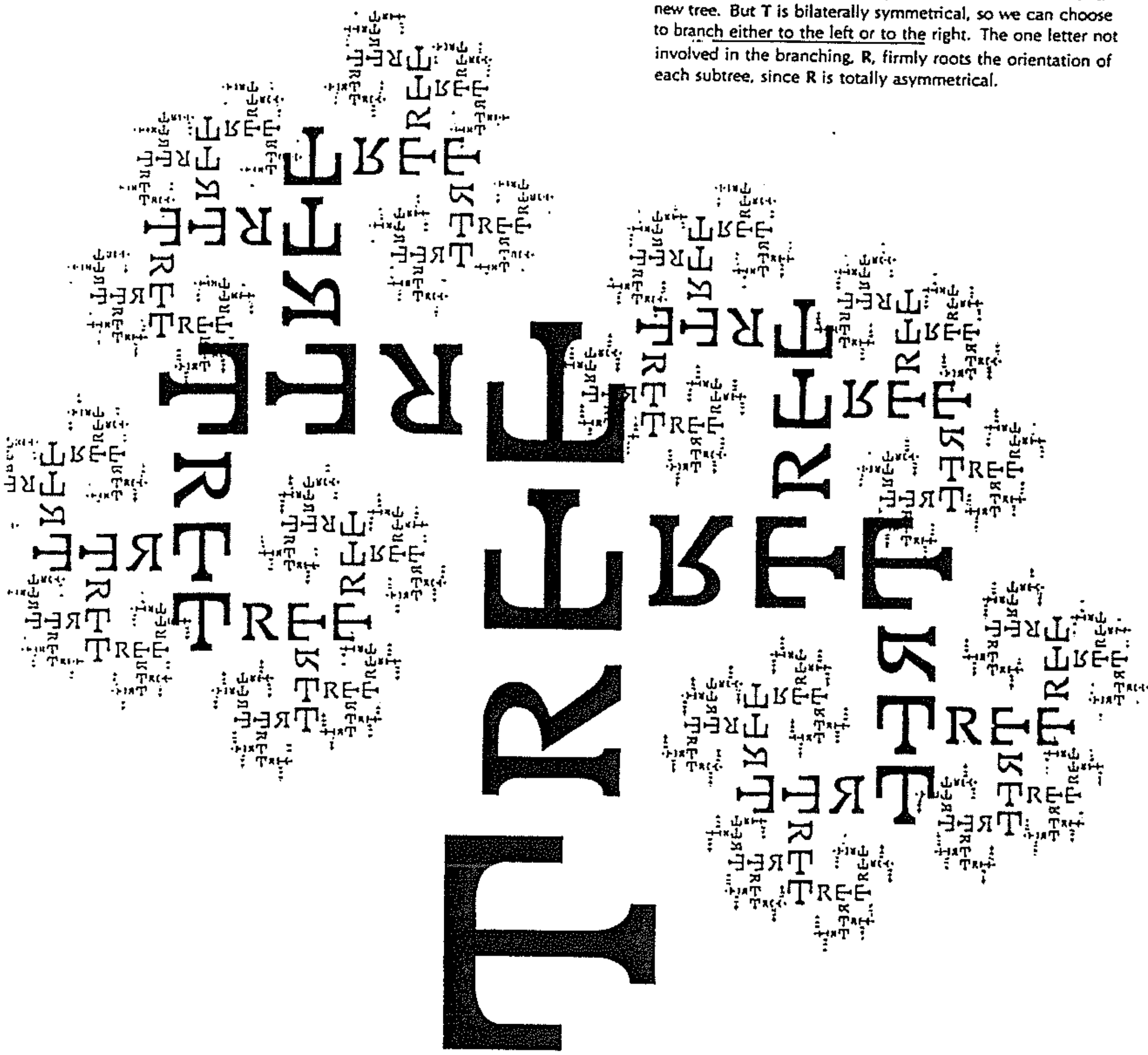


2. Apply the rules from your completed coloring look-up table to the cellular array given below. Do any geometric patterns or structures emerge?



3. Why are there 16 patterns to be accounted for in question 1? How many different ways could you have completed this coloring look-up table?

TREE (Recursion). A tree made of trees. Where do the branches end and the leaves begin? As in all recursive structures, the parts here are the same as the whole. E (with a long crossbar) turned sideways becomes T—the seed of a new tree. But T is bilaterally symmetrical, so we can choose to branch either to the left or to the right. The one letter not involved in the branching, R, firmly roots the orientation of each subtree, since R is totally asymmetrical.

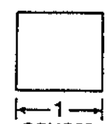


NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS November 1991

Fracturing Our Ideas about Dimension

Most of the mathematical objects you have studied have dimensions that are whole numbers. For example, such solids as cubes and icosahedrons have dimension three. Squares, triangles, and many other planar figures are two-dimensional. Lines are one-dimensional, and points have dimension zero.

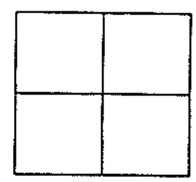
Consider a square with side of length one. Gather several of these squares by cutting them out or using patterning blocks.



square

1. What is the least number of these squares that can be put together edge to edge to form a larger square? ________

The *size* of a figure is calculated by counting the number of *replicas* (small pieces) that make it up. Here, a replica is the original square with edges of length one. The original square is made up of one small square, so its size is one.



2. What is the size of the new square? ________
3. What is the length of each edge of the new square? ________

Similar figures have the same shape but are not necessarily the same size. The *scale factor* between two similar figures can be found by calculating the ratio of corresponding edges:

$$\frac{\text{new length}}{\text{old length}}$$

4. What is the scale factor between the large square and the small square? ________
5. Find the ratio

$$\frac{\text{new size}}{\text{old size}}$$

for the two squares. ________

6. Form an even larger square that is three units long on each edge. Compare this square to the small square. What is the scale factor between the two squares? ________ What is the ratio of new size to old size? ________
7. Form an even larger square that is four units long on each edge. Compare this square to the small square. What is the scale factor between the two squares? ________ What is the ratio of the new size to the old size? ________
8. Complete the table for squares.

Scale factor	2	3	4	5	6	10
Ratio of new size to old size						

9. How are the two rows in the table related?

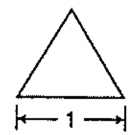
____
____
____

The editors wish to thank Tami Martin, School of Education, Boston University, Boston, MA 02215, for writing this issue of *NCTM Student Math Notes*.

Section 9: Patterns in Geometry

Fracturing Our Ideas about Dimension—Continued

Consider an equilateral triangle. The length of an edge of the triangle is one unit. The size of this triangle is one.



Equilateral triangle

10. What is the least number of equilateral triangles that can be put together edge to edge to form a similar larger triangle? ________

11. Complete the table for triangles.

Scale factor	2	3	4	5	6	10
Ratio of new size to old size						

12. How does the relationship between the two rows in this table compare with the one you found in the table for squares? ________

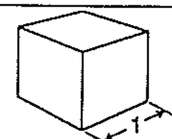
One way to define the dimension, d, of a figure relates the scale factor, the new size, and the old size:

$$(\text{scale factor})^d = \frac{\text{new size}}{\text{old size}}$$

Using a scale factor of two for squares or equilateral triangles, we can see that $2^d = 4/1$, that is, $2^d = 4$. Since $2^2 = 4$, the dimension, d, must be two. This definition of dimension confirms what we already know—that squares and equilateral triangles are two-dimensional figures.

13. Use this definition of dimension and your completed tables to confirm that the square and the equilateral triangle are two-dimensional figures for scale factors other than two. ________

Consider a cube, with edges of length one. Let the size of the cube be one.



cube

14. What is the least number of these cubes that can be put together face to face to form a larger cube? ________

15. What is the scale factor between these two cubes? ________ What is the ratio of the new size to the old size for the two cubes? ________

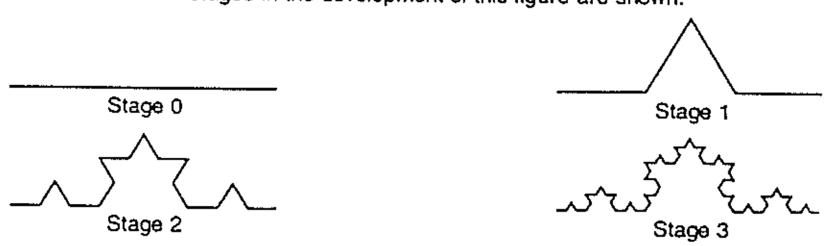
16. Complete the table for cubes.

Scale factor	2	3	4	5	6	10
Ratio of new size to old size						

17. How are the two rows in the table related?

18. Use the definition of dimension and a scale factor of two to verify that a cube is a three-dimensional object.

We have explored scale factors and sizes associated with two- and three-dimensional figures. Is it possible for mathematical objects to have fractional dimensions? Consider the following figure formed by replacing the middle third of a line segment of length one by an upside-down V, each of whose two sides are equal in length to the segment removed. The first four stages in the development of this figure are shown.



Finding the Scale Factor

Finding the scale factor for this sequence of figures is difficult because the overall length of the figure remains the same while the number of pieces increases. To simplify the procedure, follow these steps.

- Start with any stage (e.g., stage 1).
- Draw the *next* stage (e.g., stage 2) of the sequence and "blow it up" so that it contains an exact copy of the preceding stage (in this example, stage 1).

Notice that stage 2 contains four copies, or replicas, of stage 1 and is three times as long as stage 1.

Stage 1

Length = 1, size = 1 (1 replica)

Stage 2

Length = 3, size = 4 (4 replicas)

19. The scale factor is equal to the ratio

$$\frac{\text{new length}}{\text{old length}}$$

between any two consecutive stages. The scale factor between stage 1 and stage 2 is ________ .

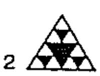
20. The size can be determined by counting the number of replicas of stage 1 found in stage 2. Old size = 1, new size = ________ .

Use the definition of dimension to compute the dimension, d, of the figure formed by this process: $3^d = 4/1$, that is, $3^d = 4$. Since $3^1 = 3$ and $3^2 = 9$, for $3^d = 4$ the dimension of the figure must be greater than one but less than two: $1 < d < 2$.

21. Use your calculator to estimate d. ________ Remember, that d is the exponent that makes 3^d equal 4. For example, since d must be between 1 and 2, try $d = 1.5$. But $3^{1.5} = 5.196\ldots$, which is greater than 4; thus d must be smaller than 1.5. Continue until you approximate d to three decimal places. (Use logarithms for an exact determination.)

The original figure was a one-dimensional line segment. By iteratively adding to the line segment, an object of dimension greater than one but less than two was generated. Objects with fractional dimension are known as *fractals*. Fractals are infinitely self-similar objects formed by repeated additions to, or removals from, a figure. The object attained at the limit of the repeated procedure is the fractal.

Next consider a two-dimensional object with sections removed iteratively. In each stage of the fractal's development, a triangle is removed from the center of each triangular region.

Stage 0  Stage 1  Stage 2  Stage 3 

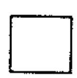
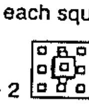
Use the process from the last example to help answer the following questions:

22. What is the scale factor of the fractal? ________ 23. Old size = 1, new size = ________ .

24. The dimension of the fractal is between what two values? ______ $< d <$ ______ .

25. Use the definition of dimension and your calculator to approximate the dimension of this fractal. ________ .

26. Find the dimension of the fractal formed by adding a cube to the center of each square region. ________

Stage 0  Stage 1  Stage 2 

A fractal with dimension between one and two can be formed in one of two ways: (1) add to a one-dimensional line and (2) remove from a two-dimensional figure.

27. How can an object of dimension between two and three be formed? ________________

28. How can an object of dimension less than one be formed? ________________

Section 9: Patterns in Geometry

Can you . . .

- determine the least number of hypercubes (the four-dimensional figure analogous to a square) needed to make a larger hypercube?
- use each of the following patterning blocks to make larger replicas of themselves?

Trapezoid Rhombus Regular hexagon

- develop a theory about which two-dimensional figures can be used to make larger replicas of themselves? Develop an analogous theory for three-dimensional figures?
- form a fractal of dimension one-half?
- find the area under a one-unit-long fractal formed by adding three sides of a square over the central third of each line segment? Compare this area to the area of the circumscribed triangle? (Hint: first find the height of the triangle.)

Stage 0 Stage 1 Stage 2

- find the volume under the fractal described in question 26? Compare this volume to the volume of the circumscribed square pyramid? (Hint: first find the height of the square pyramid.)

Did you know that . . .

- with its mountains, valleys, and oceans, the surface of the earth is a fractal of dimension approximately 2.2?
- Benoit Mandelbrot coined the word *fractal* in 1975 as a label for self-similar shapes with fractional dimension?
- W. Bolyai showed, in 1832, that one of two polygons of equal area could be decomposed into a finite set of polygons that could be reassembled into the other figure?
- M. Dane showed, in 1900, that the answer to Hilbert's third problem—whether an analogous result exists for decomposing two polyhedra of equal volume into a finite set of tetrahedra that could be reassembled into either figure—was "no"? In fact, an infinite number of tetrahedra would be required to do the job.

Mathematical Content

Integral and fractional dimension

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Answers

1. 4
2. 4
3. 2
4. 2/1, or 2
5. 4/1, or 4
6. 3/1, or 3; 9/1, or 9
7. 4/1, or 4; 16/1, or 16
8. 4, 9, 16, 25, 36, 100
9. Each ratio in the bottom row is the square of the scale factor in the top row.
10. 4
11. 4, 9, 16, 25, 36
12. The relationship is the same as in question 8. Each ratio in the bottom row is the square of the scale factor in the top row.
13. Answers will vary. Here are some examples: $3^d = 9$, thus $d = 2$; $5^d = 25$, thus $d = 2$; $4^d = 16$, thus $d = 2$; $6^d = 36$, thus $d = 2$.
14. 8
15. 2/1, or 2; 8/1, or 8
16. 8, 27, 64, 125, 216
17. Each ratio in the bottom row is the cube of the scale factor in the top row.
18. Since $2^3 = 8$, then the value of d in $2^d = 8$ must be 3.
19. 3/1, or 3
20. 4
21. 1.262, or ln4/ln3
22. 2/1, or 2
23. 3
24. $1 < d < 2$
25. The dimension calculated from $2^d = 3$. The dimension is about 1.585.
26. Scale = 3/1, or 3; size = 13. The dimension is calculated from $3^d = 13$. The dimension is about 2.335.
27. Add to a two-dimensional figure or remove from a three-dimensional figure.
28. Remove from a one-dimensional line segment; more difficult to imagine—add to a zero-dimensional point

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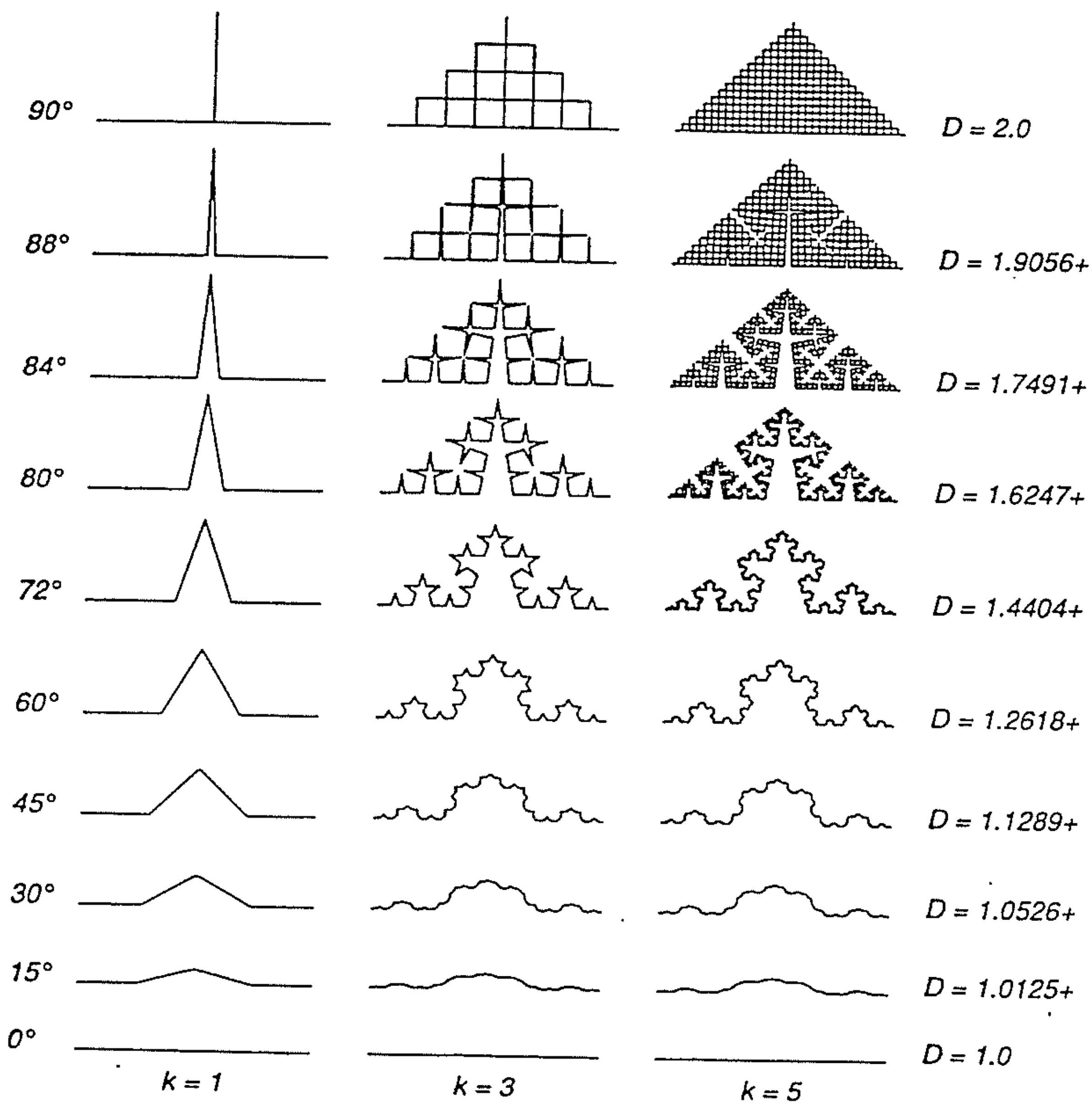
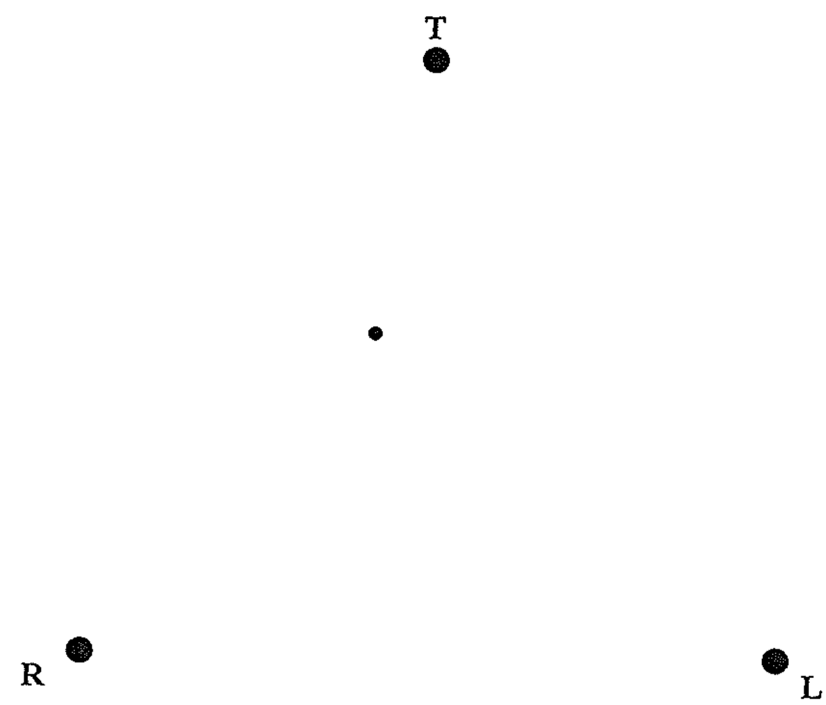


Figure 2: As base angle changes, fractal dimension varies from 1.0 (line) to 2.0 (area).

Game Board for The Chaos Game



Rules for "The Chaos Game"

1. Start with the point located in the interior of triangle labeled "TRL."

2. Toss the die.

3. A. If the result is "1" or "2", measure halfway from the current point to the vertex marked "T" and create a new point.

   B. If the result is "3" or "4", measure halfway from the current point to the vertex marked "R" and create a new point.

   C. If the result is "5" or "6", measure halfway from the current point to the vertex marked "L" and create a new point.

4. Cover the previous point with a penny so that you will remember which point is now the current point.

5. Repeat steps 2 to 4 another ten times.