# Master Document

## Workshop **8** — Number Patterns and Iteration

# LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

## Instructor's Notes

## July 15, 1999

## Workshop 8 — Number Patterns and Iteration

| Materials Needed | Allocated Time |
|---|---|
| Activity #1 — The Triangular Numbers | 40 minutes |
| • 28 linker cubes or other markers | |
| Activity #2 — Square Numbers | 10 minutes |
| • 36 square pattern blocks for each table | |
| Activity #3 — Recursive Formulas | 10 minutes |
| Activity #4 — Difference Tables | 10 minutes |
| Activity #5 — Iteration | 20 minutes |
| Activity #6 — Spirolaterals | 35 minutes |
| • Each participant should have at least 4 different colored pencils | |
| • Ruler for drawing straight lines (optional) | |
| TOTAL WORKSHOP TIME: | 125* minutes |

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Homework — Pattern blocks: each table should have 7 hexagons, 14 rhombuses, and 28 (or more) triangles.

## Activity #1 — The Triangular Numbers

(Allocated time = 40 minutes, including 15 minutes for each of A-B, C-D, E-F)

A. Distribute HO #1 (= TSP #1) and have the participants do the dots problem on that page.

*The participants will quickly realize that the numbers of dots in each row are the counting numbers, so that the number of dots in any size stack of dots will be given by* $1 + 2 + 3 + \dots$ .

After determining that they all found that there were 66 dots, introduce the triangular numbers as follows: Use a piece of paper to cover up the bottom part of the triangle so that only the top dot is showing. Ask them how many dots they see. Then reveal another row and ask them to count them again to get 3. Then 6, then 10. Make a list of these numbers and briefly mention that these are called the triangular numbers, because, as they can see, that many dots can be used to make triangular arrays. We will shortly turn to the recursive and explicit formulas for these numbers, so don't dwell on it here. A table showing the Triangular numbers will be displayed after the next activity.

B. Distribute HO #2 (= TSP #2) and ask them to try the handshake problem on that page. Give enough time for them to find two or three ways to do this problem. Since some of them will do it very quickly, encourage especially them to find other ways to get the same answer.

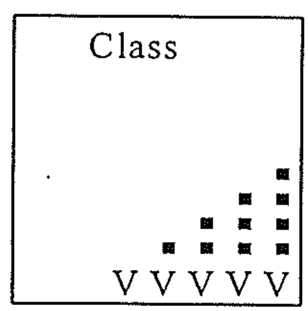
*Three ways to get the answer to this problem include: 1) The first person shakes 7 hands, then the second person shakes 6, then 5, then 4, 3, 2 and 1, for a total of 28 (as in the demonstration). 2) There are (8 Choose 2) ways to choose two people from a group of 8, and that is exactly the number of handshakes that there will be. (8 Choose 2) equals 28. 3) Each person will shake 7 hands, so there will be* $8 \times 7$ *person-handshakes. But this counts each handshake exactly twice, so we divide by 2 so that each handshake will be counted exactly once. Elicit from the participants that the handshake numbers are the triangular numbers.*

Elicit from them that the handshake numbers are the same as the triangular numbers on the list generated earlier; the next activity will explain the connection. If it hasn't come up, then mention explicitly that the handshake numbers are exactly the "choose 2" numbers, because counting the number of handshakes that will take place is the same as counting the number of ways to select 2 people to perform a handshake.

C. Marking Handshakes. (You will need 28 "commemorative markers" for this activities. Linker cubes work well, but anything that can be laid on the floor and be seen by the other participants will do.) To make the connection between the triangular numbers and the handshake numbers, do the following. Announce that we will have 8 people all shake hands with one another, but that to keep track of the handshakes, we will do it very formally. Ask for 8 volunteers to raise their hands, and assign them the ordinals "first, second, ... , eighth." Then ask the first volunteer (V) to come to the front of the room, and situate him so that he is facing the class and has room for 7 more people to his left. Ask the second volunteer to come up and shake hands with the first person. When they shake hands, you lay down a commemorative marker in front of the first person (commemorating the handshake), and then stand the 2^nd^ person to the 1^st^ person's left, also facing the classroom. Ask the 3^rd^ volunteer to come forward and shake hands with each of the two people already there. When they shake hands, you lay down commemorative markers in front of the 1^st^ and 2^nd^ volunteers, and then stand the 3^rd^ volunteer to the left of the 2^nd^ volunteer. Continue this for all 8 volunteers, building larger and larger triangular numbers with your commemorative markers. The arrangement after 5 volunteers have come forward is shown to the right.

D. Another way of commemorating handshakes is on TSP #3. As person #2 shakes hands with #1, an edge is drawn connecting vertex 2 with vertex 1. Then, as person #3 shakes hands with #1 and #2, edges are drawn connecting vertex 3 with vertices 1 and 2. Continuing around the circle, as any person shakes hands with the people with lower numbers, edges are drawn from that person's vertex to the vertices with lower numbers. The result is ... a complete graph! Conclusion. The number of edges in a complete graph $K_n$ is the same as the number of handshakes for n people. Of course, the number of edges in a complete graph corresponds to the number of ways of choosing two vertices for joining ... "n choose 2".

E. After reviewing the handshake problem, and particularly, after noting that the handshake (triangular) numbers are the same as the "choose 2" numbers (TSP #4), introduce the notation T(n) for the n'th triangular number, and calculate the triangular numbers 1, 3, 6, 10, 15, 21, and 28 explicitly from the triangles or by using sums 1+2+3+4+5+ ... . After observing that this method of calculating the triangular numbers will get tedious rather quickly, note that the choose numbers provides an easy way of calculating the triangular numbers. This is all summarized on TSP #5, where an "explicit expression" for T(n) is obtained: T(n) = (n+1)n/2. Note also that, as a by-product of having counted the number of handshakes between *n* people in two different ways,



**we discover that the sum of the counting numbers from 1 to *n* is the number of handshakes among *n+1* people, and so is equal to (n+1)n/2.**

*Note that since many teachers will never use variables in their classrooms, you should make it clear that the general rules for the explicit formulas and, later, the recursive formulas are not essential to the discussion, rather, what is most important is the structure (how the triangular numbers represent the sums of the counting numbers, so that the differences between consecutive triangular numbers should give the counting numbers back).*

**F. On TSP #6 consider the differences between the triangular numbers. Use the text on that TSP, below the table, to introduce the recursive nature of the triangular number sequence. At this point we will not be giving a general recursive expression — that will come later. But go through enough of the recursive expressions in the particular cases to give them a feel for the recursion.**

*Here might be a good place to get them to start thinking of the distinction between explicit and recursive rules. You can mention that for the triangular numbers there is a very simple recursive rule, and a mildly complicated explicit rule. When you do the square numbers activities below, keep them thinking of the distinction between recursive and explicit rules. When we turn our attention to geometric recursion tomorrow, with spirolaterals, and then also on the last two days, we see that we have very simple iterative rules which nonetheless generate hopelessly complex figures that are very difficult to describe explicitly.*

Activity #2 — Square Numbers
(Allocated time = 10 min)

*Note: If you are already behind schedule, then delete this activity about the square numbers and subsequent references to them.*

**A. Distribute HO #3 (= TSP #7) and have the participants determine how many small (orange) squares it takes to build larger squares of various sizes, and thus generate the square numbers.**

*On the hand-out they are invited to write down several expressions of the form "S(2)=4" and "S(3)=9," as well as several recursive expressions of the form "S(2)=S(1)+3" and "S(3)=S(2)+5." This is probably a bit unfamiliar to many of the* *teachers, so take care to make sure they are writing these correctly. It is likely that they will have no trouble finding the explicit rule. We will find a recursive formula below.*

**Analyze the differences of the square number sequence using TSP #8. Again, several recursive expressions are shown, but the formula will be introduced later. Show how the difference table contains this information.**

*In the homework they will be asked to do a similar investigation into the number of triangles needed to build equilateral triangles of various sizes.*

**Activity #3 — Recursive Formulas**
**(Allocated time = 10 min)**

**A. Referring to the top part of TSP #9, show how the recursive expressions for several specific cases can be generalized to a general recursive rule for the triangular numbers.**

*Experience has shown that writing down these recursive rules is quite difficult for many teachers. We have created this separate activity so that the instructors won't assume that the notation is familiar to the participants. For example, many participants do not see that T(n-1) refers to the "previous term." So take care here to introduce this notation "from the ground up," explaining the rules and really showing how they generalize the particular expressions above. They should be reassured that these recursive rules (with the "n's" in them) are not central to the workshop today, and that they will be able to understand almost all the discussion and homework without mastering it. This should help reassure those teachers who are fearful of algebra.*

**For the square numbers, you can use pattern blocks on the overhead projector to show how $S(n) = S(n-1) + 2(n-1) + 1$, and then refer to the difference tables on TSP #8 to show how this is indeed happening numerically.**

**Activity #4 — Difference Tables**
**(Allocated time = 10 min)**

**A. Here we demonstrate the power of finite difference tables regarding**

discovering and unlocking patterns in numbers. Put up TSP#10 showing the map of Ahhhs. It's called "Ahhhs" because that is hopefully what the participants will exclaim when they discover the pattern. The story here is that a thief has broken into 6 houses, proceeding up the street from house 1 to house 101 hitting the houses indicated in the order indicated. The detectives would like to be able to predict the next house that will be hit. Taking third differences does the trick! Use a difference table on a blank TSP to find that the next house is 155, and then have them find the number of the following house.

*You can point out that this method of "finite differences" works very well for many sequences, yielding eventually a constant sequence. In fact, if you start with any polynomial, like $n^2-3n+7$, and generate the first few terms 7, 5, 5, 7, 11, 17, then your students would be able to find the succeeding terms using this method. But for other sequences it does not yield any new or useful information (E.g., 1, 5, 15, 43, 71, 101, ...) For the sequence 1, 2, 4, 8, 16, 32, ..., the difference table does not eventually become constant, but still, an interesting pattern is revealed. You may also wish to mention that for sequences which grow exponentially, the difference sequence is a multiple of the sequence itself.*

Activity #5 — Iteration
(Allocated time = 20 min)

A. On tonight's exercise set, the participants will work with some iterative rules which they have likely not seen before, and in the past have been tricky for them during the homework session. So by way of introducing the notion of iteration, and helping to introduce these rules that they will see for homework, put up TSP#11 which introduces the notion and shows two examples. Explain and discuss these rules, as well as those on TSP#12, but let them try to figure out the rule in example #5 for themselves.

*The participants will be asked to analyze examples 3 and 4 extensively on the homework, and to briefly experiment with example 2. Note that tomorrow you can tell them that, while it is conjectured that the process in example 2 will always yield 1, independent of the starting point, this is still an unproven conjecture.*

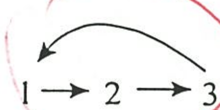
B. Ahead of time, place a small post-it on the floor and label it "START." Ask for four volunteers, and label them "first, second, third, fourth." Have the first volunteer come to the front of the room. Tell the participants that the volunteer will be acting out an iteration, and ask them to remind you what two things you need in order to iterate (a starting point and a rule for iterating). This done, reveal the first rule on TSP #13 and guide the volunteer over to the START sticker. Ask her to do one iteration, and then ask her to do a few more iterations while the class looks on. Once this is going well, you can ask the rest of the class some questions, such as "Where will she be after 10 iterations?" "Where will she be after 100 iterations? "When will she return to her starting point?" Thank the volunteer, fish for applause, and call up your second volunteer.

Guide the second volunteer through a few iterations, and when things are going well, you can pose the same kinds of questions to the rest of the class, starting with "Where will she be after 4 iterations?" "Where will she be after 8 iterations?" "Where will she be after 100 iterations?" "Where will she be after a million iterations?" and then returning to "Where will she be after 50 iterations?" Thank the volunteer, fish for applause, and call up your third volunteer.

Guide the third volunteer through their rule. When they have the idea, let them continue iterating while you ask questions of the class.

*A good way to show the variable i is to ask the volunteer to hold up that many fingers. That way the whole class can see the variable, and it doesn't look mysterious. Experience shows that participants are fairly comfortable with this. As you ask the questions, participants will gradually see that they are hard to answer. Confirm that they see the spiral path that the volunteer will travel, but confirm also that they realize that predicting where she will be after 100 iterations is a fairly tricky problem.*

*And that is one of the key ideas about chaos and fractals. You can take a very simple iteration rule and get enormous complexity after iterating the rule many times.*




Finally, have the fourth volunteer come up. Cross out the line in the third rule on TSP #13 that says "add one to *i*" and replace it with the diagram to the right. The idea is that on the first iteration they go 1 step, on the second iteration they go 2 steps, on the third iteration they go 3 steps, but then on the fourth iteration they go 1 step again, and the number of steps cycles around like that. Again, the volunteer should hold up the number of fingers to show how many steps to take at that stage. You can ask the same questions as before, but don't give away that the volunteer will indeed return to her starting point after 4 iterations. This they will discover while doing the spirolateral activity that follows.

Activity #6 — Spirolaterals
(Allocated time = 45 min)

*The next topic, spirolaterals, serves as an introduction to geometric iteration. A spirolateral is constructed as follows: Given a sequence of positive integers, say, for example, 5, 3, 4, 2, 4, pick a dot somewhere near the center of a page of dot paper, and draw a spiral that goes right, down, left, up, right, down, left, up, etc... The lengths of each step in the spiral are given by the sequence repeated over and over. An example can be found on HO#4.*

**A. Distribute the Spirolaterals—personalized version instruction sheet (HO#4 = TSP#14) and the square dot paper with the coding scheme (Handout #5 = TSP#15) and review the instructions on TSP#14. Then put up TSP #15 and pick an non-occurring odd name (i.e., a name which has an odd number of letters but is not the name of anyone in the group) to use as an example. "Joe" is a good name to use! Encode it and draw what you get from four iterations of these letters, using four different colors. That will help them get an idea of how to generate a spirolateral, and, since it returns to the starting point, of what to expect as a possible outcome.**

**When they get the idea, have them make spirolaterals with their own names. They should complete at least one good version of their own name, which they will use in the next activity.**

*The iterative nature of the construction of these spirolaterals should be emphasized. There is one rule for creating them, and we repeat this rule, building each stage from the stage before it. In the case of 4, 8, 12, etc... letters, when the spirolateral spirals off to infinity, this is especially obvious. But with the ones that close, some initially have difficulty starting a new copy of the word in the proper direction. This is an opportunity to demonstrate how each stage genuinely builds on the previous.*

**B. When each participant has a good (correct) copy of their spirolateral, walk around the room and ask the participants to discuss various similarities and differences they see among the spirolaterals they have made. Guide the excitement as they make their discoveries by first encouraging discussion among participants at the various tables, then have them test their conjectures by comparing notes with other tables, and finally, let some of the more successful observations "bubble" to the surface as a "class" observation, and ask for confirmation or contradiction of these observations.**

*One important aspect of this activity is that it introduces the notion of classification to the participants. Of course, this is very important in mathematics. You may wish to tell the participants that mathematicians do a lot of classifying, and that it is very important. Before going over the spirolaterals with them, ask them to try to think of other situations where they've classified objects according to some mathematical criterion. (You may get such answers as: shapes into triangles, squares, pentagons, circles, etc...; Triangles into obtuse, acute and right; numbers into positive, negative,*

Inserted sentences as per Val's suggestion *whole, rational, irrational, etc...; fractions into proper and improper; lines into horizontal, vertical and skew; and so on.) Now you can go over their descriptions of the spirolaterals...they should have a good idea of what you mean by classification.*

**C. After they have classified in a number of ways (and if you let them, they will be very clever and creative, and you will learn things about classification that you'd never dreamed of!) you can lead them to discover a relationship between the number of letters and the behavior. This is just one interesting way in which spirolaterals can be classified, and is probably the least subjective. You may wish to make a chart like the one below.**

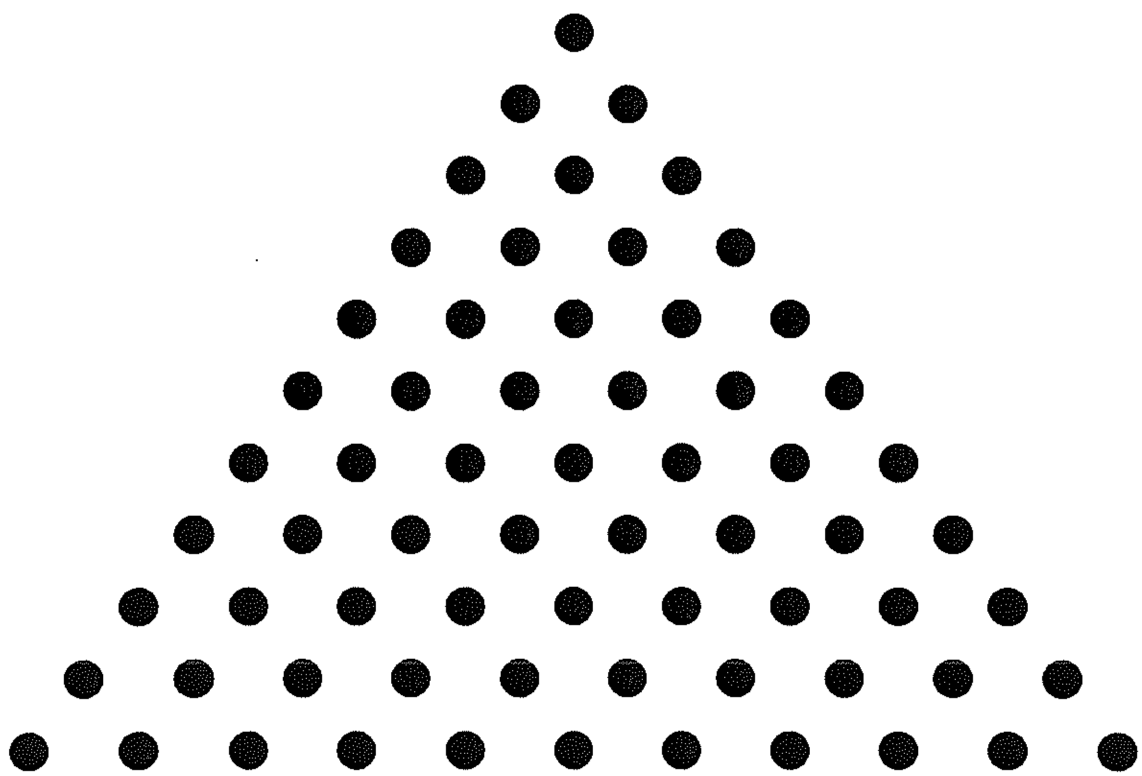
| # of letters | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # of iterations before repeating | 4 | 2 | 4 | ∞ | 4 | 2 | 4 | ∞ | 4 | 2 | 4 | ∞ | ... |

*They should see that the order depends on the greatest common divisor of the number of letters and 4.*

*Note that triangular dot paper is provided on TSP #17 in case you wish to, and have time to, explore variations. At a minimum, point out that it is in their Resource Books and that they can use the same code to generate spirolaterals here. If there is time, take a four letter word and generate two spirolaterals, one resulting from a 120 degree rotation after each edge, and the other resulting from a 60 degree rotation after each edge.*

# Handout #1 – Counting Dots

How many dots are there in this figure?



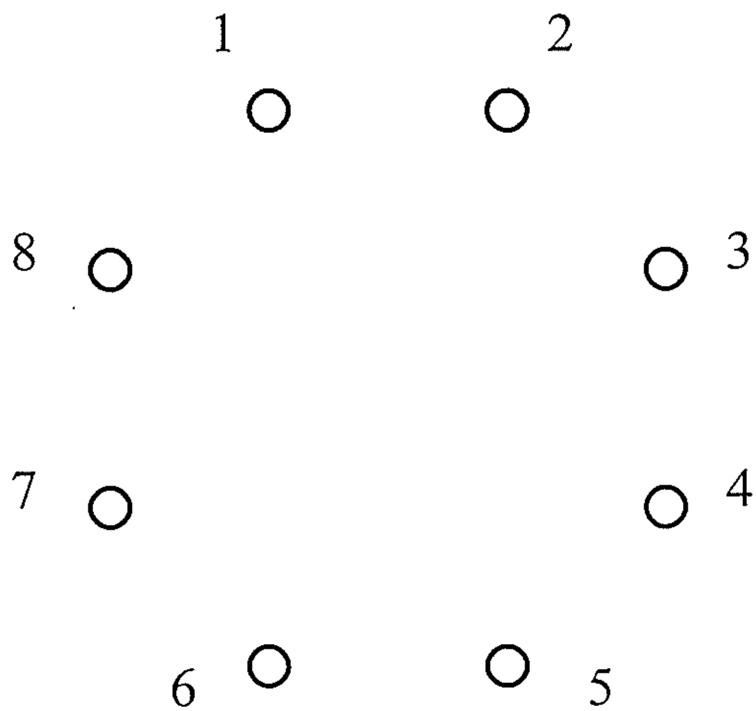
## Handout #2 — The Handshake Problem

A. There are eight people in a room and each person shakes hands with all seven other people. How many handshakes take place altogether?

(Explain your answer in several different ways.)

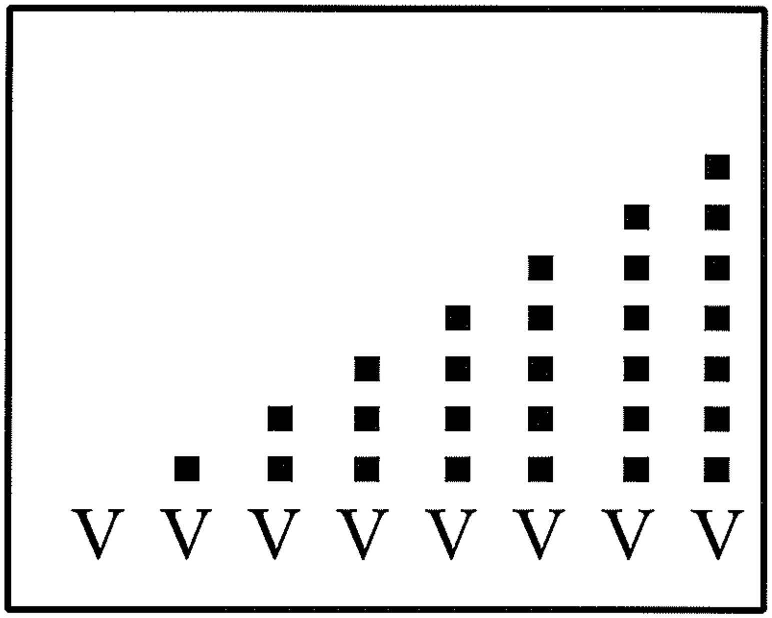
B. Make a chart which gives the number of handshakes required for 2, 3, 4, 5, 6, 7, 8, 9 and 10 people.

Commemorate handshakes by drawing edges between vertices that correspond to the people shaking hands.



# Triangular Numbers

This is the number of handshakes among 8 people:




This is also the 7$^{th}$ triangular number T(7).

So the 7$^{th}$ triangular number is equal to 8 choose 2.

T(7) = 8 choose 2 = $(8\times7)/(2\times1) = 28$.

What is the 10$^{th}$ triangular number T(10)?

What is the 100$^{th}$ triangular number T(100)?

# All Together Now!

T($n$)

=

$n^{\text{th}}$ triangular number

=

number of dots in an $n \times n \times n$ triangle

=

sum of counting numbers from 1 to $n$

=

number of handshakes for $n+1$ people

=

($n+1$) choose 2

=

$$\frac{(\mathbf{n+1}) \times \mathbf{n}}{\mathbf{2}}$$

This is the value of the $n^{\text{th}}$ triangular number T($n$).

This is an *explicit expression* for T($n$).

For example: T(11) = 12×11 / 2 = 66, is the number of dots in the 11×11×11 triangle.

# A Triangular Number Difference Table

The *differences* between the triangular numbers are the counting numbers.

| *T(1)* | | *T(2)* | | *T(3)* | | *T(4)* | | *T(5)* | | *T(6)* | | *T(7)* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 3 | | 6 | | 10 | | 15 | | 21 | | 28 |
| | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | |

$$
\begin{aligned}
T(2) &= T(1) + 2 = 1 + 2 = 3 \\
T(3) &= T(2) + 3 = 3 + 3 = 6 \\
T(4) &= T(3) + 4 = 6 + 4 = 10 \\
T(5) &= T(4) + 5 = 10 + 5 = 15 \\
T(6) &= T(5) + 6 = 15 + 6 = 21 \\
T(7) &= T(6) + 7 = 21 + 7 = 28
\end{aligned}
$$

etc.

Note that the differences of the differences, called the *second differences*, would all be 1.

# Number Sequences via Pattern Blocks

Use pattern blocks to find the number of small squares needed to build square grids of various sizes.

| Size of the grid | Number of small squares in that grid | Which can be expressed in terms of the previous number as |
|---|---|---|
| $1 \times 1$ | $S(1) =$ | |
| $2 \times 2$ | $S(2) =$ | $S(2) =$ |
| $3 \times 3$ | $S(3) =$ | $S(3) =$ |
| $4 \times 4$ | $S(4) =$ | $S(4) =$ |
| $5 \times 5$ | $S(5) =$ | $S(5) =$ |
| $6 \times 6$ | $S(6) =$ | $S(6) =$ |

Without actually building the squares, can you determine the next terms in this sequence? Can you find an explicit formula for $S(n)$?

| | |
|---|---|
| $n \times n$ | $S(n) =$ |

# A Square Number Difference Table

The *differences* between the square numbers are the odd numbers.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *S(1)* | | *S(2)* | | *S(3)* | | *S(4)* | | *S(5)* | | *S(6)* | | *S(7)* |
| 1 | | 4 | | 9 | | 16 | | 25 | | 36 | | 49 |
| | 3 | | 5 | | 7 | | 9 | | 11 | | 13 | |

$$
\begin{aligned}
S(2) &= S(1) + 3 &= 1 + 3 &= 4 \\
S(3) &= S(2) + 5 &= 4 + 5 &= 9 \\
S(4) &= S(3) + 7 &= 9 + 7 &= 16 \\
S(5) &= S(4) + 9 &= 16 + 9 &= 25 \\
S(6) &= S(5) + 11 &= 25 + 11 &= 36 \\
S(7) &= S(6) + 13 &= 36 + 13 &= 49
\end{aligned}
$$

etc.

The explicit expression is $S(n) = n^2$

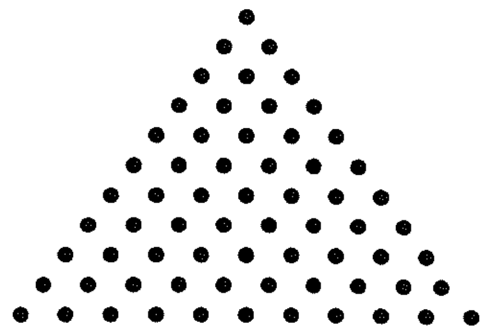
Note that the differences of the differences, called the *second differences*, would all be 2.

## Recursive Rules — Making General Formulas

We earlier saw that with the triangular numbers:

$$T(2) = T(1) + 2$$
$$T(3) = T(2) + 3$$
$$T(4) = T(3) + 4$$
$$T(5) = T(4) + 5$$

etc.



Can you see how to generalize this rule?

(current value is old value plus current size)

$$T(n) = T(n-1) + n$$

This is called a *recursive formula* for $T(n)$.

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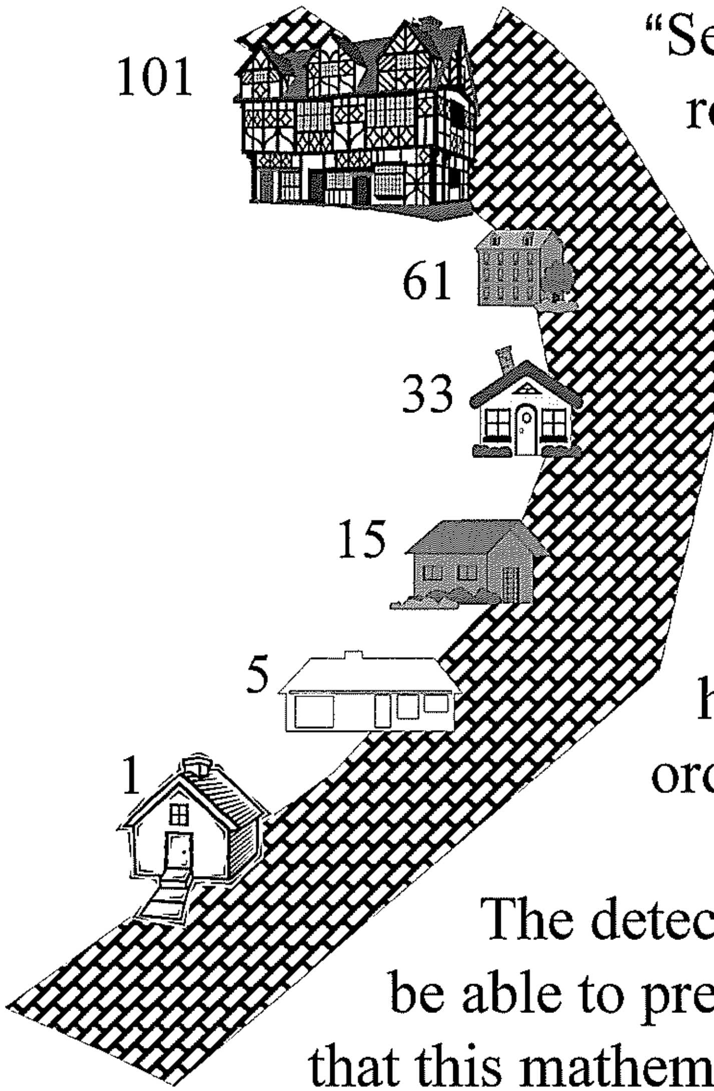
With the square numbers:

$$S(2) = S(1) + 3$$
$$S(3) = S(2) + 5$$
$$S(4) = S(3) + 7$$
$$S(5) = S(4) + 9$$

(current value is old value plus current odd number)

— as a formula, $S(n) = S(n-1) + (2n-1)$

# Welcome to the land of Ahhhs.




A thief, calling himself the "Sequencer," has been robbing houses in this peaceful land! Six weeks ago he robbed the house at 1 Brick Way, and since then has proceeded up the street robbing the houses shown in the order shown.

The detectives would like to be able to predict the next house that this mathematically inclined thief will hit.

Can you discover which house that will be?

# Iteration ...

... means repeating an action over and over, typically on the outcome of the last iteration

**Two items to consider when iterating:**

- A starting point
- Instructions for iterating

**Examples:**

1. Counting by 5
   5→10→15→20→25→30→35→40→45,
   What is the action? Where do you start?

2. Evens and Odds
   7→22→11→34→17→52→26→13→40→20→10→5→...
   What is the action?
   - If odd, triple and add 1;
   - If even, divide by two.

## Some More Examples

3. Multiply the Digits
19375→945→180→0
Pick a number.
Multiply its digits to get a new number.
Multiply its digits to get a new number.
Multiply its digits to get a new number.
Etc.

4. Numbers in Words
200→10→3→5→4→4
Pick a number. Write it in words, and count the letters to get a new number. Write it in words, and count the letters to get a new number. Etc.

5. The Look and Say Sequence
21→1211→111221→312211→13112221→1113213211→31131211131221
What is the action? What is the next number?

# Some Geometric Iteration

**Start:**

Start at "START"

**Iterate:**

- Take 2 steps forward
- Hop

---

**Start:**

Start at "START"

**Iterate:**

- Take 2 steps forward
- Turn right 90 degrees
- Hop

---

**Start:**

Start at "START" and set $i = 1$

**Iterate:**

- Take $i$ steps forward
- Turn right 90 degrees
- Add one to $i$
- Hop

# Spirolaterals — personalized version

Examples:

| | | |
|---|---|---|
| J | O | E |
| 1 | 6 | 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| V | A | L | E | R | I | E |
| 4 | 1 | 3 | 5 | 9 | 9 | 5 |

Your Name and Code:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |

Example

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | ... |
| R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | ... |

Your Code repeated Over and Over

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | |
| R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R |

| | | |
|---|---|---|
| 1-A | J | S |
| 2-B | K | T |
| 3-C | L | U |
| 4-D | M | V |
| 5-E | N | W |
| 6-F | O | X |
| 7-G | P | Y |
| 8-H | Q | Z |
| 9-I | R | |

Dr. Evan Maletsky
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| | | |
|---|---|---|
| 1-A | J | S |
| 2-B | K | T |
| 3-C | L | U |
| 4-D | M | V |
| 5-E | N | W |
| 6-F | O | X |
| 7-G | P | Y |
| 8-H | Q | Z |
| 9-I | R | |

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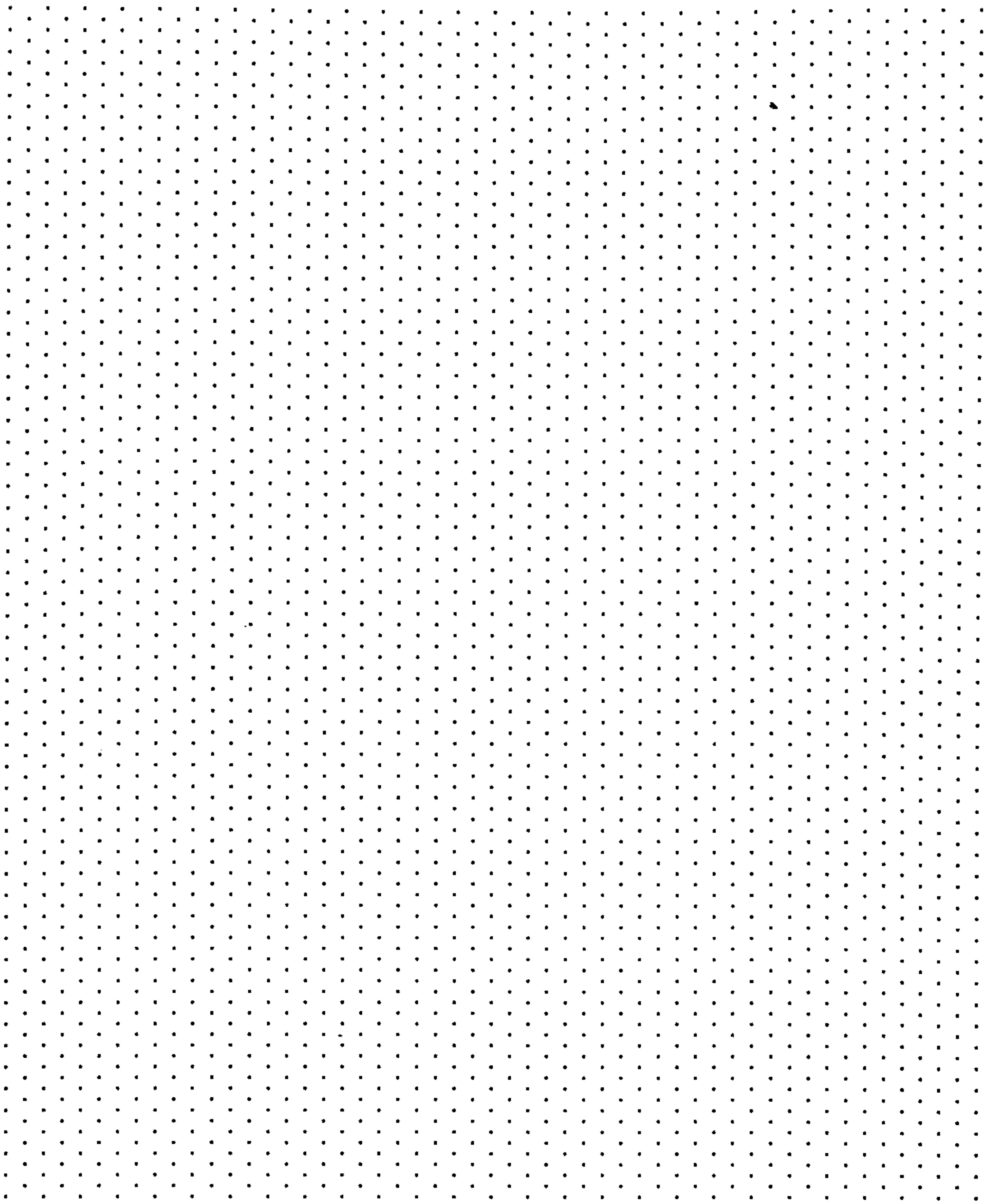


# A Triangular Number Difference Table

The *differences* between the triangular numbers are the counting numbers.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *T(1)* | | *T(2)* | | *T(3)* | | *T(4)* | | *T(5)* | | *T(6)* | | *T(7)* |
| 1 | | 3 | | 6 | | 10 | | 15 | | 21 | | 28 |
| | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | |

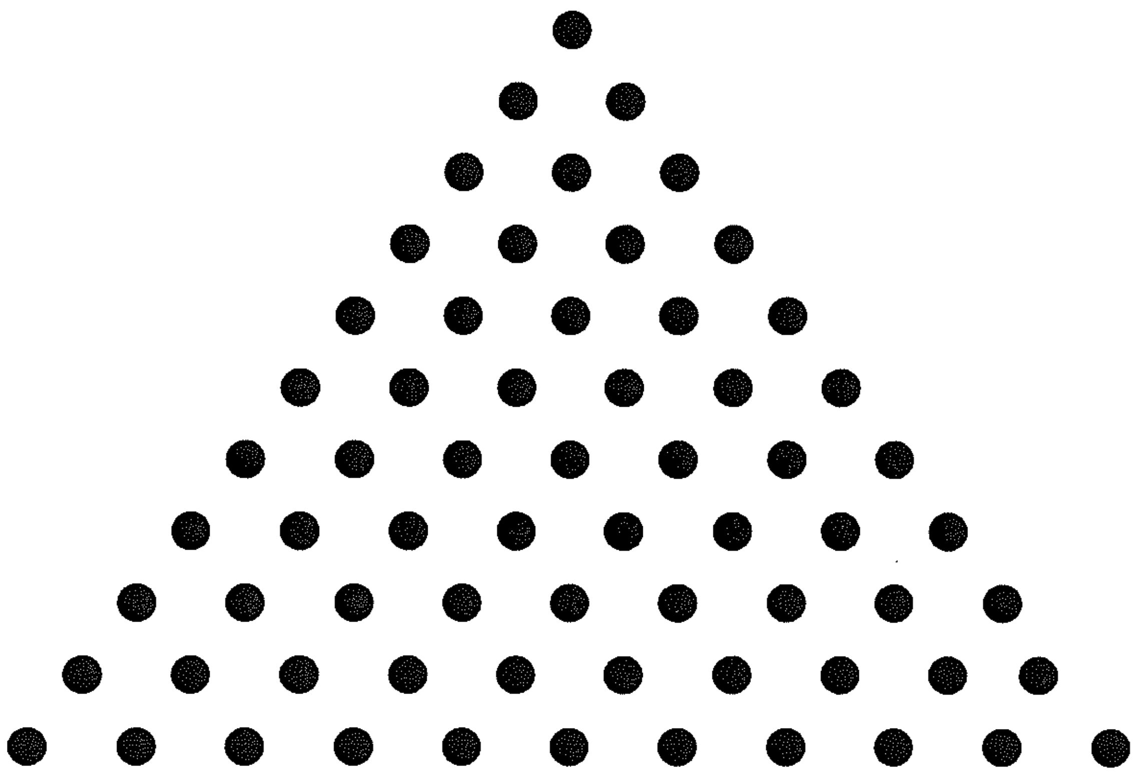
$$
\begin{aligned}
T(2) &= T(1) + 2 = 1 + 2 = 3 \\
T(3) &= T(2) + 3 = 3 + 3 = 6 \\
T(4) &= T(3) + 4 = 6 + 4 = 10 \\
T(5) &= T(4) + 5 = 10 + 5 = 15 \\
T(6) &= T(5) + 6 = 15 + 6 = 21 \\
T(7) &= T(6) + 7 = 21 + 7 = 28
\end{aligned}
$$

etc.

Note that the differences of the differences, called the *second differences*, would all be 1.

## Handout #1 — Counting Dots

How many dots are there in this figure?



## Handout #2 — The Handshake Problem

There are eight people in a room and each person shakes hands with all seven other people. How many handshakes take place altogether?

Explain your answer in two (or even three) different ways.

Make a chart which gives the number of handshakes required for 2, 3, 4, 5, 6, 7, 8, 9 and 10 people.

## Handout #3 — Number Sequences via Pattern Blocks

Use square pattern blocks to find the number of small squares, in square grids of various sizes.

| Size of the grid | Number of small squares in that grid | Which can be expressed in terms of the previous number as |
|---|---|---|
| 1 × 1 | S(1) = | |
| 2 × 2 | S(2) = | S(2) = |
| 3 × 3 | S(3) = | S(3) = |
| 4 × 4 | S(4) = | S(4) = |
| 5 × 5 | S(5) = | S(5) = |
| 6 × 6 | S(6) = | S(6) = |

Without actually building the squares, can you determine the next terms in this sequence? Can you find an explicit formula for S($n$)?

# Handout #4 — Spirolaterals — personalized version

Start with your name, and use the coding scheme on the dot paper to convert your name into your very own sequence of numbers.

Examples:

| | | |
|---|---|---|
| J | O | E |
| 1 | 6 | 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| V | A | L | E | R | I | E |
| 4 | 1 | 3 | 5 | 9 | 9 | 5 |

Your Name and Code:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |

Make a chart like that below, and enter your personal sequence of numbers on the top row starting at the left. Then enter it again (leaving no empty boxes) and again until no space is left. For example, JOE would enter 1 6 5 1 6 5 1 6 5 1 6 5 1 6 5 etc., repeating 1 6 5 over and over, as shown above.

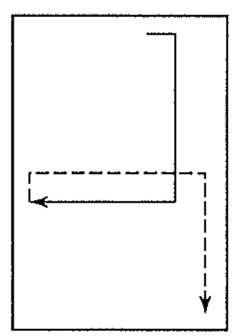
Example

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | ... |
| R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | ... |

Your Code repeated Over and Over

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | |
| R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | ... |

Now start on any dot on the graph paper, and draw a single line following the instructions in the chart, where the top row indicates the length of the line and the bottom row indicates the direction. For example, JOE would first draw a line to the right (R) one box, then continue down (D) six boxes, then left (L) five boxes, then up (U) one box, then right (R) six boxes, then down (D) five boxes, etc. The example with "165" is shown to the right, with the first "165" solid, the second dashed.



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| | | |
|---|---|---|
| 1-A | J | S |
| 2-B | K | T |
| 3-C | L | U |
| 4-D | M | V |
| 5-E | N | W |
| 6-F | O | X |
| 7-G | P | Y |
| 8-H | Q | Z |
| 9-I | R | |

**Workshop 8 — Number Patterns and Iteration — Exercises**

**Practice Problems:**

1. a. Find, by inspection, the next three numbers in the following sequence: 1, 6, 11, 16
   b. Treating these numbers as A(1), A(2), A(3), and A(4), find a recursive formula for A(n), that is, complete the formula A(n) = A(n-1) + _____.
   c. Find an explicit formula for A(n).

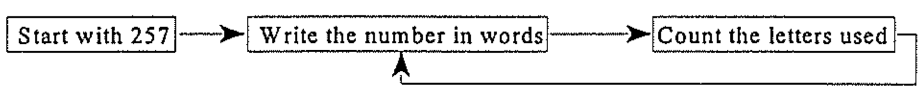
2. a. Fill in the blanks in the following difference table:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 5 | | 11 | | 24 | | ? | | ? | | 155 |
| | 4 | | 6 | | 13 | | ? | | ? | | 64 | |
| | | 2 | | 7 | | ? | | ? | | 22 | | |
| | | | 5 | | 5 | | 5 | | 5 | | | |

   b. Use difference tables to predict the next number in this sequence: 3, 4, 7, 12, 19, 28

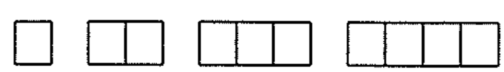
3. Follow this Procedure: Multiply the digits in the number code for your name (see Page 6). Then multiply the digits in the product. Repeat the process as long as you can.

4. Iterative procedures can be described in many ways. What happens when you follow the procedure described by this flow chart?




5. How many edges are there on the largest of the graphs on Page 7?

6. How many rectangles can you find in the sequence of figures at the right? (Note: A square counts as a rectangle.) Can you find a rule for extending the sequence of numbers that you get? Can you explain why you got that sequence of numbers?



**Study Group Problems:**

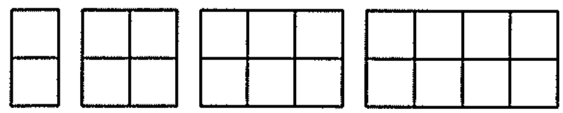
7. Procedure: multiply the digits of a number (with more than one digit) to get a new number.
   a. Show that if you iterate this procedure, starting with 4812, the result will be the number 8.
   b. Find two other starting numbers which, when this procedure is applied iteratively, will result in 8.
   c. Construct a tree that starts with 8 and traces out, in reverse order, all the numbers less than 100 that, if used as starting points, will also result in 8.

8. a. Use difference tables to predict the next two numbers in this sequence: 1, 1, 3, 9, 21, 41
   b. Use difference tables to predict the next to numbers in this sequence: 1, 0, 0, 0, 1, 6, 20, 50, 105

9. a. Using triangular pattern blocks, construct a sequence of equilateral triangles which have 1, 2, 3, 4, and then 5 pattern blocks on each side. Find the number of small triangles that make up each of the larger triangles.
   b. What are the next terms in this sequence?
   c. Can you explain why this sequence comes up in this problem?

10. Each square in each figure can be colored red or blue.
In how many ways can each of the figures be colored?
Extend the number pattern through the first eight stages.
Take first, second and third differences. What do you notice?
What would happen if we used three colors instead of two?

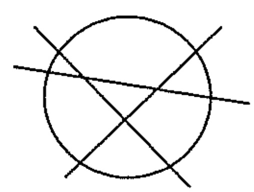
11. Draw the next three figures in the sequence to the right.
How many squares of any size can you find in each of these figures, including the three figures you drew?
Can you find the rule for extending the sequence (a recursive rule)? Can you find an explicit rule?



12. Consider the following iterative procedure:
- Start with 3 pennies, heads-up, in a row.
- At each step, find the right-most penny which is heads-up, flip it to tails, and flip all coins to its right (if any) back to heads.

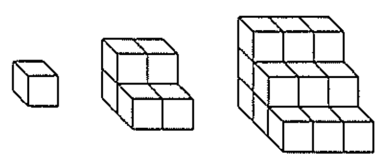
   a. Iterate this procedure as long as you can, and keep a record of the sequence at each stage. For example, starting with the 3 heads in a row above, the sequence HHH → HHT → HTH → HTT → ··· shows the first few stages of this iteration.
   b. What happens if you start with 2 pennies? 4 pennies? 5 pennies?
   c. Make a chart showing how many iterations it takes before the procedure terminates for each number of pennies from 1 to 5. Can you see the pattern?

13. The pizza to the right has been cut into 7 pieces with three straight cuts. That is the *greatest* number of pieces you can create with three straight cuts, so the third pizza number, P(3), is equal to 7. Similarly, the second pizza number is 4, and the first pizza number is 2; i.e., P(2) = 4 and P(1) = 2.



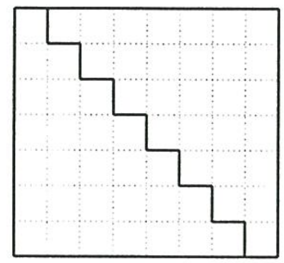
   a. Find the next four pizza numbers P(4) through P(7).
   b. Give a recursive rule for the pizza numbers, of the form $P(n) = P(n-1) +$ something.
   c. Give the reason, in geometric terms, for the recursive rule you discovered.
   d. Can you find the pizza numbers (disguised) in Pascal's triangle?

14. How many cubes are in each figure pictured? How many are in the next two figures in the sequence? Express the sequence of numbers that you get in terms of the sequence of triangular numbers.



**Extension Problems:**

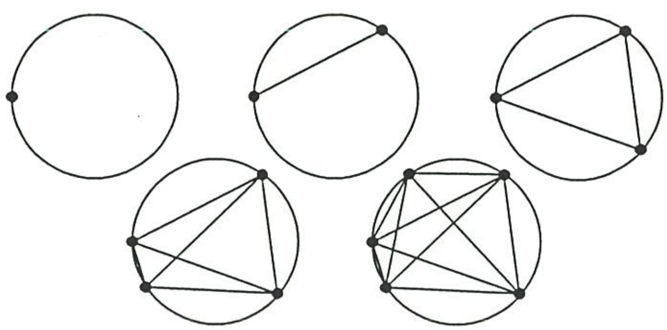
15. Consider the triangulated figure on Page 5. This figure can be covered with pattern blocks using 7 hexagons, as shown, or 42 triangles, as shown. It can also be covered with 17 pattern blocks as shown. In fact, this figure can be covered with any number of pattern blocks from 7 to 42. Can you describe in words an iterative procedure for finding such a covering for each number from 7 to 42? Ideally, the rule you give should be a single general rule to show how to go from a covering with $n$ pattern blocks to a covering with $n+1$ pattern blocks.



16. How many rectangles of any size can you find in the sequence of figures in problem 11? (Note: A square counts as a rectangle. Can you find a recursive rule for extending the sequence?

17. Can you see why the picture to the right "proves" that $1+2+\cdots+n = n(n+1)/2$?

18. The five figures below show circles with 1, 2, 3, 4, and 5 points respectively on their circumferences, and with all the chords drawn to connect each pair of points. Notice that the points are positioned on the circumference so that we never have three or more chords meet at a single point.
    a. Into how many regions is each of the circles divided?
    b. Into how many regions will a circle with 6 points (and all the chords drawn) be divided?
    c. Use a difference table to predict the answer for 7.

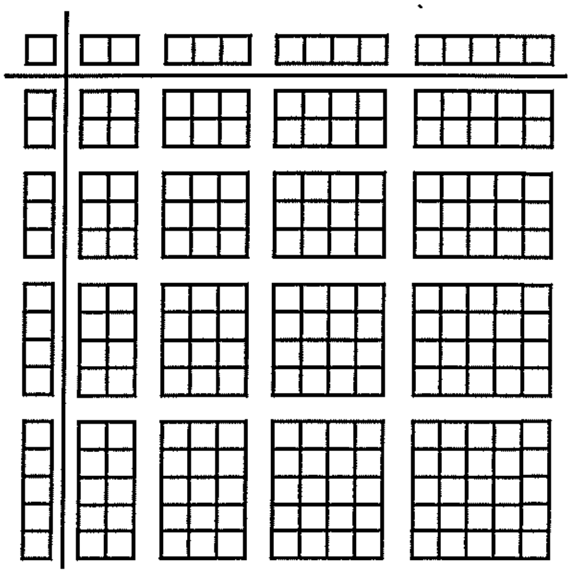


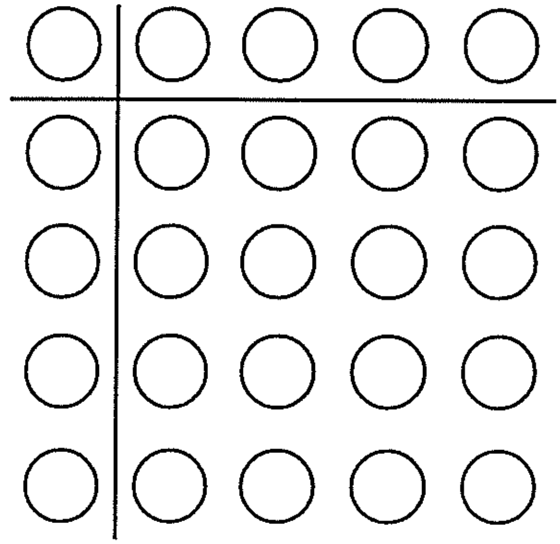
19. Describe how the sequences of these number sequences can be found embedded in Pascal's triangle. Some are "disguised", meaning that you have to add together some entries in order to find them.

| | | |
|---|---|---|
| Square numbers (disguised) | Powers of 2 (disguised) | Triangular numbers |
| Sums of triangular numbers | Choose numbers | Powers of 11 |

19. a. What is the longest sequence of different numbers that can be produced from the iterative procedure of question 7, starting with a counting number less than 100? Explain your strategy.
    b. Catalog all the counting numbers from 1 through 50 into the terminating , single-digits numbers 0 through 9 when following the procedure of question 7.
    c.

20. How many rectangles of any size (again, count squares as rectangles) can you find in each of the figures below? Put your answer in the corresponding circle in the table to the right. Of course, you should be looking for some sort of pattern in the numbers in the table. What can you find?





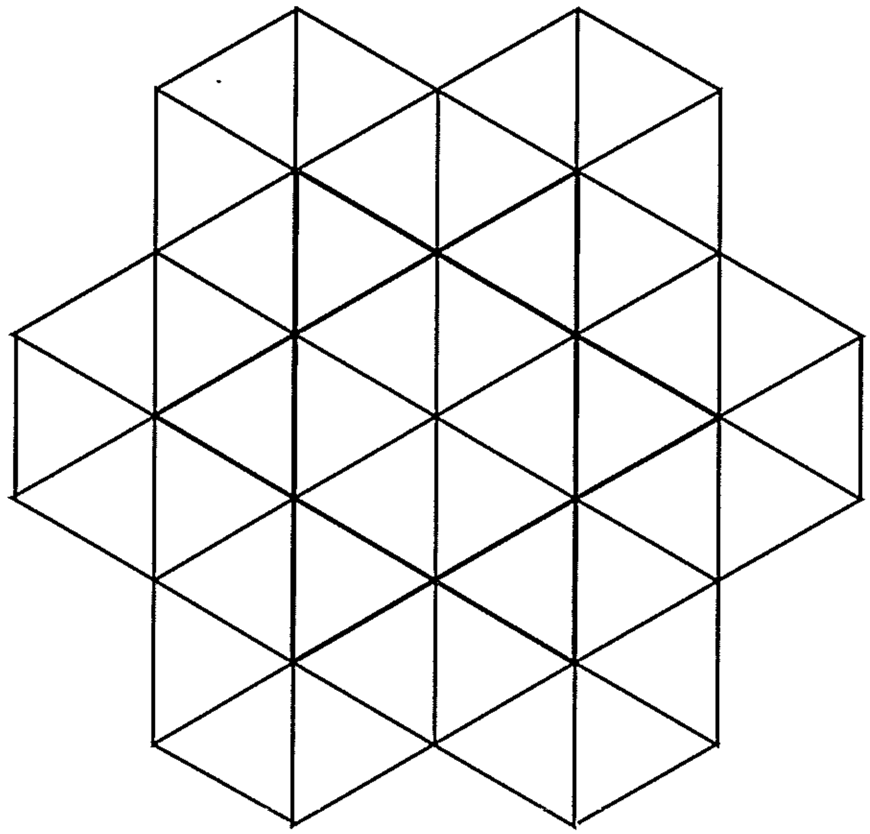
21. Suppose you start with a positive whole number and iterate the following procedure: if the number is even, divide it by 2, if the number is odd, then triple it and add 1. For example, starting with 30, we would generate the sequence: 30 → 15 → 46 → 23 → 70 → ··· .
   a. What happens if you start with 1?
   b. What happens if you start with 8?
   c. What happens if you start with 3?
   d. Try this for some other numbers of your own choosing. Do you notice any pattern?

22. A problem from earlier in this institute read: *When Alice sends Bob a card, she always adds a string of 5 X's and/or O's after her signature, with one restriction: she never puts two O's next to each other, because Bob turns them into smiley faces, and she can't stand that. Thus, for example, "XOXXO" would be allowed, but "XOOXX" would not. Systematically list all the different ways for Alice add X's and O's. How many ways are there?*
Suppose that Alice chooses to vary the number of X's and O's she signs to her cards, rather than limiting it to 5 characters as she currently does.
   a. How many different ways are there for 1, 2, 3, 4, 5, 6, 7 or 8 characters?
   b. Give a recursive rule for L($n$), the number of ways to lovingly sign a letter using $n$ characters.
   c. (optional, extra credit.) Give a reason for the recursion you found in part b.

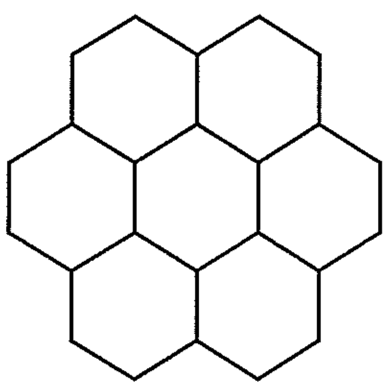
23. Extend the pattern at the right to the next row. Give the algorithm used to create this triangular array. What discoveries can you make?

1
3 5
7 9 11
13 15 17 19
— — — — —

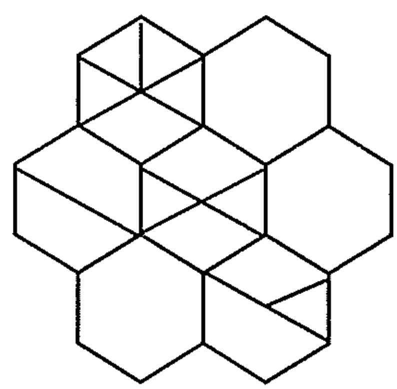
24. a. Can you find a "name" whose spirolateral encloses an area of exactly 24?
   b. Draw spirolaterals on square grid paper using the patterns "1, 2", "1, 2, 3", "1, 2, 3, 4", "1, 2, 3, 4, 5", etc..., up to "1, 2, 3, 4, 5, 6, 7, 8, 9". Compare and contrast these interesting figures.



A covering with 42 pattern blocks



A covering with 7 pattern blocks



A covering with 17 pattern blocks

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Can you show that any number of pattern blocks, from 7 to 42, can be used to cover?
Hint: Try an iterative construction, building each from a previous one.

| | | |
|---|---|---|
| 1-A | J | S |
| 2-B | K | T |
| 3-C | L | U |
| 4-D | M | V |
| 5-E | N | W |
| 6-F | O | X |
| 7-G | P | Y |
| 8-H | Q | Z |
| 9-I | R | |

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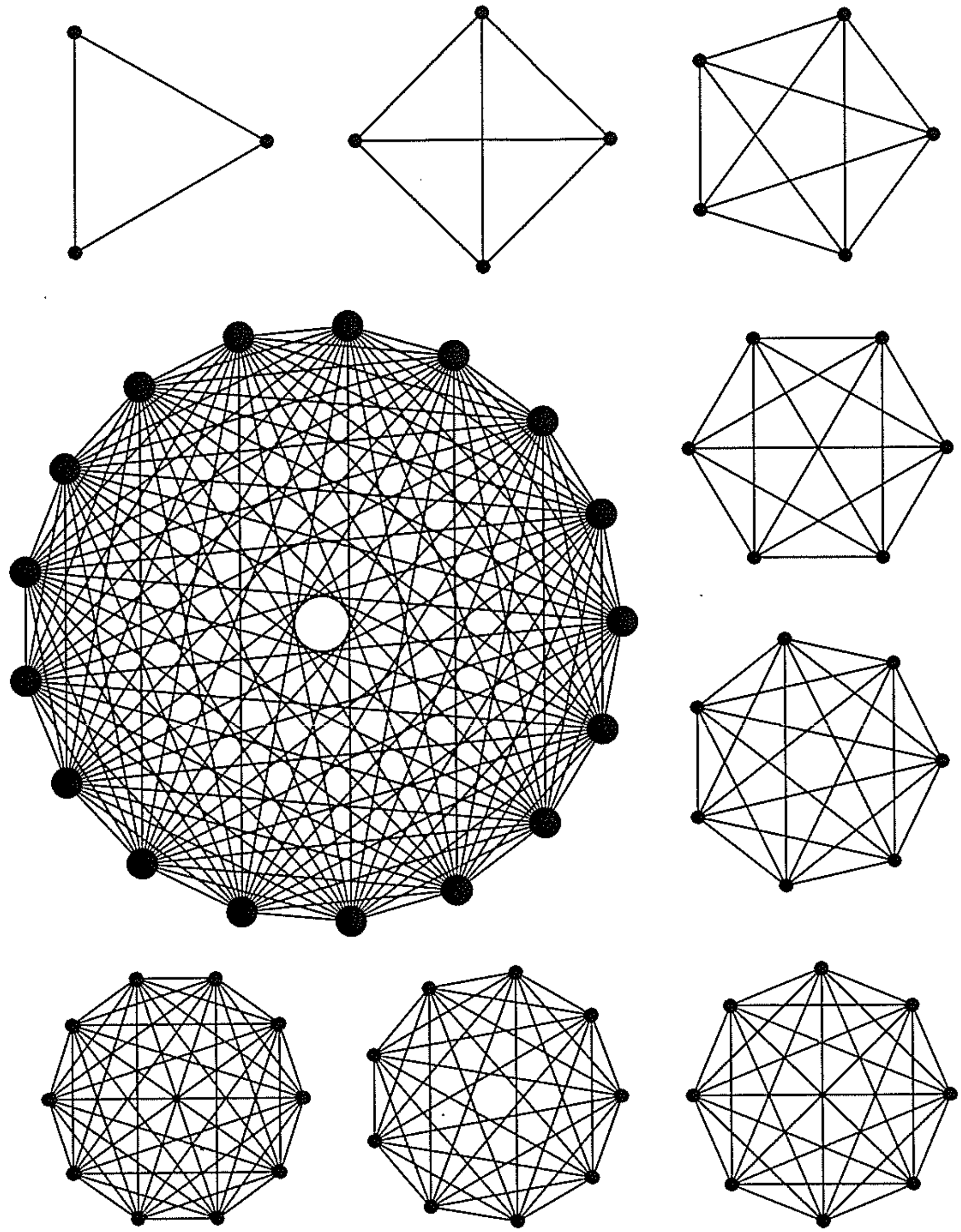
# Some Complete Graphs



How many edges does the largest graph on this page have?

# Resource Book

## Workshop 8: Number Patterns and Iteration

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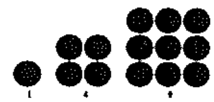
# Resource Book

## Workshop 8: Number Patterns and Iteration

**Mathematical Background**

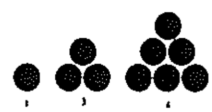
- **Square numbers:** A number is called a square number when that number of items can be arranged in a square array as shown here:

  An explicit formula for the n'th square number is:
  $S(n) = n^2 = n \times n$.

  A recursive formula for the n'th square number is:
  $S(n) = S(n-1) + (2n - 1)$

- **Triangular numbers:** A number is called a triangular number when that number of items can be arranged in a triangular array as shown here:

  An explicit formula for the n'th triangular number is:
  $T(n) = (n+1)n/2$

  A recursive formula for the n'th triangular number is:
  $T(n) = T(n - 1) + n$

- **Difference Tables:**

  When we list a sequence of numbers, then list the differences between successive number, then list the differences between successive results, etc., we arrive at what's called a difference table.

  For example, the triangular numbers yield the following difference table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Triangular numbers: | 1 | 3 | 6 | 10 | 15 | 21 | 28 | ... |
| First differences: | 2 | 3 | 4 | 5 | 6 | 7 | ... | |
| Second differences: | 1 | 1 | 1 | 1 | 1 | ... | | |
| Third differences: | 0 | 0 | 0 | 0 | ... | | | |

- **Iteration:** Repeating a process over and over again to generate a sequence or pattern. For example, making a spirolateral is an iterative process.

# Resource Book

## Workshop 8: Number Patterns and Iteration

### Workshop Outline

1. **Triangular Numbers**
   a. We counted dots in an 11×11×11 triangle of dots, and found that there were 66 of them. We then saw "66" as just one number in a sequence of numbers called the triangular numbers.
   b. Counting the number of handshakes that take place among a group of people, we found the triangular numbers again, and saw that these were precisely the "choose 2" numbers. We found an explicit rule for the traingular numbers, $T(n) = (n+1)n/2$ = "($n$+1) choose 2", and the recursive rule $T(n) = T(n-1) + n$.
   c. We used the pattern of the traingular numbers to motivate the construction of a difference table.

2. **Square Numbers**
   a. Using pattern blocks to build larger squares from smaller squares, we discovered the square numbers 1, 4, 9, 16, ... If we denote the $n$th square number by $S(n)$, then:
      i. We have the recursive rule $S(n) = S(n-1) + (2n-1)$, and
      ii. We have the explicit rule $S(n) = n^2$
   b. Using triangles to build larger triangles yielded the same result.

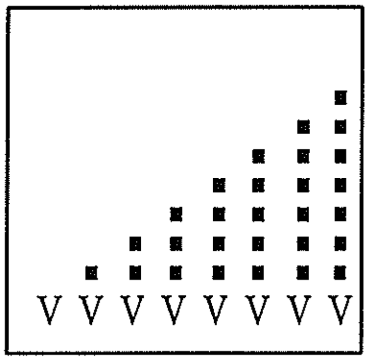
3. **Difference Tables**
   a. Considering the differences between successive terms in a sequence sometimes revealed a pattern that wasn't obvious by looking at the sequence itself. We used this technique to help catch a thief who robbed houses 1, 5, 15, 33, 61, and 101, by taking first, second and third differences and finding the pattern.

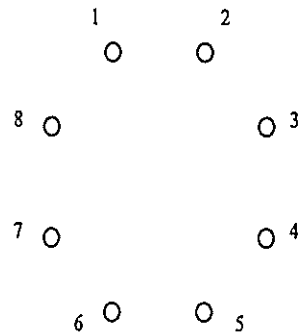
4. **Iteration and Spirolaterals**
   a. We first saw several examples of iterative procedures
   b. We generated spirolaterals by converting our names to numbers and used the resulting number sequence to determine the lengths in our spiraling pattern.
   c. Some similarities we discovered were that
      i. If a word had an odd number of letters then it would return to its starting position in exactly 4 iterations.
      ii. If a word had a number of letters which was a multiple of 4, then it would most likely spiral off the page. Some occasional 4-letter words didn't do this, like "Shaq" or "Toto" or "Adam" or "Sean" or "Swan."
      iii. If a word had an even number of letters, but not a multiple of 4, then it would return to its starting position in 2 iterations.

## Workshop 8: Number Patterns and Iteration

**The Handshake Problem**

There are eight people in a room and each person shakes hands with all seven other people. How many handshakes take place altogether? (Explain your answer in several different ways.)




You can commemorate handshakes using each of the two diagrams above. The diagram at the left represents what happens if the V at the left shakes hands with each of the people to the right, dropping a marker to commemorate each handshake, and then the next V does the same, etc. The net result is that the handshakes among these 8 V's are commemorated by piles which together are reminiscent of the triangular numbers.

Similarly, the handshakes made by vertex 1 in the diagram at the right are commemorated by drawing edges from 1 to each vertex. Then, 2 shakes hands with the remaining vertices, leaving an edge after each handshake. In commemoration of all the handshakes, we would end up with a complete graph on 8 vertices. Since the number of pairs of vertices is "8 choose 2", that is the also the number of handshakes.

Summary: The following all have the same value —

(a) the number of handshakes among 8 people;
(b) the number of edges in the complete graph with 8 vertices;
(c) the number of dots in an 8x8x8 triangle;
(d) the sum of the first 8 counting numbers;
(e) the 7'$^{th}$ triangular number;
(f) "8 choose 2"; and
(g) (8x7)/2.

In general, the following are all the same —

(a) the number of handshakes among n people;
(b) the number of edges in the complete graph with n vertices;
(c) the number of dots in an nxnxn triangle;
(d) the sum of the first n counting numbers;
(e) the (n-1)$^{st}$ triangular number; and
(f) "n choose 2"; and
(g) n(n-1)/2.

## Workshop 8: Number Patterns and Iteration

**Difference tables**

The following difference table displays the differences between successive triangular numbers; these differences are actually the counting numbers. The table also displays the "second differences", that is, the differences between successive differences; these are all 1.

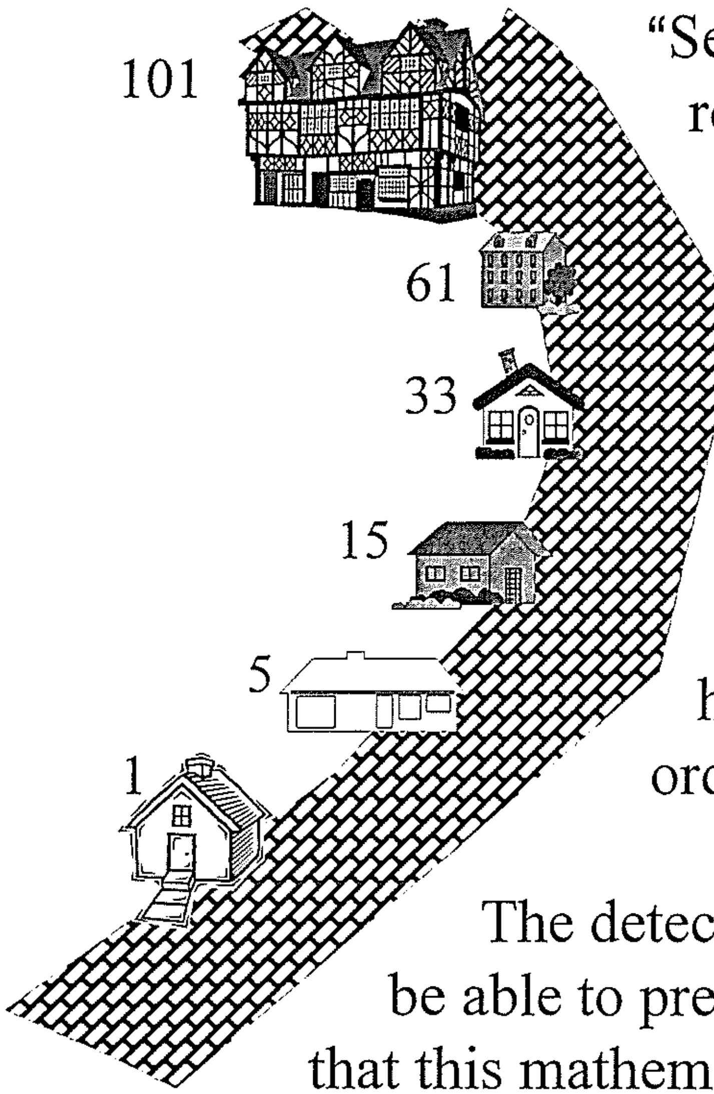
| $T(1)$ | | $T(2)$ | | $T(3)$ | | $T(4)$ | | $T(5)$ | | $T(6)$ | | $T(7)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 3 | | 6 | | 10 | | 15 | | 21 | | 28 |
| | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | |
| | | 1 | | 1 | | 1 | | 1 | | 1 | | |

Given a sequence of numbers, one possible continuation can be obtained by constructing a difference table. For example, given the sequence 1, 5, 13, 25, 61, we place them in the top row of a difference table, find the difference between successive terms, then the second differences, then the third differences, until all entries in a row are the same.

| $Q(1)$ | | $Q(2)$ | | $Q(3)$ | | $Q(4)$ | | $Q(5)$ | | $Q(6)$ | | $Q(7)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 5 | | 13 | | 25 | | 41 | | | | |
| | 4 | | 8 | | 12 | | 16 | | | | | |
| | | 4 | | 4 | | 4 | | | | | | |

Then you extend the constant row (in this case with additional 4's), and use addition to fill in the blank entries, obtaining Q(6) = 61 and Q(7) =85.

# Welcome to the land of Ahhhs.




A thief, calling himself the "Sequencer," has been robbing houses in this peaceful land! Six weeks ago he robbed the house at 1 Brick Way, and since then has proceeded up the street robbing the houses shown in the order shown.

The detectives would like to be able to predict the next house that this mathematically inclined thief will hit.

Can you discover which house that will be?

# Resource Book

## Workshop 8: Number Patterns and Iteration

### Iteration ...

... means repeating an action over and over, typically on the outcome of the last iteration

**Two items to consider when iterating:**
- A starting point
- Instructions for iterating

**Examples:**

1. Counting by 5 — 5→10→15→20→25→30→35→40→45; What is the action? Where do you start?

2. Evens and Odds — 7→22→11→34→17→52→26→13→40→20→10→5→...; What is the action? If odd, triple and add 1; if even, divide by two.

3. Multiply the Digits — 19375→945→180→0; Pick a number. Multiply its digits to get a new number. Multiply its digits to get a new number. Multiply its digits to get a new number. Etc.

4. Numbers in Words — 200→10→3→5→4→4; Pick a number. Write it in words, and count the letters to get a new number. Write it in words, and count the letters to get a new number. Etc.

5. The Look and Say Sequence — 21→1211→111221→312211→13112221→113213211→31131211131221; What is the action? What is the next number?

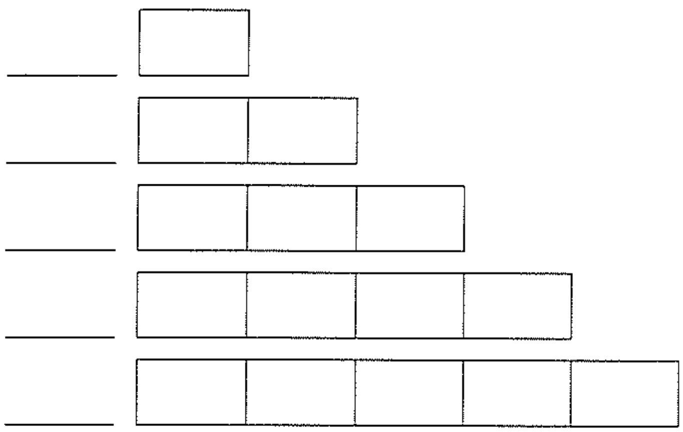
**Geometric examples:**

1. Start at "START"; iterate by taking 2 steps forward and hopping.

2. Start at "START"; iterate by taking 2 steps forward, turning right 90 degrees, and hopping.

3. Start at "START" and set $i = 1$; iterate by taking $i$ steps forward, turning right 90 degrees, adding one to i, and hopping.

# COUNTING RECTANGLES



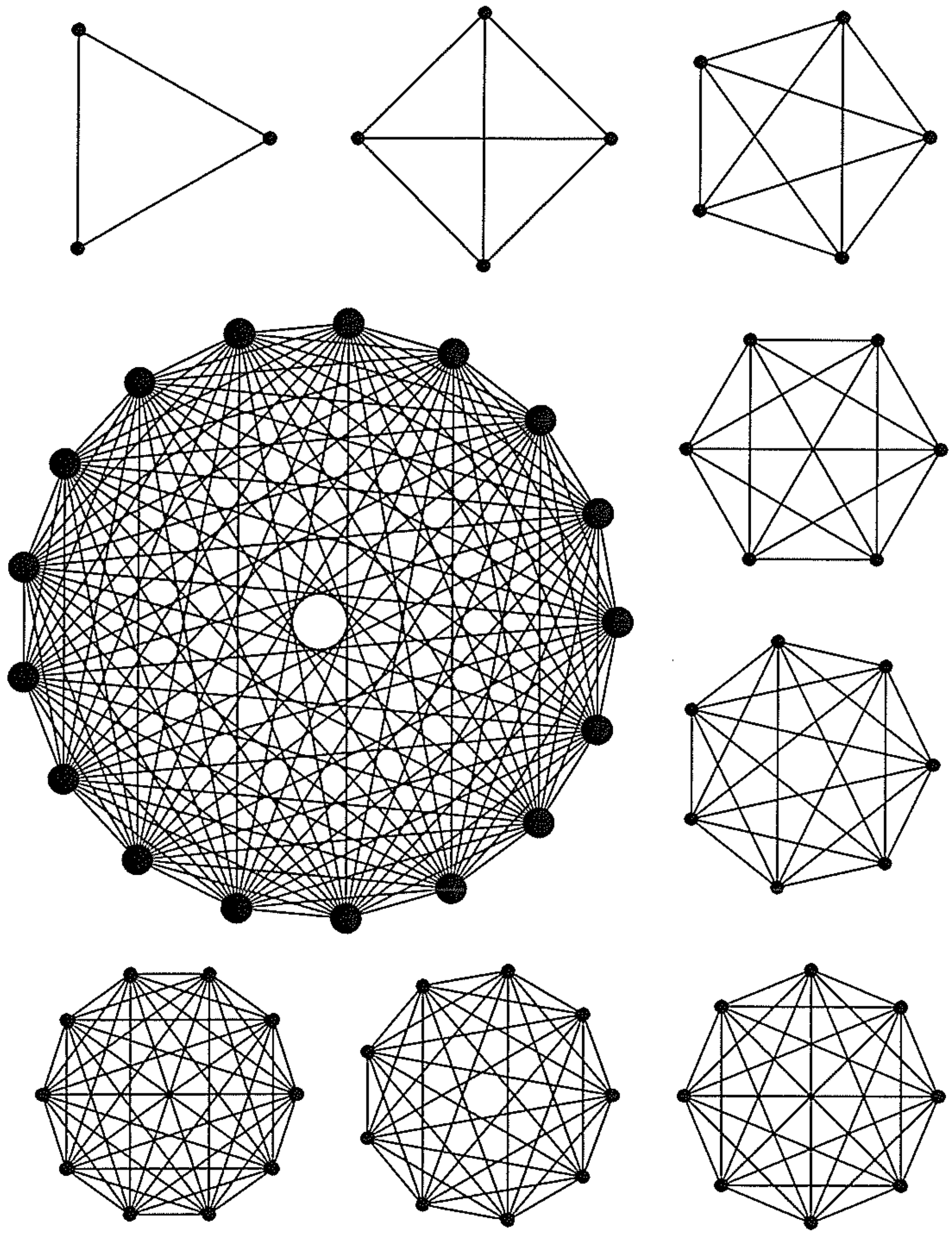
Indicate the total number of rectangles you see in each figure.

| # of small rectangles | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|
| Total number or rectangles | | | | | | | | |

# Resource Book

## Workshop 8: Number Patterns and Iteration



by Robert Hochberg — Copy as you wish

# Resource Book

## Workshop 8: Number Patterns and Iteration

### Spirolaterals — personalized version

Start with your name, and use the coding scheme on the dot paper to convert your name into your very own sequence of numbers.

Examples:

| | | |
|---|---|---|
| J | O | E |
| 1 | 6 | 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| V | A | L | E | R | I | E |
| 4 | 1 | 3 | 5 | 9 | 9 | 5 |

Your Name and Code:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | |

Make a chart like that below, and enter your personal sequence of numbers on the top row starting at the left. Then enter it again (leaving no empty boxes) and again until no space is left. For example, JOE would enter 1 6 5 1 6 5 1 6 5 1 6 5 1 6 5 etc., repeating 1 6 5 over and over, as shown above.

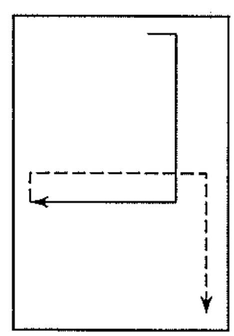
Example

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | 1 | 6 | 5 | ... |
| R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | ... |

Your Code repeated Over and Over

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | |
| R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | D | L | U | R | ... |

Now start on any dot on the graph paper, and draw a single line following the instructions in the chart, where the top row indicates the length of the line and the bottom row indicates the direction. For example, JOE would first draw a line to the right (R) one box, then continue down (D) six boxes, then left (L) five boxes, then up (U) one box, then right (R) six boxes, then down (D) five boxes, etc. The example with "165" is shown to the right, with the first "165" solid, the second dashed.



# Resource Book

## Workshop 8: Number Patterns and Iteration

Dr. Evan Maletsky
Montclair State University

| | | |
|---|---|---|
| 1-A | J | S |
| 2-B | K | T |
| 3-C | L | U |
| 4-D | M | V |
| 5-E | N | W |
| 6-F | O | X |
| 7-G | P | Y |
| 8-H | Q | Z |
| 9-I | R | |

# Resource Book

## Workshop 8: Number Patterns and Iteration

# Patterns in Powers of Digits

By RICHARD W. SHOEMAKER, University of Toledo (Emeritus), Toledo, OH 43606

We sometimes find it useful, when considering a natural number, to form some function of the digits of the number. For example, if we add the digits of 38 654 we get $3 + 8 + 6 + 5 + 4 = 26$; then do the same with this sum to get $2 + 6 = 8$. This process, "casting out nines," informs us that 38 654 is 8 more than an integral multiple of 9. We could have discarded the terms $3 + 6 = 9$ and $5 + 4 = 9$ in the first sum and arrived at 8 more efficiently. Another function involving the digits of a number is exemplified by the "casting out elevens" procedure on 38 654, where we compute $4 + 6 + 3 - (5 + 8) = 0$. This result tells us that 38 654 is divisible by 11. Hence we are on solid ground when we experiment, as we shall soon do, with other functions of the digits of various natural numbers. The work involved in this study is merely arithmetic, but in terms of saving time and eliminating drudgery it would be wise to use a calculator for the problems of squares and cubes, and possibly a computer for higher powers.

**Sums of Squares**

In our first investigation we shall, for any natural number, sum the squares of its digits to arrive at a new number (Porges 1945). We then repeat the procedure several times and hope for something interesting to happen. As an example, if we start with 35: $3^2 + 5^2 = 34$, then $3^2 + 4^2 = 25$, and from this $2^2 + 5^2 = 29$, and so on. We abbreviate these results by $35 \to 34 \to 25 \to 29$.

Suppose, for example, that we start with the number 58. The sequence will be

$$58 \to 89 \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58.$$

That is, we have returned to the original number. This result could be indicated by the pattern in figure 1, and of course if we had started with any of these numbers, the same closed eight-link chain would have been generated.

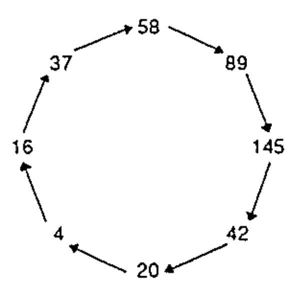



Fig 1 Sum the square of the digits

Suppose instead that we had started with 1 or 10 or 100 or ... or $10^n$; the first transformation would yield 1, and no further changes would occur. Similarly $13 \to 10 \to 1$; and

$$1247 \to 70 \to 49 \to 97 \to 130 \to 1.$$

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That is, some natural numbers will eventually transform into unity.

But what about the natural numbers not yet mentioned? These will eventually break into the eight-link closed chain in figure 1. Figure 2 indicates how a few numbers enter. To be more specific regarding figure 2, the sum of the squares of the digits of any number on the nonpointed end of an arrow equals the number on the pointed end of the same arrow. This new number is on the nonpointed side of another arrow, so repeat the procedure. The paths thus defined lead to various numbers on the terminal chain. Any number outside of the chain transforms via the path it is on to some number on the cyclic chain.

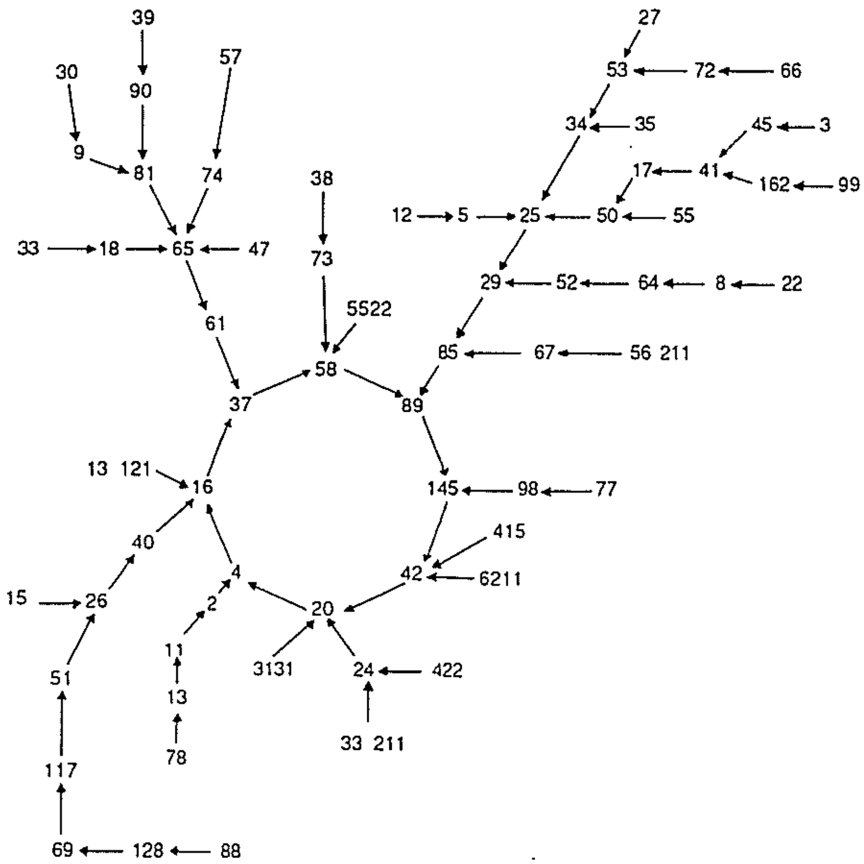


Fig 2 All numbers reduce to the eight-link chain

The interested reader may wish to try some other numbers and see how they fit into the diagram.

**Sums of Cubes**

In the next investigation the sum of the cubes, rather than squares, of the digits of any natural number is used. The results here, nine terminal states rather than the two encountered with squares, are complex enough that we are discouraged from going on to fourth, fifth, and higher powers. To categorize the nine terminal states: In five cases the terminal result is a single number (1, or 153, or 370, or 371, or 407) that reproduces itself. Two cases yield two-link closed chain terminals (919 ↔ 1459, or 136 ↔ 244); and two cases produce three-link closed chains,

$$55 \to 250 \to 133 \to 55,$$

or

$$160 \to 217 \to 352 \to 160.$$

Consider now the individual cases.

*Case 1.* If the number is of the form $3k$, for any $k$, then the terminal result is 153 (Singleton 1967). A rather lengthy example is 15. To start, $15 \to 1^3 + 5^3 = 126$; then $1^3 + 2^3 + 6^3 = 225$, and so on, so that

$$15 \to 126 \to 225 \to 141 \to 66 \to 432 \to 99 \to 1458 \to 702 \to 351 \to 153.$$

*Case 2.* If the number is of the form $3k + 1$ for certain values of $k$, the terminal result is 1. Examples:

$$10^n \to 1$$
$$211 \to 10 \to 1$$
$$1234 \to 100 \to 1$$

*Case 3.* For other values of $k$ the form $3k + 1$ leads to the terminal result 370. Examples:

$$88 \to 1024 \to 73 \to 370$$
$$7 \to 343 \to 118 \to 514 \to 190 \to 730 \to 370$$

*Case 4.* If the number is of the form $3k - 1$, for many $k$, the terminal result will be 371. Examples:

11 → 2 → 8 → 512 → 134 → 92 → 737 → 713 → 371
5 → 125 ↗ (134)     341 ↗ (92)
14 → 65 ↗ (341)

*Case 5.* For some other values of $k$, the form $3k - 1$ results in 407. Examples:

77 → 686 → 944 → 857 → 980 → 1241 → 74 → 407
42 221 → 89 ↗ (1241)

*Case 6.* For some values of $k$, natural numbers of the form $3k + 1$ terminate in the 919 ↔ 1459 chain, as shown in figure 3.

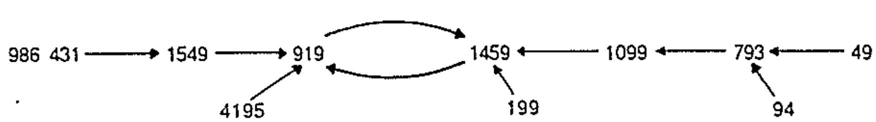


Fig. 3 Some natural numbers of the form $3k + 1$ terminate in 919 ↔ 1459

*Case 7.* For some other values of $k$, the form $3k + 1$ terminates in the 136 ↔ 244 chain shown in figure 4.

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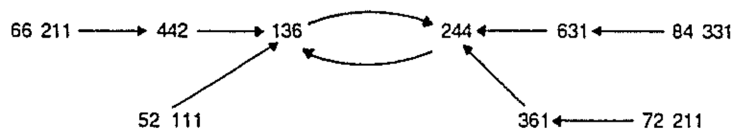


Fig. 4. Some natural numbers of the form $3k + 1$ terminate in $136 \leftrightarrow 244$.

*Case 8.* For certain values of $k$, numbers of the form $3k + 1$ terminate in the three-link chain

$$55 \rightarrow 250 \rightarrow 133 \rightarrow 55,$$

as in figure 5.

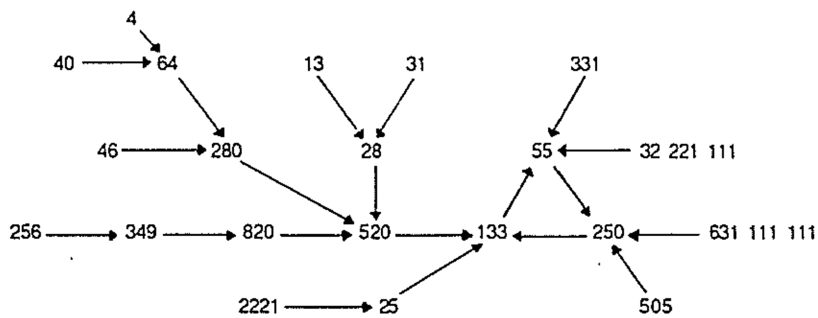


Fig. 5. Some natural numbers of the form $3k + 1$ terminate in $55 \rightarrow 250 \rightarrow 133 \rightarrow 55$.

*Case 9.* Finally, for some other values of $k$, the form $3k + 1$ results in the

$$160 \rightarrow 217 \rightarrow 352 \rightarrow 160$$

terminal chain, as in figure 6.

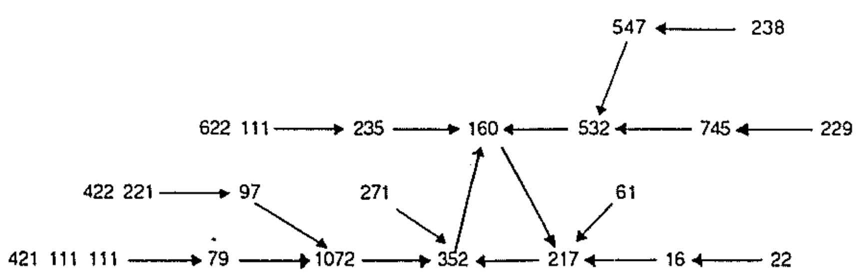


Fig. 6. Some natural numbers of the form $3k + 1$ terminate in $160 \rightarrow 217 \rightarrow 352 \rightarrow 160$.

**Conclusion**

Using elementary concepts and arithmetic-intensive methods, we have uncovered an interesting, beautiful, and perhaps unexpected aspect of mathematics. These elementary concepts and arithmetic are also available to the student who, with the aid of the hand-held calculator, could duplicate this study or could generate different functions and experience the deep satisfaction arising from individual discoveries.

The author is indebted to the referees for many helpful criticisms and suggestions.

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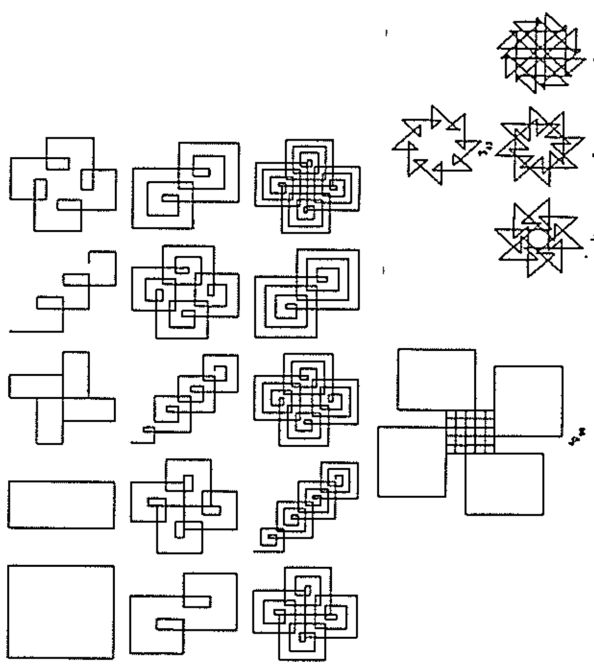

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These figures were taken off of the internet at site http://www.planetary.caltech.edu/~eww/math/node4955.html

# ACTIVITIES

## POLYGONAL NUMBERS AND RECURSION

By WILLIAM A. MILLER, Central Michigan University, Mount Pleasant, MI 48859

Teacher's Guide

*Introduction:* Recursion is a powerful mathematical process for generating sequences. In a recursively defined sequence, the initial terms are defined and each subsequent term is determined from one or more of its predecessors in the same way. Recurrence relations are the formulas that express each term as a function of its predecessors. The word *recurrence* is derived from a Latin verb meaning "to look back," and that is exactly what happens. To find the *n*th term of the sequence, one looks back to the terms that precede it in the sequence.

The sequences of polygonal numbers were first studied by the ancient Greeks and were represented geometrically by arrays of dots that formed regular polygons. These sequences are interesting examples of recursively defined sequences. In these activities, polygonal numbers are used to help students investigate recursively defined sequences.

The use of concrete materials and pictorial representations allows middle school and high school students to investigate informally the process of recursion. The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (*Standards*) (1989) recommends that "during the middle years, the study of patterns and functions should focus on the analysis, representation, and generalization of functional relationships" (p. 98). The standards for grades 9 through 12 recommend that "sequences ... receive more attention, with a greater emphasis on their descriptions in terms of recurrence relations" (p. 177). The patterns that occur within and between sequences of polygonal numbers present ample opportunity for students to analyze, represent, and generalize relationships.

*Grade levels:* 6–12

*Materials:* One set of activity sheets for each student and a transparency of sheet 1. Markers or chips for constructing the numbers are optional. Calculators are optional.

*Objectives:* Students will develop an understanding of the concepts of recursion, recursive sequences, and recurrence relations. Students will generate data, record and study their results, and discover patterns. These activities are motivational; students will enjoy discovering patterns and testing their conjectures.

*Edited by Mary Kim Prichard, University of North Carolina at Charlotte, Charlotte, NC 28223*
*Nadine Bezuk, San Diego State University, San Diego, CA 92182*
*Molly Moody, Oxford High School, Oxford, AL 36203*

*This section is designed to provide mathematical activities in reproducible formats appropriate for students in grades 7–12. This material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to the "Activities" already published, to the editorial coordinator for review. Of particular interest are activities focusing on the Council's curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers.*



*Directions:* This activity will take one or two class periods to complete, depending on the level of the class, the amount of material assigned as homework, and the number of extension activities that are completed.

*Sheet 1:* Distribute one copy of sheet 1 to each student. A transparency of sheet 1 can be used to facilitate the teacher's introduction for the entire class. Students, following the teacher's demonstration, should draw figures on their dot-paper grids or arrange markers in triangular arrays to verify their conjectures. Once the activity has been introduced, students can work individually or in small groups. Group work seems to stimulate communication and problem-solving skills. After sheet 1 has been completed and discussed, the remaining sheets can be distributed one at a time or all together. Sheets 2, 3, and 4 can be done in any order.

Many students, particularly those without formal experience in algebra, may answer problems 3c and 3f on sheet 1 incorrectly. A common answer for problem 3c is $210 - 19 = 191$, and a common answer for problem 3f is $5050 - 99 = 4951$. Some students may also answer $T_{18} + 19$ for problem 3c. Students tend to experience difficulty with the reversibility involved. Students with formal experience in algebra usually answer the problem by solving for $T_{19}$ in terms of $T_{20}$.

*Sheet 2:* When working on sheet 2, many students may recognize the general formula $S_n = n^2$ and may answer the exercises using that observation. If so, teachers have an excellent opportunity to point out and discuss the difference between sequences generated from recurrence relations and sequences generated from closed formulas (sequences where the *n*th term is a function of *n*).

*Sheet 4:* After completing sheets 1, 2, 3, and 4, students are ready to make generalizations about recursive sequences for other polygonal numbers. Students may need help in organizing their data from the activities.

The recurrence relations for other polygonal numbers are

$$HP_{n+1} = HP_n + 5n + 1$$

(heptagonal numbers),

$$O_{n+1} = O_n + 6n + 1$$

(octagonal numbers), and so on. In general,

$$_mF_{n+1} = {_mF_n} + (m - 2)n + 1,$$

where *m* denotes the number of sides in the polygon and *n* represents the number of dots on each side. Testing conjectures requires building or drawing the appropriate polygonal number. Markers and heptagonal and octagonal dot paper are helpful. (Examples are provided in the Appendix).

*Extension activities:* Students who have solved systems of equations can determine the closed formulas that produce the polygonal-number sequences. Since polygonal numbers are two-dimensional, they can be expressed as quadratic functions. To generate the closed formulas, students must find *a*, *b*, and *c* in the function

$$f(n) = an^2 + bn + c.$$

This process can be followed for any of the polygonal-number sequences when three or more terms are known. For example, for triangular numbers,

$$f(1) = a + b + c = 1,$$
$$f(2) = 4a + 2b + c = 3,$$

and

$$f(3) = 9a + 3b + c = 6.$$

Solving this system of linear equations for *a*, *b*, and *c* yields $a = 1/2$, $b = 1/2$, $c = 0$, and $f(n) = (n^2 + n)/2$.

The closed formulas for other sequences are given below. This form for the expressions helps students recognize the pattern.

$$T_n = \frac{n(n+1)}{2} \qquad S_n = \frac{n(2n+0)}{2}$$
$$P_n = \frac{n(3n-1)}{2} \qquad H_n = \frac{n(4n-2)}{2}$$
$$HP_n = \frac{n(5n-3)}{2} \qquad O_n = \frac{n(6n-4)}{2}$$
$$_mF_n = \frac{n[(m-2)n - (m-4)]}{2}$$

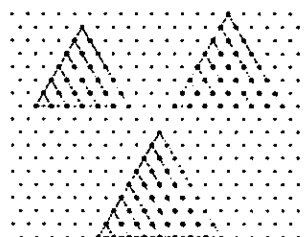
If your students are not ready to solve systems of equations, you may wish to share these formulas with them. With a few hints, more mature students will discover the formulas. For lower-level students, furnish the first two or three formulas and ask them to verify that the formulas are correct and make conjectures about formulas for other sequences. After students have generated the closed formulas, the teacher has another opportunity to compare recurrence relations and closed formulas.

If dot paper is supplied, students can generate their own sequences of numbers using oblong, trapezoidal, and star patterns, as well as patterns that they design.

*Answers: Sheet 1*

1. (a) $T_7 = T_6 + 7 = 21 + 7 = 28$
   (b) $T_8 = T_7 + 8 = 28 + 8 = 36$
   (c) $T_9 = T_8 + 9 = 36 + 9 = 45$

2.

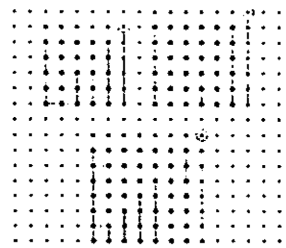


3. (a) $T_{21} = 231$ (b) $T_{22} = 253$ (c) $T_{19} = 190$
   (d) $T_{101} = 5151$ (e) $T_{102} = 5253$ (f) $T_{99} = 4950$
   (g) $T_{n+1} = T_n + n + 1$

*Sheet 2*

1. (a) $S_6 = S_5 + 2(5) + 1 = 25 + 10 + 1 = 36$
   (b) $S_7 = S_6 + 2(6) + 1 = 36 + 12 + 1 = 49$
   (c) $S_8 = S_7 + 2(7) + 1 = 49 + 14 + 1 = 64$

2.

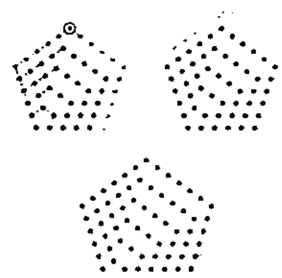


3. (a) $S_{21} = 441$ (b) $S_{22} = 484$ (c) $S_{19} = 361$
   (d) $S_{101} = 10201$ (e) $S_{102} = 10404$ (f) $S_{99} = 9801$
   (g) $S_{n+1} = S_n + 2n + 1$

*Sheet 3*

1. (a) $P_6 = P_5 + 3(5) + 1 = 35 + 15 + 1 = 51$
   (b) $P_7 = P_6 + 3(6) + 1 = 51 + 18 + 1 = 70$
   (c) $P_8 = P_7 + 3(7) + 1 = 70 + 21 + 1 = 92$

2.



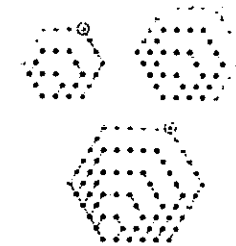
3. (a) $P_{21} = 651$ (b) $P_{22} = 715$ (c) $P_{19} = 532$
   (d) $P_{101} = 15\,251$ (e) $P_{102} = 15\,555$
   (f) $P_{99} = 14\,652$ (g) $P_{n+1} = P_n + 3n + 1$
4. $H_2 = 6, H_3 = 15, H_4 = 28$
5. $H_{n+1} = H_n + 4(n) + 1$



*Sheet 4*

1. (a) $H_5 = H_4 + 4(4) + 1 = 28 + 16 + 1 = 45$
   (b) $H_6 = H_5 + 4(5) + 1 = 45 + 20 + 1 = 66$
   (c) $H_7 = H_6 + 4(6) + 1 = 66 + 24 + 1 = 91$

2.



3. (a) $H_{21} = 861$ (b) $H_{22} = 946$ (c) $H_{19} = 703$
   (d) $H_{101} = 20\,301$ (e) $H_{102} = 20\,706$
   (f) $H_{99} = 19\,503$ (g) $H_{n+1} = H_n + 4n + 1$

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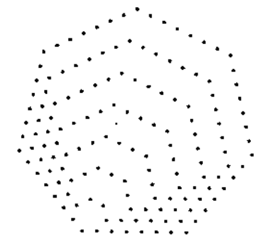
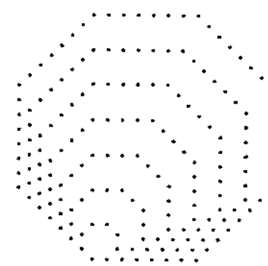
### APPENDIX





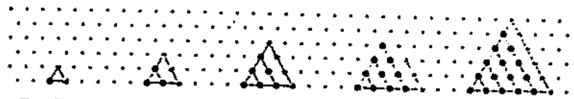
# Workshop 8: Number Patterns and Iteration

TRIANGULAR NUMBERS SHEET 1

Polygonal numbers are numbers that can be represented by a collection of dots in a polygonal array. The numbers 1, 3, 6, 10, . . . are called *triangular numbers* because they can be represented by a triangular array. Mathematicians use symbols to represent polygonal numbers. $T_n$ is used to represent a triangular number with $n$ dots on each side. Thus, $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, and so on.

$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$ $T_5 = 15$ $T_6 = 21$

As you analyze the formation of the triangular number, you will see that each new triangular number is formed from the previous one by adding a row containing one more dot than the side of the previous triangle.



$T_2 = T_1 + 2 = 1 + 2 = 3$ $T_3 = T_2 + 3 = 3 + 3 = 6$ $T_4 = T_3 + 4 = 6 + 4 = 10$ $T_5 = T_4 + 5 = 10 + 5 = 15$ $T_6 = T_5 + 6 = 15 + 6 = 21$

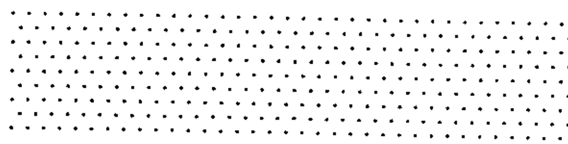
1. Compute the following:

(a) $T_7 = T_6 +$ ____ (b) $T_8 = T_7 +$ ____ (c) $T_9 = T_8 +$ ____

= ____ = ____ = ____

= ____ = ____ = ____

2. Draw figures to verify your answers to problem 1.



3. Complete the following:

$T_{20} = 210$: (a) $T_{21}$ = ____ (b) $T_{22}$ = ____ (c) $T_{19}$ = ____

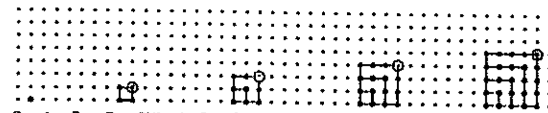
$T_{100} = 5050$: (d) $T_{101}$ = ____ (e) $T_{102}$ = ____ (f) $T_{99}$ = ____

(g) $T_{n+1} = T_n +$ ____

From the Mathematics Teacher, October 1990

SQUARE NUMBERS SHEET 2

Each square number is formed from the previous one by adding two rows of dots plus one.



$S_1 = 1$ $S_2 = S_1 + 2(1) + 1 = 1 + 2 + 1 = 4$ $S_3 = S_2 + 2(2) + 1 = 4 + 4 + 1 = 9$ $S_4 = S_3 + 2(3) + 1 = 9 + 6 + 1 = 16$ $S_5 = S_4 + 2(4) + 1 = 16 + 8 + 1 = 25$

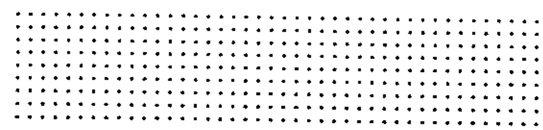
1. Compute the following:

(a) $S_6 = S_5 +$ ____ (b) $S_7 = S_6 +$ ____ (c) $S_8 = S_7 +$ ____

= ____ = ____ = ____

= ____ = ____ = ____

2. Draw figures to verify your answers to problem 1.



3. Complete the following:

$S_{20} = 400$: (a) $S_{21}$ = ____ (b) $S_{22}$ = ____ (c) $S_{19}$ = ____

$S_{100} = 10\,000$: (d) $S_{101}$ = ____ (e) $S_{102}$ = ____ (f) $S_{99}$ = ____

(g) $S_{n+1} = S_n +$ ____

### PENTAGONAL NUMBERS

SHEET

Each pentagonal number is formed from the previous one by adding thre rows of dots plus one.

$P_1 = 1$ $P_2 = P_1 + 3(1) + 1 = 1 + 3 + 1 = 5$ $P_3 = P_2 + 3(2) + 1 = 5 + 6 + 1 = 12$ $P_4 = P_3 + 3(3) + 1 = 12 + 9 + 1 = 22$ $P_5 = P_4 + 3(4) + 1 = 22 + 12 + 1 = 35$

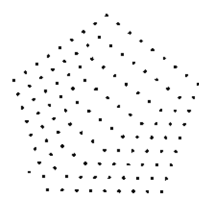
1. Compute the following:

   (a) $P_6 = P_5 +$ ______ (b) $P_7 = P_6 +$ ______ (c) $P_8 = P_7 +$ ______

   = ______ = ______ = ______

   = ______ = ______ = ______

2. Draw figures to verify your answers to problem 1.



3. Complete the following:

   $P_{20} = 590$: (a) $P_{21}$ = ______ (b) $P_{22}$ = ______ (c) $P_{19}$ = ______

   $P_{100} = 14\,950$: (d) $P_{101}$ = ______ (e) $P_{102}$ = ______ (f) $P_{99}$ = ______

   (g) $P_{n+1} = P_n +$ ______

4. The first hexagonal number is 1 ($H_1 = 1$); what do you think the nex three hexagonal numbers are?

   $H_2$ = ______ $H_3$ = ______ $H_4$ = ______

5. Can you give $H_{n+1}$ in terms of $H_n$?

   $H_{n+1} = H_n$ ______

### HEXAGONAL NUMBERS

SHEET 4

Each hexagonal number is formed from the previous one by adding four rows of dots plus one.

$H_1 = 1$ $H_2 = H_1 + 4(1) + 1 = 1 + 4 + 1 = 6$ $H_3 = H_2 + 4(2) + 1 = 6 + 8 + 1 = 15$ $H_4 = H_3 + 4(3) + 1 = 15 + 12 + 1 = 28$

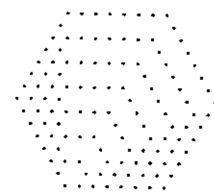
1. Compute the following:

   (a) $H_5 = H_4 +$ ______ (b) $H_6 = H_5 +$ ______ (c) $H_7 = H_6 +$ ______

   = ______ = ______ = ______

   = ______ = ______ = ______

2. Draw figures to verify your answers to problem 1.



3. Complete the following:

   $H_{20} = 780$: (a) $H_{21}$ = ______ (b) $H_{22}$ = ______ (c) $H_{19}$ = ______

   $H_{100} = 19\,900$: (d) $H_{101}$ = ______ (e) $H_{102}$ = ______ (f) $H_{99}$ = ______

   (g) $H_{n+1} = H_n +$ ______

4. Challenge:

   Organize your data from activities 1, 2, 3, and 4 and see if you can make generalizations about other polygonal numbers: $HP_n$ (heptagonal), $O_n$ (octagonal), $N_n$ (nonagonal), and so on.

## Workshop 8: Number Patterns and Iteration

# Fibonacci Numbers
# The Structure of Problems

By Robert Hochberg and Valerie DeBellis

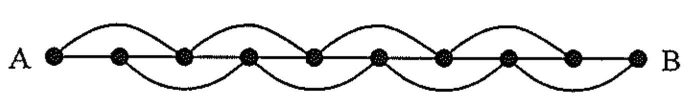
The following problems all have the same answer...the Fibonacci numbers. It often happens that a good problem arises from a good answer, and not the other way around. In the case of the Fibonacci numbers, it is their structure that enables instructors to design interesting problems. For example, the "European Village" problem was inspired by the following fact about the Fibonacci numbers:

> The number of ways to write $n$ as the sum of ones and twos is equal to the $n$th Fibonacci number.

This "structure", writing a number as the sum of ones and twos, can be addressed in many different ways by asking appropriate questions. In the last problem, the "European Village", notice that you could put a roof on either one or two houses at a time---so what you are really doing is counting the number of ways to write $n$, the number of houses, as the sum of ones and twos.

Here are several more problems for you to consider. They all have the same answer---the 10'th Fibonacci number. If you change the "10" in each problem to "$n$", then the answer becomes the $n$th Fibonacci number. Can you see the structure within the problems? Can you think of any other clever ways to ask the "same" question?

- How many different ways can you write 10 as the sum of ones and twos?
- How many different ways can a frog hop up a staircase with 10 steps if he can take hops of sizes 1 or 2 stairs at a time?
- How many different ways are there to walk from point A to point B on the following graph, moving only to the right?

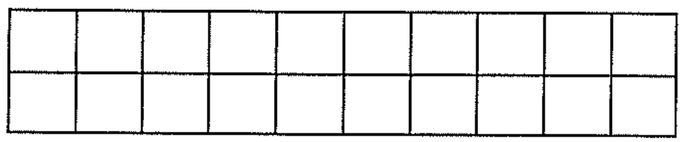


## Workshop 8: Number Patterns and Iteration

### Fibonacci Numbers
### The Structure of Problems

By Robert Hochberg and Valerie DeBellis

- How many different ways can you cover the 2x10 checkerboard below with 1x2 dominos?



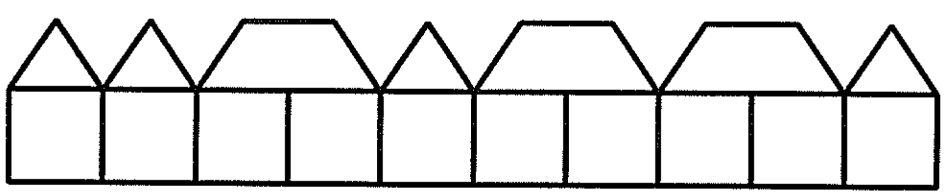
- How many different 10 digit binary numbers are there that end in '0', and don't contain a pair of adjacent '1's?

- How many different ways can you put 50¢ into a soda machine using dimes and nickels?

- How many different ways can a family have 10 children if they have either one or two children at a time? (Eg., *twins, twins, one child, one child, twins, one child, one child* is one way to do it.)

- How many different ways can Kate fulfill her 40 hour consulting contract if she works either half days (4 hours) or full days (8 hours) each day? (Eg., *full day, full day, half day, full day, half day, full day* is one way to do it.)

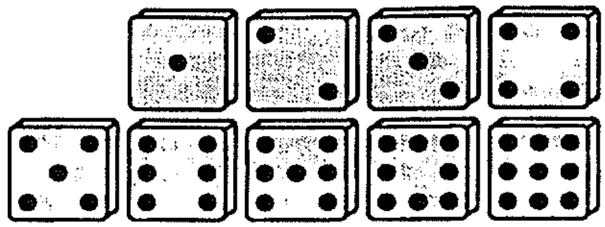
- How many ways can you put roofs on a European village with 10 houses (squares) if a roof can go on either one house (with a triangle) or two houses (with a trapezoid) at a time? One way is shown below.

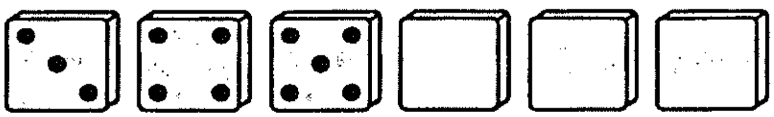


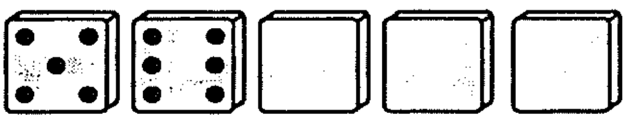
 

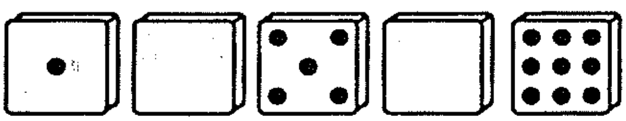
## DICE PATTERNS

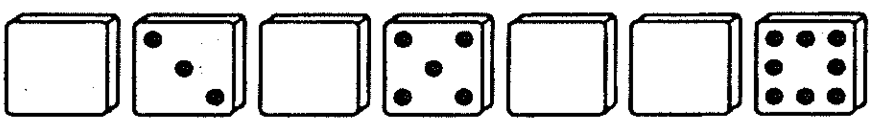
Find the pattern in each row. Fill in the dots on the blank dice.

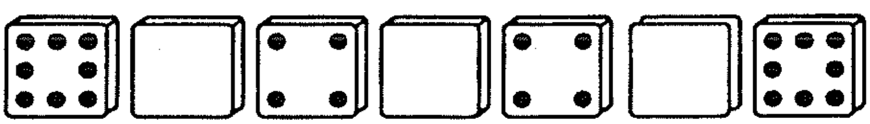


1. 

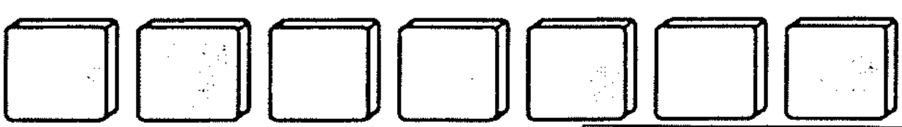
2. 

3. 

4. 

5. 

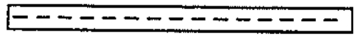
6. Make up your own pattern. Fill in the dots:



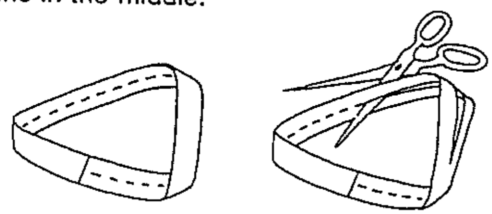
## Workshop 8: Number Patterns and Iteration

### PAPER STRIP PATTERNS ★★★

1. a. Cut a strip of paper, then draw a dotted line in the middle.



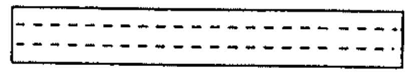
   b. Tape the ends together after making *one* twist.



   c. Cut the loop along the dotted line.

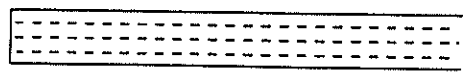
   d. What do you get? One large loop, one small loop, or two loops? ________

2. Now cut a strip of paper and draw two dotted lines on it. Follow the directions above for making a twisted loop. Remember to twist your paper once before you tape the ends together. Then cut along both dotted lines. Describe your results.



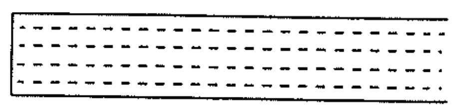
________________________________________

3. Make another twisted loop with three dotted lines. Cut along all three lines. Describe your results.



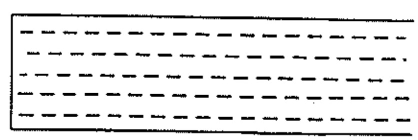
________________________________________

4. Cut another strip of paper and draw four dotted lines on it. Twist the strip and tape the ends together. Cut along all four lines. Describe your results.



________________________________________

5. Make a loop with five dotted lines. Cut along all five lines and describe your results.



________________________________________

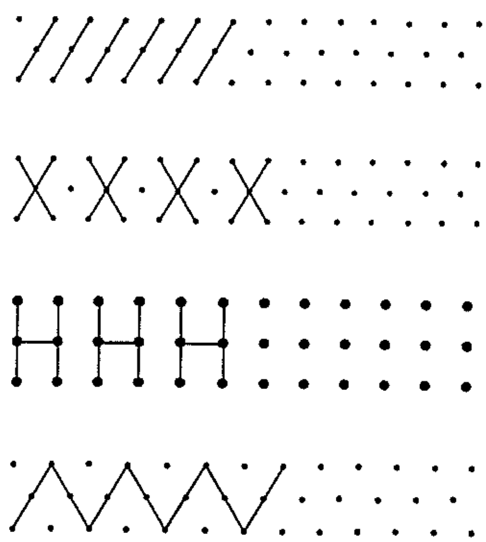
6. Predict the pattern and complete the chart below. Make and cut loops to verify your predictions.

| Number of Cuts | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Number of Large Loops | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| Number of Small Loops | 0 | 1 | 0 | 1 | | | | |

## Workshop 8: Number Patterns and Iteration

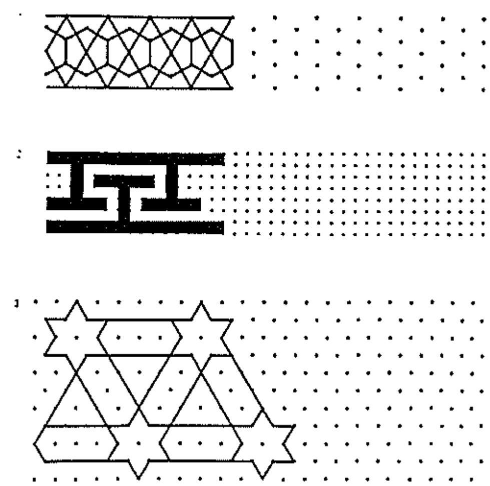
### MORE LINE PATTERNS ▼

Continue each pattern.



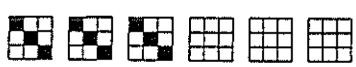
### DESIGN PATTERNS ★★★

Continue each pattern



### DRAW THE PATTERNS (I) ▼

Continue each pattern.

1. 
2. 
3. 
4. 
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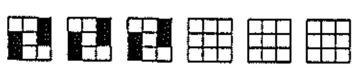
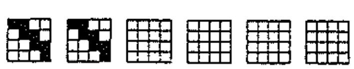
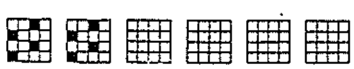
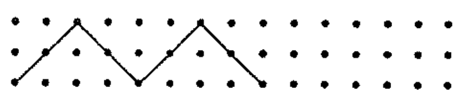
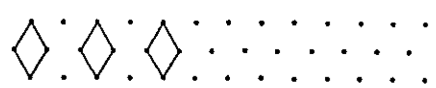
6. 
7. 
8. 

### MORE LINES AND SHAPES (I)

Continue each pattern.

1. 
2. 
3. 
4. 
5.