Master Document

Workshop 7 — The Choose Numbers

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised July 13, 1999

Workshop 7 — The Choose Numbers

Materials Needed	**Allocated Time**
Activity #1 — Review of Previous Day	5 minutes
Activity #2 — Counting Anagrams of READ, REED, DEED, and EPEE	45 minutes

- 96 3"x5" Post-It notes in 4 sets of 24. The first set contains the 24 anagrams of the word "READ" where the R is red, A is blue, E is green and D is black. Wide-tip markers should be used so that participants are able to see the Post-It notes from across the room. The second set is the same as the first, except that the blue A has changed to a blue E in each of the 24 Post-It notes, resulting in 24 anagrams of the word "REED". The third set is like the second set, except that the red R has changed to a red D, resulting in 24 anagrams of the word "DEED". The fourth set is like the third set, except that the red D has changed to a red E and the black D has changed to a black P, resulting in 24 anagrams of the word "EPEE". (Note that these Post-It notes are re-usable, so that after the activity is completed, they should be collected and returned to their original order.)

Activity #3 — The Choose Numbers	35 minutes
Activity #4 — Walking on a Grid	15 minutes
Activity #5 — The Choose Numbers and Pascal's Triangle	25 minutes
TOTAL WORKSHOP TIME:	125* minutes

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Activity #1 — Review of previous day
(Allocated time = 5 minutes)

A. Use TSP #1 and TSP #2 to review the various methods of systematic listing and counting introduced the previous day, referring as examples to problems discussed in the workshop and on the homework. Mention the two topics for today's workshop — finding the number of ways of arranging the letters in any word (not just words where all letters are different) and finding the number of ways to choose an unordered collection of objects (like three letters from the alphabet).

Activity #2 — Counting Anagrams of READ, REED, DEED, EPEE.
(Allocated time = 45 minutes, 15 minutes for parts A-B, 15 minutes for parts C-F, and 15 minutes for part G.)

Note first of all that the word "combinations" should be avoided for the duration of the institute. This word typically denotes "choose" to mathematicians, but tends to cause great confusion to those just learning the material. One reason for this confusion is that every way of arranging objects looks like a combination, and indeed, sometimes it is used that way in the "real world." For example, "combination locks" and "combinations of events" typically have inherent order which the mathematical use of "combinations" does not contain. So, it is best to avoid this word, and to tell them that you will avoid this word. The word "permutations" tends to be safe, since it implies the order that we wish it to, but we have avoided that too in these notes, since participants' prior acquaintance with this term tends to distract them from understanding the concept.

So, on the problem sets you will find that the word "select" has been used when "combinations" are being considered, and that together with the context of the problem it will be readily understood that order "doesn't matter." When we want order "to matter," we used the phrase "select and arrange," or just included the word "order." Similarly, we don't use the phrases "binomial coefficients" and "multinomial coefficients", since many participants don't have the background to understand the terms.

A. Ahead of time arrange the 96 Post-It notes with the 24 colored anagrams of the words "READ", "REED", "DEED", and "EPEE" (see instructions on the "Materials Needed" list above) on a blackboard, as shown on the last page of these notes. These should be arranged so that each group of 24 has room to its right or under it, so that the participants can place them there in a regrouped fashion. (If possible, hide all but the

first group behind sliding blackboards or overhead screens, so that one group at a time can be revealed.) Point out that the first group contains the 24 anagrams of the word "READ", and they will agree that they are listed in a nice systematic fashion, that is, alphabetically. You can also point out that you colored the letters to help you quickly distinguish them and see the patterns.

Some of the participants will not be familiar with the term "anagram". You will need to explain this by saying that an anagram of a word is another arrangement of the letters of the word. For example, "horse" is an anagram of "shore", and "extroverts" is an anagram of "vertex sort". Since there are 24 possible arrangements of the letters of the word "READ", there are 24 anagrams of the word.

B. Invite the participants to consider the second group of Post-Its with the 24 colored "anagrams" of the word "REED." In the anagrams of the word READ, the R is always red, the A is blue, the E is green, and the D is black. We have not made a new systematic list of the anagrams of the word REED; instead we have taken each of the 24 anagrams of READ and replaced its blue A by a blue E. (You should use the colors described above so that the A and E colors are distinguishable.)

Now that the "E"s are distinct, there are 24 different ways to arrange these 4 letters, just like for the word "READ." Ask them what we would see if we ignored the colors of the letters. Some of these allegedly different anagrams would become the same. Now have a participant come to the board and group together those anagrams that are really the same. When he or she is done, the group will see 12 groups each containing a pair of Post-It notes, so that there are actually 12 anagrams of "REED."

Experience has shown that if you ask them what the 12 really comes from, they will answer "24/2," which is, of course, the right way to think of that number. Take some time to explain why you are dividing by 2. Also take some time to show them that we now have a good new strategy for counting anagrams if letters are repeated. We will "overcount on purpose!" We can color the letters with different colors thereby making them all distinct. Then we can count the number of anagrams of this new colored word, which gives a factorial. Then we divide through by the number of times each anagram gets repeated.

Remind them that they drew a tree diagram yesterday for this problem, and found that there were 12 anagrams for REED, compared to 24 for READ; now they see where the 12 comes from.

C. Turn their attention to the third group of 24 containing the arrangements of the colored word "DEED." Ask: "How many groups am I going to get?" After a different volunteer has made 6 groups of 4, ask them why we divide 4! by 4, and how we could have guessed that this would happen without making all these Post-Its. Guide them through discovering that the 4 in the denominator is really 2×2, where the first "2" is for the pair of "D"s and the second is for the "E"s, or vice-versa.

> *It is typically during part C that the participants begin to see the big picture, but also to get confused. Take your time on this part, justifying over and over the denominator, and why you multiply in the denominator. Some participants see (24/2)/2 better than 24/(2×2), but some see the latter more easily. It is worth it to get this across, and to take the time. You will be rewarded shortly when they can easily count the number of anagrams of the word "Mississippi" or "Evening" or "Reveller."*

D. Finally, work through the number of anagrams of "EPEE." Ask: "How many is this going to boil down to?" Have a participant group the anagrams which are the same, and justify the denominator, which will this time be "3!"

> *It turns out to be easiest to justify the denominator as being the number of Post-Its in each of the groups. When you talk about the reason that each group has 2 or 4 or 6 Post-Its in it, they will see that it's just the multiplication rule, with which they are already familiar.*

E. Use TSP #3 and TSP #4 to summarize this entire activity.

F. Ask them to find the number of anagrams of BOOBOO as a summary activity and review this with them. List the 15 anagrams (alphabetically, of course) so that they believe that the analysis really works. (Is it an accident that the 15 anagrams, when divided up by the number of Os at the beginning, from 0 to 4, give 1+2+3+4+5?) (Both you and the participants may want to refer to this kind of problem as a "BOOBOO problem" to remind yourselves of how the problem is done. Thus for you, "BOOBOO" will mean "using multinomial coefficients".)

G. Distribute HO #1 (=TSP #5) which asks the participants to count the number of anagrams of several words, many of which have just two different letters. ✓ Review Here ✓

[Time for a 5-10 minute break]

Activity #3 — The Choose Numbers

(Allocated time = 35 minutes, 20 minutes for parts A-C and 15 minutes for D)

A. Show TSP #6 and ask whether we can answer these questions without making a list of all the possibilities. Introduce the choose number terminology using TSP #7 and note that the answer to the questions on TSP #6 are "8 choose 2" and "8 choose 3."

B. Now comes the climax of the day, where they get a simple method for calculating the "choose" numbers. They will see this in context — using TSP #8, TSP #9 and TSP #10 — the number of ways to choose 3 from 8 is the same as the number of ways to put "Y"s under the 3 that we choose, putting "N"s in the other 5 slots. And this is like counting anagrams — the BOOBOO problem — which they know how to do. After running through these ideas, you may wish to go back to the 4 choose 2 and 8 choose 2 examples, where they really listed all the ways, and compute them using the anagram ideas. This will make our technique more tangible and believable.

> *It may seem a roundabout way to introduce the choose numbers via anagrams, but it seems that the participants understand it very well this way. Another typical way to introduce them is by counting the number of ways to select, in order, some number of objects, and then dividing through by a factorial to "remove the order." This way doesn't seem to work as well in our experience, at least for the elementary teachers we address. When it comes to computing a choose number, the visual aid of the "N's and Y's" helps them to use the formula, without having to memorize one.*

C. Put up TSP #11 asking how many ways there are to select a group of 4 students from among these 10 students to form a delegation to the math fair. They should recognize from the previous discussion that this is just "10 choose 4," which is calculated, as before, using TSP #12 and TSP #13. (Note that the 6! on TSP #13 anticipates the shortcut for calculating the choose numbers that is presented later on TSP #15 .) Have the participants practice with a few more choose numbers, such as "9 choose 3", before going on to the next activity.

> *In general, we avoid algebraic formulations since most of the teachers (all those grade 5 or below) will never use them, and since they tend actually to impede their understanding of the concepts.*

D. Distribute Hand-out #2 (= TSP #14), which contains a number of "choose" problems, and review the problems after participants have had time to work on them.

Work into the discussions of these problems TSP #15 which provides a shortcut for calculating the choose numbers, and if appropriate, TSP #16, which provides an explanation of why the shortcut always works.

> *Our experience has been that presenting TSP #16, and its alternate method of computing the choose numbers, frequently muddles things up for participants who have "gotten it" via anagrams. This TSP is provided in case someone asks about the connection, or if you feel it is appropriate, and more helpful than confusing. Also provided is TSP #17 which contains "the formula", which should probably not be used, since many participants will be extremely uncomfortable with algebraic expressions like n-k+1.*

Activity #4 — Walking on a grid

(Allocated time = 15 minutes)

> *Note: If less than 35 minutes is left for the rest of the workshop, there will not be enough time to do this activity as written, with a handout. In this case, the instructor may wish to model the activity on a transparency with the participants' cooperation instead of having them complete the handout. However, if there is time, then it is certainly better to have the participants use the handout.*

A. Put up TSP #18 showing the "PATHS" grid, and explain what a spelling of PATHS involves, giving several examples. Elicit the idea that any spelling of PATHS involves a sequence of N's and E's. Label the examples that you provided with the corresponding words involving N and E.

B. Distribute Handout #3 (=TSP #19). Ask them to figure out how many ways you can spell "PATHS" and end at the middle "S". When they get 6, ask them to write down their 6 paths using "N" and "E" to denote steps North and steps East. Then ask them to find all routes to each of the S's, using N's and E's to describe their paths. (List the words in the appropriate places on TSP #19.) Referring to TSP #19, ask the participants if they can tell you, just by looking at the sequence of N's and E's, which "S" they will end up at. They should discover that it just depends on the *number* of N's (or E's). Point out, therefore, that to count the number of ways to walk to the middle "S", you simply need to count the number of ways to choose 2 of the 4 squares in which to put the N's; elicit from them that this is the same "4 choose 2" that we discussed earlier. Have the participants enumerate the number of ways to choose 2 of the 4 boxes in which to put the N's. Repeat this for a different "S".

Activity #5 — The Choose Numbers and Pascal's Triangle

(Allocated time = 25 minutes)

A. Do the first few steps of HO #4 (= TSP #20) in which participants count the number of ways to walk on a grid, and have participants work together to complete the chart. Of course, they will see that they get Pascal's triangle. Use this activity to stress the idea that each entry of Pascal's triangle records the number of paths to that entry from the entry at the top of the triangle. Hand out a copy of Pascal's Triangle (HO #5 = TSP #21 and TSP #22) so that they can refer to it in this workshop.

B. Use TSP #22 to make the connection between Pascal's triangle and the choose numbers. First review the numbering of the rows in Pascal's triangle, with the top row called the 0th row. Then review the entry numbers in each row, with the left-most entry called the 0th entry. As a result of these rules, the first entry in any numbered row is exactly the row number; for example, the first entry in the fifth row is 5 and the first entry in the eighth row is 8. Pick an entry on the triangle, say the 2nd entry, 15, on the 6th row. There are six steps from the top to 15, involving two "right" and four "left". Any path can be described by a word involving two R's and four L's. The number of such words is "6 choose 2". So the 2nd entry on the 6th row is "6 choose 2", which is exactly 15. Similarly the 4th entry on the 8th row is "8 choose 4".

C. Show TSP #23 with the following question: *How many ways are there to spell 'mathematics' on this grid and end at the circled 'S'?* They should quickly realize that the answer is "10 choose 4" (the same as the in The Math Fair problem), since this is the number of ways to picking four N's out of 10 N's and E's. The answer to The Math Fair Problem was "10 choose 4" or 210, so that 210 will be the 4th entry in the 10th row of Pascal's triangle. In general, since any entry represents the number of ways of going from the top to that entry, it corresponds to the number of ways of writing a word using R's and L's with a particular number of R's. Thus all of the entries on Pascal's triangle are choose numbers. Note that the 0th diagonal has all 1's; the first diagonal has the counting numbers; the second diagonal has the "choose 2 numbers"; the third diagonal has the "choose 3 numbers"; etc.

D. Make a chart listing the sum of each row of Pascal's triangle, until they discover that these are all powers of 2. Ask the participants to explain why it works out that way. Referring back to TSP #21, you can remind them that the total number of the entries in the 8th row represents all paths in the graph from P to the bottom. Why should that total be 2^8? At this point someone should realize that at each vertex you can go either L or R, so the total number is 2x2x2x2x2x2x2x2. Going back to TSP #18,

ask how many ways there are of spelling "PATHS", and, going back to TSP #23, ask how many ways there are to spell "MATHEMATICS".

E. Do the two problems on TSP #24, revealing them one at a time. They should be able to tell you that answer to the pizza problem is "8 choose 4" and that the answer to the hamburger problem is 2^8. They should also be able to describe the difference between these two problems — in the pizza problem, you can only have four toppings, whereas in the hamburger problem, you can have *any number* of "toppings". Point out that this activity is a good culminating experience for their classrooms as well, since the answers will be clear to those students who really do understand the choose numbers.

ADER	DAER	EADR	RADE
ADRE	DARE	EARD	RAED
AEDR	DEAR	EDAR	RDAE
AERD	DERA	EDRA	RDEA
ARDE	DRAE	ERAD	READ
ARED	DREA	ERDA	REDA

EDER	DEER	EEDR	REDE
EDRE	DERE	EERD	REED
EEDR	DEER	EDER	RDEE
EERD	DERE	EDRE	RDEE
ERDE	DREE	ERED	REED
ERED	DREE	ERDE	REDE

EDED	DEED	EEDD	DEDE
EDDE	DEDE	EEDD	DEED
EEDD	DEED	EDED	DDEE
EEDD	DEDE	EDDE	DDEE
EDDE	DDEE	EDED	DEED
EDED	DDEE	EDDE	DEDE

EPEE	PEEE	EEPE	EEPE
EPEE	PEEE	EEEP	EEEP
EEPE	PEEE	EPEE	EPEE
EEEP	PEEE	EPEE	EPEE
EEPE	PEEE	EEEP	EEEP
EEEP	PEEE	EEPE	EEPE

Systematic Listing and Counting — A Review

Techniques for listing

Make an alphabetical or numerical list

Make a tree diagram

Techniques for counting

Are there separate cases?
— Apply the Addition Rule

Is there a sequence of choices?
— Apply the Multiplication Rule

Systematic Listing and Counting — A Review

Factorial Notation

— How many five-letter words involving five given letters (each used only once) are there?

— How many ways are there of arranging five distinct objects in order?

Exponential Notation

— How many triple-decker ice cream cones can be made if 31 flavors are available?

Summary of the Anagram Counting Activity

How many anagrams of READ?

______ ______ ______ ______

Since there are 4 choices for the first letter, 3 choices for the second letter, 2 choices for the third letter, and 1 choice for the fourth letter, there are altogether

4! = 24 anagrams.

How many anagrams of REED?

______ ______ ______ ______

There are the same 24 anagrams, where the E's are distinguished as blue and green. But if we drop the distinction, we have two copies of each anagram (one with blue then green, and the other with green then blue), so the number of anagrams is altogether

$$\frac{4!}{2} = \frac{24}{2} = 12$$

Summary of the Anagram Counting Activity

How many anagrams of DEED?

There are the same 24 anagrams, where the E's are distinguished as blue and green and the D's are distinguished as red and black. But if we drop the distinction, we have four copies of each anagram (one with E blue then green, and the other with E green then blue) × (one with D red then black, and the other with D black then red), so the number of anagrams is altogether

$$\frac{4!}{2 \times 2} = \frac{24}{4} = 6$$

How many anagrams of EPEE?

There are the same 24 anagrams, where the E's are distinguished as blue, red, and green. But if we drop the distinction, we have six copies of each anagram, since there are $3! = 6$ orders of the three colors. So the number of anagrams is altogether

$$\frac{4!}{3!} = \frac{24}{6} = 4$$

Handout #1 — Anagrams of some words

Count the number of ways to arrange the letters of each of the following words.
(Ignore the fact that most of the arrangements are not real English words.)

A. ANNA

B. BOOBOO

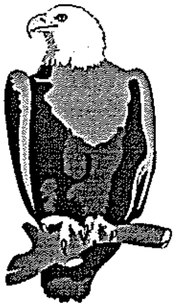
C. COCOON

D. DEEDED

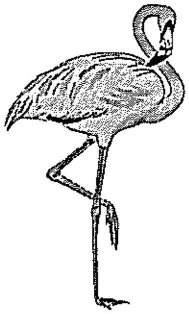
E. ESSES

F. FOOLPROOF

Eight Birds



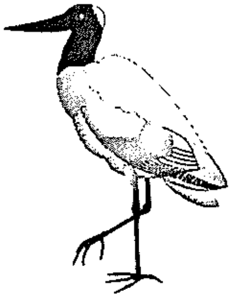
Eagle



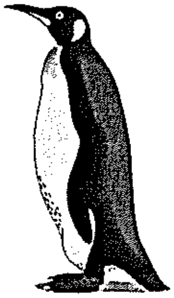
Flamingo

Grosbeak

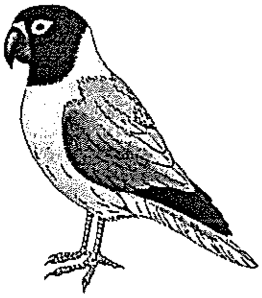
Hen



Jabiru



King Penguin



Love Bird

Macaw

How many ways are there for a student to select 2 of these birds on which to write reports? 3 birds?

The Choose Numbers

"8 choose 3"

represents the number of ways you can choose 3 objects out of a set of 8 objects.

"10 choose 4"

represents the number of ways you can choose 4 objects out of a set of 10 objects.

8 Birds — Choose 3 Birds

Here are the birds:

Eagle	Flamingo	Gros beak	Hen	Jabiru	King Penguin	Love Bird	Macaw

Here is a set of three birds:

Eagle, Hen, Macaw

Here is another way to represent that particular selection of 3 birds:

Eagle	Flamingo	Gros beak	Hen	Jabiru	King Penguin	Love Bird	Macaw
Y	N	N	Y	N	N	N	Y

So selecting 3 of these birds amounts to putting 3 Y's and 5 N's into the slots beneath the birds' names.

In other words, 8 choose 3 is equal to the number of anagrams of the word "YNNYNNNY," which we know how to calculate!

This is a "BOOBOO" problem!

The Connection between Anagrams and Choosing

“How many ways are there to select 3 out of 8 slots?”

is the same question as

“How many 8 letter words have 3 Y’s and 5 N’s?”

which is the same as

“How many anagrams are there of
YYYNNNNN?”

That’s the same kind of question as

How many anagrams are there of BOOBOO?

We found that answer to be:

$6! / (2! \times 4!) = 15.$

The Choose Numbers

The choose number

"8 choose 3"

represents the number of ways you can choose 3 objects out of a set of 8 objects.

$$\textbf{8 choose 3} = \frac{8!}{3!\times 5!}$$

$$= \frac{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(3\times 2\times 1)\times(5\times 4\times 3\times 2\times 1)}$$

$$= \frac{8\times 7\times 6}{3\times 2\times 1}$$

$$= 56$$

The Math Fair

In a certain classroom there are 10 students from which the teacher must select a delegation of 4 students to send to the math fair.

In how many different ways can she do this?

Here are the students:

Al	Bob	Cal	Deb	Ed	Fran	Gin	Hal	Ida	Joe

Here is one possibility for a delegation:

Al, Deb, Hal, Ida

Here is another way to represent that particular selection of 4 students:

Al	Bob	Cal	Deb	Ed	Fran	Gin	Hal	Ida	Joe
Y	N	N	Y	N	N	N	Y	Y	N

Thus we see that the number of ways to select 4 students from a class of size 10 is the same as the number of anagrams of "YNNYNNNYYN", which we know how to calculate!

The Choose Numbers

The choose number

"10 choose 4"

represents the number of ways you can choose 4 objects out of a set of 10 objects.

$$\mathbf{10\ choose\ 4} = \frac{10!}{4!\times 6!}$$

$$= \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)\times(6\times5\times4\times3\times2\times1)}$$

$$= \frac{10\times9\times8\times7\times6!}{(4\times3\times2\times1)\times6!}$$

$$= \frac{10\times9\times8\times7}{(4\times3\times2\times1)}$$

$$= \quad 210$$

Handout #2 — Practice with the Choose Numbers

1. You want to go to the gym on three days during the week. In how many different ways can you choose those three days?

2. Claire wrote a computer program for her Astronomy class. The person using the program enters any 4 planets, and the program figures out the next date on which those 4 planets will be aligned. How many different sets of 4 planets could the user of the program possibly enter?

3. Your class has 25 children. In how many ways can you select two children to serve as class monitors?

4. If your class has 11 boys and 14 girls, in how many ways can you select 3 boys and 3 girls for the school's representative assembly?

A Shortcut for Calculating Choose Numbers

When we calculated “8 choose 3”, we got

$$\textbf{8 choose 3} = \frac{8!}{3!\times 5!} = \frac{8\times 7\times 6}{3\times 2\times 1} = 56$$

Similarly, when we calculated “10 choose 4”, we got

$$\textbf{10 choose 4} = \frac{10!}{4!\times 6!} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 210$$

Similarly, if we want to calculate “14 choose 5”, in the numerator we would start with 14 and count down 5 terms, and in the denominator we would start with 1 and count up 5 terms.

Why the Shortcut Works

Here's another method for counting the number of ways to choose 4 out of 10 slots?

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

There are 10 slots for the first choice, 9 slots for the second choice, 8 slots for the third choice, and 7 slots for the fourth choice.

Altogether there are $\mathbf{10 \times 9 \times 8 \times 7}$ ways of choosing the four slots.

But … any particular four slots will be counted altogether **4!** times, since they could have come up in any one of $\mathbf{4 \times 3 \times 2 \times 1}$ ways.

So the actual number is

$$\frac{\mathbf{10 \times 9 \times 8 \times 7}}{\mathbf{4 \times 3 \times 2 \times 1}} = \mathbf{210}$$

This is the same as **10! / (4! × 6!)**.

The Formula

This discussion can be summarized in the following algebraic formula:

The number of ways to choose k objects from a set of n distinct objects is given by the formula:

$$\frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}{1 \times 2 \times 3 \times \cdots \times k}$$

Note: On the top, you count down k terms beginning with n, and on the bottom, you count up k terms beginning with 1. Both have k terms.

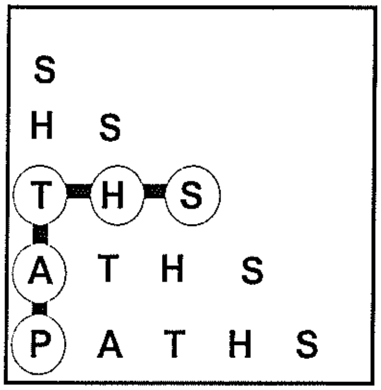
Walking on a Paths Grid

Start at "P", walk to an "A", then to an adjacent "T", then to an adjacent "H", and finally to an adjacent "S", and you will have a particular spelling of "PATHS".

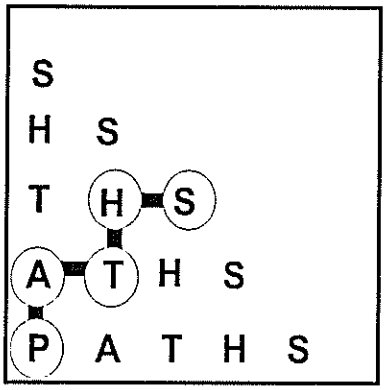
How many "PATHS" are there which end at the middle "S"?

S				
H	**S**			
T	**H**	**S**		
A	**T**	**H**	**S**	
P	**A**	**T**	**H**	**S**

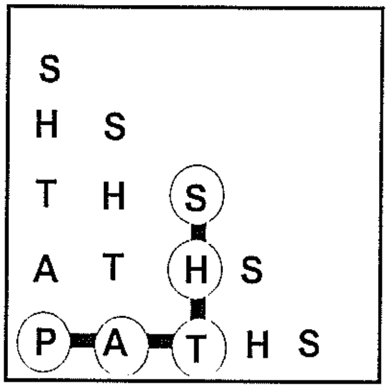
Handout #3 — Counting Paths



N N E E



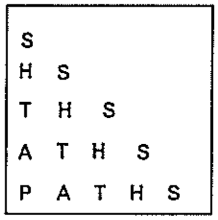
N E N E

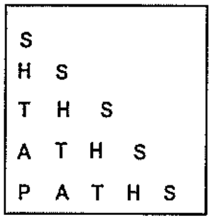


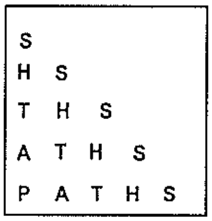
E E N N

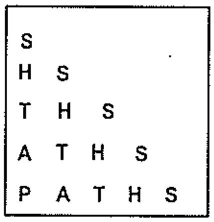
Here are three possible paths to the middle "S"

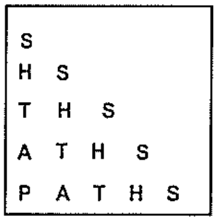
All paths to the middle "S"

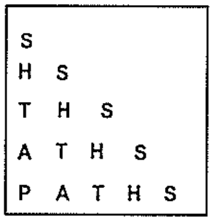


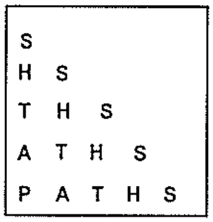


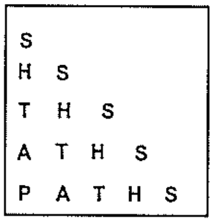












All paths to each "S"

(Here you can use the "N" and "E" notation to represent the paths.)

First S

Second S

Third S

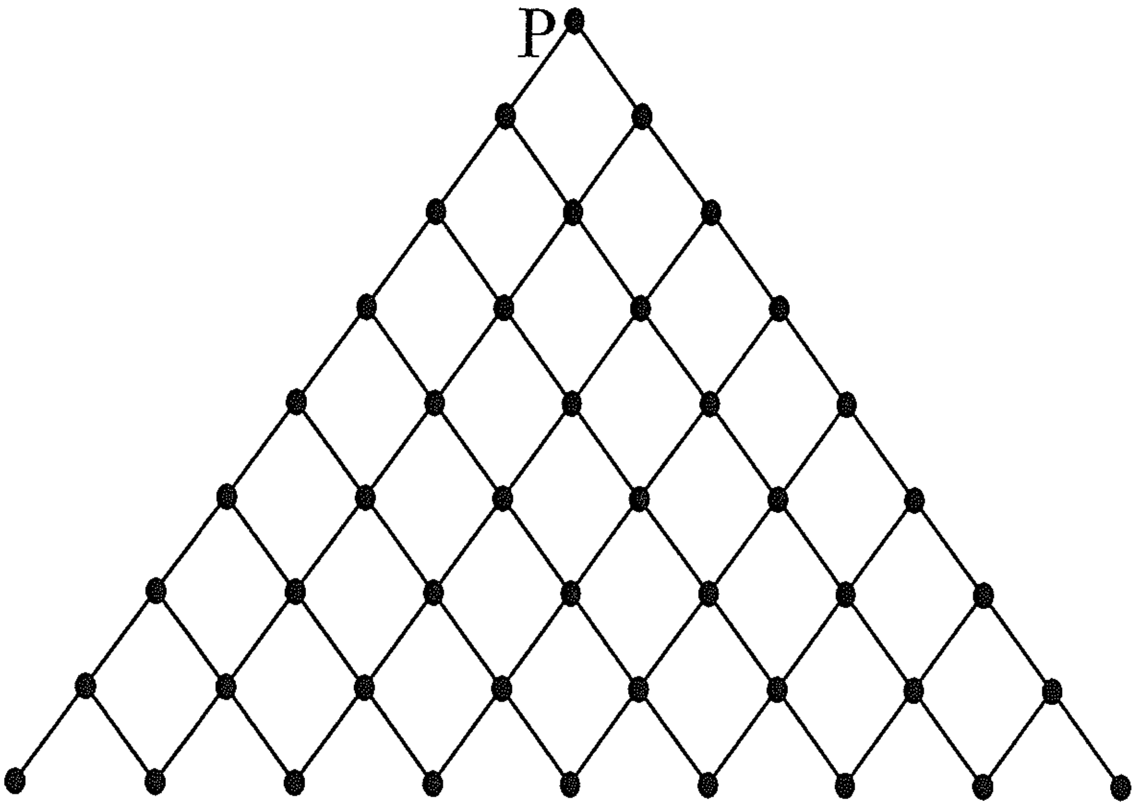
Fourth S

Fifth S

Handout #4 — Southbound Paths

How many paths are there from P to each other vertex in the graph below?

(Note: In these paths you move only along southbound edges.)



Pascal's Triangle

Row 0	1										
Row 1	1	1									
Row 2	1	2	1								
Row 3	1	3	3	1							
Row 4	1	4	6	4	1						
Row 5	1	5	10	10	5	1					
Row 6	1	6	15	20	15	6	1				
Row 7	1	7	21	35	35	21	7	1			
Row 8	1	8	28	56	70	56	28	8	1		
Row 9	1	9	36	84	126	126	84	36	9	1	
Row 10	1	10	45	120	210	252	210	120	45	10	1

Pascal's Triangle and the Choose Numbers

- Each entry represents the number of downward paths from the top to that entry.

- Count rows beginning with the 0th row —
  the row 1 4 6 4 1 is then the 4th row

- In each row, count entries beginning with the 0'th entry at the left — the 0th entry in the 4th row is 1, the first entry is 4, the second entry is 6, etc.

- The 0th diagonal has all 1's.
  The 1st diagonal has the counting numbers.
  The 2nd diagonal has the "choose 2 numbers",
  The 3rd diagonal has the "choose 3 numbers", etc.
  The sum of the entries in each row is a power of 2; e.g., the sum of the entries in the 5th row is 2^5.

  (Note: There are 2^5 five-letter words consisting of R's and L's.)

S										
C	S									
I	C	S								
T	I	C	S							
A	T	I	C	S						
M	A	T	I	C	S					
E	M	A	T	I	C	Ⓢ				
H	E	M	A	T	I	C	S			
T	H	E	M	A	T	I	C	S		
A	T	H	E	M	A	T	I	C	S	
M	A	T	H	E	M	A	T	I	C	S

How many ways will end at the circled “S”?

Handout #1 — Anagrams of some words

Count the number of ways to arrange the letters of each of the following words.
(Ignore the fact that most of the arrangements are not real English words.)

A. ANNA

B. BOOBOO ✓ ✓

C. COCOON

D. DEEDED

E. ESSES

F. FOOLPROOF

Handout #2 — Practice with Choose Numbers

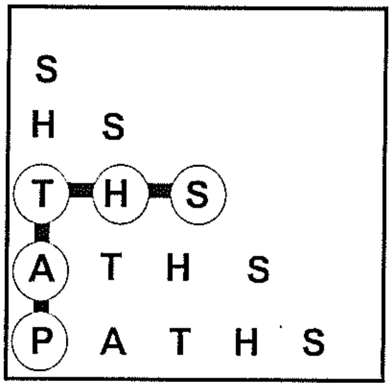
1. You want to go to the gym on three days during the week. In how many different ways can you choose those three days?

2. Claire wrote a computer program for her Astronomy class. The person using the program enters any 4 planets, and the program figures out the next date on which those 4 planets will be aligned. How many different sets of 4 planets could the user of the program possibly enter?

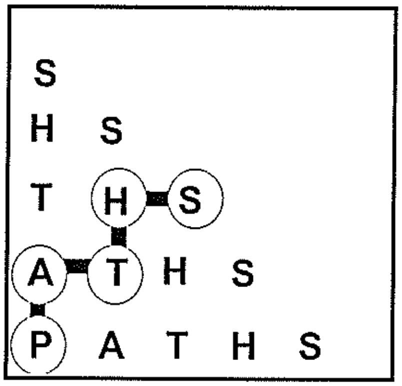
3. Your class has 25 children. In how many ways can you select two children to serve as class monitors?

4. If your class has 11 boys and 14 girls, in how many ways can you select 3 boys and 3 girls for the school's representative assembly?

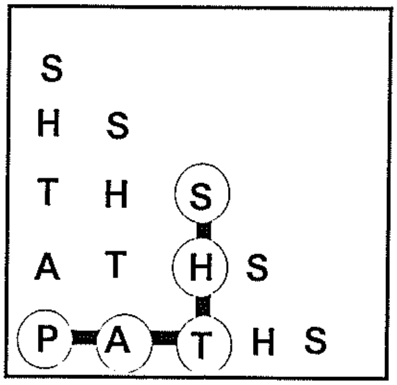
Handout #3 – Counting Paths



N N E E



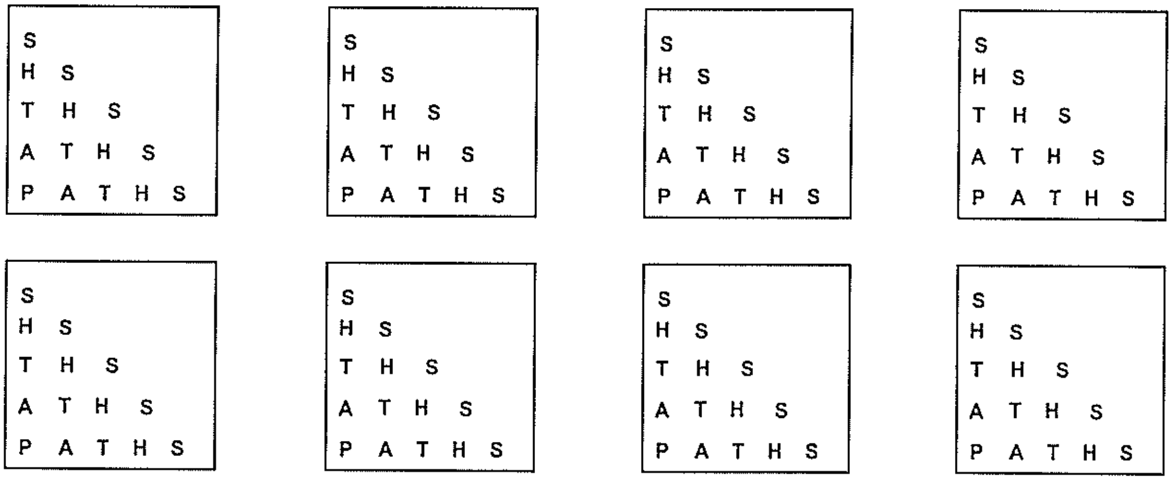
N E N E



E E N N

Here are three possible paths to the middle "S"

All paths to the middle "S"



All paths to each "S"

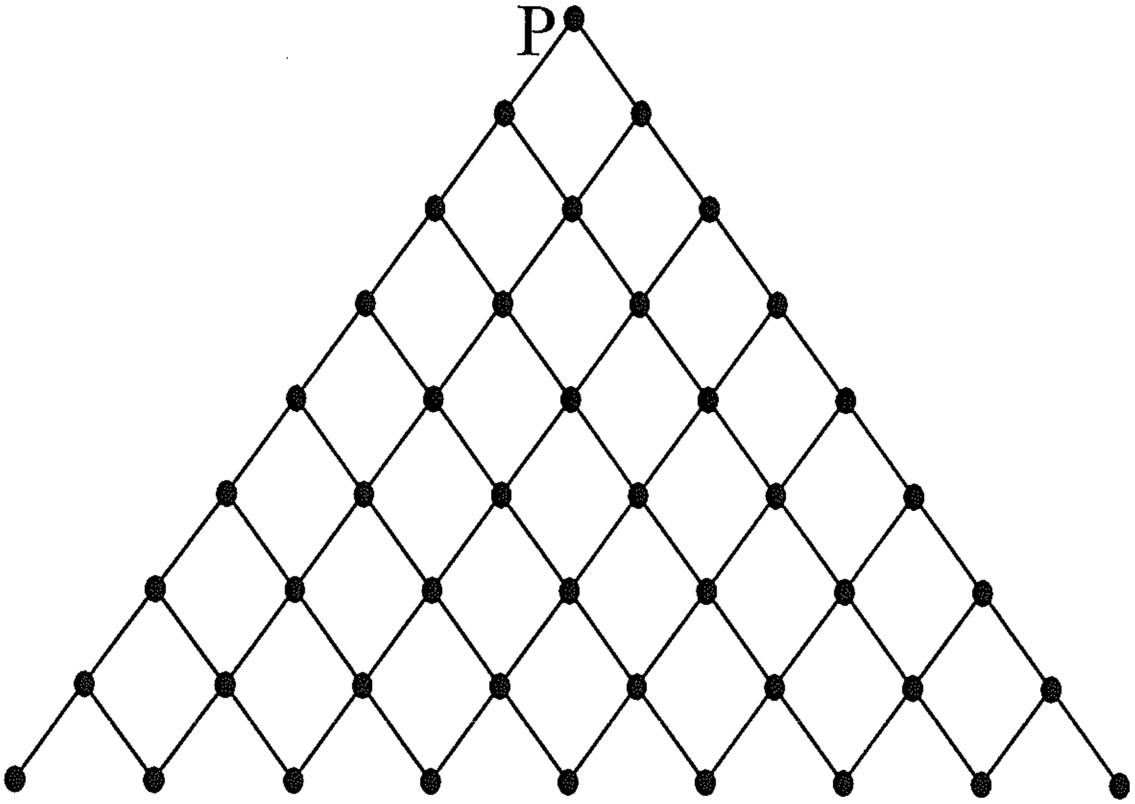
(Here you can use the "N" and "E" notation to represent the paths.)

First S Second S Third S Fourth S Fifth S

Handout #4 — Southbound Paths

How many paths are there from P to each other vertex in the graph below?

(Note: In these paths you move only along southbound edges.)



Handout #5 — Pascal's Triangle and the Choose Numbers

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
Row 5	1 5 10 10 5 1
Row 6	1 6 15 20 15 6 1
Row 7	1 7 21 35 35 21 7 1
Row 8	1 8 28 56 70 56 28 8 1
Row 9	1 9 36 84 126 126 84 36 9 1
Row 10	1 10 45 120 210 252 210 120 45 10 1

- Each entry represents the number of downward paths from the top to that entry.
- Count rows beginning with the 0th row — the row 1 4 6 4 1 is then the 4th row
- In each row, count entries beginning with the 0th entry at the left — the 0th entry in the 4th row is 1, the first entry is 4, the second entry is 6, etc.
- The 0th diagonal has all 1's.
  The 1st diagonal has the counting numbers.
  The 2nd diagonal has the "choose 2 numbers",
  The 3rd diagonal has the "choose 3 numbers", etc.
  The sum of the entries in each row is a power of 2; e.g., the sum of the entries in the 5th row is 2^5.

  (Note: There are 2^5 five-letter words consisting of R's and L's.)

Workshop 7 — The Choose Numbers — Exercises

Practice Problems:

1. Print your first name. How many arrangements of the letters are possible, if we consider that letters of the same type are indistinguishable? For example, there are 6 ways for Eva to arrange the letters of her name, but only 3 ways for Eve.

2. How many ways can you arrange the letters in the word "Kentucky"? In the word "Tennessee"?

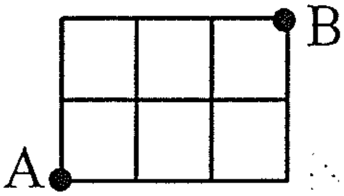
3. Ben Franklin has 9 kites and 12 keys at home, and wants to select three of each to bring with him when he does his next experiment. How many different ways are there for him to do this?

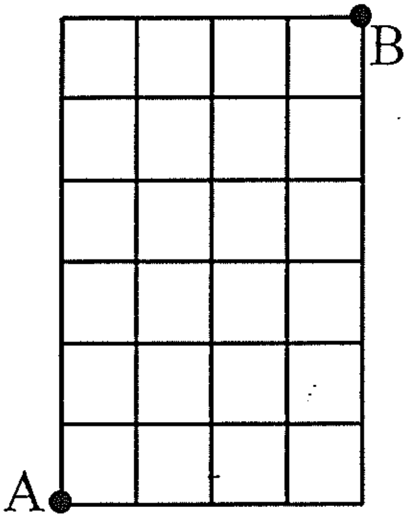
4. Ned was sent to *Bob's Ice Cream Sundaes* to get a sundae for Nora, but when he arrived, he couldn't remember which three flavors of ice cream Nora had asked for, though he was sure that they were three different flavors. Looking over the list of 14 available flavors didn't help him either, so he decided that he would get one sundae for each possible selection of three flavors. How many sundaes will poor Ned have to buy?

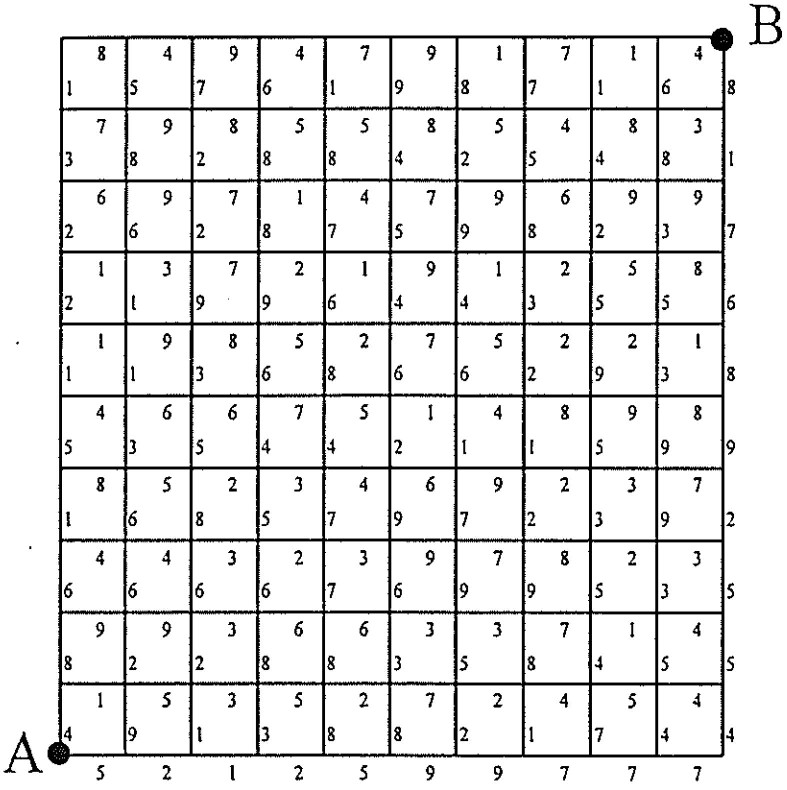
Study Group Problems:

5. How many anagrams are there of each of the following words: "levee", "teeth", "monsoon", and "Mississippi?

6. How many routes are there from A to B in each of the following grids? (A "route" follows the lines and always goes either north (N) or east (E). Ignore the weights in the large grid unless you want to find the shortest route.)







7. You want to see the top six movies.
   a. In how many sequences would that be possible?
   b. If you have money for only three of them, how many ways are there to pick three of these movies on three consecutive nights?
   c. How many different sets of three movies could you choose to see?

8. a. In how many ways can you select a three-letter initial?
   b. In how many ways can you select a three-letter initial consisting of different letters?
   c. In how many ways can you select a set of three different letters from the alphabet?

9. King Lewser unjustly condemns eight of his subjects to death and locks them in a prison cell deep beneath his castle. Immediately, the prisoners start digging an escape tunnel which will take them 65 days to complete. According to the rules of the land, at sunrise of each day the prisoners may send a delegation from among themselves to plead for their lives, but subject to two conditions: a) The delegation must be the same size each day, and b) the delegation must never consist of the same set of people as on any previous day. When they run out of delegations, King Lewser has the option of executing them. The prisoners may select the size of delegation they send. What size should they choose? Can they save themselves?

10. How many ways are there for Carol to add X's and O's to her letters to Dave if she uses 7 characters altogether? (Note that Carol and Dave don't have any problem with two O's appearing next to each other.) How many ways are there if she uses exactly 2 X's and the rest are O's? How many ways are there if she uses exactly 5 X's and the rest are O's?

11. A teacher wants to assign 6 students to three teams; a red team, a green team and a blue team, so that each team gets two students. How many different ways are there to do this?

[illegible]

✓

12. You have two pieces of candy to distribute to Althea, Bill, and Corazon. Make a systematic list of all the possible ways you can distribute the candy. Make similar lists for the situations where you have three pieces, four pieces, and five pieces of candy. Count the total number of possibilities for two, three, four, and five pieces of candy. Find the totals on Pascal's triangle.

Extension Problems:

13. Can you explain why the answers to Exercise 12 are in those locations on Pascal's triangle?

14. A CD player is programmed to shuffle the order in which it plays the eight pieces on one disk and the 11 pieces on another disk without repeating. How many ways is this possible if all pieces on the first disk are played before the second starts? if the machine shuffles between disks as well as pieces?

15. A hexagon, as shown to the right, has 9 diagonals. Let's determine the number of diagonals that a convex polygon with 75 vertices will have:

   a. Since there are 75 vertices, there will be 75 choose 2, which equals _______, lines altogether, including the sides of the polygon. Thus the number of diagonals will be _____ .
   b. There are 75 vertices, and each vertex is the endpoint of _____ diagonals, so there will be _____ endpoints of diagonals altogether. When we correct for over-counting, we discover that there will be _________ diagonals altogether.

16. There are some men and 21 women in a room. No person shakes hands with someone of the same sex, but it is noted that each man shakes hands with 3 women, and each woman shakes hands with 5 men. How many men are in the room?

Resource Book

Workshop 7 : The Choose Numbers

Table of Contents

Resource Book

Workshop 7 : The Choose Numbers

Mathematical Background

- **Arrangements and Anagrams**

  When counting the number of ways to arrange a number of objects, some of which are repeated, you can temporarily act as if the objects were all distinct, and then divide through by a factor which takes into consideration the repetitions.

  For example, to count the number of anagrams of the word "peace," you can act as if the two e's were distinct (perhaps different colors) and say that there were 5! = 120 anagrams. Then when you "uncolor" the e's and make them the same again, you see that each anagram now appears twice, so that 120 should be divided by 2. Thus the number of anagrams of "peace" is actually 120/2 = 60.

- **The general rule for counting anagrams**

  This yields the general rule that to count the number of anagrams of a word, you start with the factorial of the number of letters, and then, for each letter in that word, divide by the factorial of the number of repetitions of that letter. For example, the number of anagrams of "Mississippi" would be $11! / (1!\times4!\times4!\times2!)$.

  In the case where there are only two distinct letters, we have what we called "the BOOBOO problem", where the number of anagrams is $6!/(4!x2!)$. In general, if a word with n letters has k of one letter and the remaining n-k of the second letter, the number of anagrams is $n! / k!x(n-k)!$

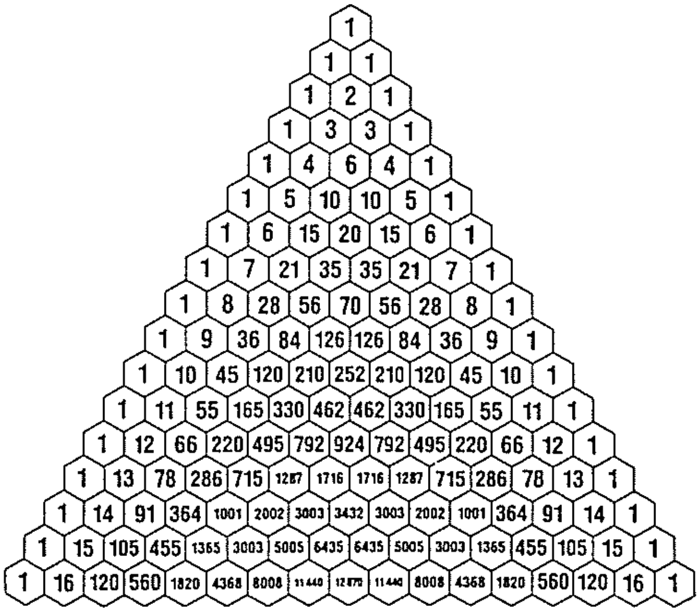
- **The Choose Numbers**

  The number of ways to choose k objects from a set of n objects is called the "choose number" — n choose k.

  Since this corresponds to labeling k objects "Y" and the remaining n-k objects "N", this is the same as a BOOBOO problem with k Ys and n-k Ns, so the value of "n choose k" is given by the formula "n choose k" = $n! / k!\times(n-k)!$

  For example, the number of ways to choose which 2 fingers from your left hand to put rings on would be "5 choose 2" = $5!/2!\times3!$ = 10.

- **Pascal's Triangle:** All the choose numbers can be found in Pascal's triangle, shown here to the right. To find "n choose k" you would go to row n (which is the row that has '1, n' as its first two entries and go to the k^{th} entry in that row, where "1" is considered to be the 0^{th} entry, and "n" is the first entry. Thus, on the Pascal's triangle to the right, you find that 10 choose 3 equals 120.



Resource Book

Workshop 7 : The Choose Numbers

Outline of Workshop

1. **Counting Anagrams**
   - a. Our sets of Post-Its were arranged on the wall
     - i. The first set showed all 24 arrangements of the letters of the word “READ” where each letter was a different color.
     - ii. The second set showed 24 arrangements of the letters of the word “REED” where the letters were still four different colors;
       - (1) but when we disregarded the colors we saw that this list of 24 contained many repeats;
       - (2) when we grouped the repeats together, we got 12 groups of 2;
       - (3) thus we saw that there were really 24/2 (which is 4!/2!) ways to arrange the letters of the word “REED”.
     - iii. The third set showed 24 arrangements of the word “DEED” where again the letters were 4 different colors.
       - (1) Again there were repeats when we disregarded color.
       - (2) When we grouped repeats together, disregarding color, we found 6 groups of 4.
       - (3) Thus we discovered that there were 24/4 (which is 4!/(2!×2!)) ways to arrange the letters of the word “DEED”.
     - iv. Finally, the fourth set showed 24 arrangements of the word “EPEE” where again the letters were 4 different colors.
       - (1) Here, via similar considerations, we found that there were 4!/3! arrangements of the letters of the word “EPEE”.
   - a. As a summary, we did the BOOBOO problem, which has 6!/(4! x 2!) = 15 anagrams. In general, to find the number of ways to arrange the letters of a word with *n* letters, you consider the letters initially as different, in which case there would be *n!* arrangements. But then you divide that quantity by *k!* for each repeated letter, assuming it repeats *k* times, since there are *k*! arrangements of the repeated letters. For example, the number of arrangements of the letters of “reveller” would be 8!/(1!×2!×2!×3!) = 1,680.

2. **The Choose Numbers.**
   - a. We discussed the number of ways of choosing two of eight birds, and then three of eight birds, and realized that there were getting to be too many possibilities to list. We introduced the terminology “8 choose 3" to represent this number.
   - b. We translated the birds problem into a BOOBOO problem, the number of anagrams of YYYNNNNN, so that the answer is “8 choose 3" = 8! / 3! x 5! = 56.
   - c. We discussed The Math Fair problem, involving “10 choose 4” and then did a number of additional choose problems.

3. **Walking on a grid and other problems**
   - a. We counted walks on a “PATHS” grid, going North and East, and discovered that these could be counted by considering arrangements of the letters of words like “NEENE.” As a result, we were able to use choose numbers to count the number of paths to any destination.

Workshop 7 : The Choose Numbers

3. **Walking on a grid and other problems (cont.)**
   b. Thus, for example, the number of ways to walk to (6,4) from (0,0), heading only North and East, one unit at a time, is the same as the number of arrangements of the letters "NNNNEEEEEE." This quantity is 10!/(4!×6!).
   c. We discussed the number of ways of spelling MATHEMATICS on a grid, ending at a particular S, and recognized that this was the same problem with the same answer as The Math Fair problem.

4. **Choose Numbers and Pascal's Triangle**
   a. We did a sequence of problems to review the counting techniques discussed previously.
   b. We made the connection between the choose numbers and Pascal's triangle.
   c. We found that the sum of the numbers in any row of Pascal's triangle was a power of 2, and understood that as the total number of paths from the top to that row. We concluded that the number of paths in PATHS was 2^5 and that the number of ways of spelling MATHEMATICS was 2^{11}.
   c. As a summary activity, we discussed the hamburger and pizza problems.

Resource Book

Workshop 7 : The Choose Numbers

Summary of the Anagram Counting Activity

How many anagrams of READ?

______ ______ ______ ______

Since there are 4 choices for the first letter, 3 choices for the second letter, 2 choices for the third letter, and 1 choice for the fourth letter, there are altogether

$$4! = 24 \text{ anagrams.}$$

How many anagrams of REED?

______ ______ ______ ______

There are the same 24 anagrams, where the Es are distinguished as blue and green. But if we drop the distinction, we have two copies of each anagram (one with blue then green, and the other with green then blue), so there are altogether

$$4!/2! = 24/2 = 12 \text{ anagrams}$$

How many anagrams of DEED?

______ ______ ______ ______

There are the same 24 anagrams, where the Es are distinguished as blue and green and the Ds are distinguished as red and black. But if we drop the distinction, we have four copies of each anagram (one with E blue then green, and the other with E green then blue) × (one with D red then black, and the other with D black then red), so there are altogether

$$4!/2!2! = 24/4 = 6 \text{ anagrams}$$

How many anagrams of EPEE?

______ ______ ______ ______

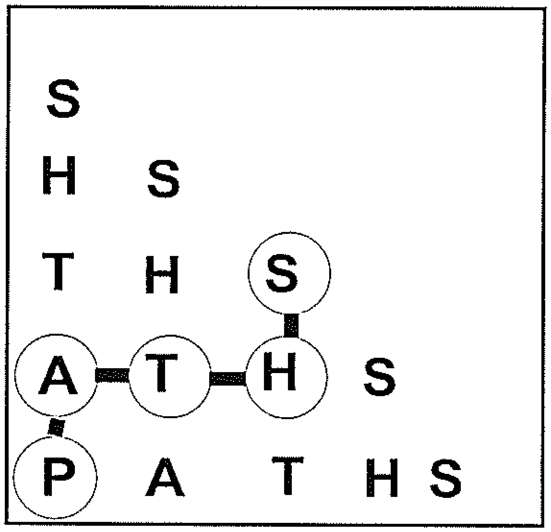
There are the same 24 anagrams, where the Es are distinguished as blue, red, and black. But if we drop the distinction, we have six copies of each anagram, since there are 3! = 6 orders of the three colors. So there are altogether

$$4!/3! = 24/6 = 4 \text{ anagrams}$$

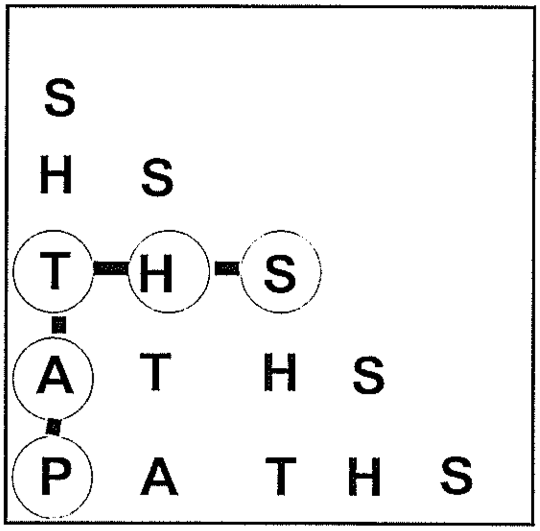
S				
H	S			
T	H	S		
A	T	H	S	
P	A	T	H	S

How many ways are there to spell "PATHS" on this grid?

You can think of a particular spelling as a walking route from the "P" to an "S".



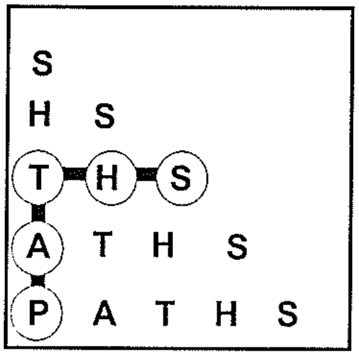
N E E N

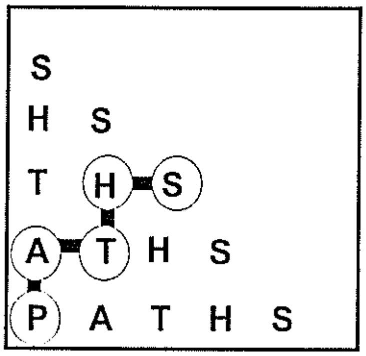


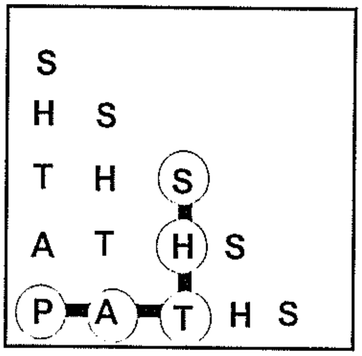
N N E E

Two routes to the middle "S"

Workshop 7: The Choose Numbers

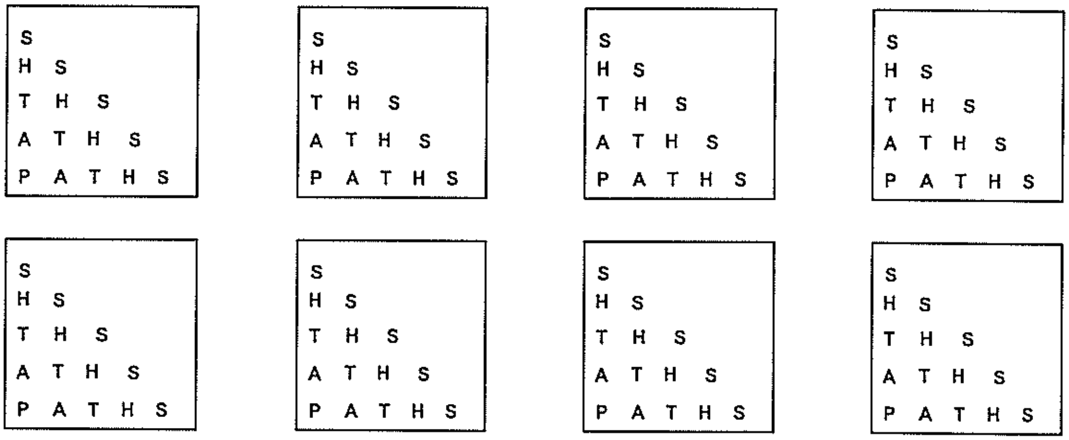

N N E E


N E N E


E E N N

Here are three possible paths to the middle "S"

All paths to the middle "S"



All paths to each "S"

(Here you can use the "N" and "E" notation to represent the paths.)

First S   Second S   Third S   Fourth S   Fifth S

Resource Book

Workshop 7: The Choose Numbers

How many ways to spell "Mathematics" on this grid?

S										
C	S									
I	C	S								
T	I	C	S							
A	T	I	C	S						
M	A	T	I	C	S					
E	M	A	T	I	C	Ⓢ				
H	E	M	A	T	I	C	S			
T	H	E	M	A	T	I	C	S		
A	T	H	E	M	A	T	I	C	S	
M	A	T	H	E	M	A	T	I	C	S

How many ways will end at the circled "S"?

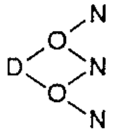
Resource Book

Workshop 7 : The Choose Numbers

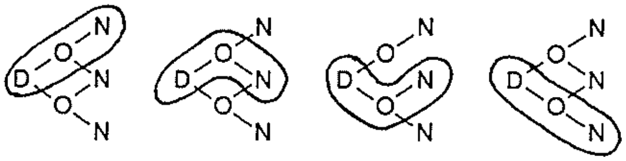
LETTER COMBINATION PATTERNS

Read each letter triangle from left to right. Use a different path to spell the same words. Circling the paths may help you find how many there are.

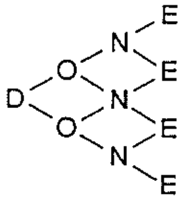
Example:



4 paths

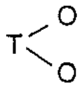


1.



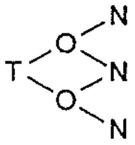
______ paths

2.



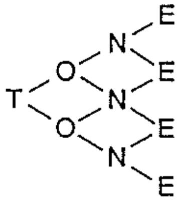
______ paths

3.



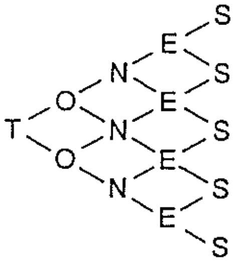
______ paths

4.



______ paths

5.



______ paths

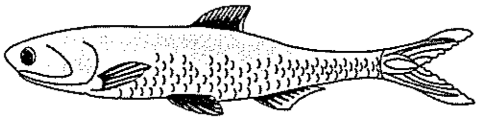
6. Make a letter puzzle of your first name.
   How many paths can you take to spell it? ______

7. Summarize your results by completing the table.

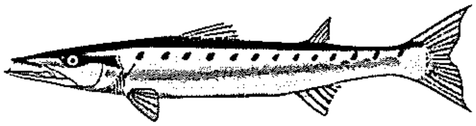
Number of letters in each word	2	3	4	5	6	7	8
Number of Paths	2	4					

From "Critical Thinking Activities, Grades K-3" by D. Seymour and E. Beardslee. ©1988 by by Dale Seymour Publications.

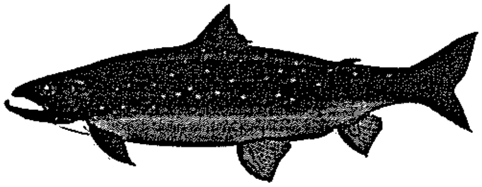
One Fish, Two Fish, Out of Four Fish



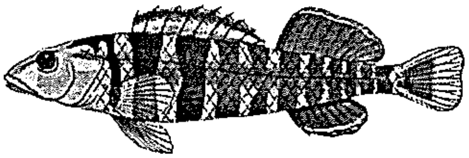
Anchovy



Barracuda



Char

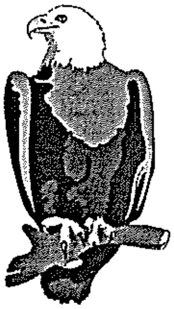


Darter

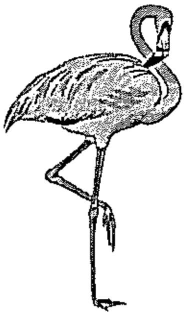
Mr. Poisson, the science teacher, has asked each student to write a report on two of these fish.

How many different ways are there for little Suzie, a student, to select two fish on which to write reports?

Eight Birds



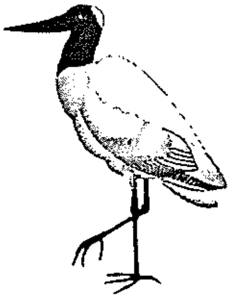
Eagle



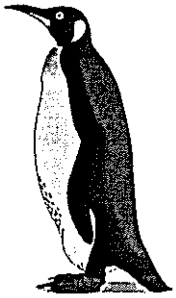
Flamingo

Grosbeak

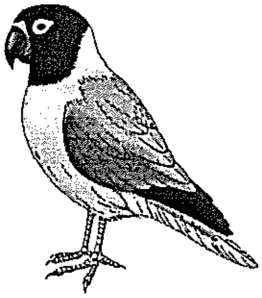
Hen



Jabiru



King Penguin



Love Bird

Macaw

How many ways are there for a student to select 2 of these birds on which to write reports? 3 birds?

The Math Fair

In a certain classroom there are 10 students from which the teacher must select a delegation of 4 students to send to the math fair.

In how many different ways can she do this?

Here are the students:

Al	Bob	Cal	Deb	Ed	Fran	Gin	Hal	Ida	Joe

Here is one possibility for a delegation:

Al, Deb, Hal, Ida

Here is another way to represent that particular selection of 4 students:

Al	Bob	Cal	Deb	Ed	Fran	Gin	Hal	Ida	Joe
Y	N	N	Y	N	N	N	Y	Y	N

Thus we see that the number of ways to select 4 students from a class of size 10 is the same as the number of anagrams of "YNNYNNNYYN", which we know how to calculate!

Workshop 7 : The Choose Numbers

The Connection between Anagrams and Choosing

"How many ways are there to select 4 out of 10 slots?"

which is the same question as

"How many 10 letter words have 4 N's and 6 E's?"

which is the same as

"How many anagrams are there of NNEEENNEEE?"

That's the same kind of question as

How many anagrams are there of BOOBOO?
We found that answer to be:
6! / (3! × 3!) = 20.

Here, the answer is 10! / (4! × 6!) = 210.

Another method:

How many ways are there to choose 4 out of 10 slots?

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

There are 10 slots for the first choice, 9 slots for the second choice, 8 slots for the third choice, and 7 slots for the fourth choice.

Altogether there are 10 × 9 × 8 × 7 ways of choosing the four slots.

But ... any particular four slots will be counted altogether 4! times, since they could have come up in any one of 4 × 3 × 2 × 1 ways.

So the actual number is

$$\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

This is the same as 10! / (4! × 6!).

Workshop 7: The Choose Numbers

The Choose Numbers

The choose number

"10 choose 4"

represents the number of ways you can choose 4 objects out of a set of 10 objects.

$$10 \text{ choose } 4 = \frac{10!}{4!\times 6!} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 210$$

A Formula for *n* choose *k*

The number of ways to choose *k* objects from a set of *n* distinct objects is given by the formula:

$$\frac{n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)}{1 \times 2 \times 3 \times \cdots \times k}$$

Note: On the top, you count down *k* terms, beginning with *n*, and on the bottom, you count up *k* terms beginning with 1. Both have *k* terms.

For example: The number of ways to hold up two fingers on your right hand is:

$$\frac{5 \times 4}{1 \times 2}$$

Which equals 10.

Practice with the Choose Numbers

1. You want to go to the gym on three days during the week. In how many different ways can you choose those three days?

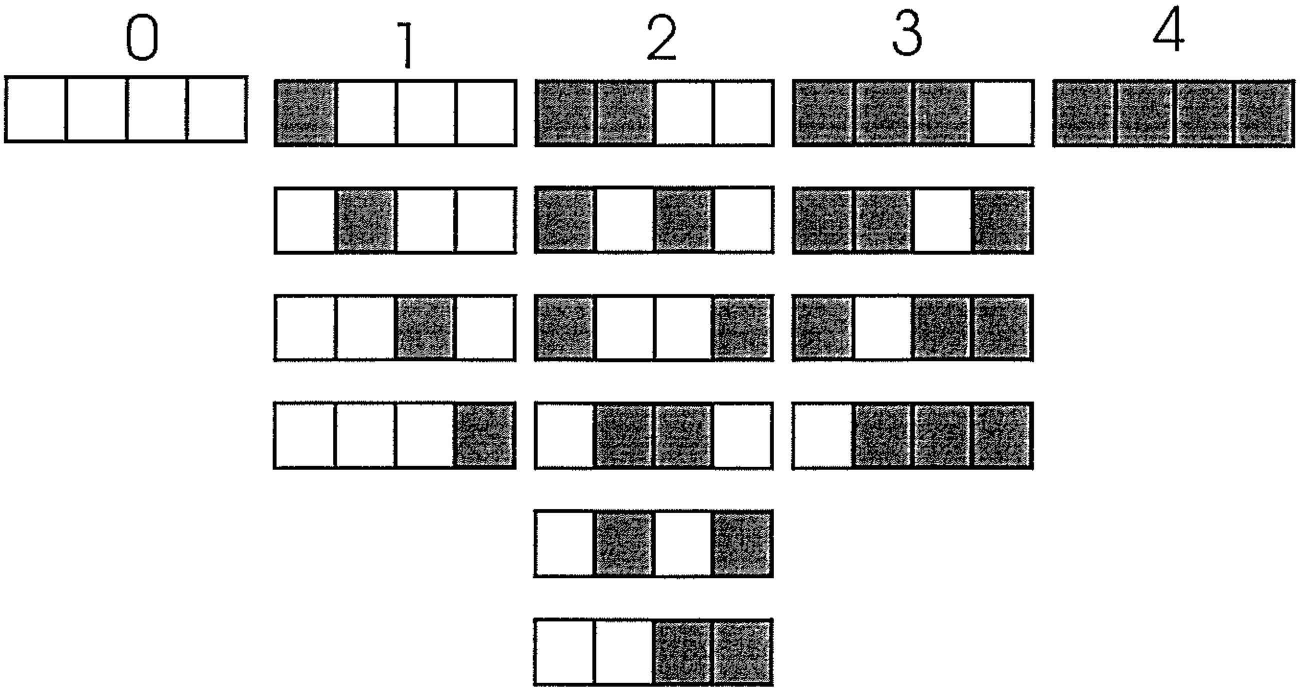
2. In how many ways can you select a three-letter initial? In how many ways can you select a three-letter initial consisting of different letters? In how many ways can you select a set of three different letters from the alphabet?

3. Your class has 25 children. In how many ways can you select two children to serve as class monitors? In how many ways can you select one child to erase the first blackboard and a second child to erase the second blackboard?

4. If your class has 11 boys and 14 girls, in how many ways can you select 3 boys and 3 girls for the school's representative assembly?

THE 16 WAYS TO SHADE IN A ROW OF 4 SQUARES

LISTED ACCORDING TO NUMBER OF SHADED SQUARES



Handout #1 — Anagrams of some words

Count the number of ways to arrange the letters of each of the following words. (Ignore the fact that most of the arrangements are not real English words.)

G. ANNA

H. BOOBOO

I. COCOON

J. DEEDED

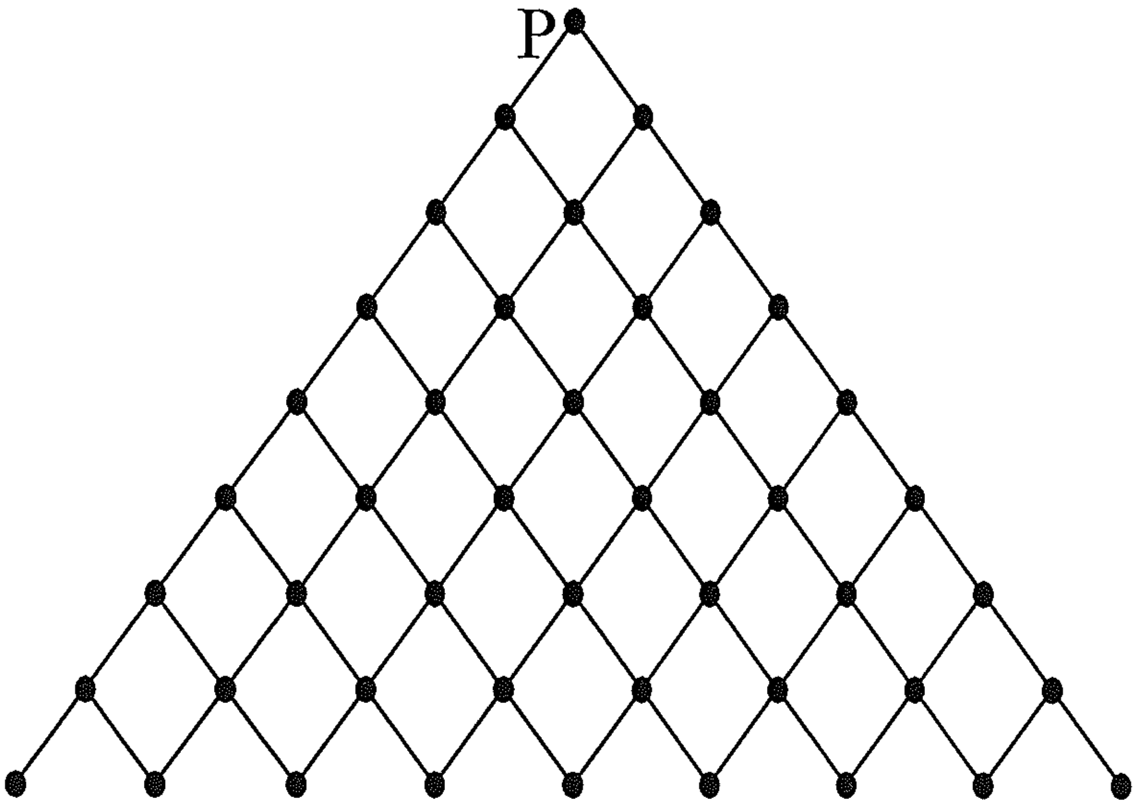
K. ESSES

L. FOOLPROOF

Southbound Paths

How many paths are there from P to each other vertex in the graph below?

(Note: In these paths you move only along southbound edges.)



Pascal's Triangle and the Choose Numbers

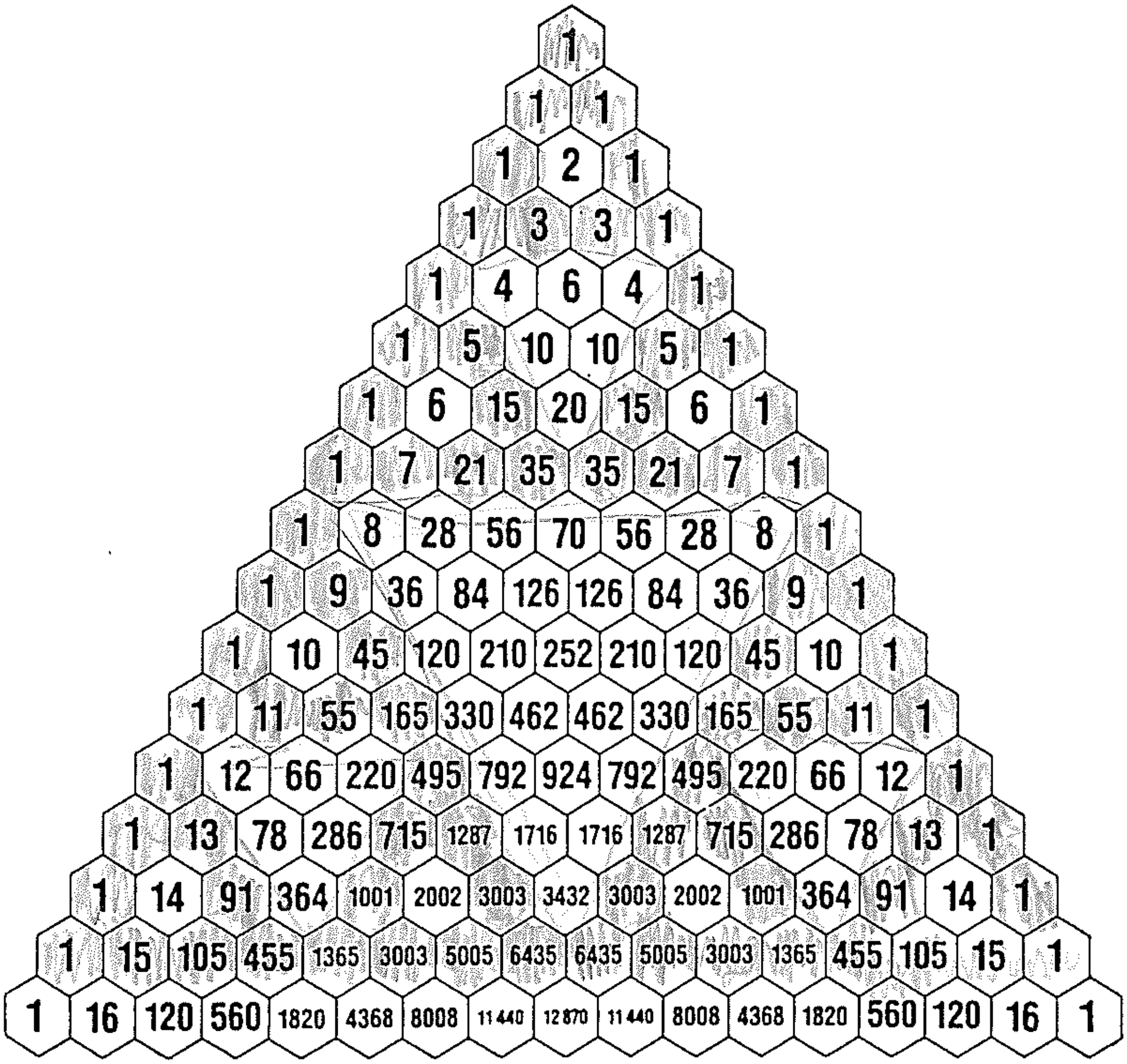
Row 0											1										
Row 1										1		1									
Row 2									1		2		1								
Row 3								1		3		3		1							
Row 4							1		4		6		4		1						
Row 5						1		5		10		10		5		1					
Row 6					1		6		15		20		15		6		1				
Row 7				1		7		21		35		35		21		7		1			
Row 8			1		8		28		56		70		56		28		8		1		
Row 9		1		9		36		84		126		126		84		36		9		1	
Row 10	1		10		45		120		210		252		210		120		45		10		1

- Each entry represents the number of downward paths from the top to that entry.
- Count rows beginning with the 0th row — the row 1 4 6 4 1 is then the 4th row
- In each row, count entries beginning with the 0th entry at the left — the 0th entry in the 4th row is 1, the first entry is 4, the second entry is 6, etc.
- The 0th diagonal has all 1's.
  The 1st diagonal has the counting numbers.
  The 2nd diagonal has the "choose 2 numbers",
  The 3rd diagonal has the "choose 3 numbers", etc.
  The sum of the entries in each row is a power of 2; e.g., the sum of the entries in the 5th row is 2^5.

  (Note: There are 2^5 five-letter words consisting of R's and L's.)

Workshop 7 : The Choose Numbers

PASCAL'S TRIANGLE



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

From "Critical Thinking Activities, Grades K-3" by D. Seymour and E. Beardslee. ©1988 by by Dale Seymour Publications.

A Culminating Activity on The Choose Numbers

The following two problems form an excellent culminating activity for a unit on systematic listing and counting, because if your students understand the concept of "choose numbers" and the display in Pascal's triangle, they will be able to answer these questions, but if they don't, they will be mystified by them.

The Pizza Problem:

How many different pizzas can be made with four out of the following eight toppings:

Broccoli, Olives, Onion, Tomato, Cheese, Anchovies, Mushrooms, Peppers

Hamburger Problem:

How many different hamburgers can be made if you have the following eight options:

Sesame (on the bun), ketchup, mustard, relish, onion, tomato, cheese, lettuce

The number of different pizzas is just "8 choose 4" (which is 70), since each pizza involves a choice of four of the eight toppings, and there are "8 choose 4" such choices.

The number of different hamburgers is $2^8 = 256$, since there are eight items and for each item you can answer "yes" or "no". Since there are two choices for the first item, two choices for the second item, etc., there are altogether 2x2x2x2x2x2x2x2 possibilities, by the multiplication rule for counting.

But why are the answers so different?

Your students should be able to tell you that in the pizza problem, you have to have exactly 4 toppings, whereas in the hamburger problem, you can select any number of the eight options. If you selected exactly four of them, there would be "8 choose 4" possibilities. But you could select 0 options, 1 option, 2 options, 3 options, and so on, up to 8 options. So the total is "8 choose 0" plus "8 choose 1" plus "8 choose 2" and so on, up to "8 choose 8", by the addition rule for counting. These are all the numbers in the 8th row of Pascal's triangle, and their sum is 2^8, or 256.