Master Document

Workshop 5 — What's the Shortest Route?

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

revised August 1, 1999 (and, most recently, June 28, 2011)

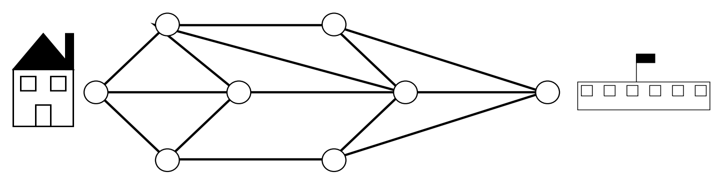
Workshop 5 — What's the Shortest Route?

Materials and Pre-Workshop Preparations | **Allocated Time**

Activity #1 — Overview of Graph Problems . . . . **5 minutes**

Activity #2 — "From Home to School" Activity . . . . **20 minutes**

- Diagram of a graph with eight vertices and fourteen edges made from masking tape and the centers of paper plates (removing the rim permits the plates to lie flat); the distance from "home" to "school" should be at least 12 feet, large enough so that participants can comfortably walk along the edges. Draw "home" and "school" on the appropriate vertices.



- Linker cubes (or other countable objects) to be placed along edges; about 400 should be provided, an average of 25 per edge.
- Twelve arrows, each about 20"×2", in fluorescent colors, if possible.

Activity #3 — How Many Paths Are There in a Grid? . . . . **50 minutes**

Activity #4 — Finding the Shortest Path . . . . **50 minutes**

- Use diagram on the floor as above in Activity #2; two different color markers; large arrows

. . . . TOTAL WORKSHOP TIME: **125* minutes**

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

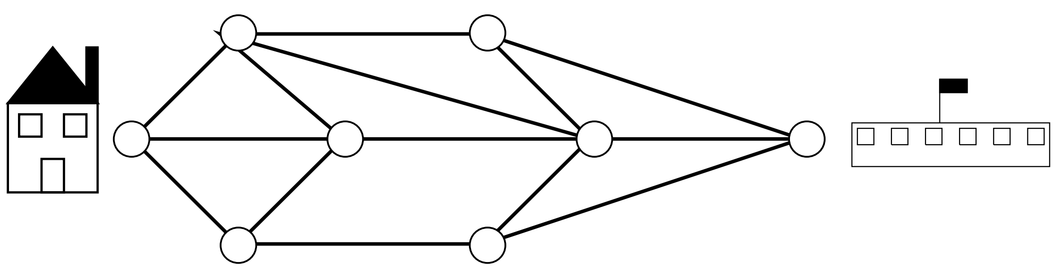
Word Wall – Number of Paths from A to B, Shortest Path from A to B, Pascal's Triangle, Dijkstra's Algorithm

Activity #1 — Overview of Graph Problems
(Allocated time = 5 minutes)

Use TSP #1 to introduce this session. In each of workshops 3-5, we deal with optimal solutions and participants are typically confused among the three types of situations dealt with — best cycle, best tree, best path. So we start with an explicit discussion of the three types of situations. While reviewing the mathematical differences presented in TSP #1, remind participants of the types of applications that lead to each of these three types of problems.

Activity #2 — From Home to School
(Allocated time = 20 minutes)

A. For this activity, we draw on the floor (using the centers of paper plates for vertices and masking tape for edges) the graph below (that occurs on the institute brochure, labeled "What's the Shortest Route?"):



Have the participants gather around the graph, and invite them to put lots of cubes (or other countable objects) on the edges to indicate the weight of each edge; 400 linker cubes are provided so that the number on each edge can be between 20 and 30. Ask each person to count the number of cubes on one edge, and then mark the tape on that edge with the number of cubes; also ask a lead teacher to record the number of cubes on each edge on a copy of the graph. Ask "How do we find the shortest route from home to school?" Elicit the "trial-and-error" strategy, and invite participants to walk through different paths on the graph. Elicit also the "list all possibilities" strategy, and the idea of using a tree diagram to list all possibilities. After participants return to their seats, show TSP #2 on which the eight vertices are labeled. Using TSP #3 (= HO #0)construct a tree diagram below the graph to list all eight paths from home

to school. (Emphasize that you can only go in the appropriate direction; back-tracking is not allowed.)

Further discussion may be needed about "back-tracking" since some participants may raise objections to including a "no back-tracking" constraint, since in some graphs, it may be that back-tracking results in a shorter path. That is, of course, possible. But for now, to simplify our discussion, we will assume that back-tracking is not allowed. However this stipulation is ambiguous because in any particular situation (such as TSP #6 below), there may be different opinions about whether a particular route involves back-tracking. So it is best not to dwell on this point, or to be sidetracked by controversy, particularly since Dijkstra's algorithm takes back-tracking into account; participants can be told that we will return to this issue at the end of the workshop.

Record on the graph the number of counters on each edge in the floor diagram, and ask each table to add the total on one path, so that you will be able to determine the shortest path.

Another strategy will be discussed later in the workshop (when we will again use the graph on the floor), but first we'll try the "trial-and-error" and "list all paths" strategies on another problem.

Note that the diagram on the floor can be used for many problems, since different sets of weights result in different situations. Reinforce what was said on previous days: students can be assigned problems like this as a review of addition — of counting numbers, of two- and three-digit numbers, of fractions, of decimals — which is much more fun than simply adding a list of numbers.

B. Distribute Hand-out #1 (= TSP #4) with the Cozzens/Porter example of finding the shortest route from Portland to Salt Lake City. (Actually, the numbers in the problem here are simplified versions of the actual numbers — they are all rounded to the nearest 10 — but participants have the original problem in their copy of HiMAP's Module 6, *Problem Solving Using Graphs* by Margaret B. Cozzens and Richard D. Porter.) Ask participants to first use a "trial-and-error" strategy, and then a "list all paths" strategy. On TSP #5 provide a tree diagram which contains the solution to this problem. Note that neither the greedy algorithm (start by going to Eugene) nor the direct path algorithm (start by going to Burns) gives the optimal solution; these are two strategies that most people think of first.

C. Show them TSP #6 and ask whether they think either the trial-and-error or list-all-paths strategy will work for this problem. (You might ask participants to estimate the total number of possible routes — there seem to be about 200 altogether.)

For any doubters, show TSP #7 and ask the same question.

Since estimation is an important topic taught in grades K-8, you might spend a few minutes thinking about how you might estimate the number of paths in TSP #6. Ask whether the number of paths is in the tens, hundreds, or thousands — see how many participants vote for each — and note that it would be helpful to know what order of magnitude is correct. Try to elicit the strategy of dividing the map up into "layers" using vertical lines and estimating successively how many paths there are from Poughkeepsie to the end of each layer. That would lead to 3 x 3 x 3 x 3 x 2 paths since each path to the end of one layer (except the last) can be continued in about 3 ways through the next layer (show examples!). This gives about 160 paths. We will see soon how to get an exact number of non-back-tracking paths. Then note that they will be asked to estimate the number of paths in TSP #7 *on the homework — but ask them to guess now whether we would be talking about hundreds, thousands, ten-thousands, hundred-thousands, or millions of paths. (Note that next week we will see how to find an exact answer.)*

Activity #3 — How many paths are there in a grid?
(Allocated time = 50 minutes)

A. Just to see how hard it would be to list all the paths, let's count the number of paths in a grid. This will also be an introduction to the first topic of next week — systematic counting. Show TSP #8 (which will soon be Hand-out #2) and ask them how would we find the number of paths from A to B on this grid. (Note: Paths in this grid always go either north or east!) Allow participants to think about this question, then ask, "What strategies might we use?" Elicit the answer that we should try the "look-at-a-simpler-problem" strategy.

If participants have difficulty, ask them what makes the problem on TSP #8 *hard? A typical response might be "too many squares" implying that the more squares there are in the grid, the more possibilities that have to be considered. Continue by asking, "How can we make the problem easier?" until you have elicited, "Look at one square".*

So let's start with the 1x1 grid on TSP #9. How many ways are there to get from A to B (two diagonally opposite corners)? Now let's try to do the same for a 1x2 grid (also on TSP #9). How many ways are there to get from one corner to the diagonally opposite corner? (Have participants do this problem themselves.) After people come up with the answer, ask if there's a way to use the answer to the first question to get an answer to the second question, and lead them to the following conclusion (described on TSP #10) — that the number of paths from A to B in the 1x2 grid is the sum of the number of paths from A to the point below B and the number of paths from A to the point to the left of B.

Remind them that in referring to the grids on TSP #9 as 1x1 and 1x2 grids, the numbers refer to the squares. We could have called them 2x2 and 2x3 grids, referring to the number of vertices rather than the number of squares, but that wouldn't have squared with common usage. Similarly the grid in TSP #8 is a 2x3 grid although the number of vertices is 3x4.

B. Distribute Hand-out #2 (= TSP #8) and ask them to determine how many paths there are from A to B on this grid, and to list all the paths systematically. Survey the answers you get and the strategies participants used to obtain those answers. Use the "look-at-a-simpler-problem" strategy and work through the process of labeling each vertex with the number of paths from A to that vertex — to arrive at the conclusion that there are altogether 10 paths from A to B. (If they are having difficulties understanding the basic principle used, show them TSP #11).

Our experience is often that, despite the discussion of the 1x1 and 1x2 grids, many participants are unable to determine that ten is the correct number of paths in the 2x3 grid; thus the careful working out of that problem on the board is essential. Moreover, even if they determine that there are ten different paths, many participants have difficulty enumerating them all systematically; hence we have introduced the following activity.

C. In order to convey visually that there actually are ten paths from A to B, and to lay the groundwork for next week's discussion of systematic counting, (use the phrase "systematically count"), review the tree-diagram in TSP #12 and then the ten paths in TSP #13 which are presented in the same order as the ten branches in the tree diagram. The ten paths are labeled from E-E-N-N-N to N-N-N-E-E. Ask them to indicate what they find similar about these labels, expecting that someone will notice that all have three Ns and two Es. Ask them if they notice any patterns in the order of the ten labels, expecting that someone will notice that they are in alphabetical order. Show TSP #14 so that they can see that the paths are in alphabetical order. Remind them that what we have been doing is making a systematic list of all possible paths, and that systematic listing will be a theme of next week's workshops.

D. Distribute Hand-out #3 (= TSP #15) and ask them to determine how many paths there are from one corner to the diagonally opposite corner on a 4x4 grid, and to write next to each vertex the number of paths from A to that vertex. When they complete this activity, ask if they notice any patterns. If not, rotate the transparency so that vertex A is located at the top of the screen and ask again. If still no response, mention that some of them may have seen the pattern formed by the numbers labeling the vertices, especially when the grid is rotated; this pattern, which will be discussed in more detail next week, is called Pascal's triangle!

It has been our experience that someone always recognizes this pattern without the instructor saying, "Pascal's Triangle" and after "Pascal's Triangle" is mentioned, many more of the participants (perhaps half) will recognize the connection. For your information: Next week's agenda will include revisiting the question of the number of paths in a 4x4 grid, with the explanation that there are "8 choose 4" paths because to get from A to B you have to walk 8 blocks, of which 4 are north and 4 are east, so the number of paths equals the number of ways of choosing the 4 north blocks from the 8 blocks.

E. Show TSP #16 and discuss Pascal's triangle, noting that each entry is the sum of the two entries above it, one to the right and one to the left, just as, in the 4x4 grid, the number of paths from A to each vertex is the sum of the numbers of paths to the two preceding vertices.

[Time for a 5-10 minute break]

Activity #4 — Finding the shortest path

(Allocated time = 50 minutes, including 25 minutes for parts A and B, and 25 minutes for parts C and D.)

A. Find the number of paths in the graph of Activity #2 using the method above (on TSP #17 = TSP #2) and verify that the number of paths corresponds to the number of paths in the tree diagram generated earlier.

B. Since we have used the "look-at-a-simpler-problem" strategy to count the number of paths, let's try to use a similar strategy to find the shortest path. In the previous activity, we labeled each vertex with the number of paths from START to that vertex; now we will label each vertex with the <u>length</u> of the <u>shortest</u> path from START to that vertex. First, considering each edge, going from left to right on the diagram, label each edge (writing near the end of each tape entering a vertex) with the shortest path from the house to the vertex which concludes with that edge. Have the group return to the graph on the floor, and elicit the participants' aid in labeling each vertex (writing directly on the plate) with a number indicating the shortest distance from the house to that vertex. To do this, you will first need to label each edge entering a given vertex with the shortest distance using that edge, and then take the minimum of the shortest distances over all edges entering the vertex.

Note: Some participants may have difficulty grasping what the different numbers indicate, so you may have to go slowly. You should also use different colors; i.e., use a blue color to write on the masking tape (edge) and a red color to write on each plate (vertex). This helps remind participants that the numbers stand for different things.

At the end of the activity, you will have the <u>length</u> of the shortest path, but you won't actually have the shortest path. Elicit from the participants the idea that we should be keeping track not only of the shortest distance to each vertex, but also the shortest path to each vertex — in other words, that we should designate certain edges as contributing to a shortest path. Mark those edges appropriately, by placing a large arrow on top of each edge which contributes to the shortest path. Note that although we were interested only in finding the shortest path from home to school, we have actually found the shortest path from home to every other vertex.

B. Review this procedure for finding the shortest path from A to B using the 2x3 grid on TSP #18. (Provide participants with Hand-out #<u>4</u> so that they can follow this procedure.) Keep track of the selected paths as you go through this activity; as in the previous example, use one color to label each edge into a vertex with the minimum path length using that edge, and another color to label each vertex with the minimum path length to that vertex.

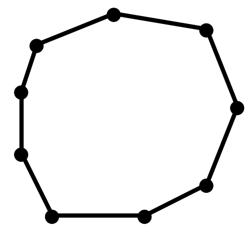
Ask participants in what directions they are assuming to travel; this is the first time they are going south and east instead of north and east.

C. Distribute Hand-out #<u>5</u> (= TSP #19) with a 4x4 grid; remind participants that they need to take all vertices into consideration (since many participants tend to converge prematurely to a solution). After the groups have had an opportunity to work on their solutions, draw the edges used on a blank transparency over TSP #19, and then remove the underside to reveal a ... spanning tree — this spanning tree can be called the "shortest paths from A" spanning tree. It not only provides the shortest path from A to B, but also the shortest path from A to every vertex of the graph. An important observation is that in order to find the shortest path from A to B, you have to determine and take into account the shortest path from A to every other point as well; there is no easier way to find the shorter route from A to B by looking only at some select group of vertices. From every vertex V on the graph you can develop a "shortest paths from V" spanning tree. These will also be different; for example, the "shortest paths from B" spanning tree will include the edges to both neighbors of B. Point out that this method of labeling vertices and designating edges is called Dijkstra's algorithm; this algorithm does take into consideration paths that involve back-tracking. If it seems appropriate for a particular group of participants, review the description of Dijkstra's algorithm on TSP #20. (TSP #20 and TSP #21 have been combined into HO #6.)

D. Where else have we seen spanning trees? Is this the same as a minimum weight spanning tree? Discuss the two different problems using TSP #21 — minimum weight networks and shortest paths — and note that they need not be the same. (This will be further explored on a homework problem.)

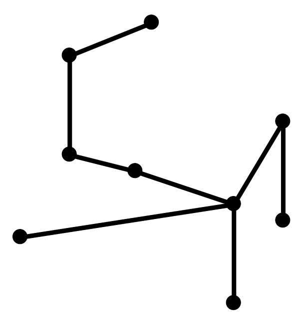
Three Separate Problems Involving Weighted Graphs

I. Find the best cycle



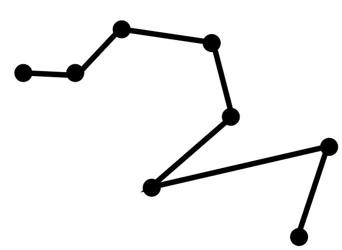
Traveling Salesperson Problem
(workshop 3)

II. Find the best network



Minimum Weight Spanning Tree Problem
(workshop 4)

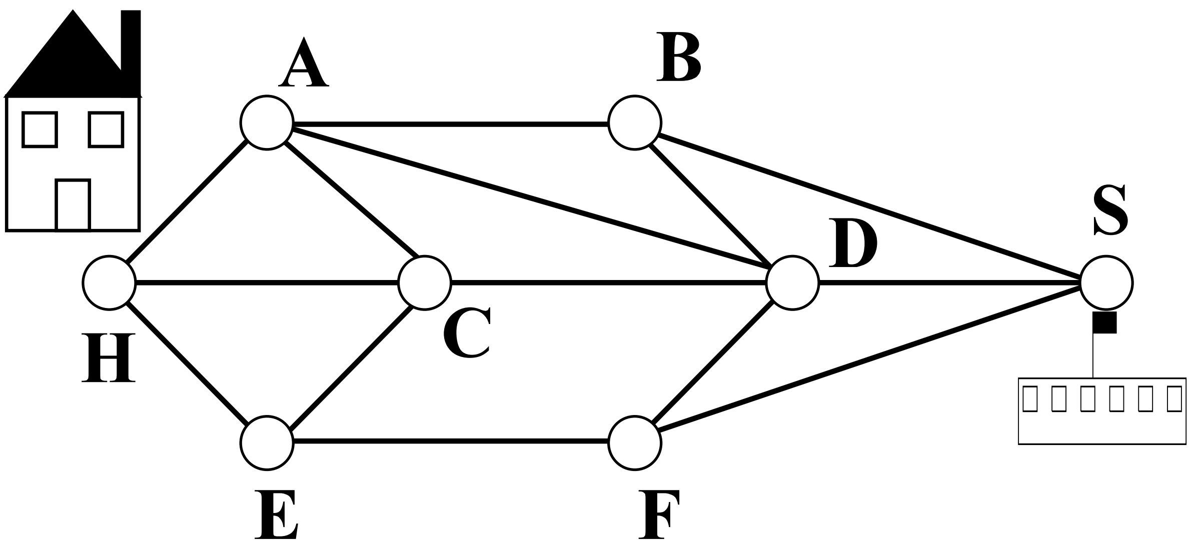
III. Find the best path



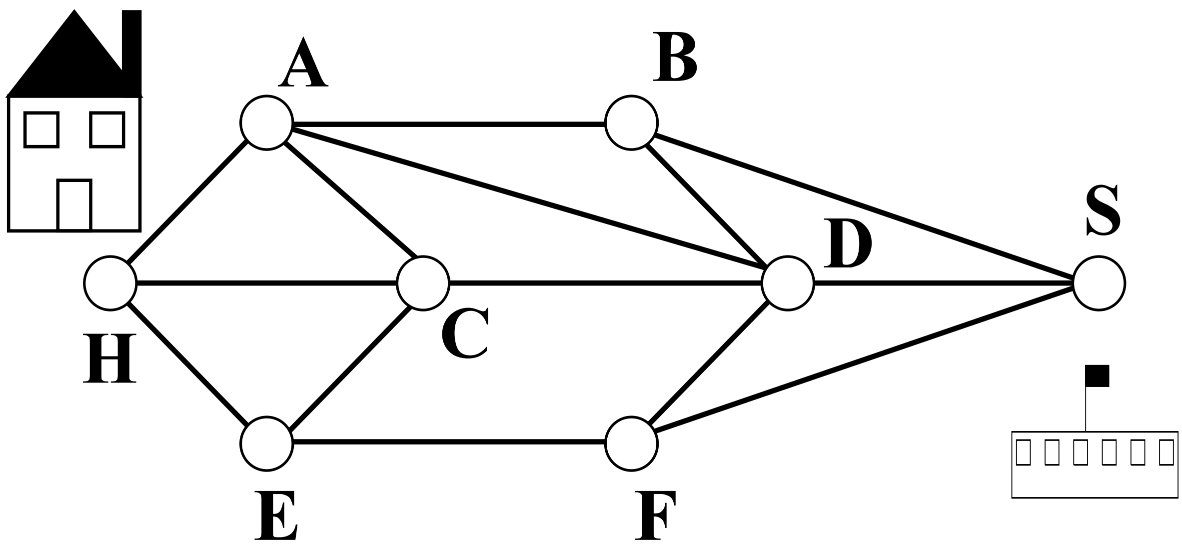
Shortest Path Problem
(workshop 5)

What’s the shortest route from home to school?

Use a tree-diagram to list all the possible routes from home to school in the diagram below, and find the shortest route.

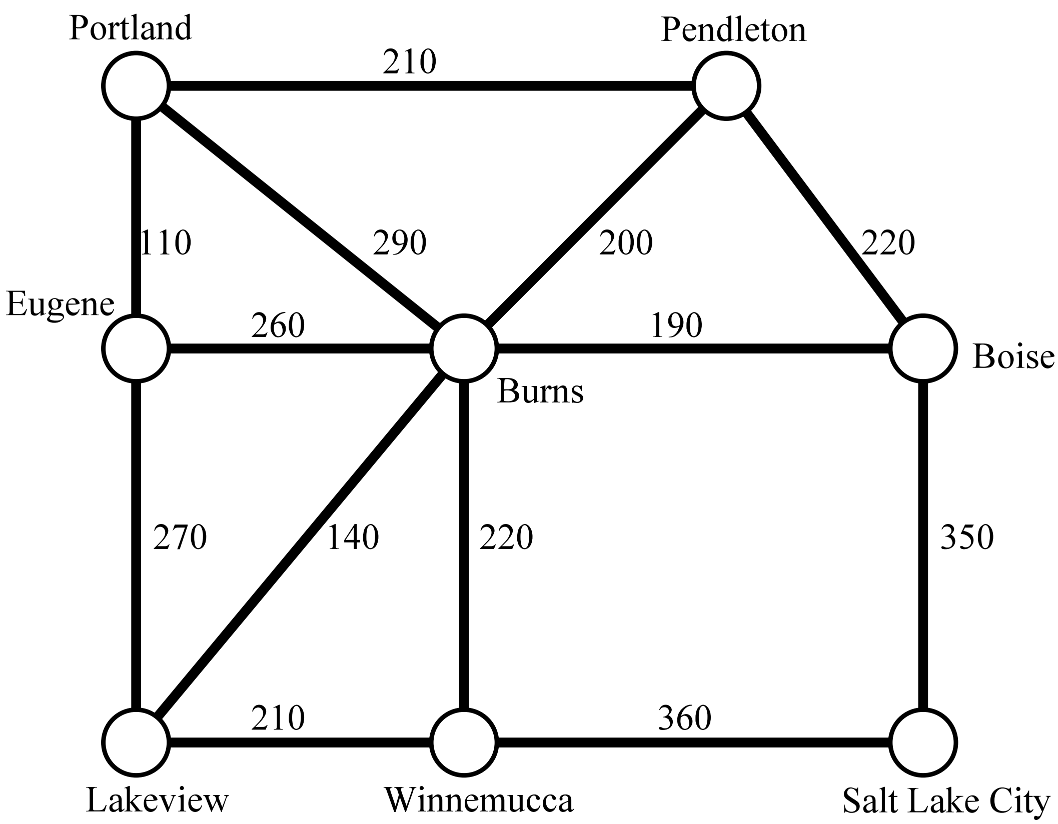


Tree-diagram of all possible routes

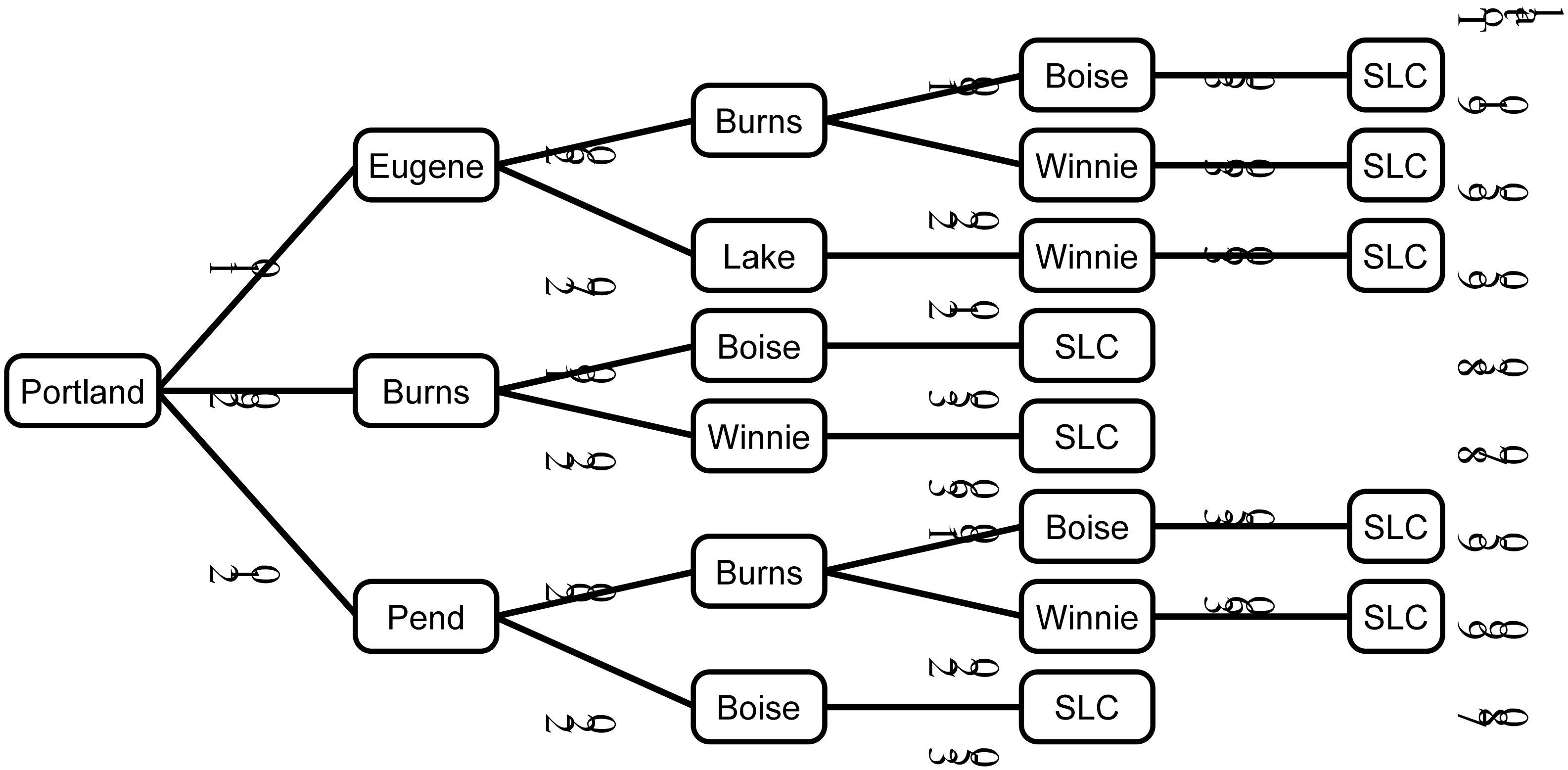


Hand-out #1: On the road to Salt Lake City

A family in Portland, Oregon is planning a drive to Salt Lake City. The weights on the edges in the figure give the mileage between various cities. What route should they take if they want to drive as few miles as possible? (From Cozzens and Porter, Module #6, COMAP.)

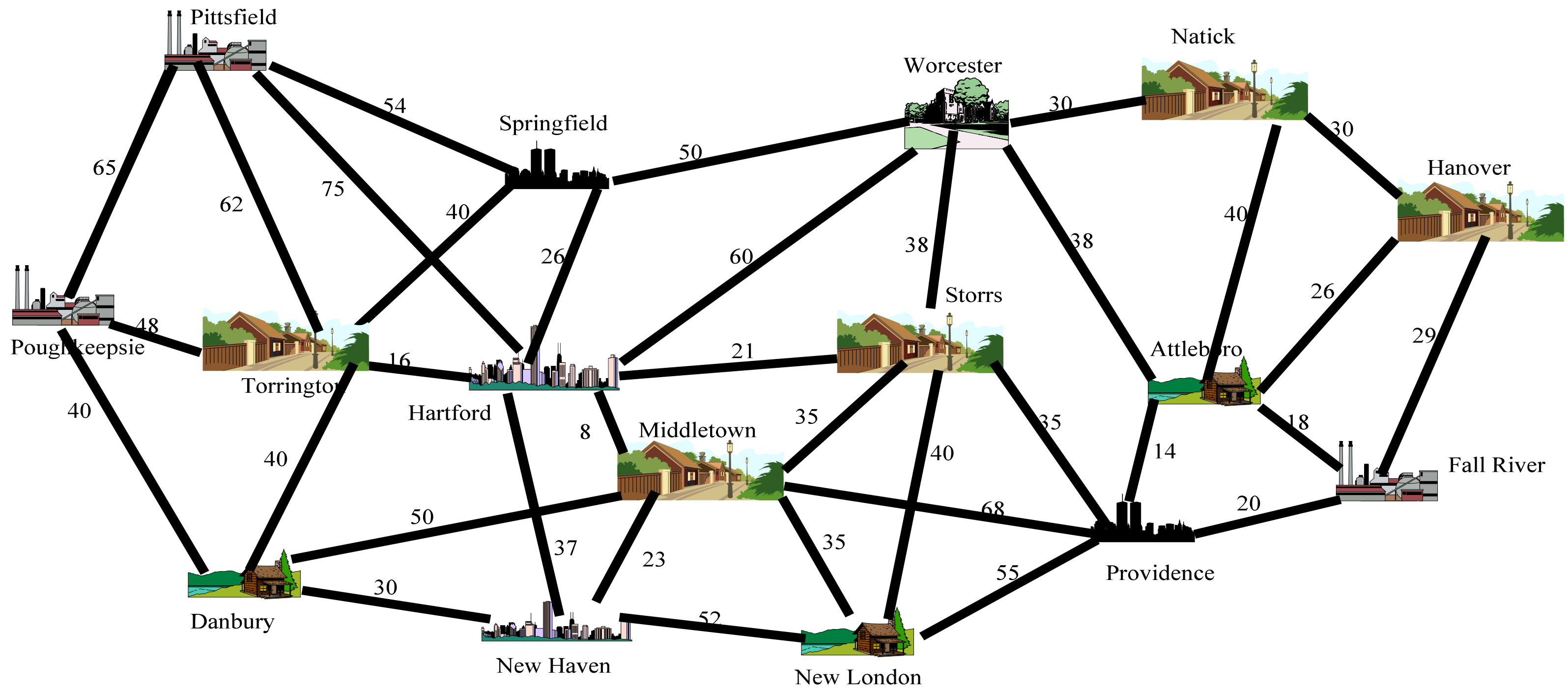


On the Road to Salt Lake City
Tree diagram of all possible routes from Portland to Salt Lake City



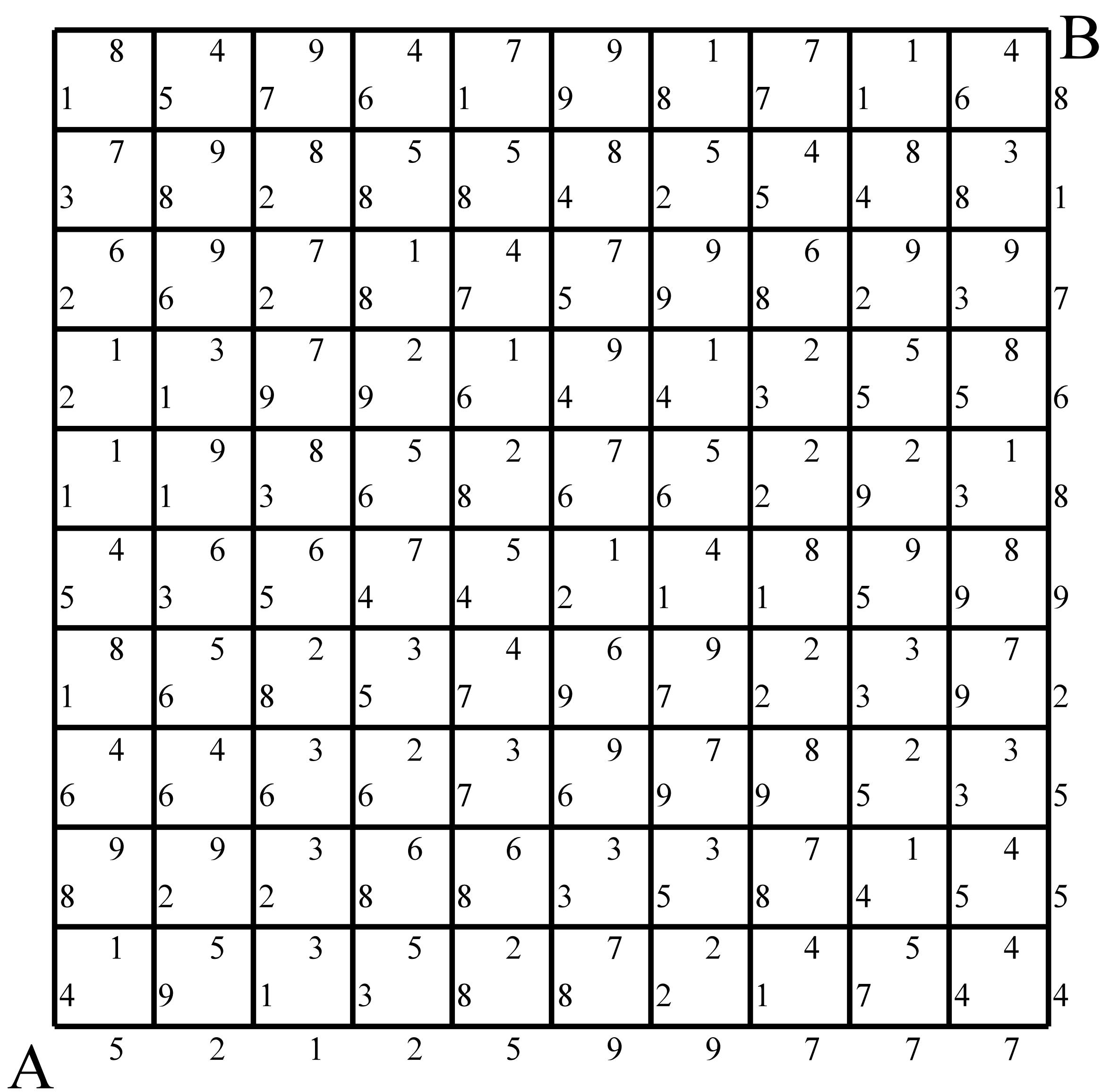
On the Road to Hanover

Peter Sampson lives in Hanover (MA) and Roberta Simons lives in Poughkeepsie (NY), and they spend a lot of time traveling back-and-forth between those cities. Use the map below to help them find the shortest route from Poughkeepsie to Hanover. (From Cozzens and Porter, Module #6, COMAP.)



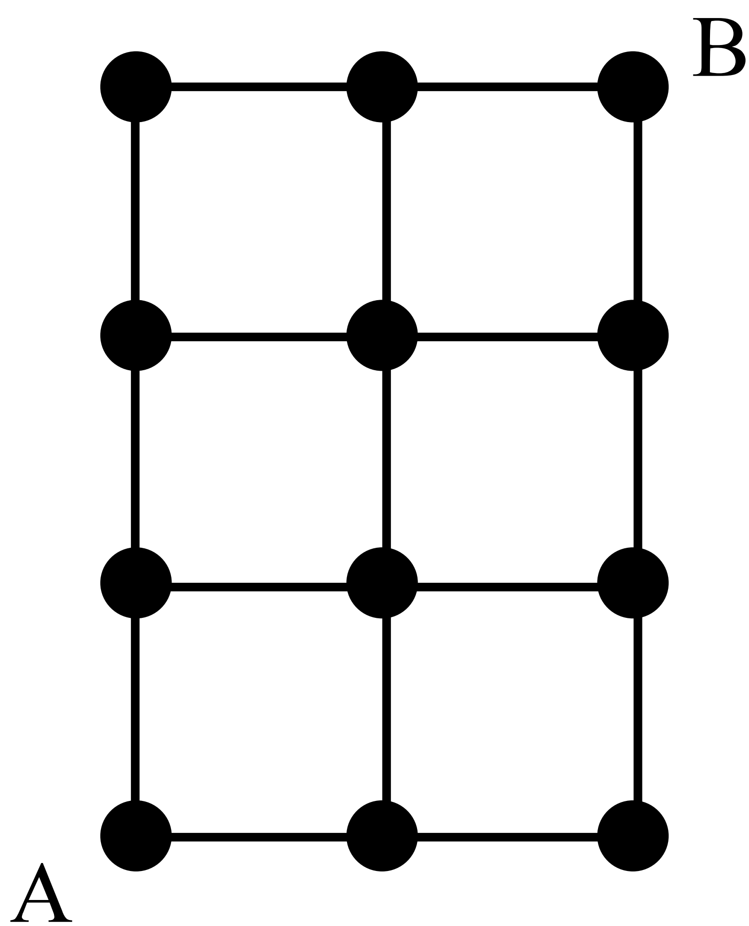
Find the Shortest Path in a 10x10 Grid

Find the shortest path from A to B in the 10x10 grid below. (This problem appears in an article on "dynamic programming" by Sloyer et al. in Mathematics Teacher, February 1985.)



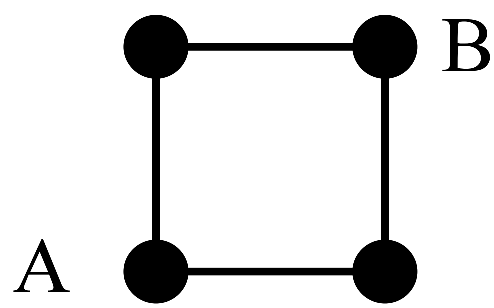
Hand-out #2: Number of paths on a grid

How many paths are there from A to B on the grid below? (Assume that you always go North or East.) Make a systematic list of these paths.

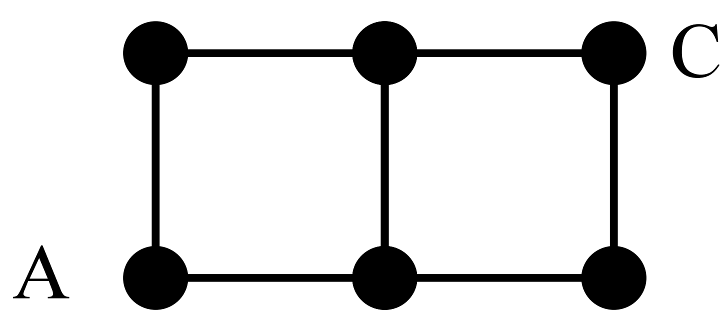


Number of paths on a small grid

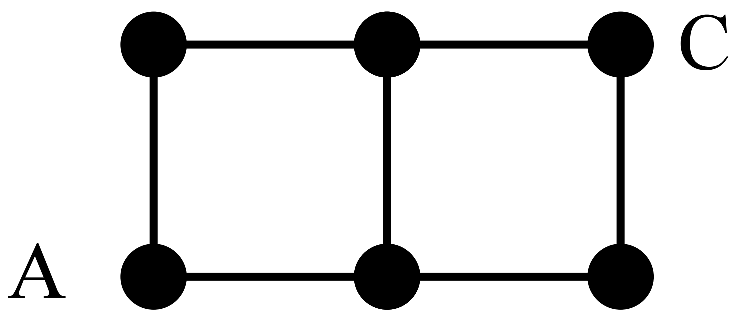
How many paths are there from A to B on the grid below?
(Assume that you always go North or East.)



How many paths are there from A to C on the grid below?
(Assume that you always go North or East.)



How many paths are there from A to C on the grid below?



The number of ways of getting from A to C is the sum of

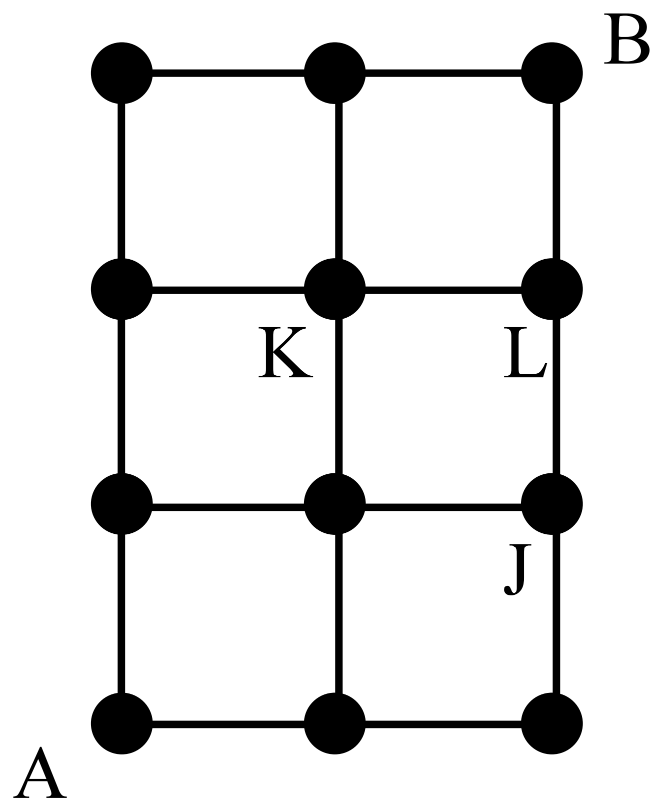
the number of ways of getting from A to the vertex to the left of C

and

the number of ways of getting from A to the vertex below C

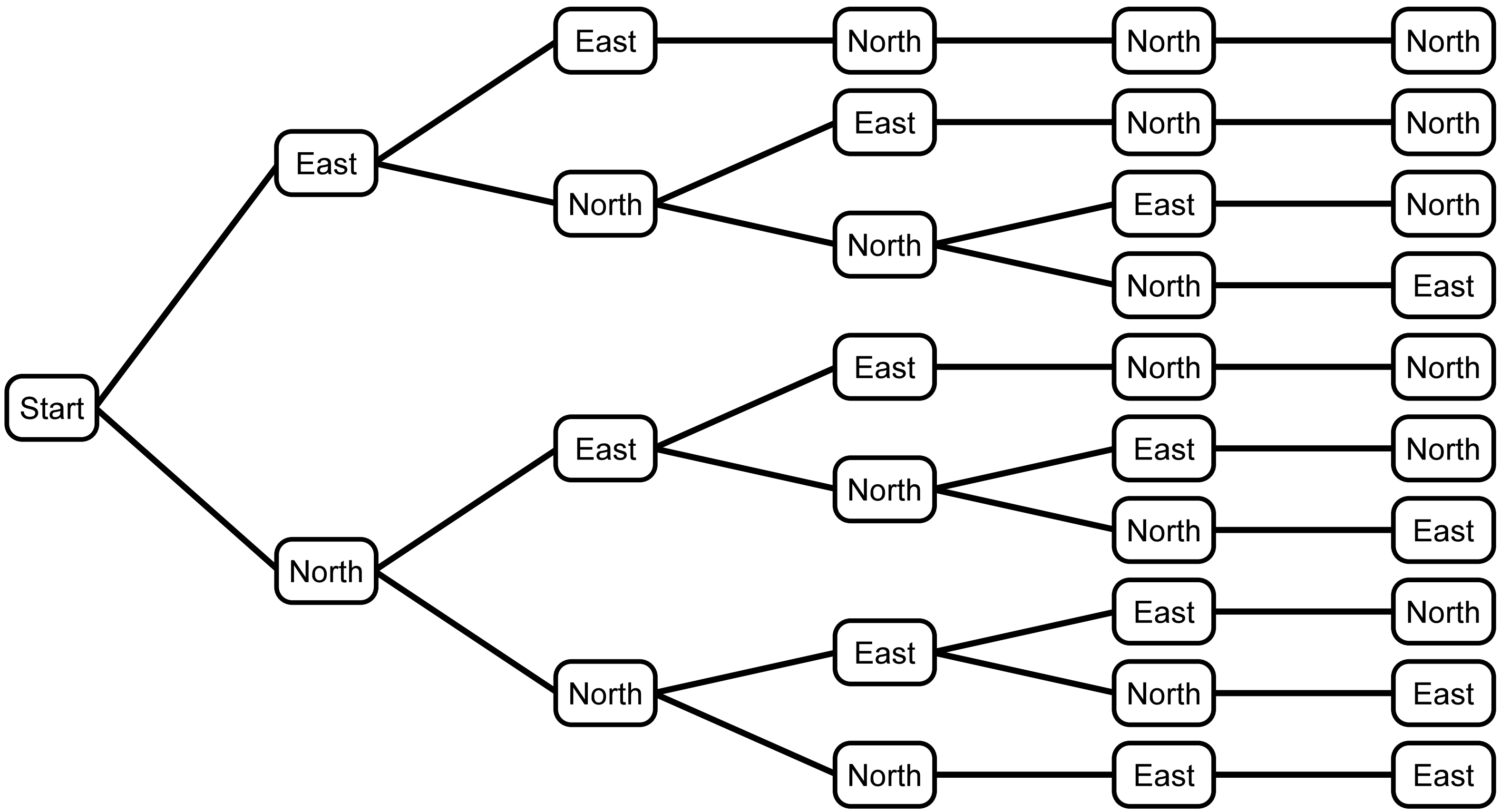
because the only way to get from A to C is via the vertex to the left of C or the vertex below C

In a grid graph where the bottom left vertex is A, the number of ways of getting from A to L is the

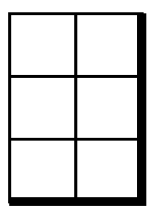


sum of the # of ways of going from A to J (where J is the vertex just below L) and
the # of ways of going from A to K (where K is the vertex just left of L).

The Ten Paths in a 2x3 Grid



The 10 paths on a 2×3 grid
Shown Pictorially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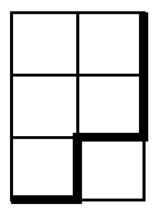
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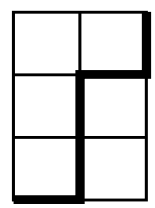
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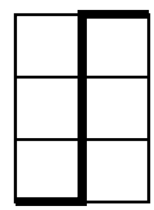
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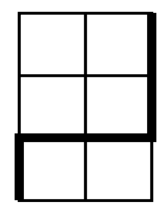
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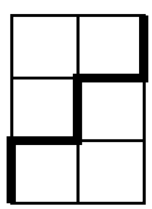
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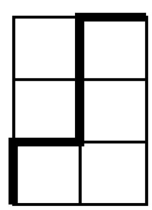
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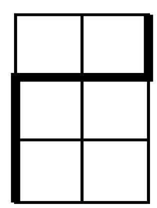
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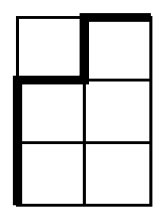
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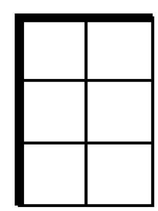

NNNEE

Alphabetical List of the ten paths in a 2x3 grid

E-E-N-N-N

E-N-E-N-N

E-N-N-E-N

E-N-N-N-E

N-E-E-N-N

N-E-N-E-N

N-E-N-N-E

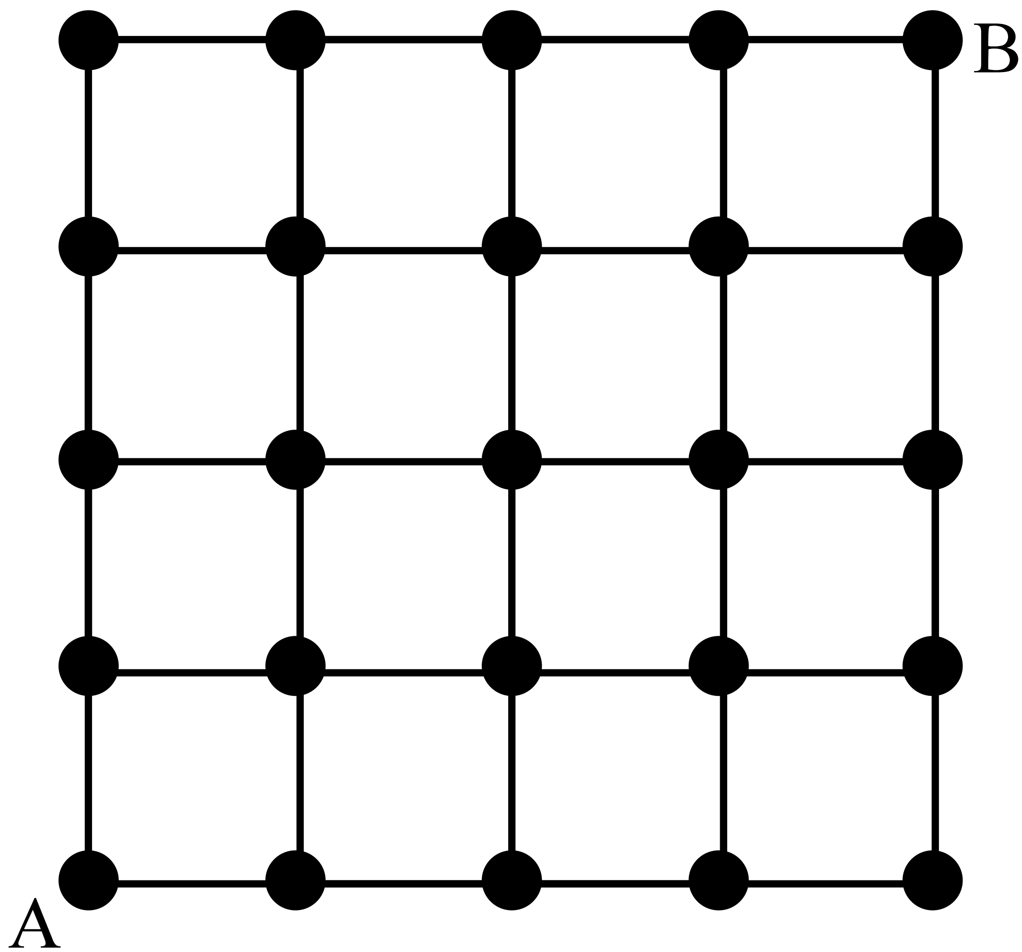
N-N-E-E-N

N-N-E-N-E

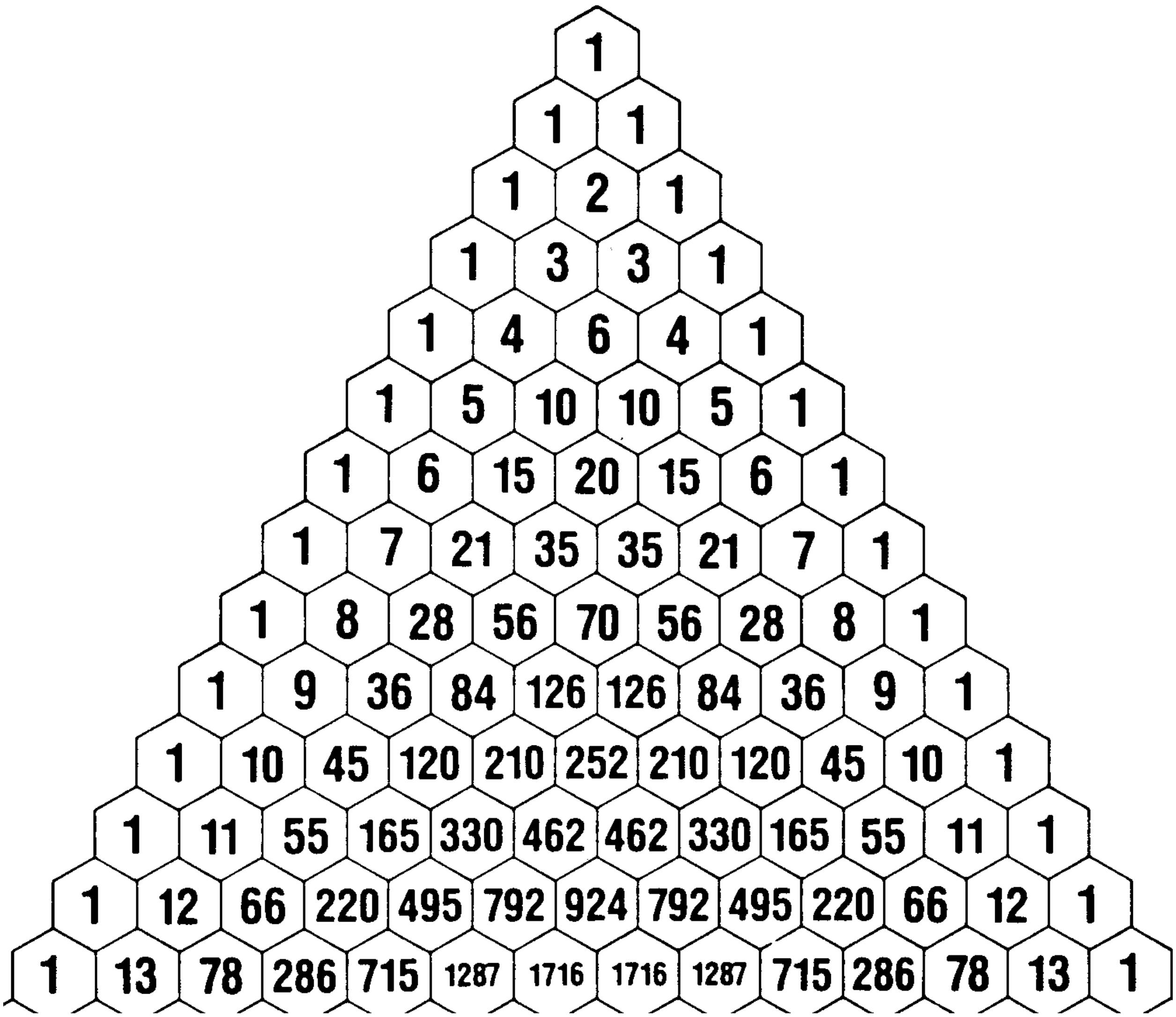
N-N-N-E-E

Hand-out #3: Number of paths on a grid (continued)

How many paths are there from A to B on the grid below? (Assume that you always go North or East.)

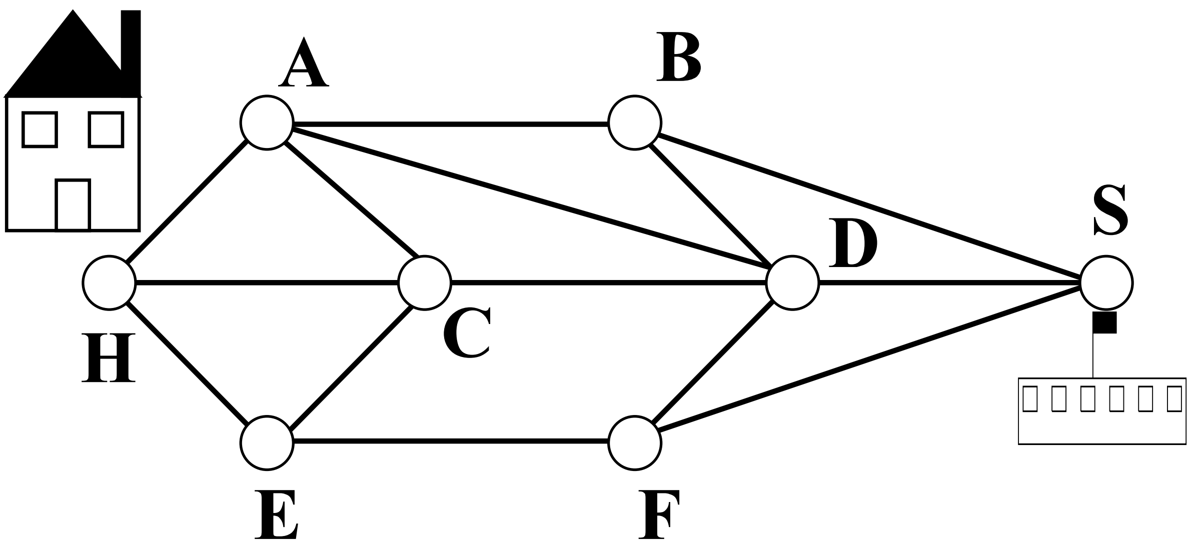


Pascal's Triangle



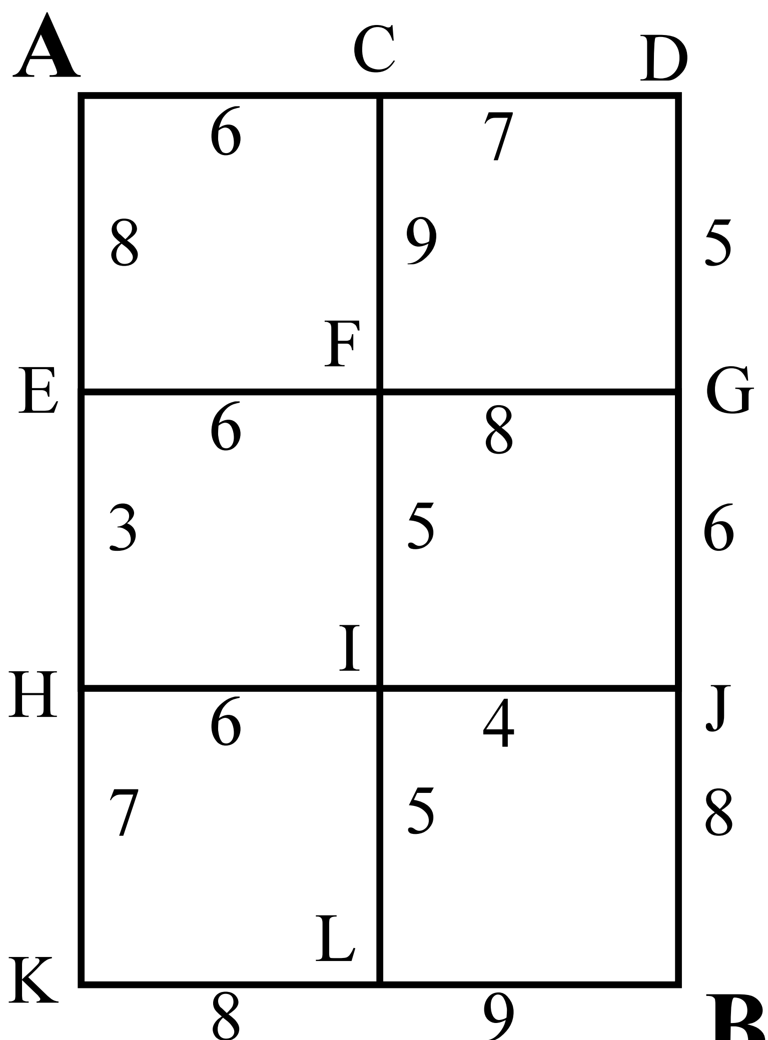
Each entry (except 1) is the sum of the two entries above it (one to the right and one to the left). Each entry also represents the number of paths from the top 1 to that entry.

How many routes are there from home to school?



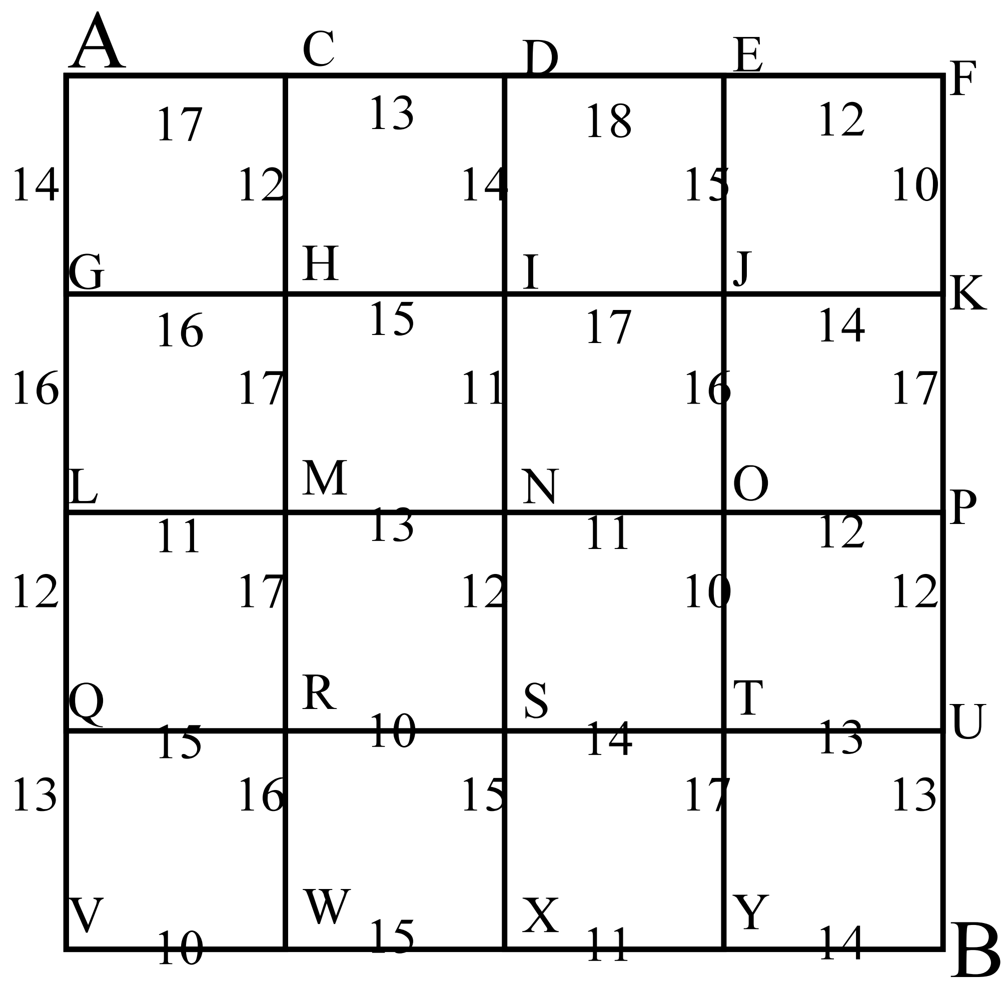
Hand-out #4: Traveling on a Grid

What is the shortest way of getting from A to B in the grid below, where each edge is labeled with the number of miles between the vertices. (Note, in this grid you always go either South or East.)



Hand-out #5: Traveling on a Grid

What is the shortest way of getting from A to B in the grid below, where each edge is labeled with the number of miles between the vertices. (Note, in this grid you always go either South or East.)



Dijkstra's Algorithm

Select the edge of minimal weight which contains the vertex A.

At each stage, a tree has been generated such that the path from A to any vertex B on the tree is a path of minimum weight to B.

At each stage, consider all the vertices not yet on the tree which are reachable from the tree by one edge. Find one for which the path from A to this vertex via the tree and this one edge has minimal weight, and select this edge for the tree.

Continue until all vertices of the graph are on the tree.

> Dijkstra's algorithm finds a spanning tree so that for any vertex B in the weighted graph G, the path from A to B in the spanning tree is a path of minimal weight from A to B.

Two different kinds of spanning trees

1. Minimum weight spanning trees — spanning trees which provide a network with smallest total weight.

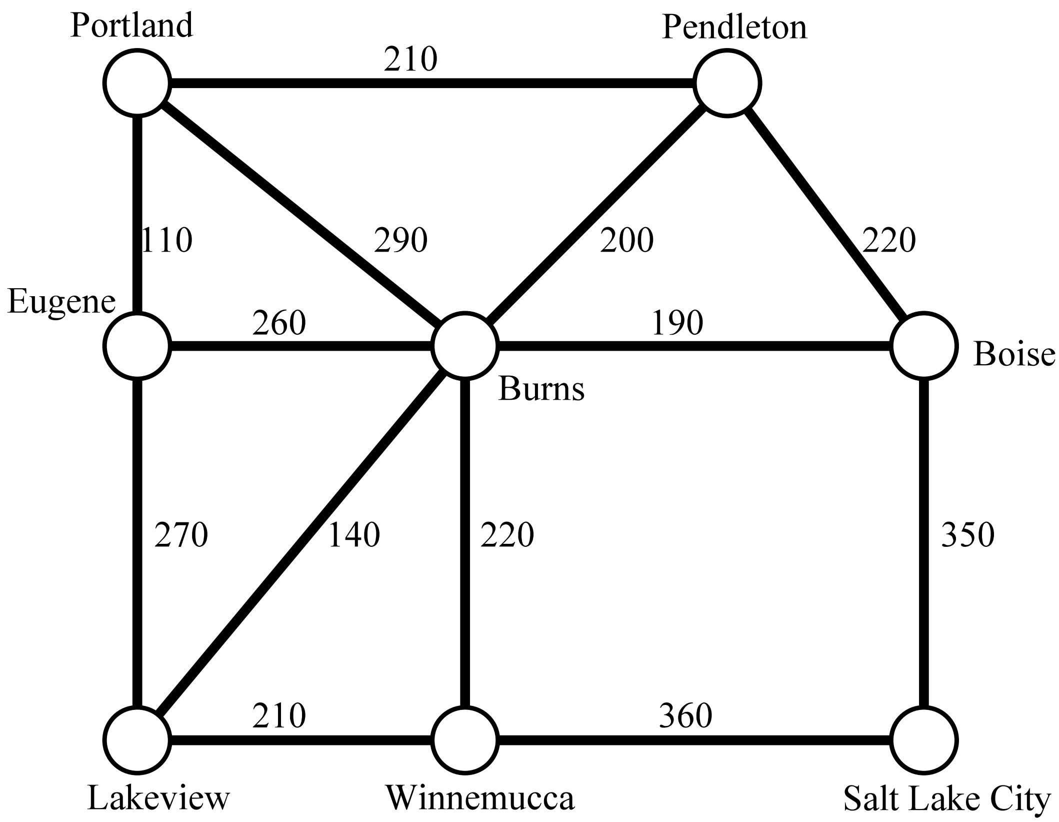
2. Shortest path spanning trees — spanning trees which provide the shortest path from a given vertex to all other vertices.

Since in the construction of a shortest path spanning tree, you are focused on one particular vertex and how to get quickly from that vertex to all other vertices, it is not surprising that a shortest path spanning tree will not be a minimum weight spanning tree.

Since in the construction of a minimum weight spanning tree, you are focused on minimizing the total weight, it is not surprising that you may have to take a rather circuitous route to get from one particular vertex to another.

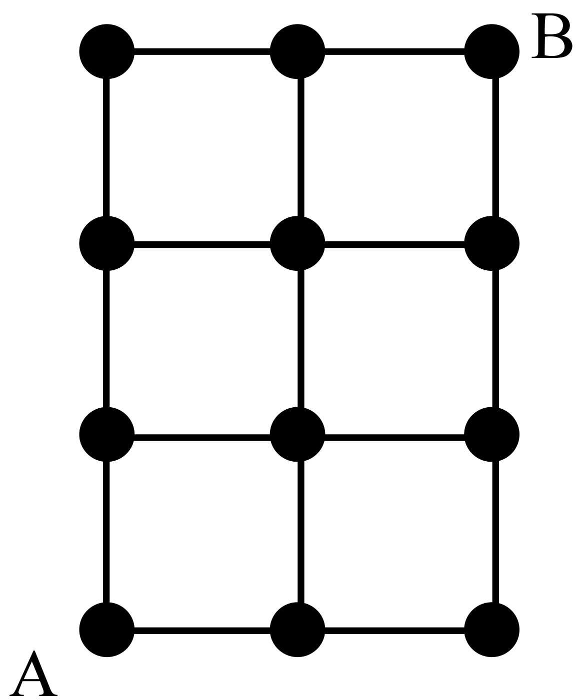
Hand-out #1: On the road to Salt Lake City

A family in Portland, Oregon is planning a drive to Salt Lake City. The weights on the edges in the figure give the mileage between various cities. What route should they take if they want to drive as few miles as possible? (From Cozzens and Porter, Module #6, COMAP.)



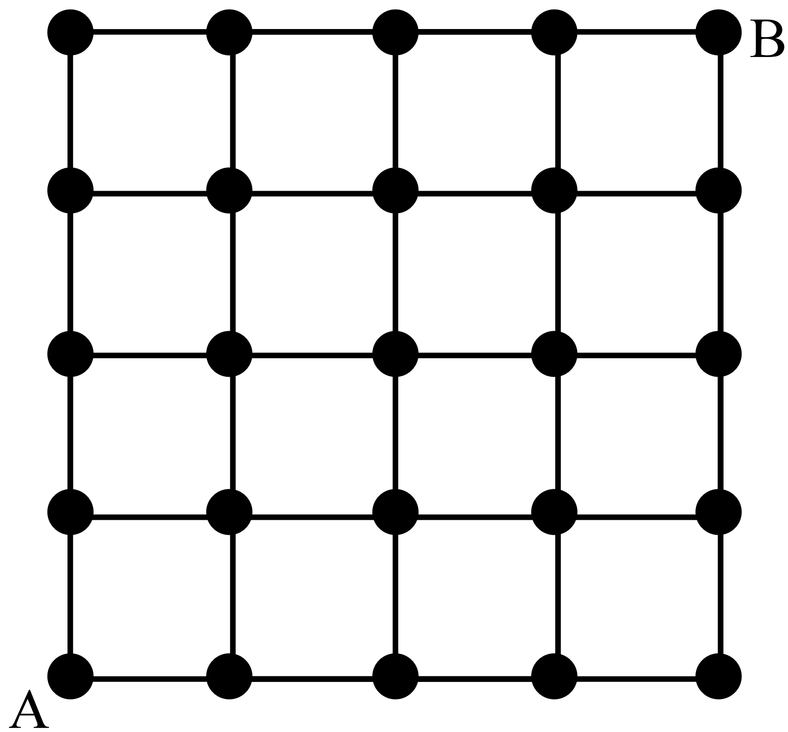
Hand-out #2: Number of paths on a grid

How many paths are there from A to B on the 2x3 grid below? (Assume that you always go North or East.) Make a systematic list of these paths.



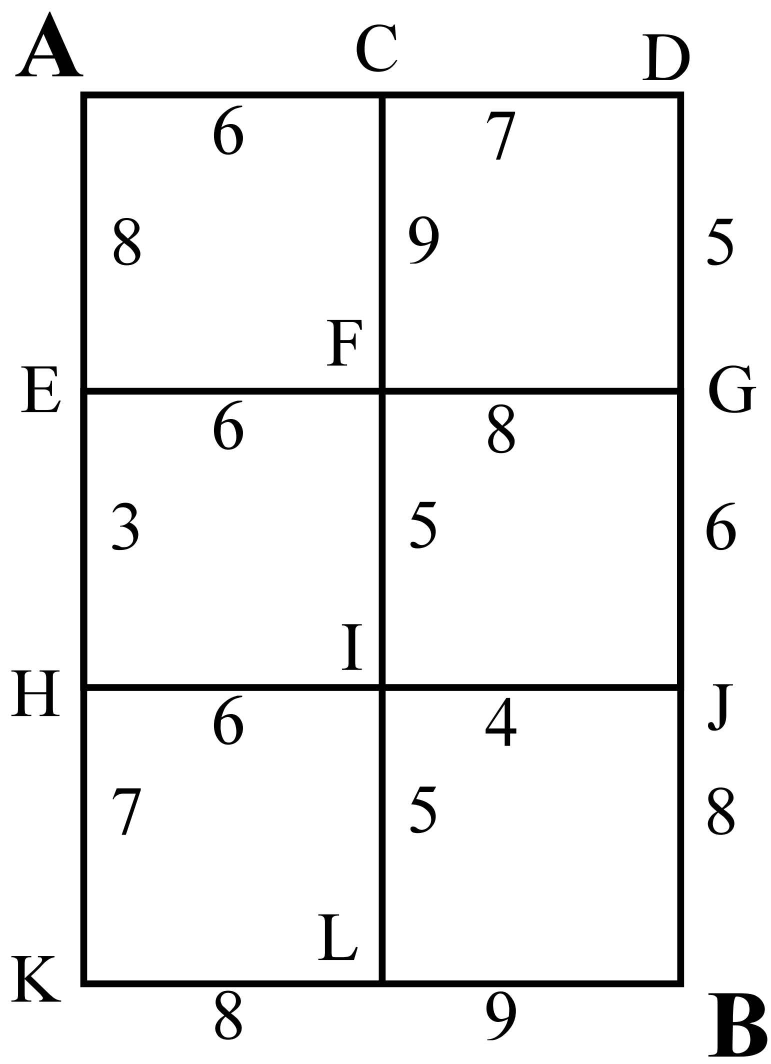
Hand-out #3: Number of paths on a grid (continued)

How many paths are there from A to B on the 4x4 grid below? (Assume that you always go North or East.)



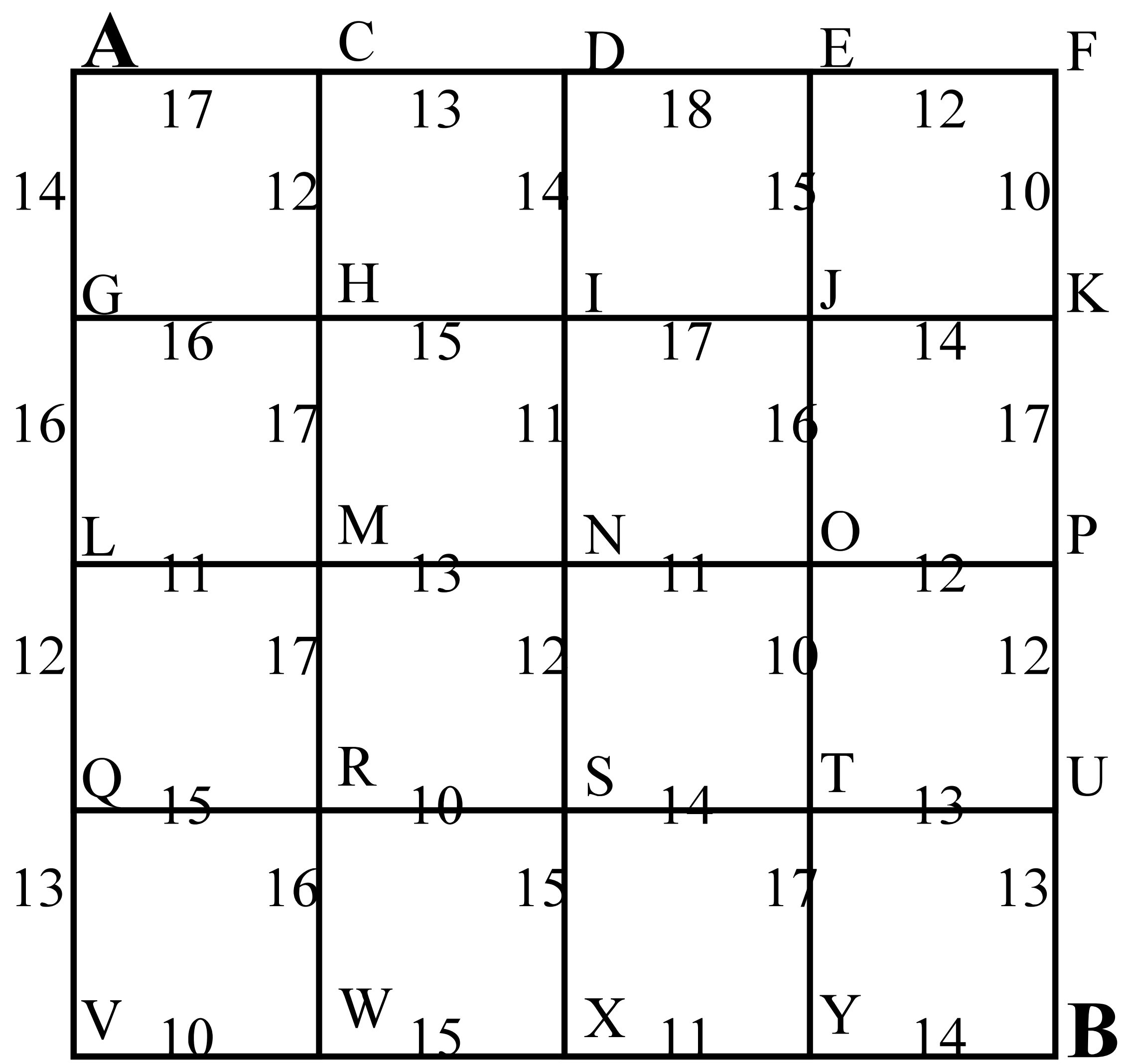
Hand-out #4: Traveling on a Grid

What is the shortest way of getting from A to B in the grid below, where each edge is labeled with the number of miles between the vertices. (Note, in this grid you always go either South or East.)



Hand-out #5: Traveling on a Grid

What is the shortest way of getting from A to B in the grid below, where each edge is labeled with the number of miles between the vertices. (Note that in this grid, you are always traveling South or East.)



Hand-out #6: Dijkstra's Algorithm

Select the edge of minimal weight which contains the vertex A.

At each stage, a tree has been generated such that the path from A to any vertex B on the tree is a path of minimum weight to B.

At each stage, consider all the vertices not yet on the tree which are reachable from the tree by one edge. Find one for which the path from A to this vertex via the tree and this one edge has minimal weight, and select this edge for the tree.

Continue until all vertices of the graph are on the tree.

> Dijkstra's algorithm finds a spanning tree so that for any vertex B in the weighted graph G, the path from A to B in the spanning tree is a path of minimal weight from A to B.

Two different kinds of spanning trees

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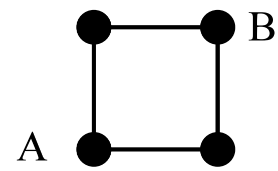
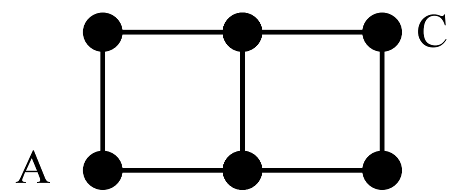
Since in the construction of a minimum weight spanning tree, you are focused on minimizing the total weight, it is not surprising that you may have to take a rather circuitous route to get from one particular vertex to another.

Number of paths on a small grid



How many paths are there from A to B on the first grid?
(Assume that you always go North or East.)

How many paths are there from A to C on the second grid?
(Assume that you always go North or East.)



The number of ways of getting from A to C is the sum of

the number of ways of getting from A to the vertex X to the left of C

and

the number of ways of getting from A to the vertex Y below C

because the only way to get from A to C is

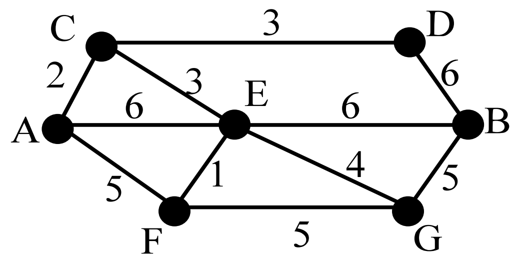
via the vertex X to the left of C

or the vertex Y below C

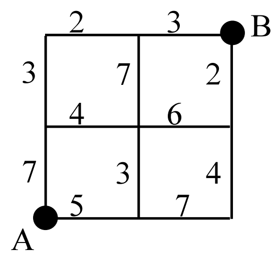
Workshop 5 — What's the Shortest Route — Exercises

Practice problems:

1. List all paths from A to B in the diagram below and find the shortest path.

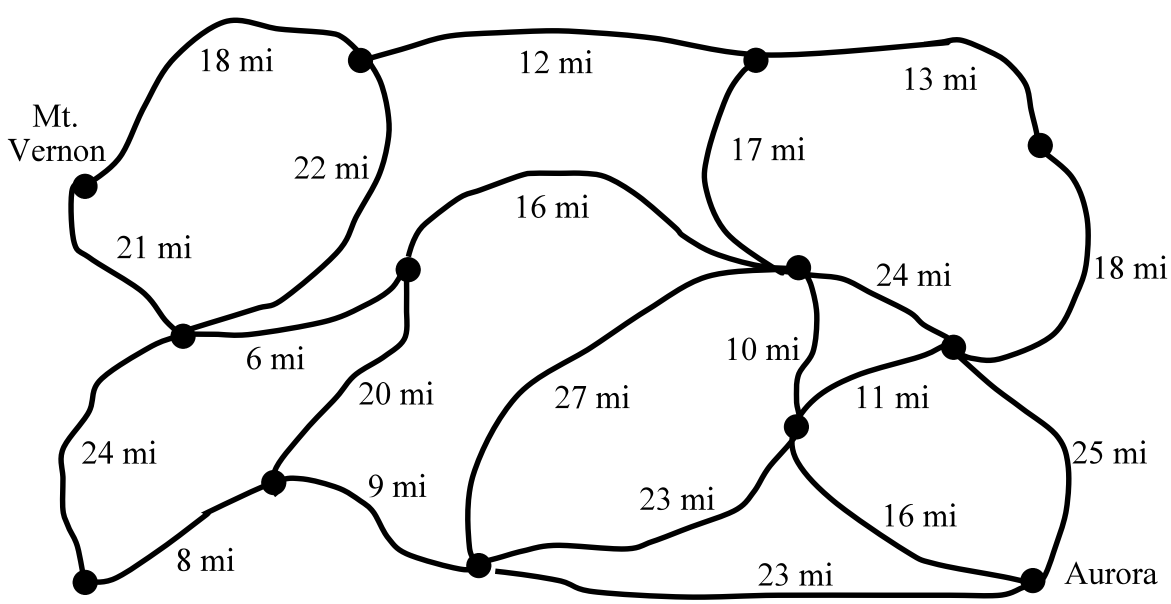


2. Find the shortest path from A to B in the diagram below, using a tree diagram and using Dijksra's algorithm..

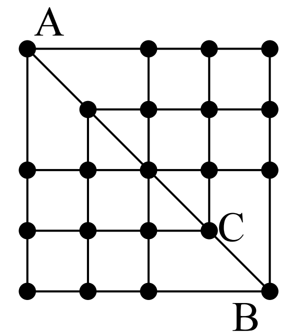


Study group problems:

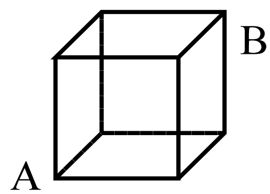
3. Find the shortest route from Mt. Vernon to Aurora using Dijkstra's algorithm. (You can interpret "mi" as "minutes", rather than "miles".)

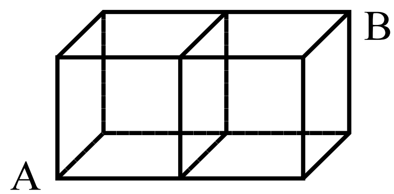


4. a. How many routes are there from A to B in the diagram to the right which always travel south, east, or southeast?



b. How many routes are there from A to B in each of the following two 3-dimensional diagrams? (You may want to first try it out with actual cubes.) How many routes would there be if there were 3 (or 4) cubes?



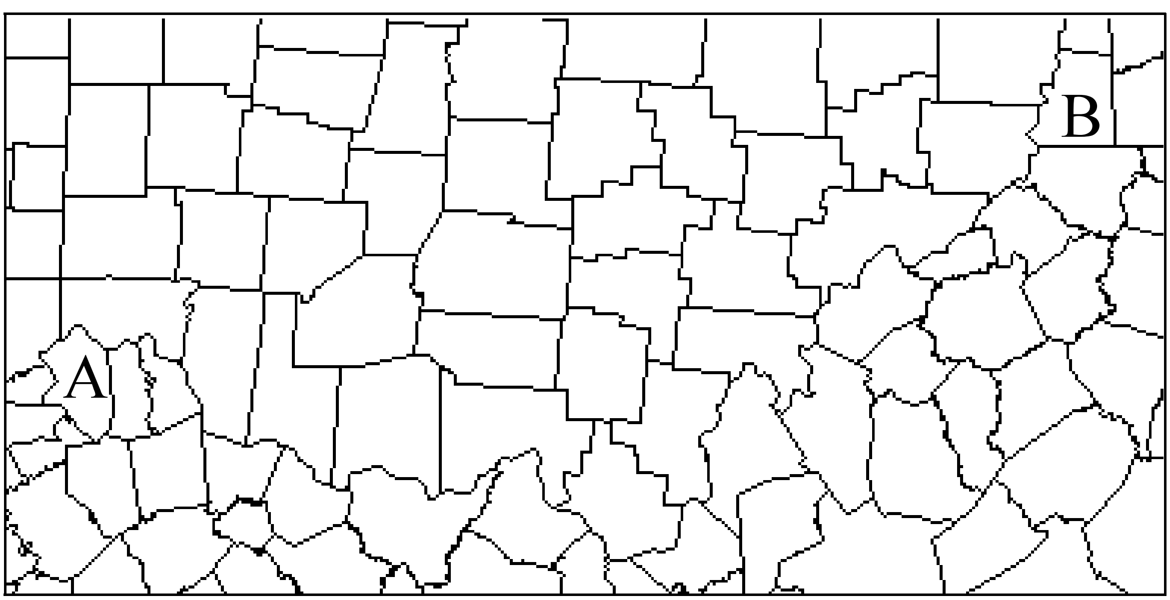


5. Find the total number of routes from Poughkeepsie to Hanover on Page 6. Assume that you always travel in an easterly direction, even if the road essentially goes north or south.

6. Find the shortest route from Poughkeepsie to Hanover on Page 6. Assume that you always travel in an easterly direction, even if the road essentially goes north or south.

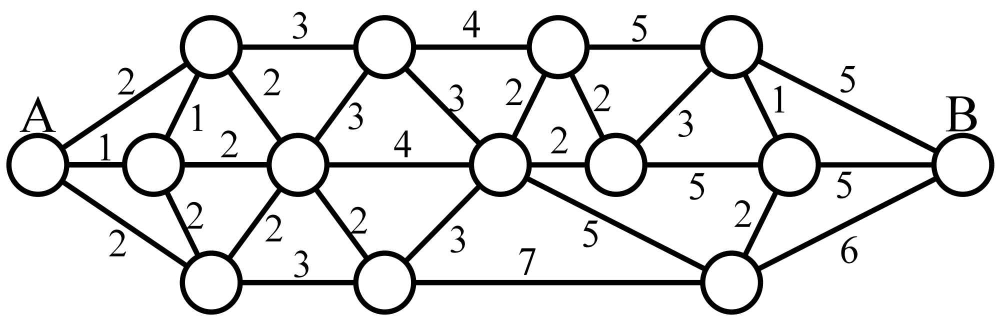
7. Make a systematic list (using E and N) of all paths from A to B in the graph of problem #2 above. Then make a systematic list (using S, E, and SE — for "southeast") of all paths from A to C in the graph of problem #4 above; extra copies of this graph appear on Page 5.

8. Below is a map of the counties of southern Ohio and surrounding states. As president of the Vital Expectations Road Trotters Inter-County Exercise Society, it is your job this year to plan a foot-race route from county A to county B. Since each county charges a $1250 fee for use of its roads, you seek to find a route that passes through as few counties as possible, including A and B. What is the cheapest route you can find? (Assume that the distance between two adjacent counties is always 1, and apply Dijkstra's Algorithm.)



9. a. How many paths are there from A to B in the diagram below? (Assume that you can only travel in an easterly direction.)

   b. Use Dijkstra's algorithm to find the shortest route from A to B in the diagram below.



10. a. Find the "shortest path from A" spanning tree for the graph below — that is, the spanning tree which provides the shortest path from A to every other vertex.

    b. Find the "shortest path from B" spanning tree for the graph below — that is, the spanning tree which provides the shortest path from B to every other vertex.

    c. Do the two spanning trees both give the shortest route from A to B? Do they both give the shortest route from A to the vertex at the bottom right corner of the graph?

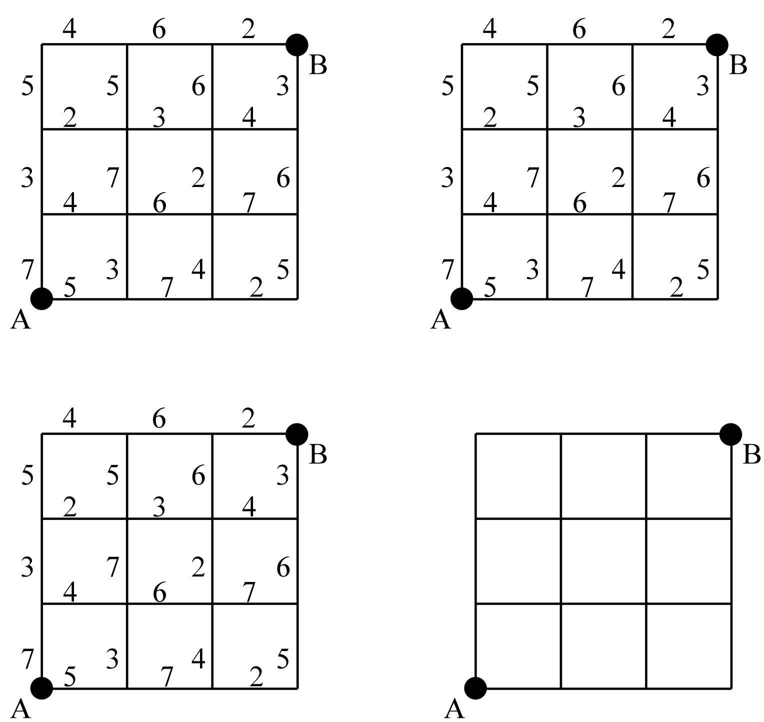
    d. Find the minimum weight spanning tree for the graph below using one of yesterday's algorithms.

    e. How many edges are in each of these spanning trees? Is this a coincidence?

    f. What is the total weight of each of the spanning trees? Is this a coincidence?

    g. What is the distance from A to B in each of the spanning trees? Is this a surprise?

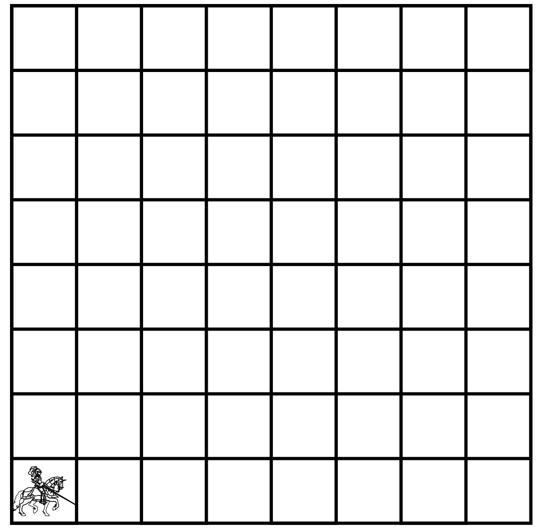
    h. Explain why the three spanning trees you found are all different.

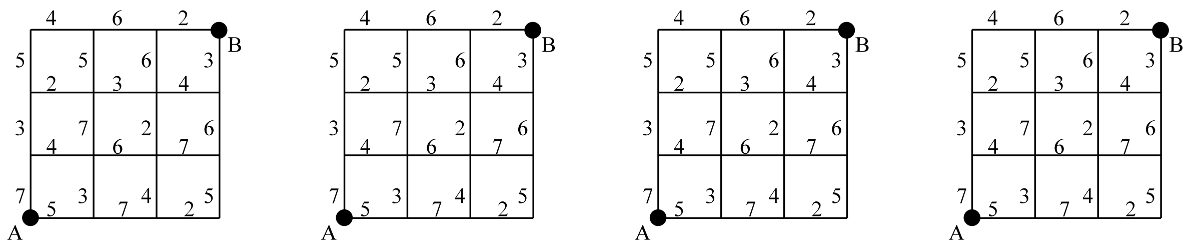


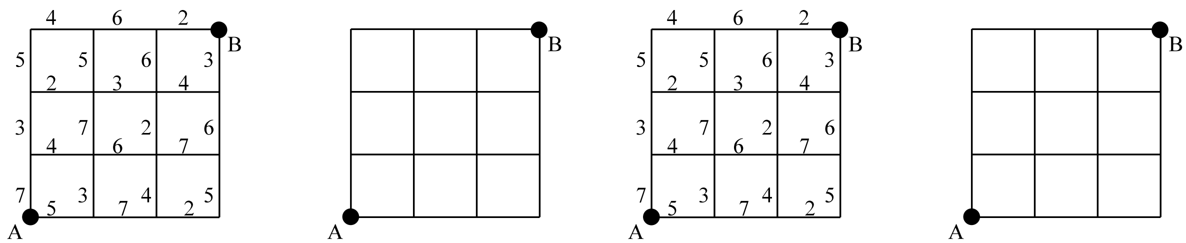
Extension Problems

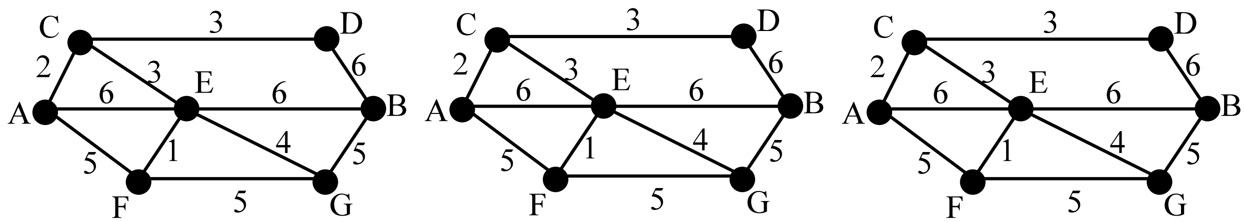
11. In exercise 2, the shortest route from A to B always proceeded up or to the right. Is it possible that, with different weights on the edges of that grid, the shortest path from A to B actually does some backtracking? If you think so, give an example.

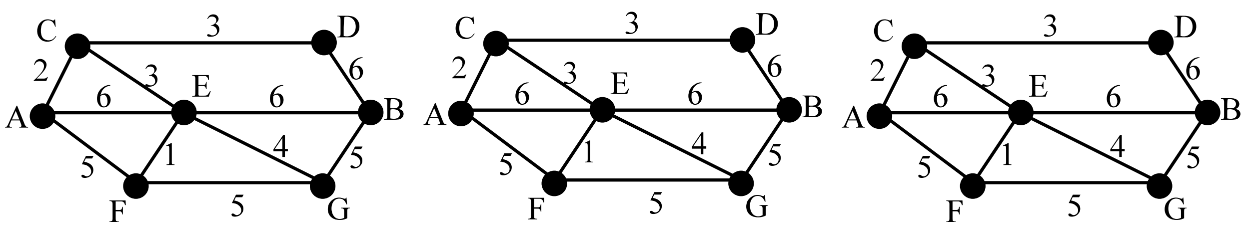
12. To the right is an 8×8 chessboard, with a knight in the lower left corner. What is the least number of moves the knight could make to get to the opposite corner? If you think of this as a "shortest path on a graph" problem, where the graph has 64 vertices representing the 64 squares of the checkerboard, then what would the edges of the graph correspond to? How many edges would this graph have?




A
B
4 6 2
5 5 6 3
2 3 4
3 7 2 6
4 6 7
7 3 4 5
5 7 2


A
B
4 6 2
5 5 6 3
2 3 4
3 7 2 6
4 6 7
7 3 4 5
5 7 2


A
B
C
D
E
F
G
3
2 3 6
6 6
5 1 4 5
5


A
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C
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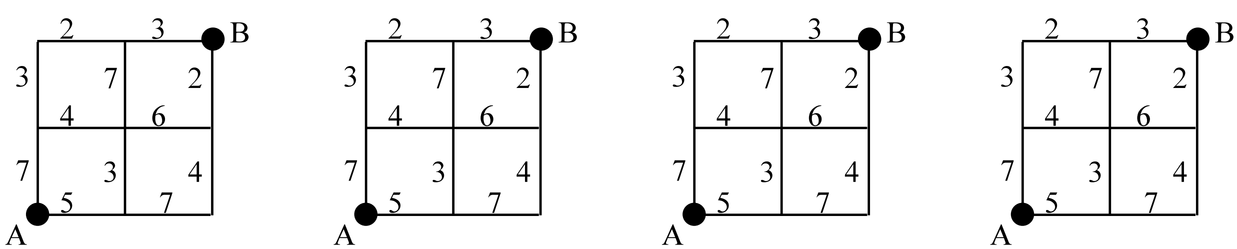
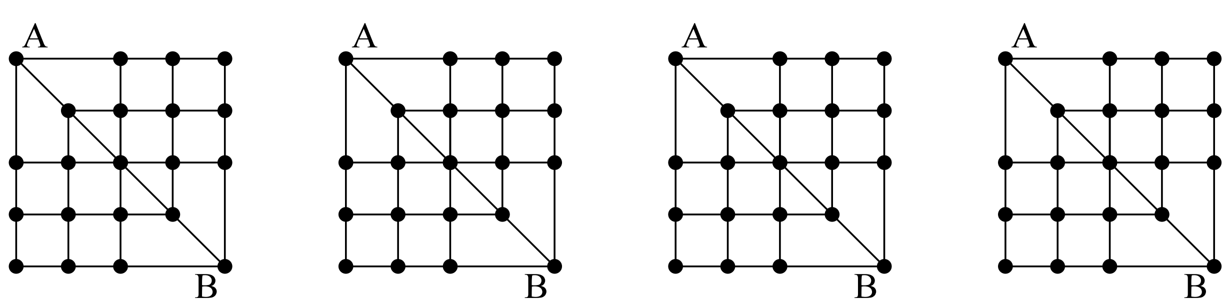
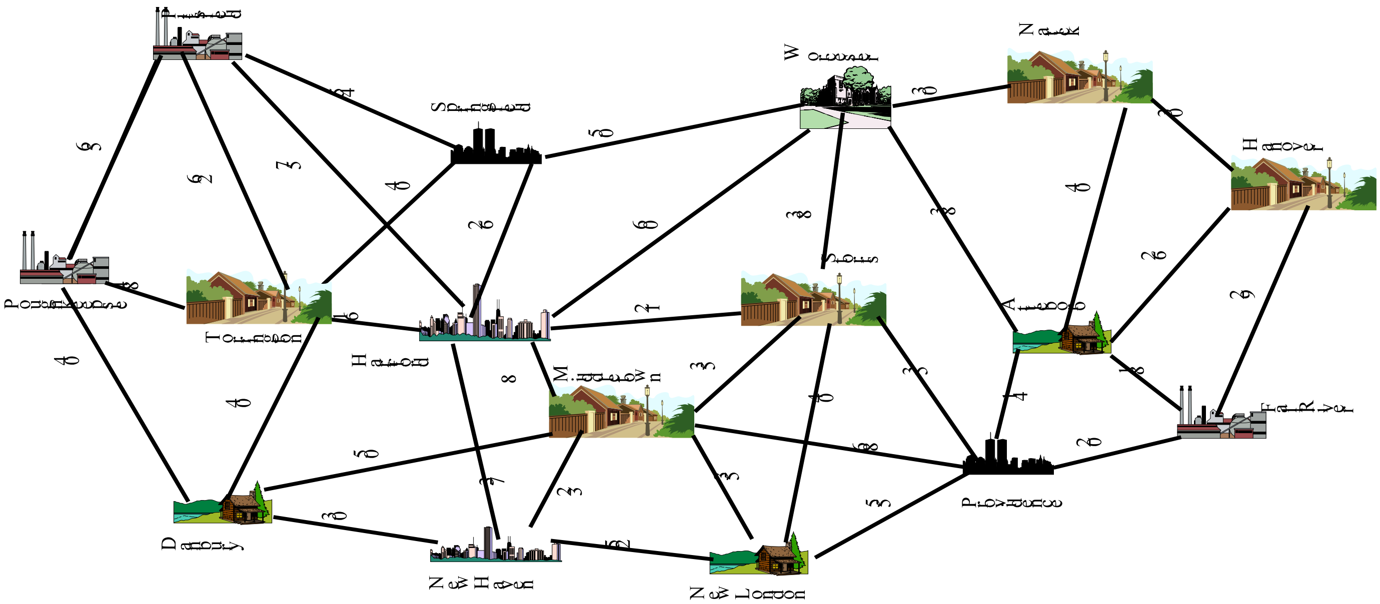
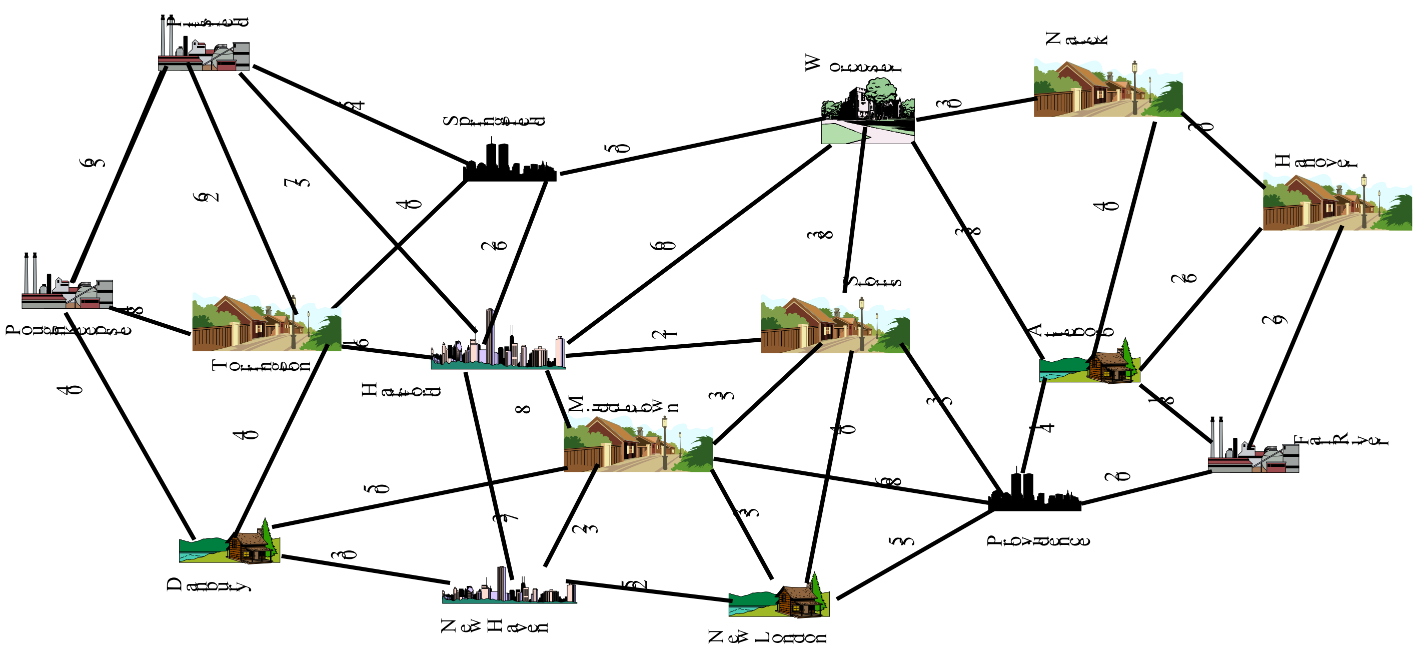
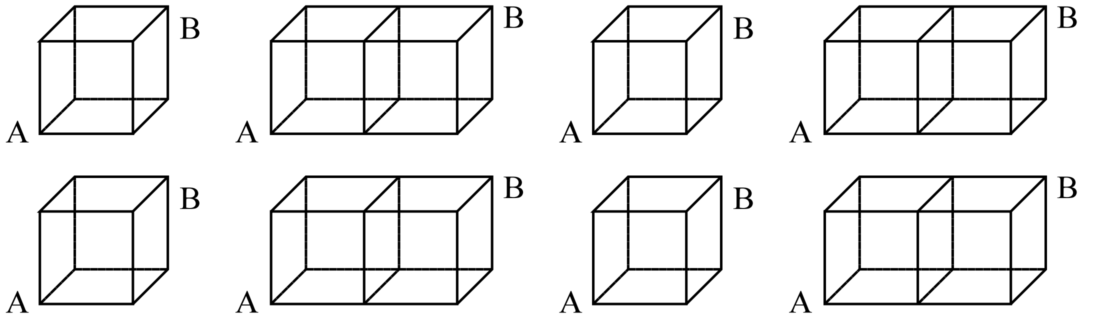
G
3
2 3 6
6 6
5 1 4 5
5

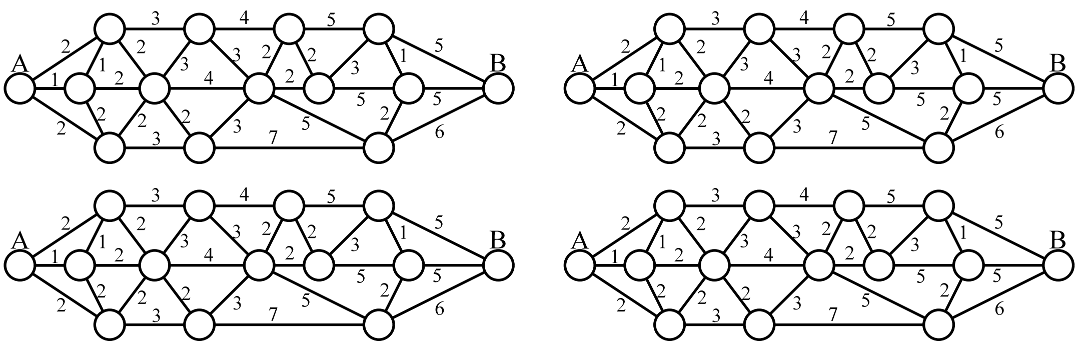

A
B
2 3
3 7 2
4 6
7 3 4
5 7

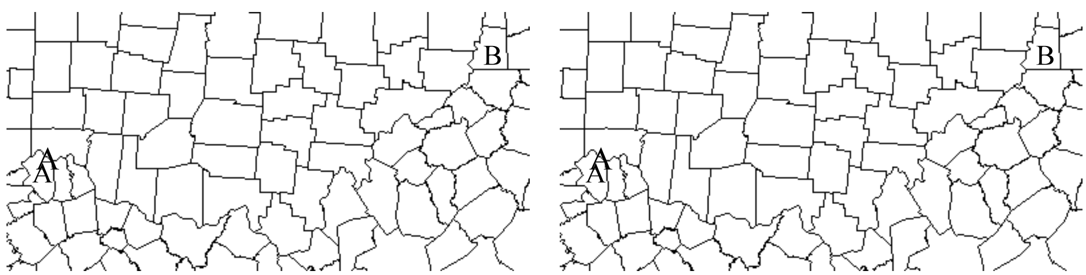

A
B

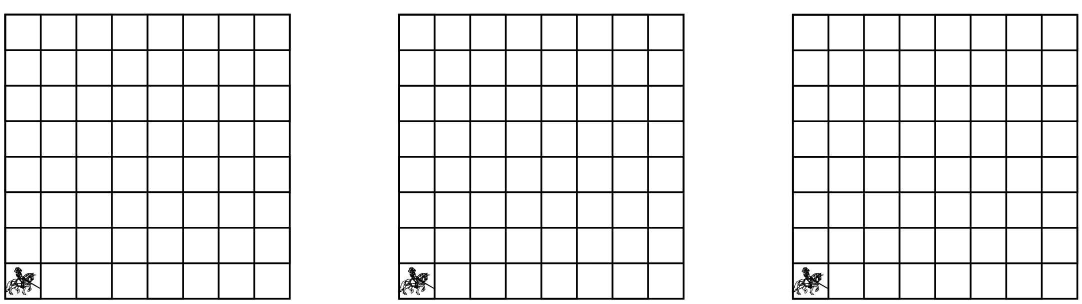





A
B


A
B


A
B



Resource Book

Workshop 5: What's the Shortest Route?

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Workshop 5: What's the Shortest Route?

Mathematical Background:

✓ **Weighted graph** — a graph in which every edge has been assigned a "weight", a number which could refer, for example, to its length or its cost.

✓ **Weight of a path** — in a weighted graph, each path has a weight which is the sum of the weights of the edges in the path.

✓ **Shortest route from A to B** — in a weighted graph, a path from A to B whose weight is a minimum.

✓ **Dijkstra's algorithm** — a method for finding the shortest route from A to B which is based on a "try-a-simpler-problem" strategy — Select the edge of minimal weight which contains the vertex A. At each stage, a tree has been generated such that the path from A to any vertex B on the tree is a path of minimum weight to B. At each stage, consider all the vertices not yet on the tree which are reachable from the tree by one edge. Find one for which the path from A to this vertex via the tree and this one edge has minimal weight, and select this edge for the tree. Continue until all vertices of the graph are on the tree. Dijkstra's algorithm finds a spanning tree so that for any vertex B in the weighted graph G, the path from A to B in the spanning tree is a path of minimal weight from A to B.

✓ **Shortest path from A spanning tree** — if you use Dijkstra's algorithm, you find the shortest path from A to every other vertex; these paths together from the "shortest path from A spanning tree".

✓ **Pascal's Triangle** — A triangular array of numbers where the entries on the border are each 1 and each other entry is the sum of the two entries above it; each entry in Pascal's triangle also represents the number of paths from the top 1 to that entry. Pascal's triangle emerges in a variety of situations where systematic counting is used.

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Workshop 5: What's the Shortest Route?

Workshop Outline

1. **Introduction**
   a. Review the similarities and differences between the three types of problems discussed in workshops 3-5: finding the best cycle (Traveling Salesperson Problem), finding the best network (Minimum Weight Spanning Tree Problem), and finding the best route (today's topic). The first deals with cycles, the second with trees, the third with routes.

2. **From Home to School**
   a. Brainstorm strategies (trial-and-error, tree diagram) for finding the shortest route from home to school on a diagram on the floor, where the distances of each portion of the route is determined by the number of linker cubes placed on that portion.
   b. Use a tree-diagram strategy to find the shortest route from home to school.
   c. Apply the above strategies to the Portland to Salt Lake City problem.
   d. Recognize that these strategies will not work if the problem has a larger number of cities.

3. **How many paths are in a grid?**
   a. Count the number of paths in a 1x1 grid and 1x2 grid. Using a "look at a simpler problem" strategy, participants count the number of paths in a 2x3 grid and enumerate them systematically.
   b. Enumerate the paths in a 2x3 grid in a tree diagram, in a (geometric) list of all paths, and in an alphabetical list of names of the paths.
   c. Count the number of paths in a 4x4 grid using a "look at a simpler problem" strategy, and recognize the appearance of a portion of Pascal's triangle.

4. **Finding the shortest path.**
   a. Use a "look at a simpler problem" strategy to determine the shortest path in the "from home to school" problem on the floor.
   b. Review this strategy on a 2x3 grid.
   c. Participants use this strategy on a 4x4 grid.
   d. Notice that this strategy (called Dijkstra's algorithm) results

in a spanning tree which provides the shortest route from A to every other vertex.

e. Observe that there is no way to find the shortest route from A to B without considering the possibility that that route goes through any other vertex, so a greedy algorithm cannot work for finding the shortest path.

f. Discuss difference between the "shortest paths from A" spanning tree and the minimum weight spanning tree.

g. Find the shortest route from Poughkeepsie to Hanover.

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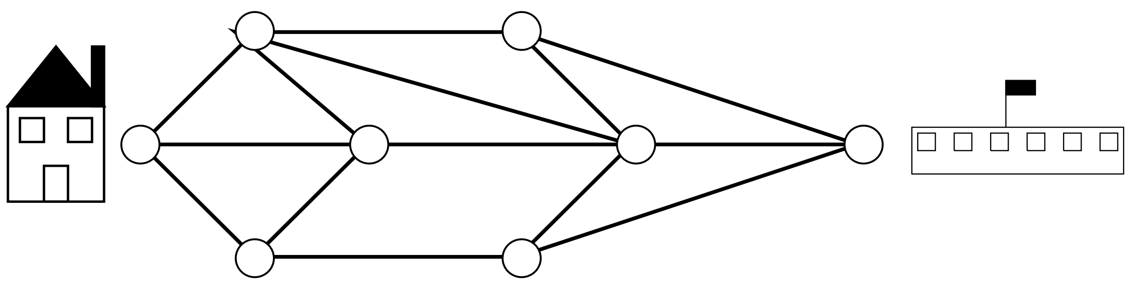
Workshop 5: What's the Shortest Route?

What's the shortest route from home to school?

Use a tree-diagram to list all the possible routes from home to school in the diagram below, and find the shortest route. (In order to find the shortest route, the distance represented by each edge must be given.)



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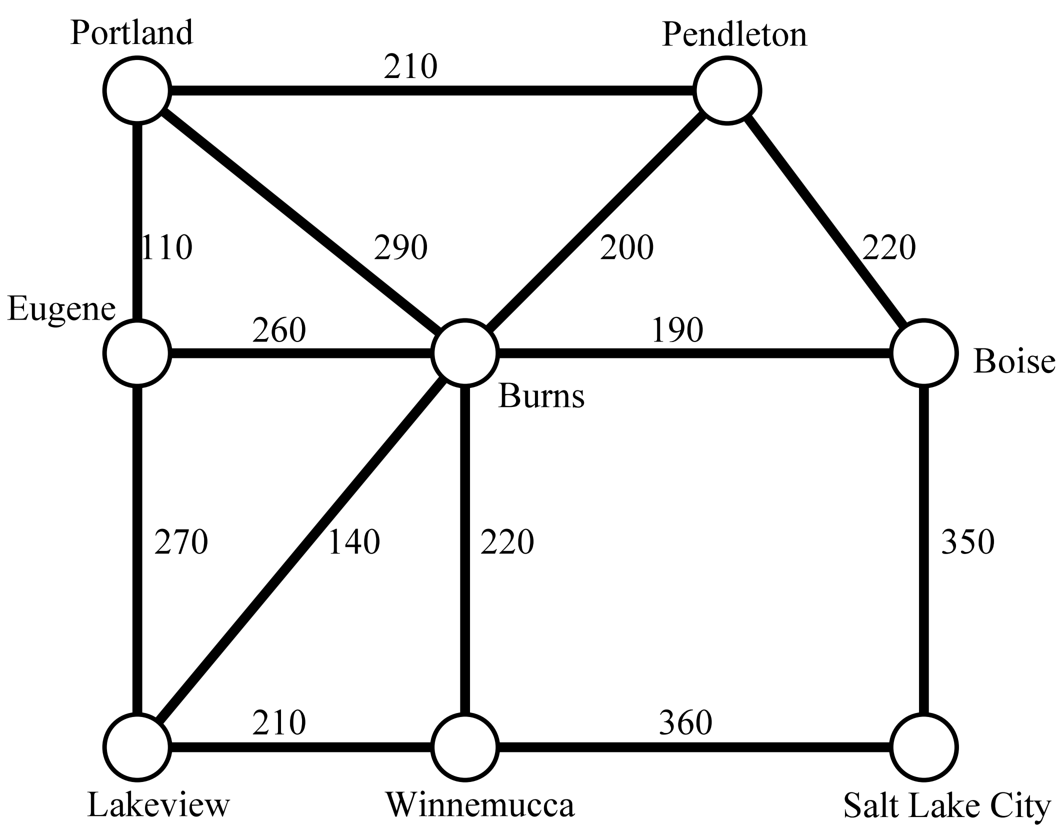
Workshop 5: What's the Shortest Route?

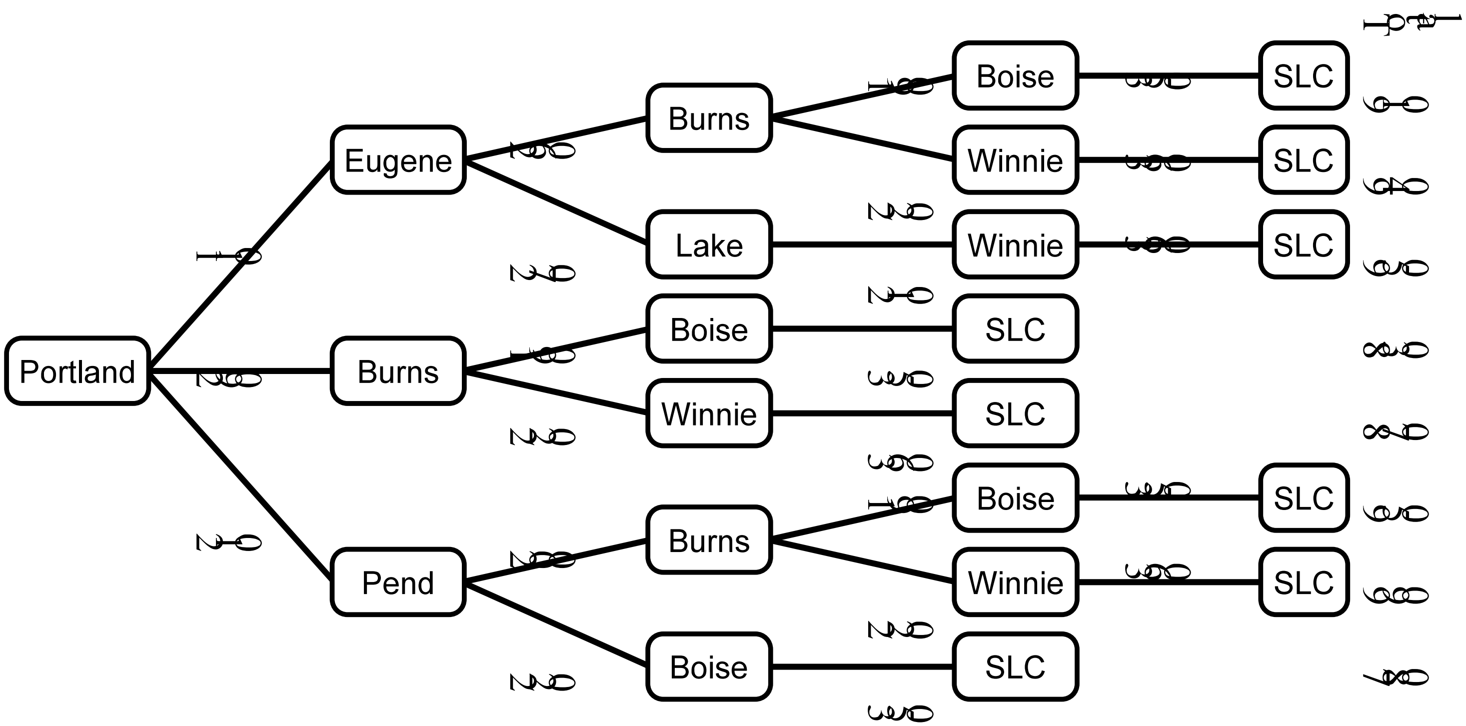
On the road to Salt Lake City

A family in Portland, Oregon is planning a drive to Salt Lake City. The weights on the edges in the figure give the mileage between various cities. What route should they take if they want to drive as few miles as possible? (From Cozzens and Porter, Module #6, COMAP.)



On the Road to Salt Lake City

Tree diagram of all possible routes from Portland to Salt Lake City



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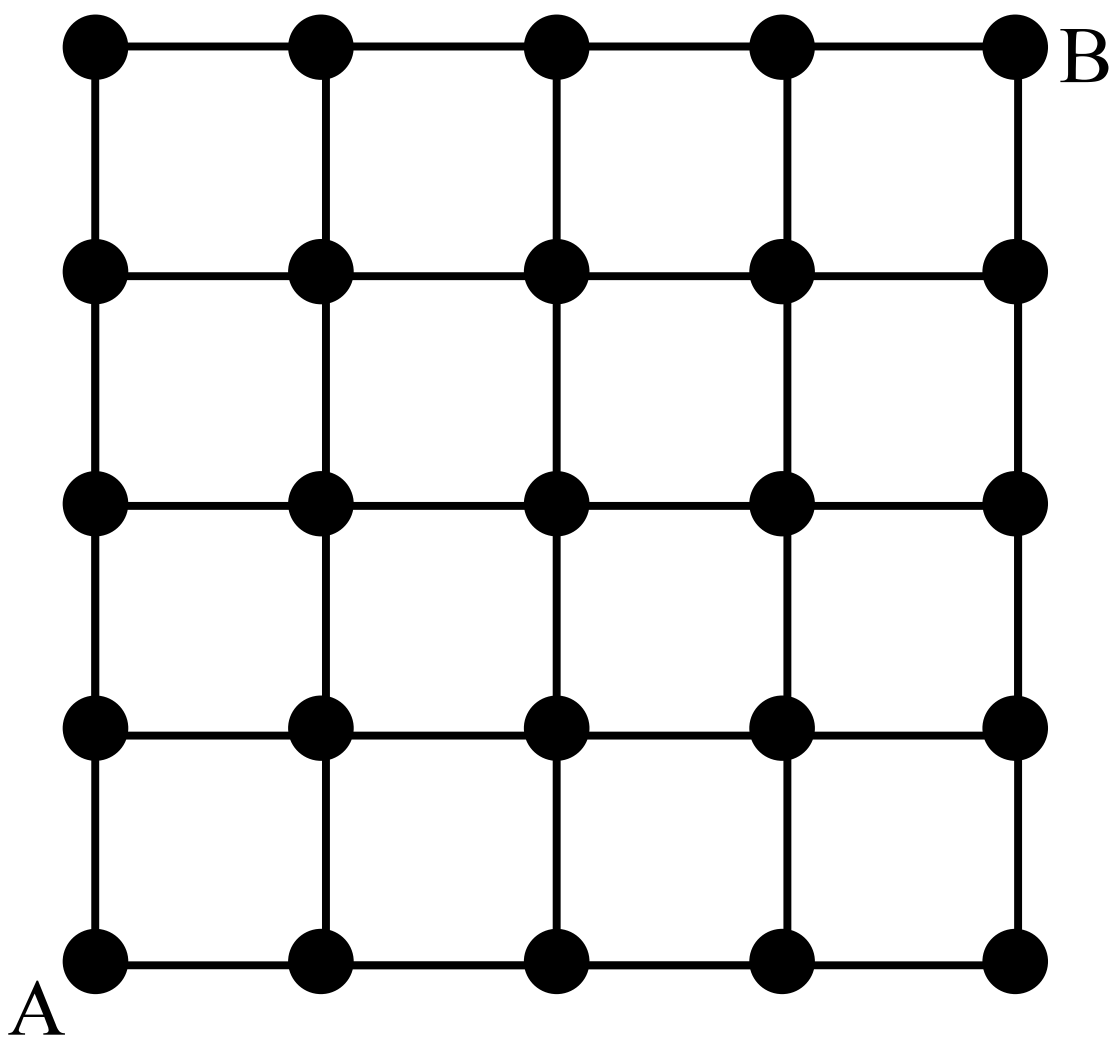
Workshop 5: What's the Shortest Route?

Number of paths on a grid

How many paths are there from A to B on the grid below? (Assume that you always go North or East.)



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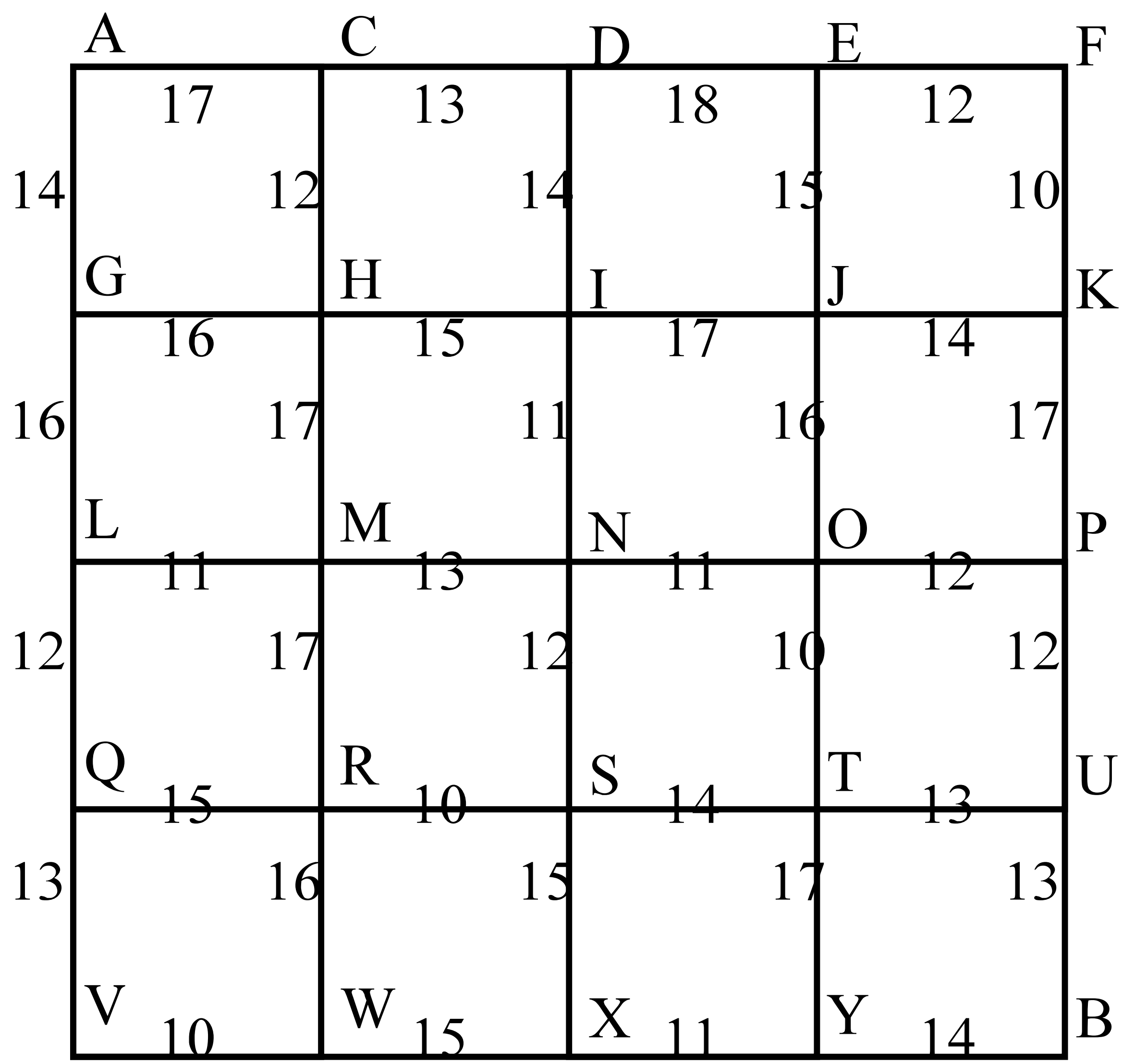
Workshop 5: What's the Shortest Route?

Traveling on a Grid

What is the shortest way of getting from A to B in the grid below, where each edge is labeled with the number of miles between the vertices.



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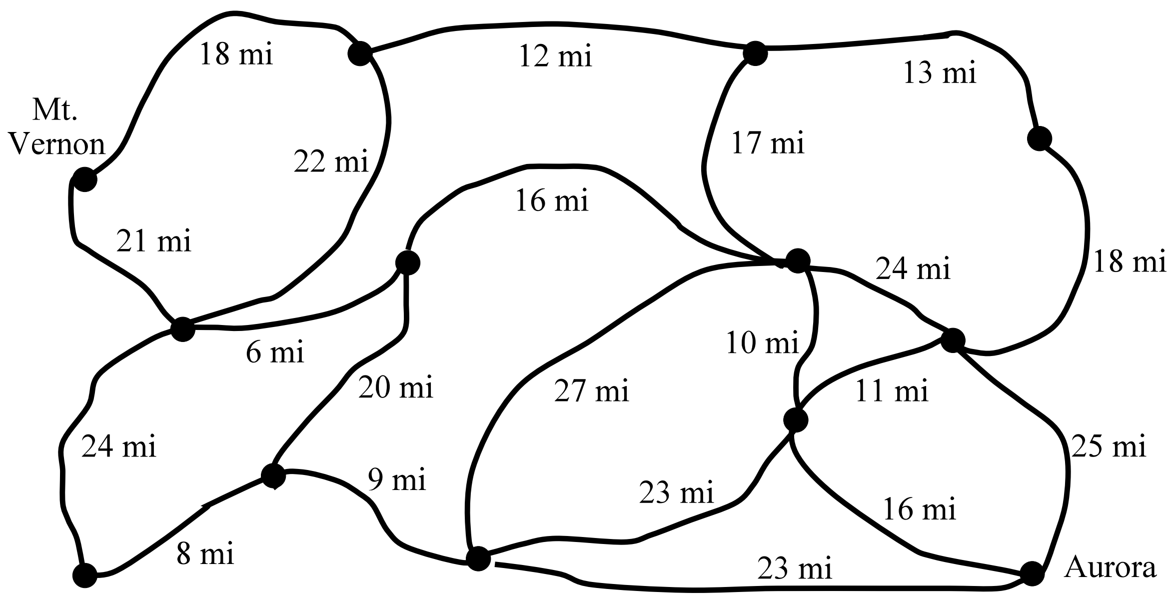
Workshop 5: What's the Shortest Route?

Find the shortest route from Mt. Vernon to Aurora.



Developed by Dave Fogle Leadership Program 1993

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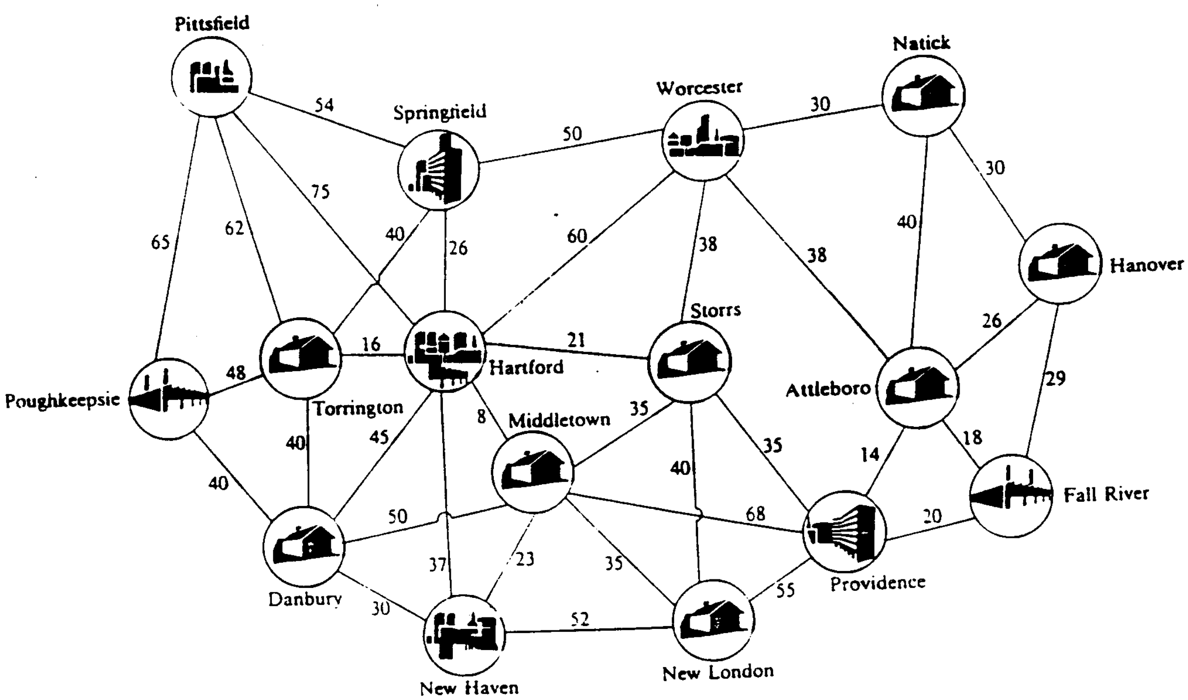
On the Road to Hanover

Peter Sampson lives in Hanover (MA) and Roberta Simons lives in Poughkeepsie (NY), and they spend a lot of time traveling back-and-forth between those cities. Use the map below to help them find the shortest route from Poughkeepsie to Hanover. (From Cozzens and Porter, Module #6, COMAP.)



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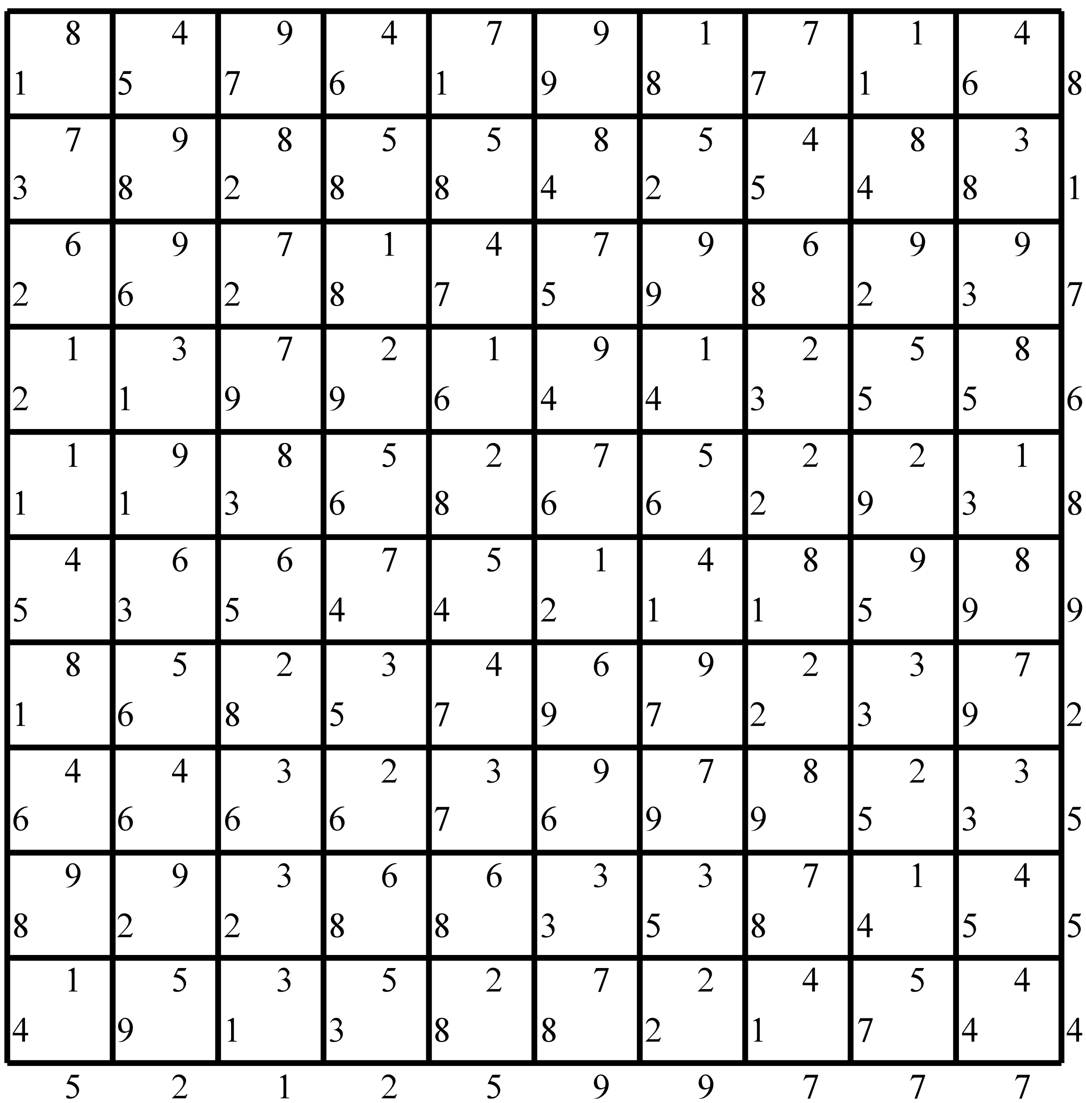
Find the Shortest Path in a 10x10 Grid

Find the shortest path from A to B in the 10x10 grid below. (This problem appears in an article on "dynamic programming" by Sloyer et al. in Mathematics Teacher, February 1985.)



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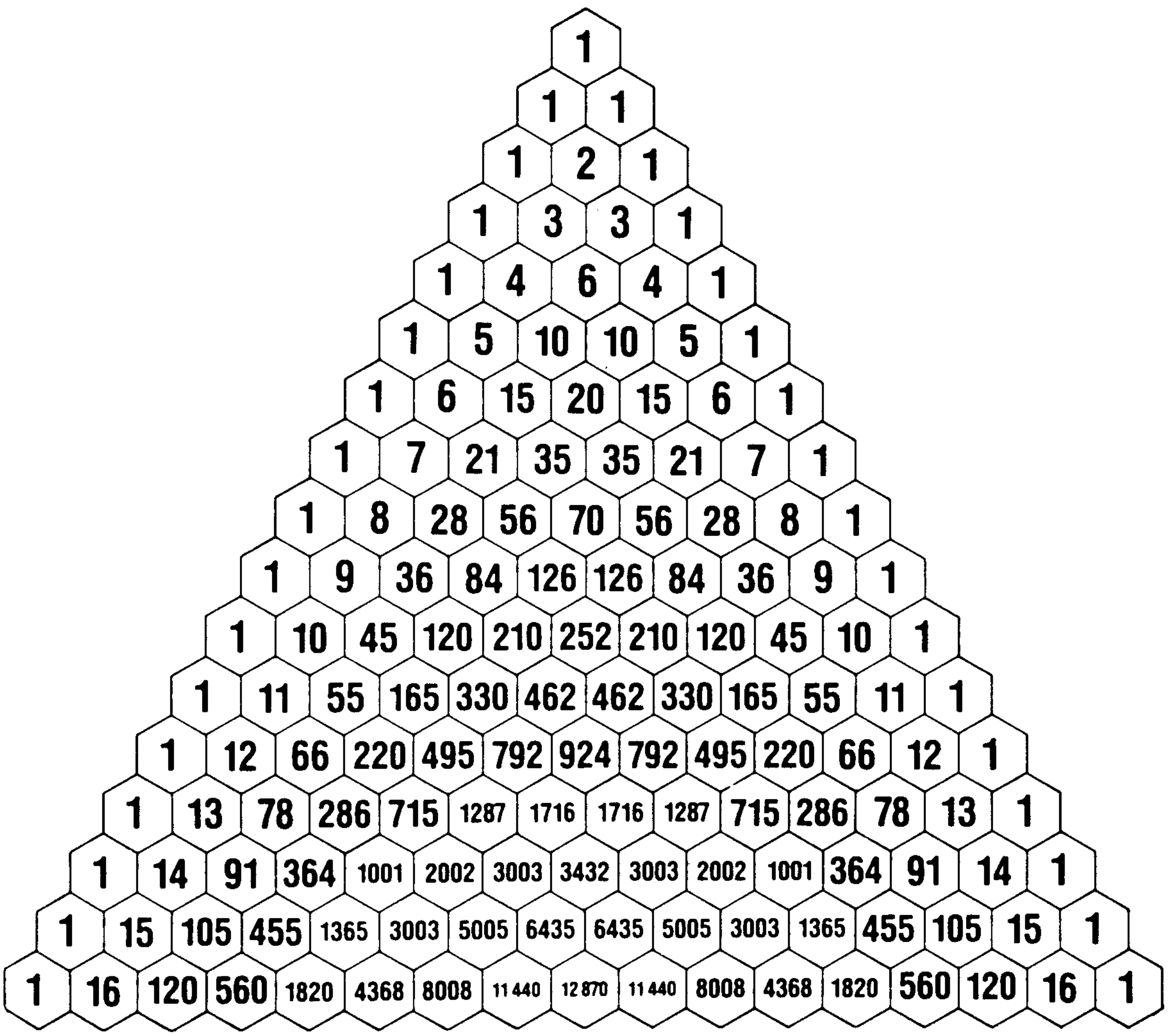
Workshop 5: What's the Shortest Route?

Pascal's Triangle



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