Master Document

Workshop 4 — Making the Right Connections

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

revised August 1, 1999
(additional revisions – June 25, 2007 and June 24, 2011)

Workshop 4 — Making the right connections

Materials and Pre-Workshop Preparations — **Time Allotted**

Activity #1 — Muddy City Problem #1 . . . . **15 minutes**
- One "Communicator" for each participant

Activity #2 – Problem Solving Strategies . . . . **15 minutes**
- One "Tower of Hanoi" setup for each two participants

Activity #3 — Muddy City Problem #2 . . . . **15 minutes**

Activity #4 — Four Different Algorithms . . . . **50 minutes**
- One "Communicator" for each participant
- Thirty toothpicks for each table

Activity #5 — Finding Shortest Connections using Soap Films . . . . **30 minutes**
- Each participant should have a ruler to measure distances on a page.
- Soapy water with glycerine and several templates (see supplementary notes) to be dipped into the soapy solution.

Activity #6 — Connecting County Seats . . . . **15 minutes**
- Each participant should have a ruler to measure distances on a page.

. . . . TOTAL WORKSHOP TIME: **125* minutes**

Activity #6 should be regarded as optional

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Word Wall – Tree, Spanning Tree, Minimum Weight Spanning Tree, Look for a Simpler Problem, Make a Table, Find a Pattern, Take the Cheapest!, Throw Away the Worst!, Stay Connected!, "Break Those Cycles!, Steiner Point

Activity #1 — Muddy City Problem #1
(Allocated time = 15 minutes)

A. Ask participants to work in pairs on Hand-out #1, after first reviewing it on TSP #1. The town has six major buildings and twelve roads connecting them. How many need to be paved so that it is possible to get from any one of the six buildings to any other one exclusively on paved roads?

It will take each pair no more than five minutes to decide that this can be done with five roads. While they are working on this, draw about five diagrams (vertices, no edges) of the six locations on the blackboard; ask for volunteers to come up and draw their solutions, and ask some of them to explain why they chose the roads they did. [Having the vertices on the board in advance will save lots of time; having them use the board rather than the overhead projector will enable all the solutions to be viewed at one time. And having the teachers share their solutions will serve as a model for them asking their students to share their solutions.] Summarize by comparing the various solutions — there will be "stars" with a central hub at the firehouse (or at the bank for those who highly value that convenience), there will be "linear" solutions; elicit the observation that none of the solutions have cycles, as well as justification for why this is so!

Some solutions will be based on participants' attempts to optimize cost on the basis of perceived length of the roads — that will be the focus of the next activity.

Ask the participants what all of the solutions have in common, besides all having six vertices and five edges. One property that they share is that each is connected, you can get from any vertex to any other vertex; point to "connected" on the word wall – this was introduced in the second workshop. A second property that they share is that none contains a cycle; point to "cycle" on the word wall – this was introduced in the first workshop. Tell the participants that a graph that is connected and has no cycles is called a "tree"and explain how it is that each real-life tree also has these two properties. Link this notion of tree to the previously discussed notion of "tree-diagram" (from the third workshop).

B. After observing that no one found a solution with only four roads, ask the participants why the problem cannot be solved with four roads.

Ask them to imagine that "You bring in your proposed construction plan and the City

Council tells you that you have to justify why five roads are needed; how would you do that?" Suggest that they use the "look at a simpler problem" strategy (which may have been discussed at a problem-solving session earlier in the program) by considering other muddy cities with fewer major buildings. After someone suggests looking at the situation with two sites, where just one edge is needed, ask them to make a chart and then look for a pattern. Many groups will develop a pattern which will lead them to the conclusion (perhaps not stated abstractly) that if there are n buildings, then n-1 roads are needed.

Ask the participants for what grade level they think that the muddy city problem is appropriate; after they have an opportunity to respond, let them know that participants in the program have used it even with second and sometimes first graders. Ask the participants at what grade level can students conclude that the number of edges needed for a network is one less than the number of sites in the network; after they have an opportunity to respond, let them know that participants in the program have done this with fourth graders and sometimes with third graders.

The muddy city problem is nice for a number of reasons – one is that it is an easily understood problem where you can use "try a simpler problem," "make a chart," and "look for a pattern." Another is that it works well with all sorts of group – with teachers, with kids, with principals and administrators, and with parents. Try it sometime at back-to-school night.

Activity #2 – Problem Solving Strategies

A. Review the problem solving strategies used in the preceding activity using TSP #2.

B. Tower of Hanoi problem. Provide each participant with a Towers of Hanoi set-up and show TSP #3 that describes the Towers of Hanoi. Ask them to use these problem strategies to determine the solution to the Towers of Hanoi problem.

Activity #3 – Muddy City Problem #2
(Allocated time = 15 minutes)

A. Ask participants to work in different pairs on the second muddy city problem on Hand-out #2 (= TSP #4). Explain that in this version of the problem, each road has a numerical value, the cost of that road, and the task is to find five roads which not only do the job, but do it at minimal cost. After completing the problem, participants should compare their solutions with others at their table.

B. After agreeing on the optimal solution (without discussion of how you would

prove that it is optimal), ask participants to explain how they obtained their solutions.

The basic response that you will get is Kruskal's algorithm — that at each step you "take the cheapest edge" if that edge is needed. Clarify with the participants that "if that edge is needed" means "if no cycle is formed". This method which we will call the "Take the Cheapest!" method is not the only method for finding an optimal solution; indicate that in the next activity we will explore a number of other methods. You should elicit that this method is similar to the "cheapest link" algorithm for the TSP and that this is a "greedy" algorithm. [Note that "cheapest link" has been de-emphasized in the updated workshop #3, so you will want to make this connection here only if you discussed "cheapest link" previously.]

Explain that in reviewing the result of an algorithm it is helpful to have a record of what happened at each step. Review TSP #5 which records how the "Take the Cheapest!" method resulted in the solution that was obtained; the purpose of this TSP is to encourage participants to apply systematically their algorithm in the next activity.

Activity #4 — Four different algorithms
(Allocated time = 50 minutes, including 40 minutes for parts A and B.)

A. Ask participants to solve the Desert Tours problem on Handout #3 (= TSP #6) using the two algorithms specified there, "Take the Cheapest!" and "Throw away the worst!," where they write out all the steps using a systematic approach like that in TSP #5. Tell them that they can either use their "Communicators" or toothpicks to assist in solving these problems.

B. Review the solutions of these two problems. Use toothpicks on the edges so that participants can see how they can be used in a classroom setting as an inexpensive alternative to the "Communicators."

C. Hand out Handout #4 (= TSP #7) and ask half the tables to use the "Break those cycles!" algorithm on the Desert Tours problem and the other half to use the "Stay Connected!" algorithm.

(They don't need to write out all the steps, although it might be useful for the instructor to do so in reviewing the problem; note that unlike the other two algorithms, when you use either of these algorithms, there is no unique order for the steps.)

D. Note (see TSP #8) the striking result that all four algorithms give the same result. Note that the "Stay Connected!" algorithm is called Prim's algorithm, and the "Take the Cheapest!" algorithm is called Kruskal's Algorithm; these problems and methods were first studied in the 1950s.

That all algorithms give the same result can become more apparent if, for example, in both "Break the Cycles!" and "Throw Away the Worst!" you use red to mark up the discarded edges and blue for the retained edges, and then superimpose the transparencies to demonstrate that the two algorithms give exactly the same result

E. Introduce terminology — tree, spanning tree (which reaches out to all vertices in the graph), weighted graph, minimum weight spanning tree, and conclude that they now know how to find a minimum weight spanning tree in any weighted graph! (Use TSP #9, the material on which is in the Resource Book — so the Resource Book may be distributed here.)

Note that students can be assigned problems like this as a review of addition — of counting numbers, of two- and three-digit numbers, of fractions, of decimals — which is much more fun than simply adding a list of numbers.

E. Comparison to Traveling Salesperson Problem. (See TSP #10) Point out the differences between the"Nearest Neighbor" algorithm and the "Stay Connected!" algorithm; note particularly that in this problem, you are not looking to find a path or a cycle (as on previous days), but rather a network. (If the "cheapest link" algorithm was discussed in workshop #3, then point out the differences between the "Cheapest Link" algorithm for the TSP and the "Take the Cheapest!" algorithm here.)

[Time for a 5-10 minute break]

Activity #5 — Finding Shortest Connections using Soap Films
(Allocated time = 30 minutes)

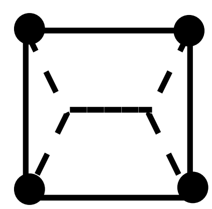
A. Ask participants to draw three dots on a piece of paper (demonstrate on a transparency) and find the shortest way of connecting the three dots. After everyone agrees that you measure the three edges and take the shortest two, ask them to draw a dot somewhere in the middle of the triangle, and measure (using their rulers) the three distances from the dot to the three vertices. Have each person give you two numbers which you put in a two column chart on a blank transparency, one for the sum of the lengths of the two initial segments and the other for the sum of the lengths of the three

new segments — surprise! Connecting the three vertices using the point in the middle gives (usually) a smaller total than the two shortest edges. Introduce the idea of a Steiner point as one which gives the smallest total network, called a Steiner tree, and relate Steiner trees to minimum weight spanning trees.

Ask the participants to make the measurements in millimeters, to the closest half of a millimeter. For a variety of reasons, many (perhaps even half) will conclude that the two measurements are approximately the same, and some will even conclude that the second measurement is larger. Don't dwell on this point. But after you complete and explain part B below, you can say that if the measurements are made carefully, then the second measurement will in fact be smaller, but not by much.

B. How can we find the Steiner point? Introduce the soap film demonstration with the story of Bell Telephone Company and Delta Airlines; Delta was told by Bell that it would charge Delta for its phone network based on the shortest distances used to link its offices (then in NYC, Chicago, and Atlanta); so Delta opened a branch office in a small town in Ohio and asked that the charges be calculated on that basis. Why?

Review the two pages on this topic in the Resource Book (see attached pages IN #8 and IN #9) as background for this demonstration. Do the demonstration that leads to the Steiner point for the Chicago-NYC-Atlanta configuration. Because participants have difficulty understanding how lines show up on the screen, pass around several plates which have been dipped in the solution so that they can see the soap film and come to believe that that is responsible for the lines on the screen. Then provide an explanation of why the soap films actually give the optimal solution — that nature tries to minimize everything; show that at the Steiner point here (and in the subsequent demonstrations), all angles are 120 degrees — which is why bees make hexagonal honeycombs. (Note that many kids believe that bubbles are round because the wand used has a round hole; suggest that they use a wand with a triangular or square hole with their students, and provide a sample wand.)



C. Ask the participants what is the best way of connecting four sites at the corners of a square; many will propose, based on the previous activity, that the center of the square is the Steiner point. Do the demo and get a surprise! The shortest total comes about from using two points in the square, as in the configuration at the right.

You might ask for a volunteer to do this. That is, you provide the three-sides configuration and ask the participant to blow the soap film off the two internal vertices. Then ask the volunteer to blow the two new vertices together. Of course the volunteer will blow too hard and the two vertices will separate again. Ask the volunteer to blow less energetically.

Continue a few more times until the volunteer (and everyone else) realizes that it can't be done. Then point out the Steiner point has to connect to three vertices and have 120 degree angles, but the center of the square would connect to four vertices and create 90 degree angles. So the task was impossible. Apologize to the volunteer for pulling his/her leg ... or not!

D. Do the demonstration with a rectangle of dimensions Y by (Y × root 3); make sure that the three sides at which the soap film resides include the two longer edges of the rectangle. When you blow the soap film off the two vertices, the configuration is unstable, and so you get a "Surprise!" which is guaranteed to extract "ooh" from the audience. Repeat this one so that they can see it again and understand what happened.

E. Review TSP #11. Point out that in solving any minimum spanning tree problem, you can also consider the associated Steiner tree problem. In any situation, there are two problems, "What is the minimum network using only the given vertices?" and "What is the minimum network if you are allowed to add vertices?" Sometimes the latter is impractical, but sometimes, as in the case of Delta airlines, it is very practical.

Activity #5 — Connecting New Jersey County Seats OR
Connecting Arizona County Seats
(Allocated time = 15 minutes)

Note: There are two County Seats Problems provided in the instructor's packet; one for New Jersey and one for Arizona. Only one of these two problems is to be completed during the workshop.

A. Distribute either New Jersey's County Seat Problem (Hand-out #7 = TSP #12) or Arizona's County Seat Problem (Hand-out #8 = TSP #13) . Ask participants to create a minimum weight spanning tree connecting the 21 county seats in New Jersey or the 15 county seats in Arizona.

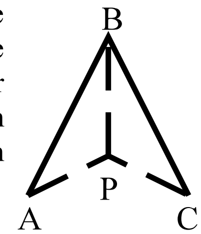
Participants should notice quickly that, unlike with earlier problems, distances are not provided. They may thus decide to calculate all distances, which will take a long time. Hopefully, however, they will realize that the minimum spanning tree will automatically include some distances, and exclude others, and that by a mixture of estimating and measuring, they will arrive at a minimum spanning tree in a reasonably short time. Review this problem using TSP #12 or TSP #13, noting that the total number of measurements you really needed to do was no more than a dozen, and that you only needed to measure in order to decide which of two edges to add to the tree.

1. Using Soap Films to Demonstrate Minimum Networks

Mathematical Background

The minimum spanning tree for three cities consists of the two shortest edges connecting the cities. However, if a point is selected in the interior of the triangle formed by the cities, then the three distances from that point to the cities add up to less than the two shortest edges. In the diagram at the right, the sum of the lengths of AP, BP, and CP is less than the sum of the lengths of AB and BC.

The point P is called a Steiner point for the triangle if it results in the smallest sum for the three line segments. The Steiner tree for the three cities consists of the line segments from the Steiner point to the three cities; the three line segments in the Steiner tree make three angles of 120°. The three line segments in the Steiner tree form a minimum network connecting the three cities. You can demonstrate Steiner points in your classroom using soap films.



Steiner point demonstration

In the 1950s, the telephone company charged multi-location customers for service based on the shortest way of connecting the customer's locations, that is, they thought, based on a minimum weight spanning tree. Delta Airlines, with offices in Chicago, New York, and Atlanta opened up a new office on the border of Ohio and Kentucky, at the Steiner point of the triangle formed by its existing three offices. This resulted in substantially smaller fees for telephone service.

To demonstrate the location of Delta's fourth office using soap films, for example, you need to construct a model (see next page for instructions) for a triangle whose vertices correspond to Chicago, New York, and Atlanta. Before beginning the construction, make a transparency map of Eastern United States, and locate the three cities on the map. Drill the holes at the locations of Chicago, New York, and Atlanta on the transparency. Attach the transparency map to the model by passing the bolt through the transparency before the plexiglass.

Materials for demonstration: Overhead projector, dish pan, detergent (Joy is recommended by some people), glycerine (to slow the movement of the soap film), straws, towels

Prepare the solution by filling the dish pan with about 1" of water at room temperature, adding 1/8 cup of detergent, and 1 or 2 caps full of glycerine; stir the solution, and then let it settle down. (Because different detergents and waters behave differently, try the following steps before the actual demonstration to make sure that the soap solution works.) Dip the model in the water so that soap films are formed between the bolts representing Atlanta and Chicago and between the bolts representing Chicago and New York; place the model on the overhead projector.

Use a straw to blow the soap films gently off the Chicago bolt, so that the two soap films become one, and then seek a minimum total length. When that happens, you see the Steiner point for the network, and can demonstrate that all angles at that point are 120° by placing an opaque angle of 120° over each angle at the Steiner point. This demonstration works because the soap film minimizes its total length. Other demonstrations involve models where four cities form a square, or where four cities form a rectangle of dimensions 2" x 3.5" (actually, 2 times the square root of 3).

2. Constructing Models for Soap Film Demonstrations

These transparent models are designed to demonstrate, using an overhead projector, where Steiner points are for various configurations of cities, and how nature finds those Steiner points. The models typically involve 3-6 cities arranged in triangles, quadrilaterals, pentagons, and hexagons of various shapes. They can be constructed in a short time and will last for years. If you are hesitant about making them yourself, you can often find a willing student.

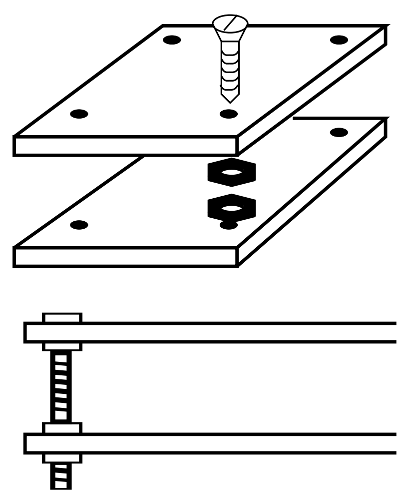
Materials

Plexiglass (1/8 inch) - 5" x 6" (two pieces of approximately this size for each model)
Plexiglass knife
Brass bolts (1") - 3-5 for each model (try to make them very narrow)
Brass nuts - 9-15 for each model (three for each bolt)
Ruler
Screwdriver
Drill (with bit diameter - bolt diameter)
Sandpaper (220 grit)

If you are making a number of models, you may have to start with a larger piece of plexiglass and cut it into smaller pieces as described in Step 1; if your plexiglass is already in 5" x 6" rectangles, then you begin with Step 2.

STEP 1: Cut the plexiglass into 5" x 6" rectangles. Plexiglass can be cut by scoring the surface repeatedly (three or more times) with a plexiglass knife. The sheet should be placed on a flat table and strapped along the scored line. Sand the edges lightly.

STEP 2: First mark the points you want to drill on one piece of plexiglass. (You will probably start with a triangle.) Place another piece of plexiglass below the marked piece, and drill through both pieces at the same time, using a drill bit whose diameter is the same as that of the bolts. Place a bolt through the first hole to hold the pieces from shifting. Continue to drill until all holes are drilled.



STEP 3: Separate the plexiglass pieces. Insert bolts through the holes and then secure them with a nut. Screw another nut on each bolt. Measure to be sure that the second nut is ½ inch from the base. (See diagrams at right.)

STEP 4: Place the second piece of plexiglass on the first. It will rest on the nuts and should be ½ inch above the other piece. Secure this piece in place with the third bolt.

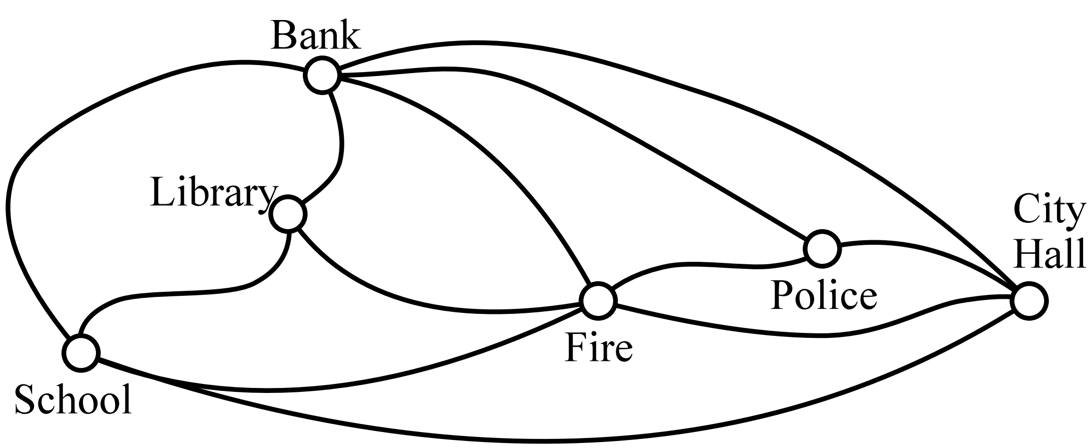
Hand-out #1: The Muddy City Problem

Muddy City has six major building and twelve roads connecting these buildings.

Unfortunately, when it rains, all of the roads in Muddy City turn to mud — for none of the roads in Muddy City is paved!

The Muddy City Council decided to pave all of the roads, but the voters decided that that would be too expensive. So the Council decided that it would pave just enough roads so that it would be possible to get from any one of the six major buildings in Muddy City to any other one by traveling exclusively on paved roads.

How many roads does the Council have to pave?



Problem by Joseph G. Rosenstein, based on story by Mike Fellows.

Three Problem-Solving Strategies

Look at a simpler problem

For example, if you were asked to find the number of regions that would be created if 7 lines were drawn in the plane, ask yourself how many regions would be created if 1 line was drawn, if 2 lines were drawn, if 3 lines were drawn, etc.

Make a table

With the example on Handout #1, the table has two columns – number of vertices and number of edges needed. With the above example, the table has two columns – number of lines and number of regions.

Find a Pattern

You can often use a table to find a pattern. We did this with Euler's Theorem, recognizing that there were Euler circuits only if the graph had no odd vertices. In the example on Handout #1, we recognized from the table that the minimum number of edges was one less than the number of vertices.

(Caution: The pattern you found may not be the right pattern. You need to prove that the pattern you found is actually correct.)

Towers of Hanoi

There are three posts (towers).

On one of the posts there are five discs of different sizes,

ranging from the largest disc, which is on the bottom,

to the smallest disc, which is on the top.

Move the five discs to

another post. Here are the rules:

1. You can move only one disc at a time.

2. You may not place a disc on top of a smaller disc.

What is the minimum number of moves that are needed?

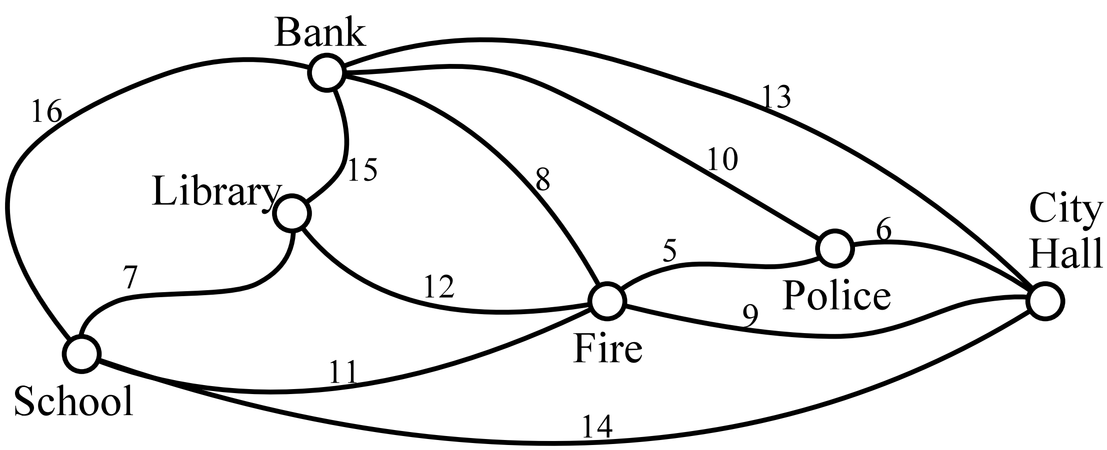
Hand-out #2: More on The Muddy City Problem

If the Muddy City Council can only pave five roads, which five roads should it pave?

The Council plans to make its decision only on the basis of cost — that is, it will pave the five roads whose total cost is the least.

The Muddy City Council obtains estimates for paving each of the twelve roads; the cost depends not only on the length of the roads, but also on other factors, such as hills, curves, and drainage. (The estimates are indicated on the map; all are multiples of $100,000.)

Which five roads connecting the major buildings would result in the lowest total cost?



Problem by Joseph G. Rosenstein, based on story by Mike Fellows.

Steps taken in obtaining the solution to the Muddy City Problem

1. Look at F-P road labeled 5 — accept it.
2. Look at C-P road labeled 6 — accept it.
3. Look at L-S road labeled 7 — accept it.
4. Look at F-B road labeled 8 — accept it.
5. Look at C-F road labeled 9 — reject it;
   it creates a cycle.
6. Look at P-B road labeled 10 — reject it;
   it creates a cycle.
7. Look at F-S road labeled 11 — accept it.

Algorithm ends; five roads have been identified.

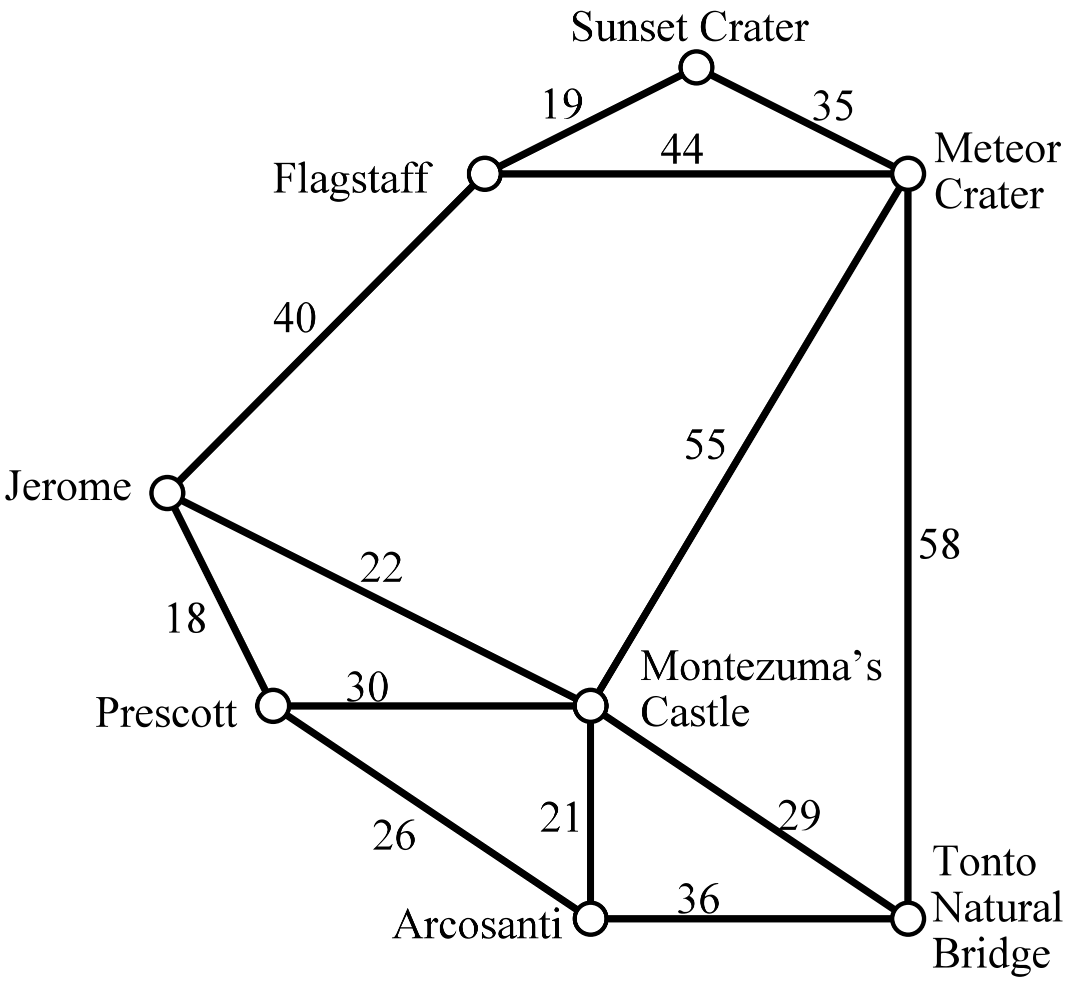
Hand-out #3: **Desert Tours, Inc.**

Desert Tours, Inc. needs to set up a communications network between the various sites of interest that they include on their tours for visitors to Arizona. The graph below provides the cost of establishing communications between each pair of sites.

Use the following two algorithms to connecting all eight sites at minimum total cost. For each algorithm, provide a list of steps that you performed in carrying out the algorithm:

Take the cheapest: Select the cheapest edge in the graph and hold on to it, but be careful not to create a cycle. Keep doing this until you have the right number of edges.

Throw away the worst: Select the worst edge in the graph and remove it, but be careful not to disconnect the graph. Repeat until you have the right number of edges.



Developed by Linda Boland, Leadership Program 1994

Four methods for solving the Desert Tours (and other similar) problems.

1. **"Throw away the worst!" method.**
   Select the worst edge in the graph and remove it, but be careful not to disconnect the graph. Keep doing this until you have the right number of edges.

2. **"Break those cycles!" method.**
   Find a cycle and discard the worst edge in the cycle. Keep doing this until there are no more cycles.

3. **"Take the cheapest!" method.**
   Select the cheapest edge in the graph and hold on to it, but be careful not to create any cycles. Keep doing this until you have the right number of edges.

4. **"Stay connected!" method.**
   Grab any vertex. Select the cheapest edge in the graph which is connected to the part that you are holding on to, but which, if added, would not create any cycles. Add that edge to the part you are holding onto. Keep doing this until you have the right number of edges.

Conclusions

All four algorithms give the same result — this will happen for any connected graph.

Greedy algorithms sometimes work! (They don't with the Traveling Salesperson Problem.)

Kruskal's algorithm ("Take the Cheapest!") and Prim's algorithm ("Stay Connected!") were first described in the 1950's.

Terminology

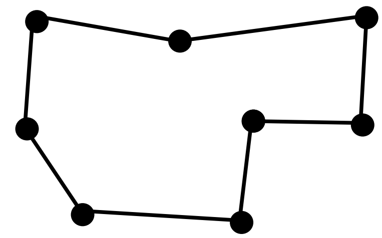
A "tree" is a connected graph that has no cycles.

A "spanning tree" is a tree that is part of a larger graph and that contains every vertex of the larger graph.

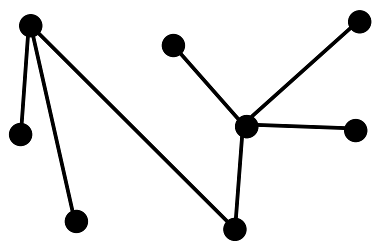
A "weighted graph" is a graph in which each edge has an assigned number, called its "weight". The weight could represent distance, time, cost, etc.

A "minimum weight spanning tree" is a spanning tree for which the sum of the weights of its edges is as small as possible.

Each of the four algorithms will efficiently find a minimum weight spanning tree in any weighted graph!



For the Traveling Salesperson problem, you're looking for a cycle.



For the Muddy City problem, you're looking for a tree.

In a cycle, every vertex has degree 2; in a tree, more than two edges may meet at a vertex.

"Nearest Neighbor" algorithm for TSP
vs.
"Stay Connected!" algorithm for MWSTP

Similar, except that you cannot have a vertex of degree 3 with "Nearest Neighbor"

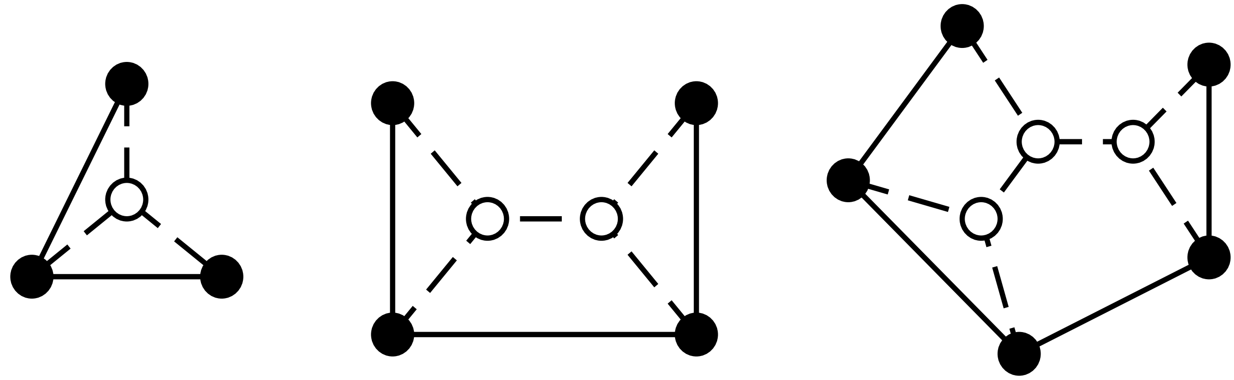
"Cheapest Link" algorithm for TSP
vs.
"Take the Cheapest!" algorithm for MWSTP

Similar, except that you cannot have a vertex of degree 3 with "Cheapest Link"

A "Steiner tree" for a given set of cities is a tree whose vertices include all the cities, as well as some additional cities (called "Steiner points"), and for which the total weight of its edges is a minimum.

In each example below, the Steiner tree (dotted lines) has smaller total weight than a minimum weight spanning tree (solid lines). Solid circles represent the original cities and open circles the Steiner points.

In a Steiner tree, each Steiner point has degree 3, and the three angles formed by the edges at each Steiner point all measure 120°.

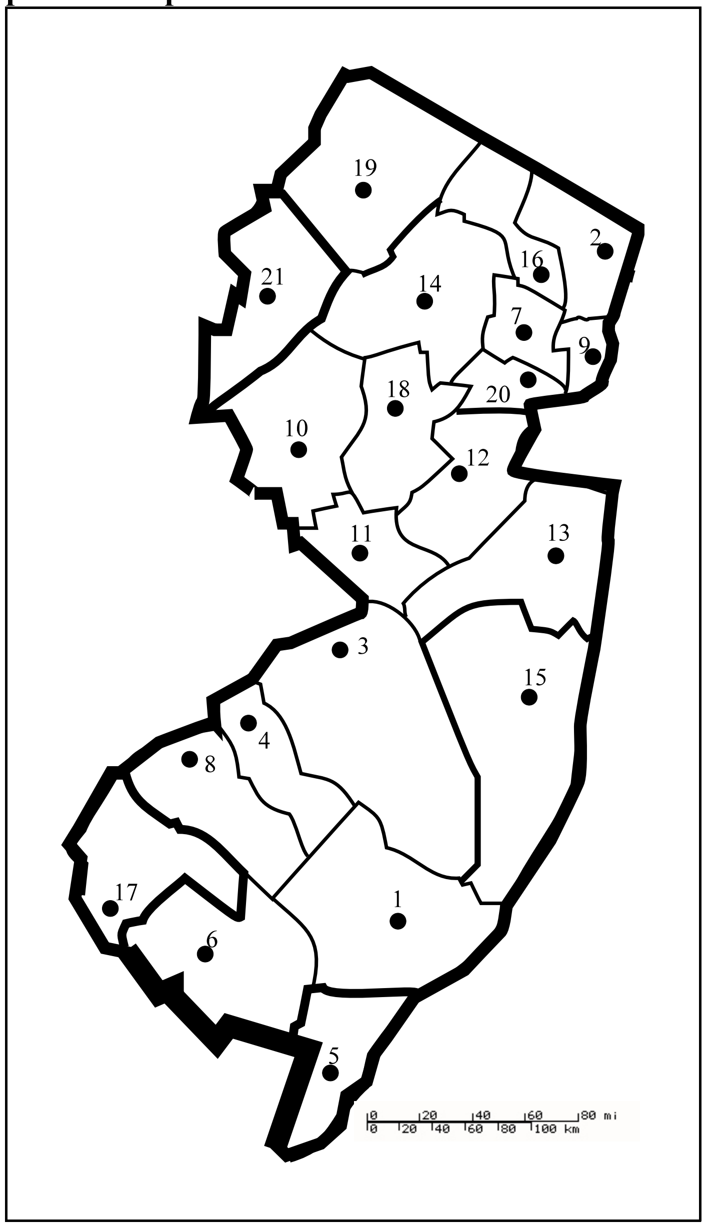


Hand-out #7

: Connecting New Jersey County Seats

In problems that don't appear in textbooks, we aren't always provided with the information we need. The New Jersey County map gives the locations of the 21 county seats, but not the distances between them. How would you construct a network consisting of the county seats which involves the minimum total distance? Discuss this question in a group, keeping track of the questions you asked, the methods you used, and the decisions you made in solving this problem; using this information write a report on the problem.

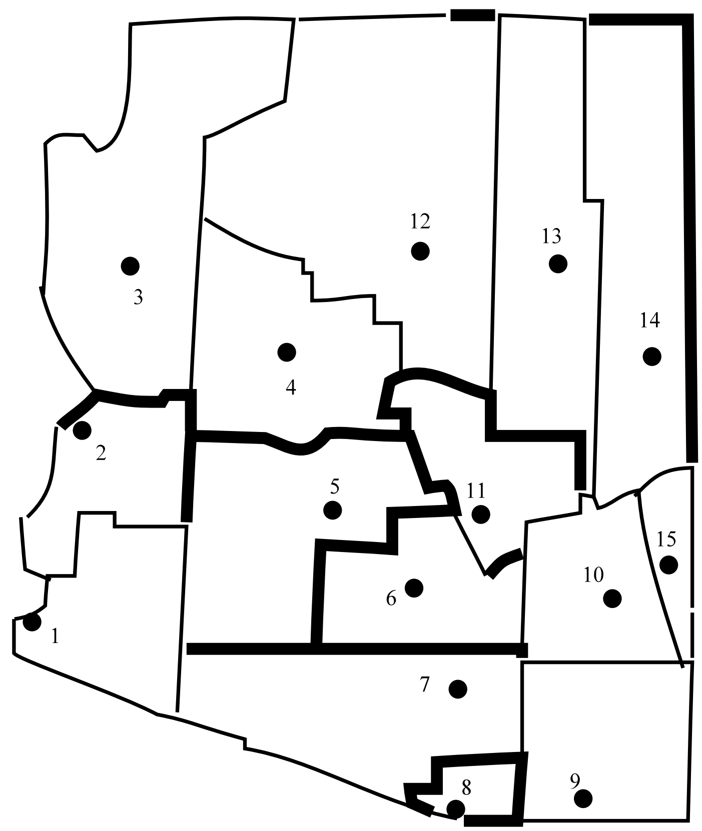
1. **Alantic County — Mays Landing**
2. **Bergen County — Hackensack**
3. **Burlington County — Mount Holly**
4. **Camden County — Camden**
5. **Cape May County — Cape May**
6. **Cumberland County — Bridgeton**
7. **Essex County — Newark**
8. **Gloucester County — Woodbury**
9. **Hudson County — Jersey City**
10. **Hunterdon County — Flemington**
11. **Mercer County — Trenton**
12. **Middlesex County — New Brunswick**
13. **Monmouth County — Freehold**
14. **Morris County — Morristown**
15. **Ocean County — Toms River**
16. **Passaic County — Paterson**
17. **Salem County — Salem**
18. **Somerset County — Somerville**
19. **Sussex County — Newton**
20. **Union County — Elizabeth**
21. **Warren County — Belvedere**



Hand-out # 8: Connecting Arizona County Seats

In problems that don't appear in textbooks, we aren't always provided with the information we need. The Arizona county map gives the locations of the 15 county seats, but not the distances between them. How would you construct a network consisting of the county seats which involves the minimum total distance? Discuss this question in a group, keeping track of the questions you asked, the methods you used, and the decisions you made in solving this problem; using this information write a report on the problem.

1. **Yuma County - Yuma**



2. **La Paz County - Parker**
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14. **Apache County - St. Johns**
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One inch = 20 miles

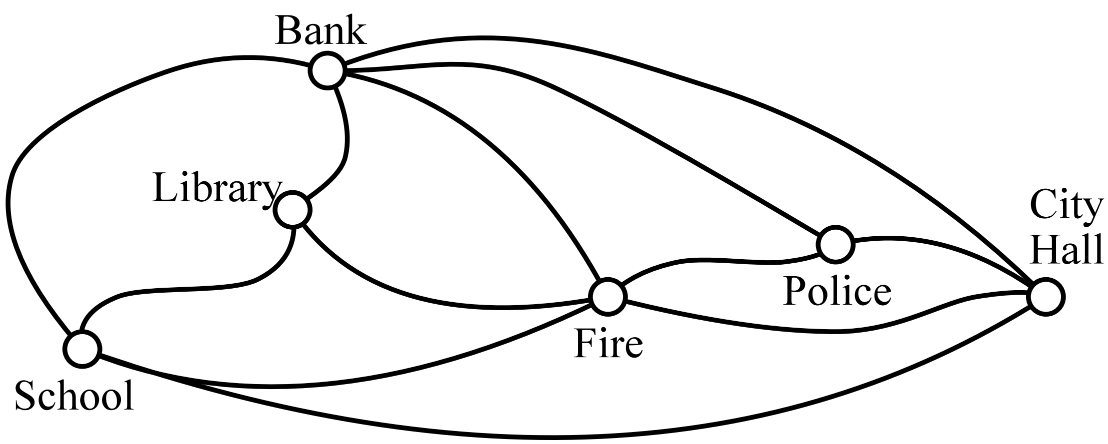
Hand-out #1: The Muddy City Problem

Muddy City has six major building and twelve roads connecting these buildings.

Unfortunately, when it rains, all of the roads in Muddy City turn to mud — for none of the roads in Muddy City is paved!

The Muddy City Council decided to pave all of the roads, but the voters decided that that would be too expensive. So the Council decided that it would pave just enough roads so that it would be possible to get from any one of the six major buildings in Muddy City to any other one by traveling exclusively on paved roads.

How many roads does the Council have to pave?



Problem by Joseph G. Rosenstein, based on story by Mike Fellows.

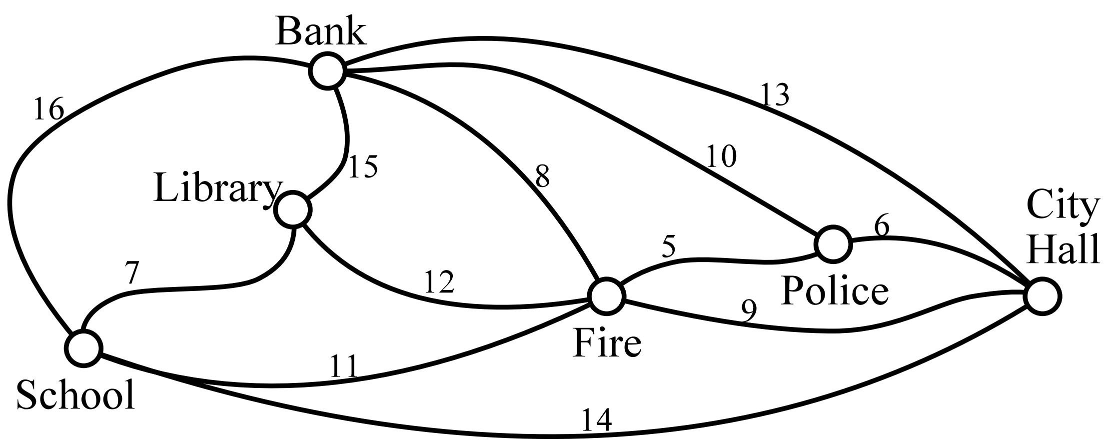
Hand-out #2: More on The Muddy City Problem

If the Muddy City Council can only pave five roads, which five roads should it pave?

The Council plans to make its decision only on the basis of cost — that is, it will pave the five roads whose total cost is the least.

The Muddy City Council obtains estimates for paving each of the twelve roads; the cost depends not only on the length of the roads, but also on other factors, such as hills, curves, and drainage. (The estimates are indicated on the map; all are multiples of $100,000.)

Which five roads connecting the major buildings would result in the lowest total cost?



Problem by Joseph G. Rosenstein, based on story by Mike Fellows

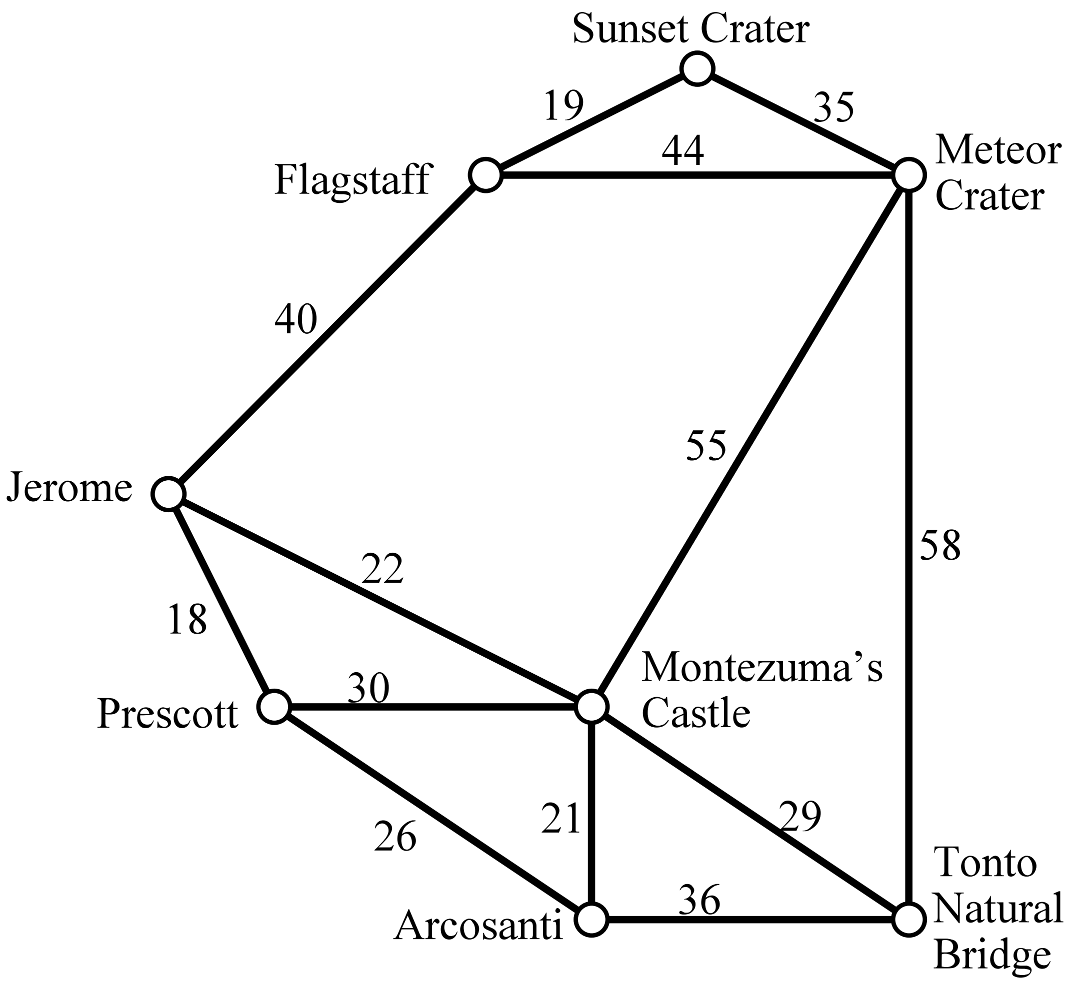
Hand-out #3: Desert Tours, Inc.

Desert Tours, Inc. needs to set up a communications network between the various sites of interest that they include on their tours for visitors to Arizona. The graph below provides the cost of establishing communications between each pair of sites.

Use the following two algorithms to connecting all eight sites at minimum total cost. For each algorithm, provide a list of steps that you performed in carrying out the algorithm:

Take the cheapest: **Select the cheapest edge in the graph and hold on to it, but be careful not to create a cycle. Keep doing this until you have the right number of edges.**

Throw away the worst: **Select the worst edge in the graph and remove it, but be careful not to disconnect the graph. Repeat until you have the right number of edges.**



Developed by Linda Boland, Leadership Program 1994

Handout #4: Four methods for solving the Desert Tours (and other similar) problems.

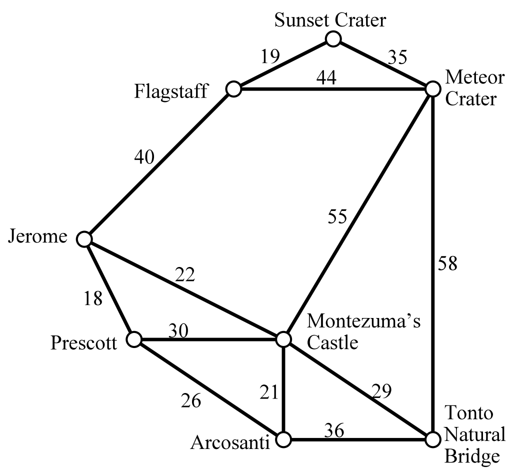
1. **"Throw away the worst!" method.**
   Select the worst edge in the graph and remove it, but be careful not to disconnect the graph. Keep doing this until you have the right number of edges.

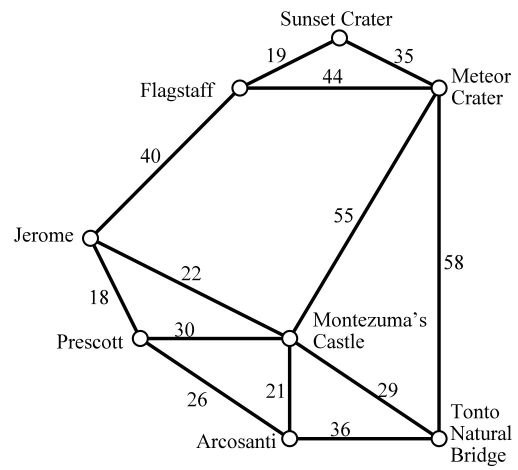
2. **"Break those cycles!" method.**
   Find a cycle and discard the worst edge in the cycle. Keep doing this until there are no more cycles.

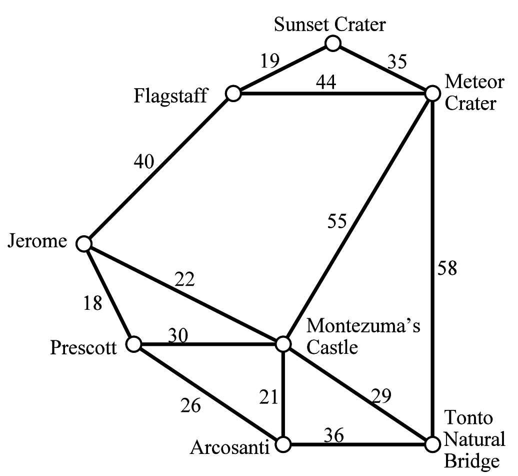
3. **"Take the cheapest!" method.**
   Select the cheapest edge in the graph and hold on to it, but be careful not to create any cycles. Keep doing this until you have the right number of edges.

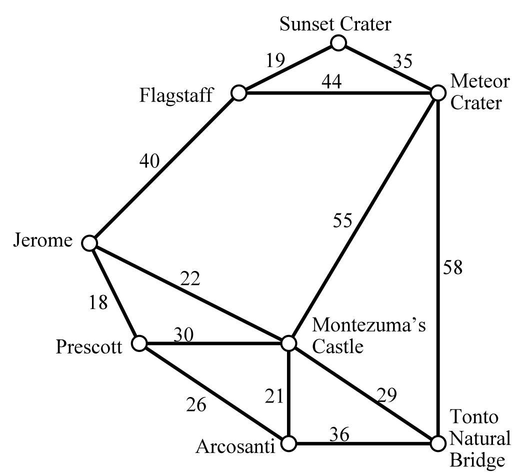
4. **"Stay connected!" method.**
   Grab any vertex. Select the cheapest edge in the graph which is connected to the part that you are holding on to, but doesn't create any cycles, and add that edge to the part you are holding onto. Keep doing this until you have the right number of edges.

Hand-out #5 — More Desert Tours



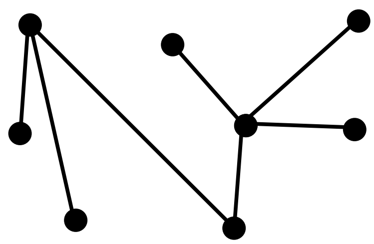


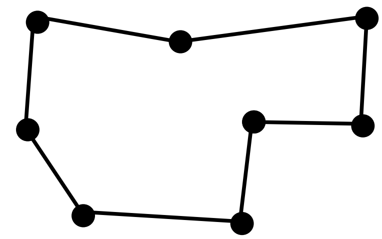




Hand-out #6: The Traveling Salesperson Problem and the Muddy City Problem.

For the Traveling Salesperson problem, you're looking for a cycle.





For the Muddy City problem, you're looking for a tree.

In a cycle, every vertex has degree 2; in a tree, more than two edges may meet at a vertex.

"Nearest Neighbor" algorithm for TSP
vs
"Stay Connected!" algorithm for MWSTP

These are similar, except that you cannot have a vertex of degree 3 with "Nearest Neighbor".

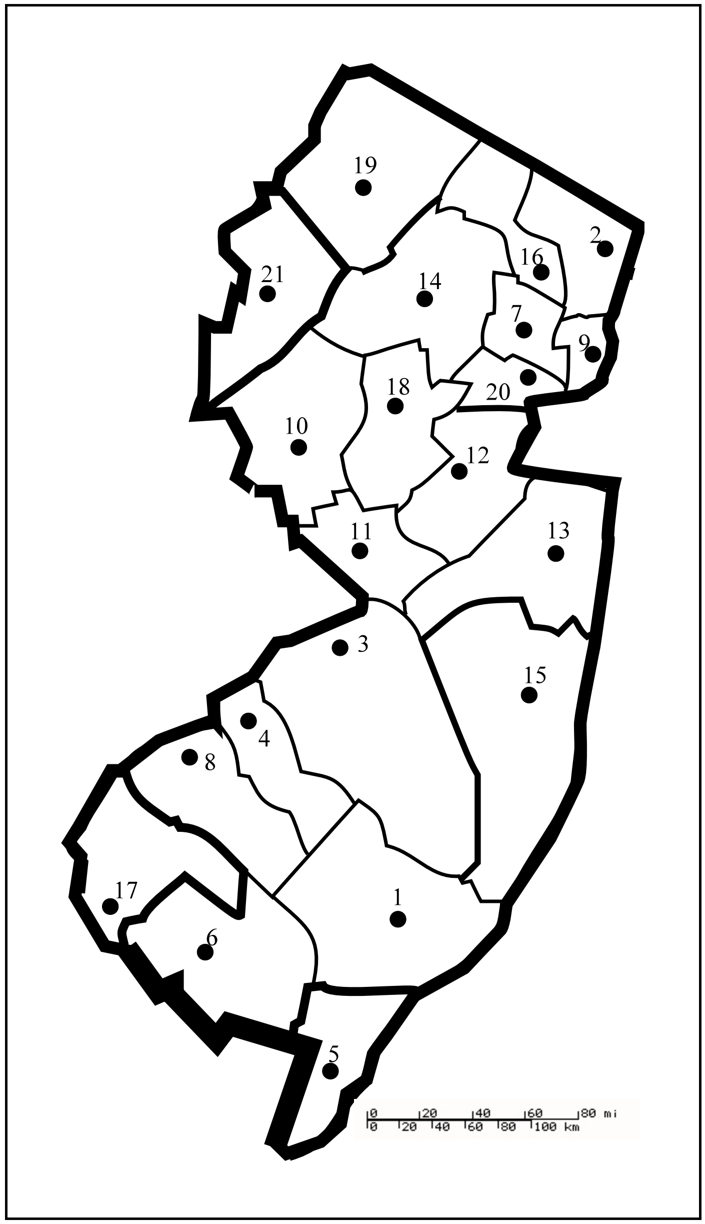
"Cheapest Link" algorithm for TSP
vs
"Take the Cheapest!" algorithm for MWSTP

These are similar, except that you cannot have a vertex of degree 3 in "Cheapest Link".

Hand-out #7: Connecting New Jersey County Seats

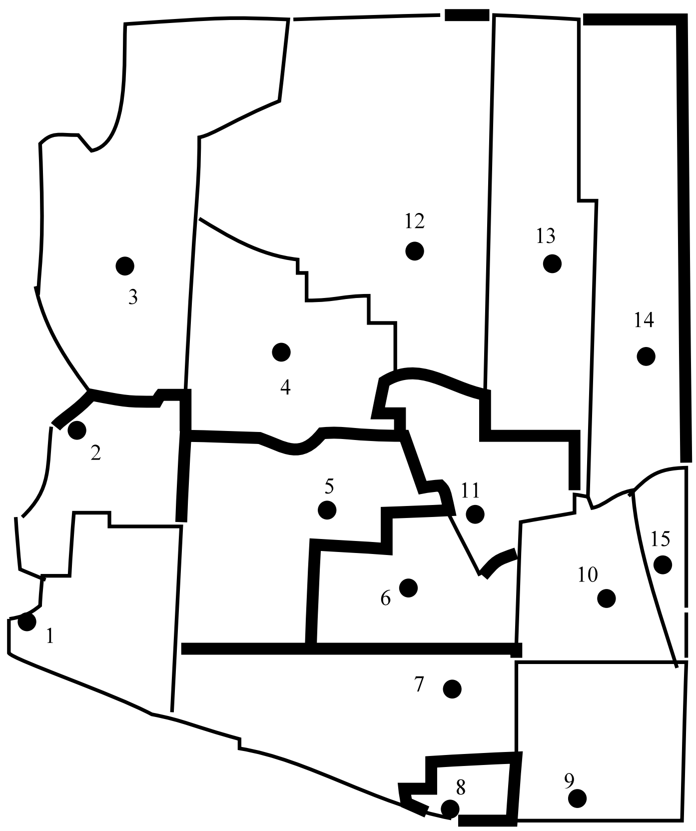
In problems that don't appear in textbooks, we aren't always provided with the information we need. The New Jersey County map gives the locations of the 21 county seats, but not the distances between them. How would you construct a network consisting of the county seats which involves the minimum total distance? Discuss this question in a group, keeping track of the questions you asked, the methods you used, and the decisions you made in solving this problem; using this information write a report on the problem.

1. **Atlantic County — Mays Landing**
2. **Bergen County — Hackensack**
3. **Burlington County — Mount Holly**
4. **Camden County — Camden**
5. **Cape May County — Cape May**
6. **Cumberland County — Bridgeton**
7. **Essex County — Newark**
8. **Gloucester County — Woodbury**
9. **Hudson County — Jersey City**
10. **Hunterdon County — Flemington**
11. **Mercer County — Trenton**
12. **Middlesex County — New Brunswick**
13. **Monmouth County — Freehold**
14. **Morris County — Morristown**
15. **Ocean County — Toms River**
16. **Passaic County — Paterson**
17. **Salem County — Salem**
18. **Somerset County — Somerville**
19. **Sussex County — Newton**
20. **Union County — Elizabeth**
21. **Warren County — Belvedere**



Hand-out # 8: Connecting Arizona County Seats

In problems that don't appear in textbooks, we aren't always provided with the information we need. The Arizona county map gives the locations of the 15 county seats, but not the distances between them. How would you construct a network consisting of the county seats which involves the minimum total distance? Discuss this question in a group, keeping track of the questions you asked, the methods you used, and the decisions you made in solving this problem; using this information write a report on the problem.



1. **Yuma County - Yuma**
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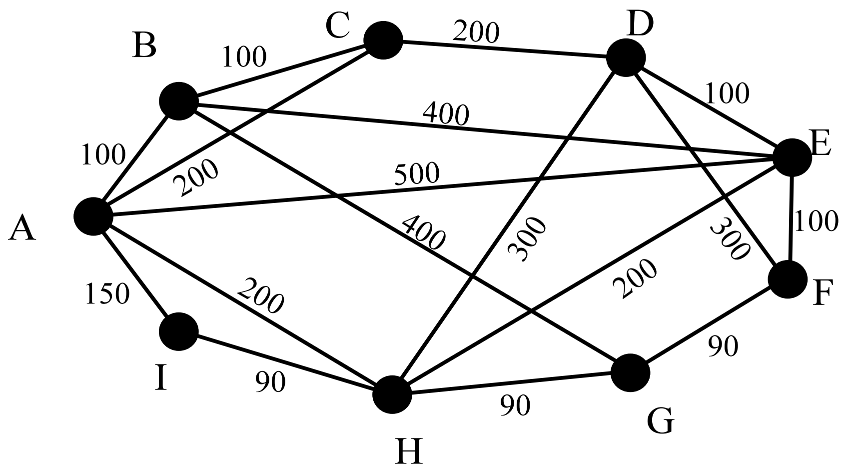
One inch = 20 miles

Workshop 4 — Making the Right Connections — Exercises

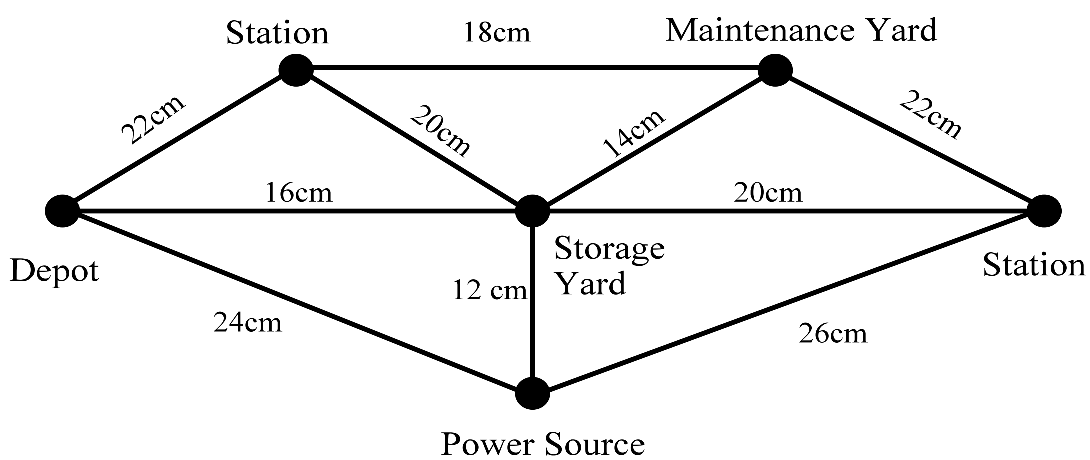
Practice problems:

1. **Suppose that you are planning to build a pipeline system which will take oil across the northern part of Canada. The various depots for storing oil have already been determined. The possible routes available are shown in the graph to the right, and the cost of construction of each portion of the pipeline (in millions of dollars) is the weight attached to each edge. Develop a pipeline system which will allow shipping between any pair of depots and which will incur the minimum construction cost.**

   (Adapted from Cozzens and Porter, *Mathematics and Its Applications to Management, Life and Social Sciences*, D. C. Heath and Company, 1987.)

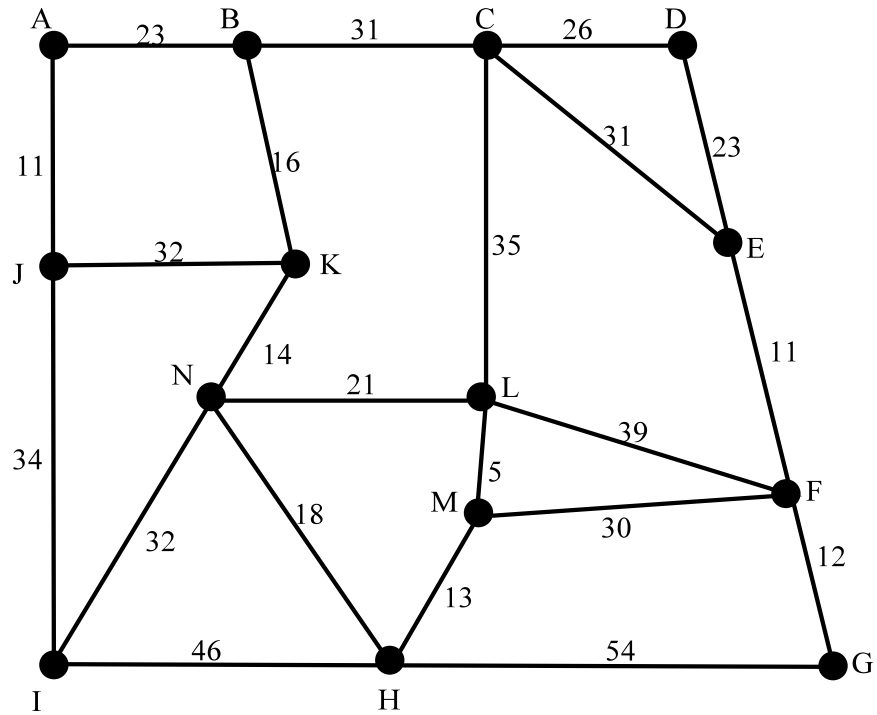


2. **Sheena received a model railroad set which had belonged to her uncle Ray when he was young. The layout includes two stations, a storage yard, a train depot and a maintenance yard, all fixed on a large board. She wants to be able to light all of them, so they need to be connected to a power source. The distances are shown in the diagram below. Can Sheena wire the system using 80cm of wire?**

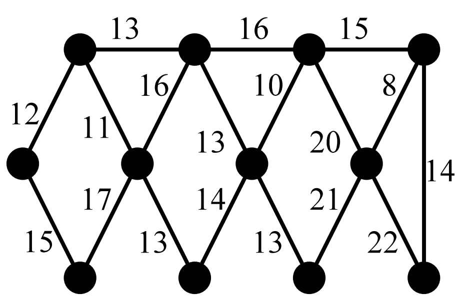


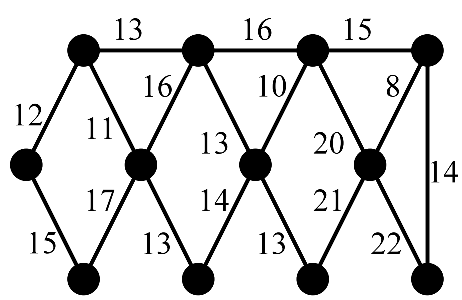
Study group problems:

3. A gardener wants to lay down a network of pipes so that water from A, the main water source, can get to each of the sprinkler heads (all the other vertices). He also wants to do the job as cheaply as possible. He has determined that the cost of laying each pipe is the number given in the diagram (in hundreds of dollars.) Make a list of steps giving the decisions that the gardener makes, and determine the minimum cost for the network of pipes.

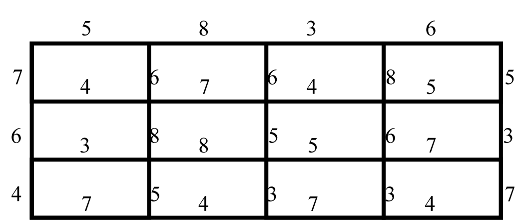


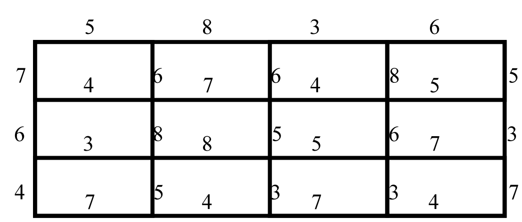
4. Find a minimum weight spanning tree for the weighted graph below. Use two different methods, in each case giving a list of the decisions you made. (Two copies of the graph are provided.)





5. How many different minimum weight spanning trees are there in the weighted graph below? Note that all grid points are vertices in the graph. (Hint: Use "take the cheapest.") (An extra copy is provided.)



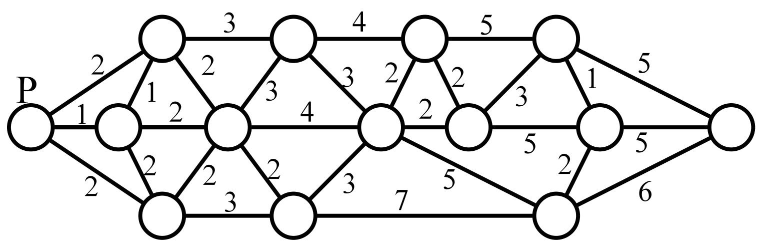


6. **There are mailboxes at every corner in the map below. The United States Postal Service wishes to install a system of air-suction tubes (like the one that drive-in tellers use) linking each mailbox to the post office, located at P.**

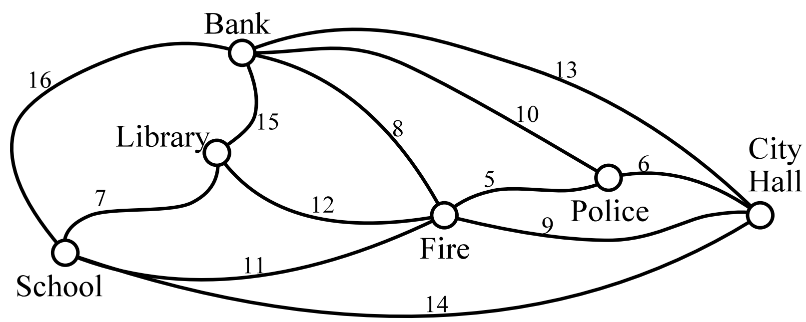
   a. If the numbers represent the cost, in thousands of dollars, to build an air-suction tube along the indicated stretch of road, what is the cheapest possible network?

   b. Does it make a difference where the post office is located? Would the cost be cheaper if the post office were relocated from the left-most vertex to the right-most vertex?

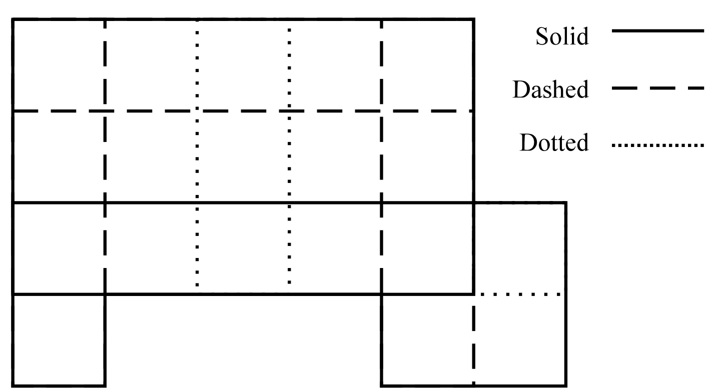
   c. The USPS also considered having the air-suction tubes form a Hamilton circuit. What would be the cheapest cost of building that configuration?



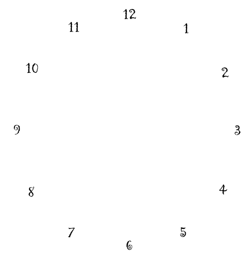
7. **The Muddy City Mayor lives on the road joining the bank and city hall, and she wants to make sure that that road is paved. Find a spanning tree containing this road which is of minimum weight (among those which contain this road).**



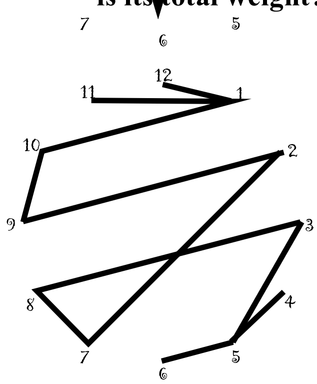
8. **The land of Mennifentia, shown to the right, consisted of 19 kingdoms, each square in shape, and each with great big, high walls built around it. When peace came to the land and the people decided that it was time to take down the walls, they realized that it would be very expensive to do so, and instead decided to just put gates in the walls, and then only just as many gates as was absolutely necessary so that a resident could get from any kingdom to any other. Because of differences in construction, it cost 2000, 1000 and 500 Mennibucks to put a gate in a solid, dashed and dotted wall respectively. What is the least the project could have cost?**



9. The clock face shown to the right should be thought of as 12 vertices, with an edge between each and every pair of vertices. Your job is to find a minimum weight spanning tree in each of the four following cases. In each case, there is a different rule that gives the weights on the edges, so the spanning tree that gives the minimum weight may be different in each case.

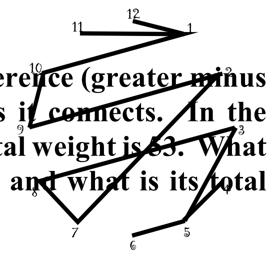


a. The weight of an edge is equal to the sum of the numbers on the vertices it connects. In the example shown of a spanning tree, therefore, the total weight is 129. What is the minimum weight spanning tree ... and what is its total weight?



Example

b. The weight of an edge is equal to the difference (greater minus smaller) of the numbers on the vertices it connects. In the example shown of a spanning tree, the total weight is 53. What is the minimum weight spanning tree ... and what is its total weight?



Example

10. See the instructions for problem #9.

a. The weight of an edge is equal to the *sum in clock-arithmetic* of the numbers on the vertices it connects. Clock arithmetic figures sums like on a clock face, so that 1+1=2, as usual, and 3+6=9, as usual, but 7+9 does not equal 16; rather, it equals 4. This is because, on a clock, 9 hours after 7 o'clock is 4 o'clock. Similarly, 8+5=1 and 7+7=2 and 11+1=12. In the example shown of a spanning tree, the total weight is 93. What is the minimum weight spanning tree ... and what is its total weight?

b. The weight of an edge is the units digit (the "ones place") of the number you get by multiplying together the numbers on the vertices the edge connects. In the example shown of a spanning tree, the total weight is 30. What is the minimum weight spanning tree ... and what is its total weight?

11. In the solution of the Towers of Hanoi problem, how many times does the smallest disc move?

12. Construct a minimum weight spanning tree that connects the county seats of New Jersey's 21 counties. (See next page.)

Extension problems:

13. a. Iliana says that a minimum weight spanning tree must contain every edge of the graph that has the smallest weight? Is she correct?

b. Mephistopheles says that a minimum weight spanning tree can't contain the edge in the graph that has the largest weight? Is he correct?

14. What pipeline system in problem #1 would result in the maximum cost? (Imagine that the company is Gouge-U Oil Company, which is looking for ways to charge more at the pumps.)

15. Is it possible for a weighted graph to have the property that a minimum weight spanning tree has greater weight than the minimum weight Hamilton circuit? Can they be equal?

16. If we assign the same weight to all of the edges of a connected graph, then it's easy to see that *every* spanning tree will have exactly the same weight as the minimum weight spanning tree. Can you

construct a graph where all spanning trees have the same weight, and yet not all the edges have the same weight?

Connecting New Jersey County Seats

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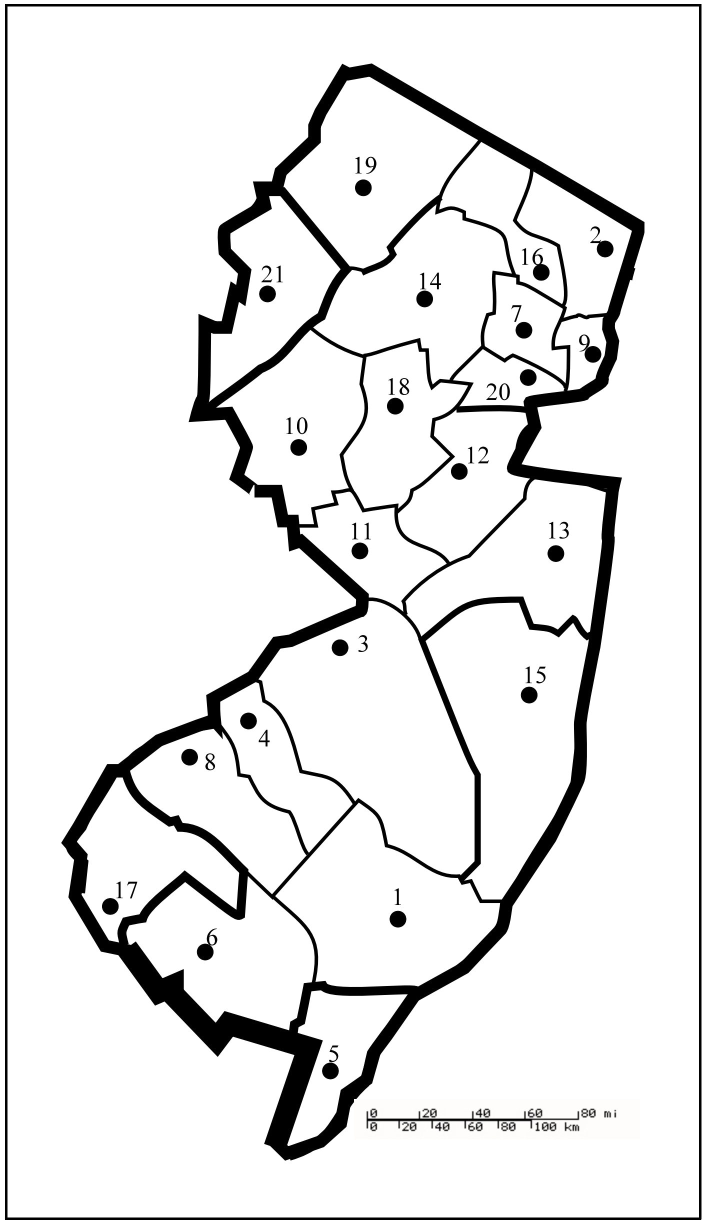

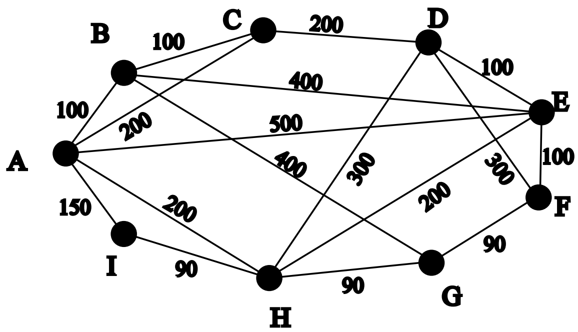
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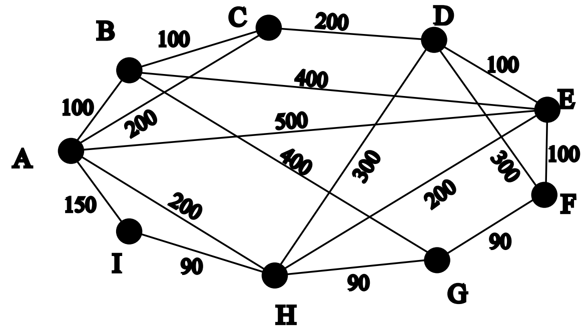
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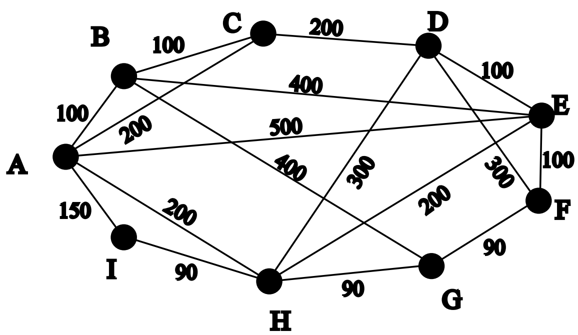
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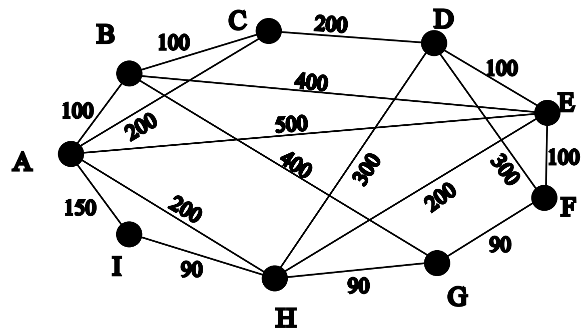
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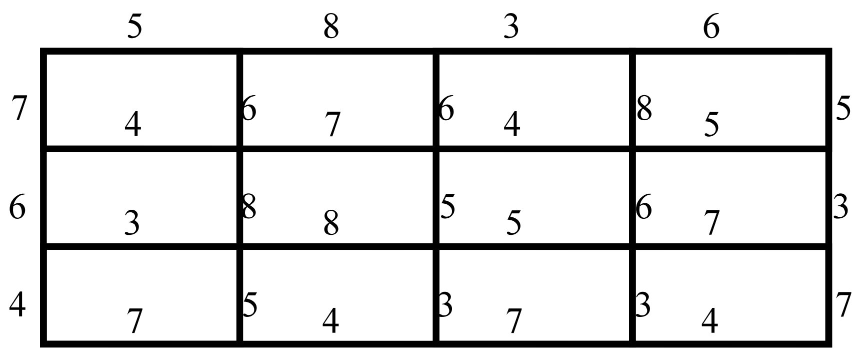
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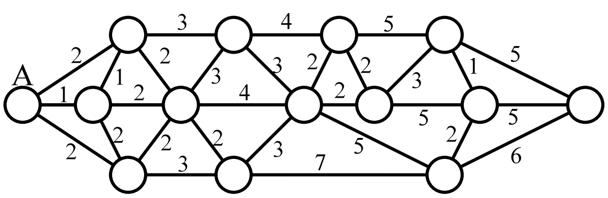
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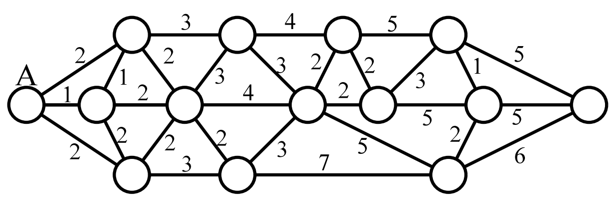
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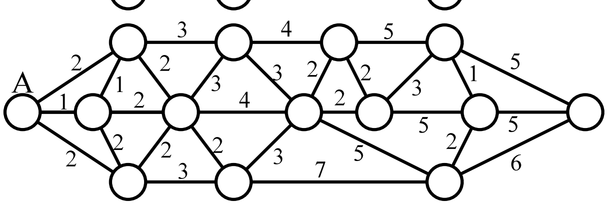
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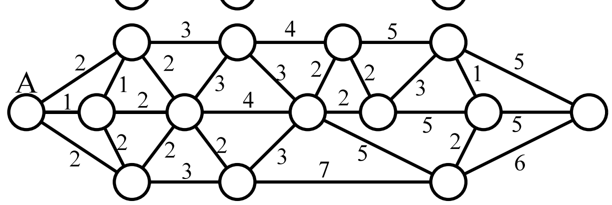
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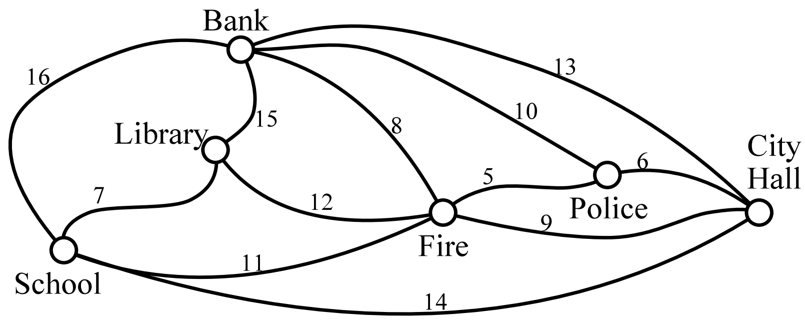
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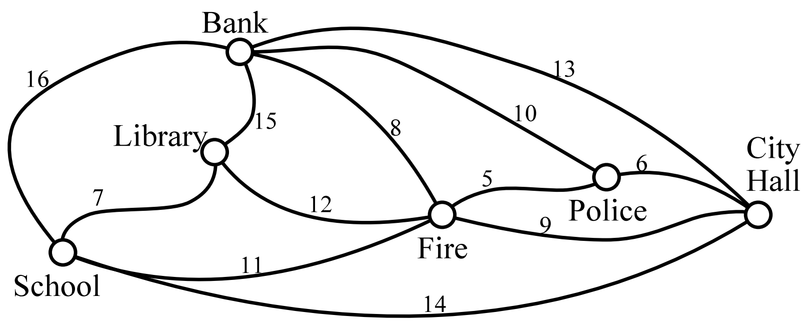
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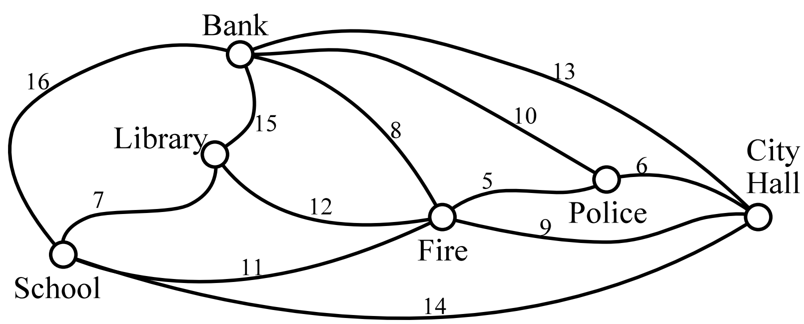
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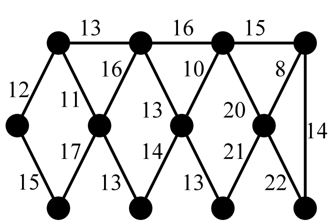
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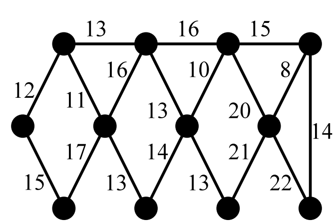
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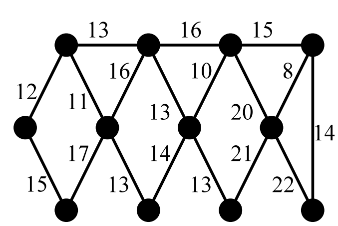
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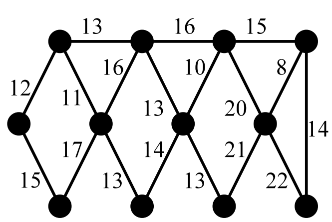
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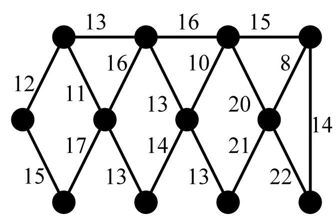
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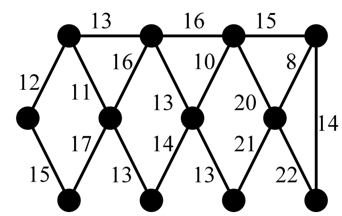
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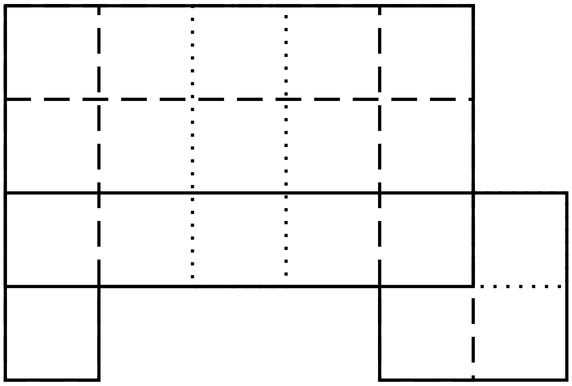

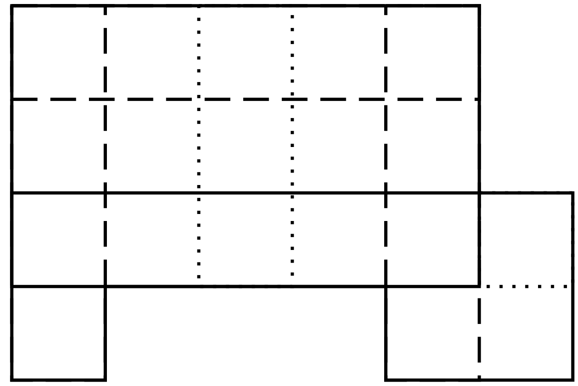

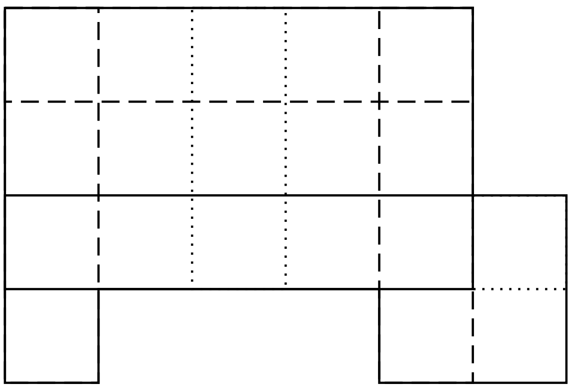

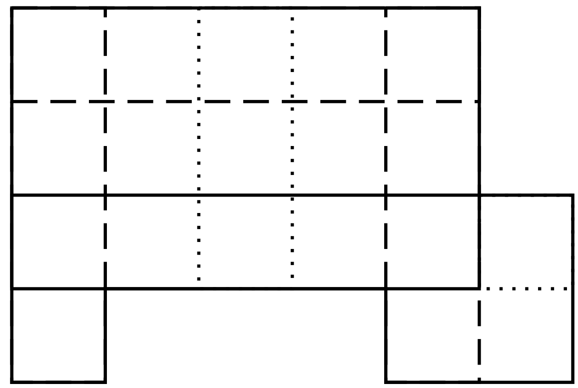

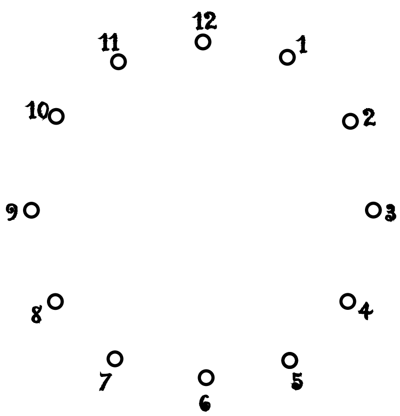
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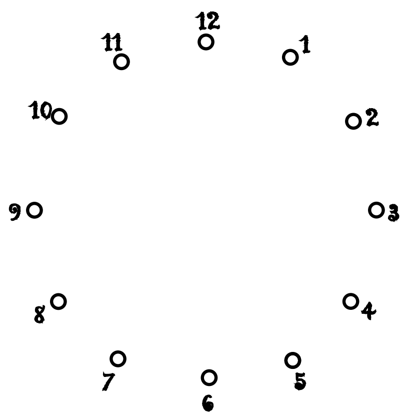
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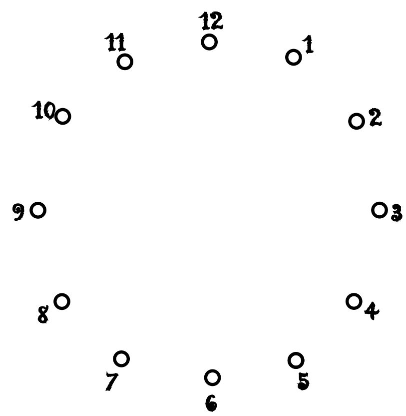
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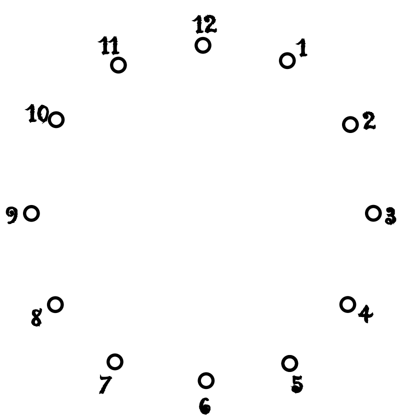
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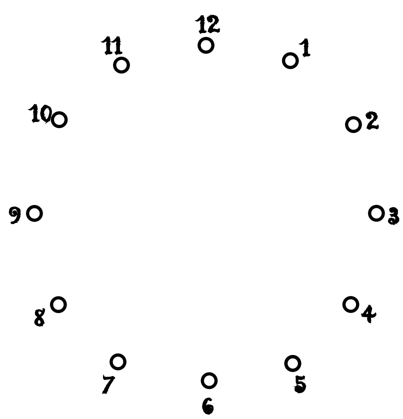
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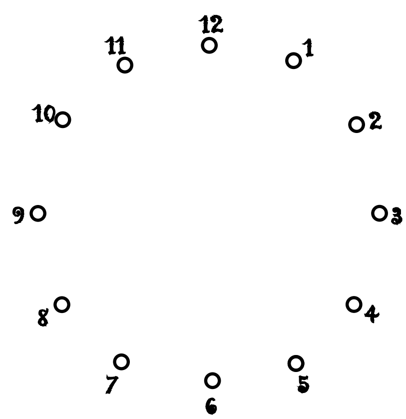
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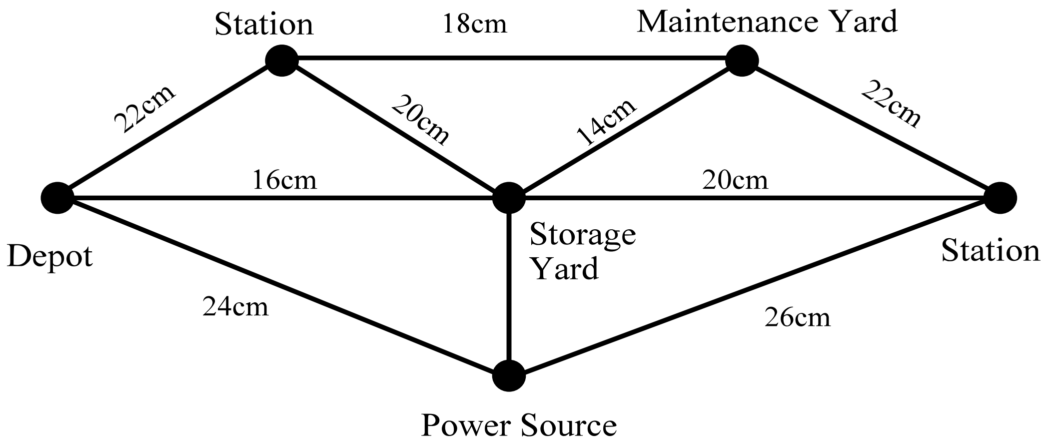
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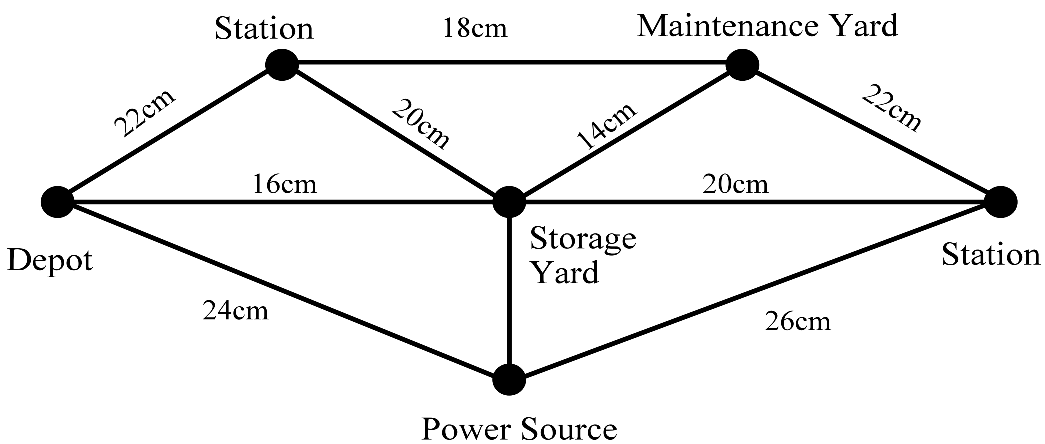
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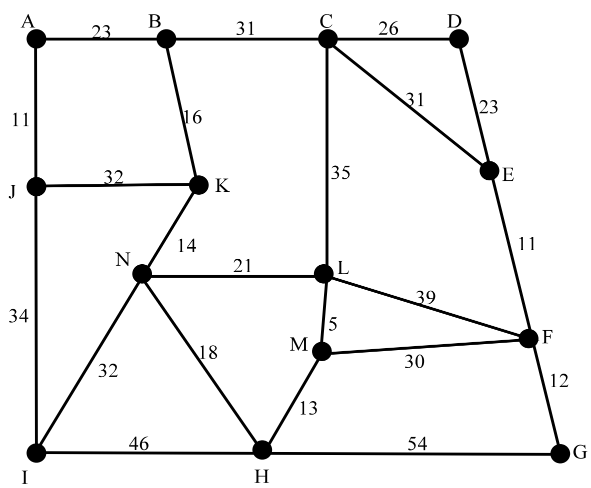
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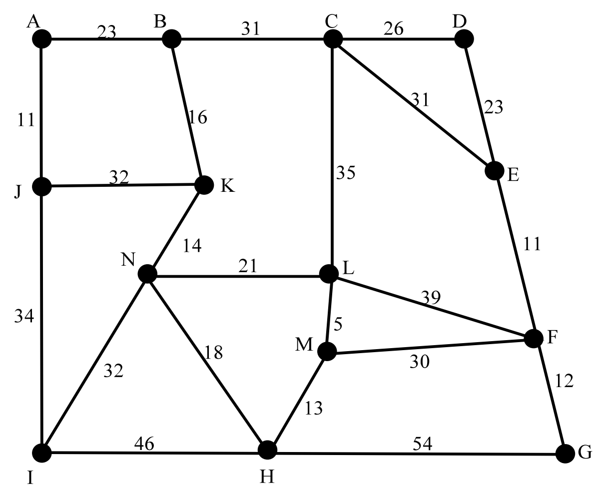
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Resource Book

Workshop 4: Making the Right Connections

Table of Contents

The Resource Book contains activities that teachers can use in their classes in addition to those discussed in the institute workshop on the topic of coloring graphs and maps, and the applications of graph coloring to resolving conflict situations.

Page 2, in the section "Mathematical Background", contains the terminology introduced in this workshop. Page 3 contains an outline of the workshop on "Making the Right Connections".

Pages 4-20 contain a variety of problems involving finding minimum weight spanning trees. Students should be asked to do each problem using at least two of the algorithms provided below. In the problems on pages 4-18 the weights are given for each edge; in the problems on pages 19-20, the weights have to be determined.

Pages 21-23 deal with Steiner points. Page 21 is a map of eastern United States that can be used to demonstrate the historical situation in which Steiner points were applied, pages 22-23 are instructions for creating the materials to demonstrate Steiner trees using soap bubbles, and describes how those materials can be used to demonstrate Steiner points.

Resource Book

Workshop 4: Making the Right Connections

Mathematical Background

A tree is a connected graph that has no cycles.

A spanning tree is a tree that is part of a larger graph and that contains every vertex of the larger graph.

A weighted graph is a graph in which each edge has an assigned number, called its "weight". The weight could represent distance, time, cost, etc.

A minimum weight spanning tree is a spanning tree for which the sum of the weights of its edges is as small as possible.

Minimum weight spanning tree algorithms

Four algorithms for finding minimum weight spanning trees are the following:

1. "Throw away the worst!" method. Select the worst edge in the graph and remove it, but be careful not to disconnect the graph. Keep doing this until you have the right number of edges.

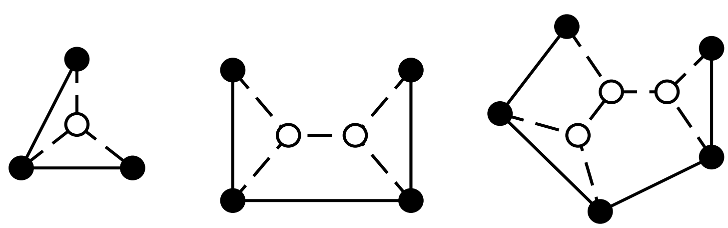
2. "Break those cycles!" method. Find a cycle and discard the worst edge in the cycle. Keep doing this until there are no more cycles.

3. "Take the cheapest!" method. Select the best edge in the graph and hold on to it, but be careful not to create any cycles. Keep doing this until you have the right number of edges.

4. "Stay connected!" method. Grab any vertex. Select the best edge in the graph which is connected to the tree that you are building, but doesn't create any cycles, and add that edge to the tree. Keep doing this until you have the right number of edges.

The "Stay connected!" method is called Prim's Algorithm, and the "Take the cheapest!" method is called Kruskal's Algorithm. These problems and methods were first studied in the 1950s.

A Steiner tree for a given set of cities is a tree whose vertices include all the cities, as well as some additional cities (called Steiner points), and for which the total weight of its edges is a minimum. In a Steiner tree, each Steiner point has degree 3, and the three angles formed by the edges at each Steiner point all measure 120°.



Resource Book

Workshop 4: Making the Right Connections

Workshop Outline

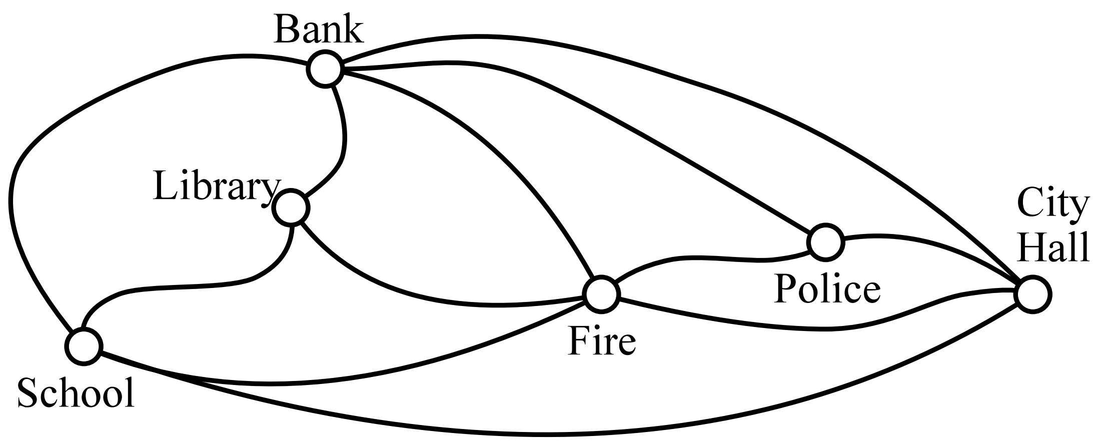
1. **Muddy City Problem #1**
   a. **Participants discover that by paving five roads it becomes possible to get from any one of the six buildings to any other one exclusively on paved roads; many configurations of five roads are possible, ranging from linear to star patterns.**
   b. **Are five roads actually necessary, or is it possible to pave only four roads? Using a "look at a simpler problem" strategy, participants find that, in general, the number of roads that need to be paved is one less than the number of buildings, and can explain why it is not possible to do it with fewer roads.**
2. **Muddy City Problem #2**
   a. **Participants find a configuration of five roads whose total cost is minimal.**
   b. **Discuss the methods used to obtain a minimal configuration, and connect to discussions of algorithms in previous workshops.**
3. **Minimum weight spanning trees**
   a. **Participants work on one problem with four different algorithms, and then discuss the outcome— that all 4 algorithms give the same results. Two of the algorithms are Kruskal's algorithm and Prim's algorithm.**
   b. **Introduce terminology of tree, spanning tree, weighted graph, minimum weight spanning tree, and conclude that participants now know how to find a minimum weight spanning tree in any weighted graph.**
   c. **Participants devise way to find a minimum weight spanning tree for a graph (NJ or AZ county Seats) for which weights are not given.**
4. **Finding shortest connections using soap films.**
   a. **Participants asked to find shortest way to connect three dots on paper. Improve on the two edge solution by having participants draw dot in middle of triangle and measure and compare the total lengths of the three segments from the central dot with the total of the two edges.**
   b. **Introduce idea of Steiner point as one which gives smallest total network (Steiner tree) and relate Steiner trees to minimum weight spanning trees.**
   c. **Finding Steiner points using soap films — show solution of Delta Airline problem involving opening a "branch office" near the Ohio border.**
   d. **Find Steiner tree solution for four points in a square and for four points in a rectangle; each is surprising, the first because there are two Steiner points rather than just one in the center, and the second because the two expected Steiner points transform into two different ones.**

Note: In the Traveling Salesperson Problem, you're looking for a cycle, but in the minimum weight spanning tree problem, you're looking for a tree. The "Nearest Neighbor" algorithm for the Traveling Salesperson Problem is similar to the "Stay Connected!" algorithm for the minimum weight spanning tree problem — except that you cannot have a vertex of degree 3 in a solution to the TSP. Similarly, the "Cheapest Link" algorithm for the Traveling Salesperson Problem is similar to the "Take the Cheapest!" algorithm for the minimum weight spanning tree problem — except that you cannot have a vertex of degree 3 in a solution to the TSP.

Workshop 4: Making the Right Connections

Hand-out #1: The Muddy City Problem

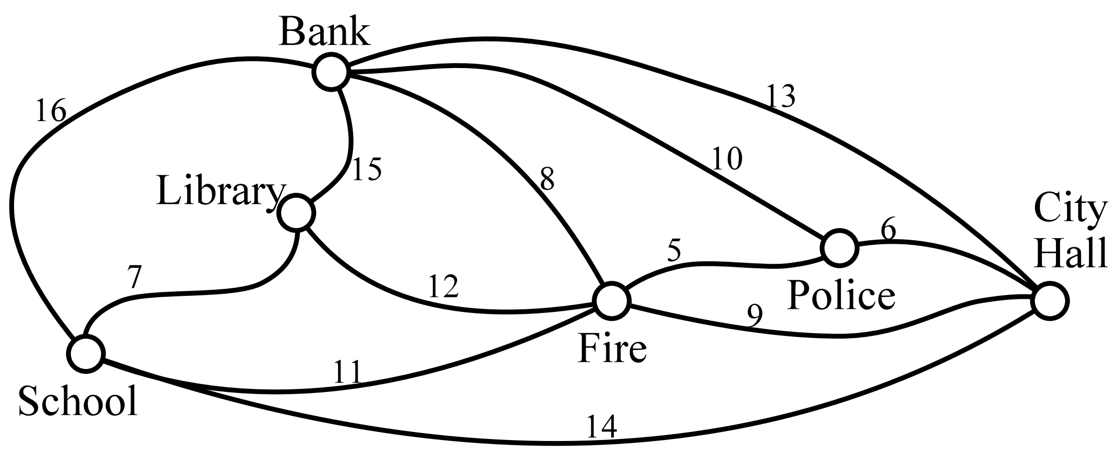
Muddy City has six major building and twelve roads connecting these buildings.
Unfortunately, when it rains, all of the roads in Muddy City turn to mud — for none of the roads in Muddy City is paved!
The Muddy City Council decided to pave all of the roads, but the voters decided that that would be too expensive. So the Council decided that it would pave just enough roads so that it would be possible to get from any one of the six major buildings in Muddy City to any other one by traveling exclusively on paved roads.
How many roads does the Council have to pave?



Problem by Joseph G. Rosenstein, based on story by Mike Fellows.

Hand-out #2: More on The Muddy City Problem
If the Muddy City Council can only pave five roads, which five roads should it pave?
The Council plans to make its decision only on the basis of cost — that is, it will pave the five roads whose total cost is the least.
The Muddy City Council obtains estimates for paving each of the twelve roads; the cost depends not only on the length of the roads, but also on other

factors, such as hills, curves, and drainage. (The estimates are indicated on the map; all are multiples of $100,000.)
Which five roads connecting the major buildings would result in the lowest total cost?



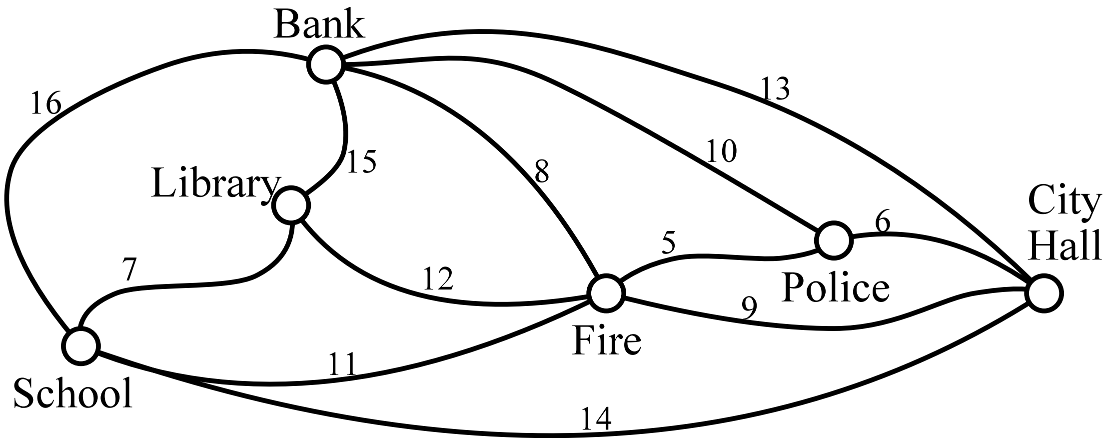
Problem by Joseph G. Rosenstein, based on story by Mike Fellows.

Still More on The Muddy City Problem

Some people said: "If you're going through all this trouble to pave the roads, why not have the paved roads form a loop." In mathematical terminology, what they wanted was to connect the six buildings into a cycle. How many roads would the Muddy City Council have to pave? And which roads should they pave so that the total cost is as small as possible?

Resource Book

Workshop 4: Making the Right Connections



Problem by Joseph G. Rosenstein, based on story by Mike Fellows.

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Workshop 4: Making the Right Connections

Resource Book

Workshop 4: Making the Right Connections

Long Island Tours

Connect the fourteen cities on the Long Island map into a network of minimum total distance. (You can use the distances on the mileage chart on the next page or measure the distances directly.)

Developed by Bob Eldi and Doug Schumacher, Leadership Program 1989

Resource Book

Workshop 4: Making the Right Connections

Section 4: Making the Right Connections

Version 2 — July 20, 1995

Resource Book

Workshop 4: Making the Right Connections

Making the right connections ... Using Steiner points
In the 1950s, the telephone company charged multi-location customers for service based on the shortest way of connecting the customer's locations, that is, based on a minimum weight spanning tree. Delta Airlines, with offices in Chicago, New York, and Atlanta opened up a new office on the border of Ohio and Kentucky, at the so-called Steiner point of the triangle formed by its existing three offices. Referring to the map below, explain why Delta decided to do this.

Workshop 4: Making the Right Connections

SHARING PROTOCOL

Step 1.	**Two teams — Team A and Team B — work on the problem. Team A's members are A-1 and A-2, and Team B's members are B-1 and B-2.**
Step 2.	**A-1 explains to B-1 what Team A did; at the same time, B-2 explains to A-2 what Team B did.**

Team A		**Team B**
A-1	**explains to**	**B-1**
A-2	**explains to**	**B-2**

Step 3.	**B-1 or A-2 (the "listeners") explain to everyone what both Team A and Team B did.**

VOCABULARY

PRIM'S ALGORITHM (1957)

1. **Find the shortest edge of the graph. Darken it and circle its two vertices. (Break any ties arbitrarily.)**
2. **Find the shortest remaining undarkened edge having one circled vertex and one uncircled vertex. Darken this edge and circle its uncircled vertex.**
3. **Repeat step 2 until all vertices are circled.**

Hand-out #6: Connecting New Jersey County Seats

Connect the county seats of New Jersey's 21 counties into a minimum weight spanning tree.

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Workshop 4: Making the Right Connections

Making Emergency Repairs

Resource Book

Workshop 4: Making the Right Connections

Resource Book

Workshop 4: Making the Right Connections

Resource Book

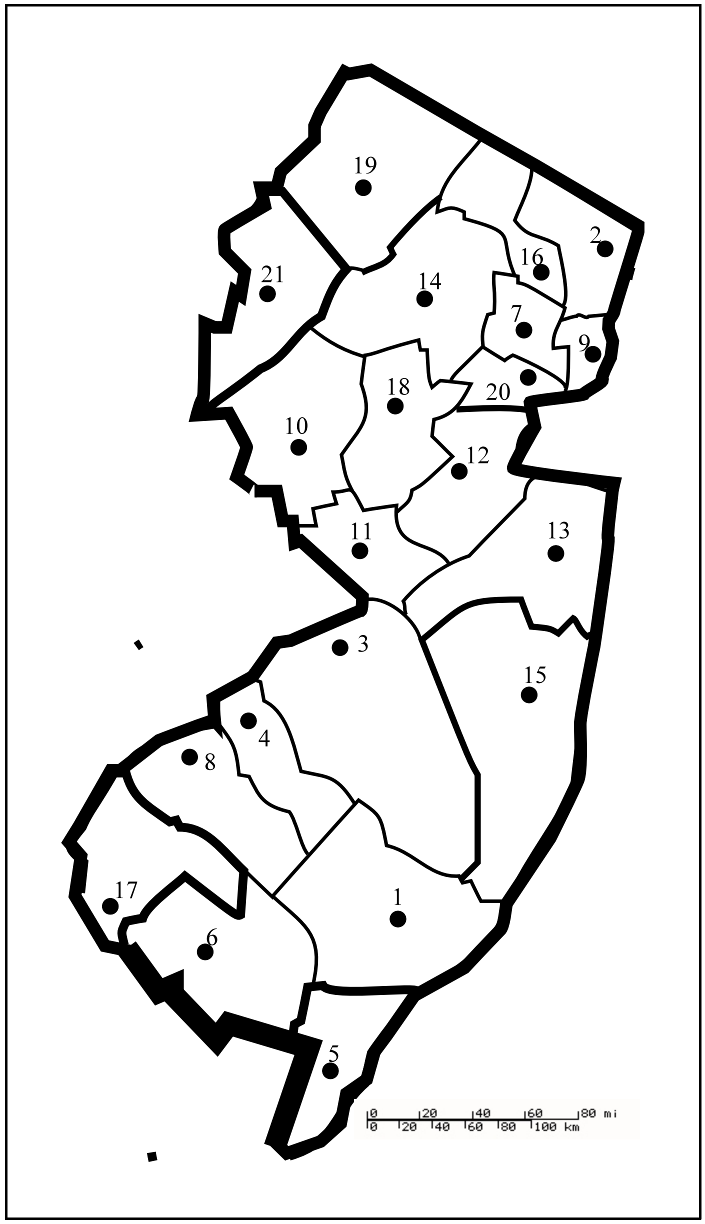
Workshop 4: Making the Right Connections

Resource Book

Workshop 4: Making the Right Connections

In problems that don't appear in textbooks, we aren't always provided with the information we need. The New Jersey County map gives the locations of the 21 county seats, but not the distances between them. How would you construct a network consisting of the county seats which involves the minimum total distance? Discuss this question in a group, keeping track of the questions you asked, the methods you used, and the decisions you made in solving this problem; using this information write a report on the problem.

1. **Atlantic County — Mays Landing**
2. **Bergen County — Hackensack**
3. **Burlington County — Mount Holly**
4. **Camden County — Camden**
5. **Cape May County — Cape May**
6. **Cumberland County — Bridgeton**
7. **Essex County — Newark**
8. **Gloucester County — Woodbury**
9. **Hudson County — Jersey City**
10. **Hunterdon County — Flemington**
11. **Mercer County — Trenton**
12. **Middlesex County — New Brunswick**
13. **Monmouth County — Freehold**
14. **Morris County — Morristown**
15. **Ocean County — Toms River**
16. **Passaic County — Paterson**
17. **Salem County — Salem**
18. **Somerset County — Somerville**
19. **Sussex County — Newton**
20. **Union County — Elizabeth**
21. **Warren County — Belvedere**



Resource Book

Workshop 4: Making the Right Connections

Resource Book

Workshop 4: Making the Right Connections

Resource Book

Workshop 4: Making the Right Connections

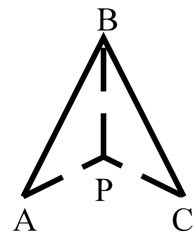
Using Soap Films to Demonstrate Minimum Networks

Mathematical Background

The minimum spanning tree for three cities consists of the two shortest edges connecting the cities. However, if a point is selected in the interior of the triangle formed by the cities, then the three distances from that point to the cities add up to less than the two shortest edges. In the diagram at the right, the sum of the lengths of AP, BP, and CP is less than the sum of the lengths of AB and BC.

The point P is called a Steiner point for the triangle if it results in the smallest sum for the three line segments. The Steiner tree for the three cities consists of the line segments from the Steiner point to the three cities; the three line segments in the Steiner tree make three angles of 120°. The three line segments in the Steiner tree form a minimum network connecting the three cities. You can demonstrate Steiner points in your classroom using soap films.

Steiner point demonstration



In the 1950s, the telephone company charged multi-location customers for service based on the shortest way of connecting the customer's locations, that is, they thought, based on a minimum weight spanning tree. Delta Airlines, with offices in Chicago, New York, and Atlanta opened up a new office on the border of Ohio and Kentucky, at the Steiner point of the triangle formed by its existing three offices. This resulted in substantially smaller fees for telephone service.

To demonstrate the location of Delta's fourth office using soap films, for example, you need to construct a model (see next page for instructions) for a triangle whose vertices correspond to Chicago, New York, and Atlanta. Before beginning the construction, make a transparency map of Eastern United States, and locate the three cities on the map. Drill the holes at the locations of Chicago, New York, and Atlanta on the transparency. Attach the transparency map to the model by passing the bolt through the transparency before the plexiglass.

Materials for demonstration: Overhead projector, dish pan, detergent (regular, but not concentrated, Joy is recommended), glycerine (to slow the movement of the soap film), straws, towels

Prepare the solution by filling the dish pan with about 1" of water at room temperature, adding 1/4 cup of detergent, and 1 or 2 capsful of glycerine; stir the solution, and then let it settle down. (Try the following steps before the actual demonstration to make sure that the soap solution works.) Dip the model in the water so that soap films are formed between the bolts representing Atlanta and Chicago and between the bolts representing Chicago and New York; place the model on the overhead projector.

Use a straw to blow the soap films gently off the Chicago bolt, so that the two soap films become one, and then seek a minimum total length. When that happens, you see the Steiner point for the network, and can demonstrate that all angles at that point are 120° by placing an opaque angle of 120° over each angle at the Steiner point. This demonstration works because the soap film minimizes its total length. Other demonstrations involve models where four cities form a square, or where four cities form a rectangle of dimensions 2" x 3.5" (actually, 2 times the square root of 3).

Resource Book

Workshop 4: Making the Right Connections

Constructing Models for Soap Film Demonstrations

These transparent models are designed to demonstrate, using an overhead projector, where Steiner points are for various configurations of cities, and how nature finds those Steiner points. The models typically involve 3-6 cities arranged in triangles, quadrilaterals, pentagons, and hexagons of various shapes. They can be constructed in a short time and will last for years. If you are hesitant about making them yourself, you can often find a willing student.

Materials

- **Plexiglass (1/8 inch) - 5" x 6" (two pieces of approximately this size for each model)**
- **Plexiglass knife**
- **Brass bolts (1") - 3-5 for each model (try to make them very narrow)**
- **Brass nuts - 9-15 for each model (three for each bolt)**
- **Ruler**
- **Screwdriver**
- **Drill (with bit diameter - bolt diameter)**
- **Sandpaper (220 grit)**

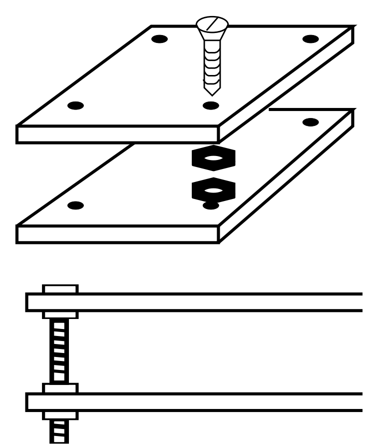
If you are making a number of models, you may have to start with a larger piece of plexiglass and cut it into smaller pieces as described in Step 1; if your plexiglass is already in 5" x 6" rectangles, then you begin with Step 2.

STEP 1: **Cut the plexiglass into 5" x 6" rectangles. Plexiglass can be cut by scoring the surface repeatedly (three or more times) with a plexiglass knife. The sheet should be placed on a flat table and strapped along the scored line. Sand the edges lightly.**

STEP 2: **First mark the points you want to drill on one piece of plexiglass. (You will probably start with a triangle.) Place another piece of plexiglass below the marked piece, and drill through both pieces at the same time, using a drill bit whose diameter is the same as that of the bolts. Place a bolt through the first hole to hold the pieces from shifting. Continue to drill until all holes are drilled.**

STEP 3: **Separate the plexiglass pieces. Insert bolts through the holes and then secure them with a nut. Screw another nut on each bolt. Measure to be sure that the second nut is ½ inch from the base. (See diagrams at right.)**

STEP 4: **Place the second piece of plexiglass on the first. It will rest on the nuts and should be ½ inch above the other piece. Secure this piece in place with the third bolt.**



You can make several models by repeating the steps. Suggested models are a triangle, a square, a pentagon, a hexagon, a map of your state with important cities networked, and randomly drilled holes that will allow you to experiment.

When using the models you will find that the soap solution adheres to the bolts. If you gently blow the soap the minimum surfaces will form. With a little creativity these models can be used in many ways. There are many nice applications with three dimensional models.