Master Document

Workshop 3 — Hamilton Circuits and the Traveling Salesperson Problem

Note that TSP #21-27 are not electronic

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

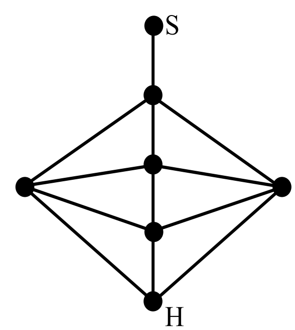
Instructor's Notes

Revised July 30, 1999 (and, most recently, June 22, 2011)

Workshop 3 — Hamilton Circuits and the Traveling Salesperson Problem

Materials and Pre-Workshop Preparations	**Allocated Time**
Activity #0 — How Much is a Million?	**5 minutes**
Activity #1 — Traffic Light Inspector's Problem	**35 minutes**
Activity #2 — Going Home from School	**10 minutes**

- Diagram of a graph with seven vertices and ten edges made from masking tape and the centers of paper plates (removing the rim permits the plates to lie flat); the distance from H to S should be at least 10 feet, large enough so that participants can comfortably walk along the edges. Draw the following buildings, one per plate: a school building, "your home", and five different homes where each "friend" lives.
- Twelve arrows, each about 20"×2", in fluorescent colors, if possible.



Activity #3 — Introducing the Traveling Salesperson Problem	**35 minutes**
Activity #4 — Algorithms for the Traveling Salesperson Problem	**40 minutes**
TOTAL WORKSHOP TIME:	**125* minutes**

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Word Wall: Hamilton path, Hamilton circuit, Weighted graph, Tree diagram, Factorial, Algorithm, Nearest Neighbor Algorithm, Greedy Algorithm, Traveling Salesperson Problem, Four Point Switch, Multiplication Principle of Counting

Activity #0 — How Long is a Million Seconds
(Allocated time = 5 minutes)

A. Put up TSP #1, which displays vertically the words hour, day, week, month, year, decade, century, and millenium, and ask participants where they think one million seconds would go within this list — that is, would one million seconds be expressed in terms of hours, days, ... , or millenia. Ask for a show of hands for each possibility. After they guess, have them estimate the answer — an hour is about 60x60 = 3600 seconds, so a day is about 25x3600 or 90,000 seconds, and that goes into a million about 11 times; the actual answer is 11 days and 14 hours. Then ask the same question for a billion seconds. That turns out to be about 31.7 years. Point out that this is a good way to help students build their number sense, and it is a calculation which we will need for later in today's workshop.

Activity #1 — Traffic Light Inspector's Problem
(Allocated time = 35 minutes, including 15 minutes for part A, 10 for part B, and 10 for parts C-D.)

A. Traffic light inspector's problem: Provide participants with Hand-out #1 (=TSP #2) and ask them to solve the problems for the three indicated graphs.

Explain that the traffic inspector problem is different from the letter carrier's problem in that the inspector does not have to cover each road, just each intersection. Discuss their solutions to these problems — in the first case, there is a solution; in the second and third cases, there are no solutions that end where they begin (that is, no Hamilton circuit), but there is a solution which doesn't (that is, a Hamilton path). A homework problem asks for an explanation of why there is no Hamilton circuit in the second case, so requests for an explanation here should be deferred.

B. Introduce the ideas of Hamilton circuit and Hamilton path (TSP #3). How can you tell whether or not a graph has a Hamilton circuit? For example, try to determine whether or not each of the three graphs in Hand-out #2 (=TSP #4) has a Hamilton circuit.

Because of the constraints in the graph, there is a unique Hamilton circuit in graph #2, unlike the previous case (where there are many Hamilton circuits), and there is no Hamilton circuit in graph #3 even though, as you should point out, it is almost identical to graph #2. Discuss briefly why graph #3 does not have a Hamilton circuit, leaving a more thorough discussion for the homework review. (Any Hamilton circuit *must include the seven points (excluding top left and bottom right) in one of four configurations, none of which can be completed to form a Hamilton circuit because the vertices at the top left and bottom right corners are separated from each other by these configurations; the four configurations all start with the two slanted lines not including the central vertex, and then include the central vertex in four possible ways.) This explanation depends very much on the particular characteristics of this graph — unlike the general way of telling when a graph has no Euler circuit. Moreover, graphs #2 and #3 demonstrate that two graphs with the same number and degrees of vertices may be very different with respect to whether there are Hamilton circuits; that is, the kind of criterion that was used to determine whether there are Euler circuits can't be available for determining whether there are Hamilton circuits.*

C. How can you tell whether a graph has a Hamilton circuit, or a Hamilton path? There is in fact no efficient way of determining for an arbitrary graph whether or not it has a Hamilton path. This fact leads to all sorts of puzzles involving paths through graphs — what is a puzzle? a problem to which usual methods don't apply — three of which are in the Resource Book, a bicycle puzzle by Sam Lloyd (TSP #5) (he was the most famous puzzle-maker of the 19th century), and a GAMES Magazine puzzle (TSP #6), and another, about a homicidal necrophobe, is in the problem set.

D. Discuss Sam Lloyd's puzzle (TSP #5) filling in edges 14-18-22, and 6-12-E, and 5-11-16-19-10. Use this as introduction to the idea that in a Hamilton path or circuit you can't have three edges at any vertex. (If there is interest in continuing the problem, distribute a copy (Handout #3) and challenge them to complete it later.) Continuing the solution, we then connect 2-7-13, then 2-9 (otherwise get cycle), then 4-8, then 3-14 (otherwise 13-3-8 creates a cycle), then 17-13 (since 6-17-21 results in leaving 20 out), then 3-8 and 9-5 and 10-4, Phil-15-22, and finally 17-21-20-6.

E. Road inspector's problem: Show TSP #7 with the road inspector's problem, ask participants how they might solve this problem, and elicit the response that this is exactly the same as the Euler problems discussed the previous day.

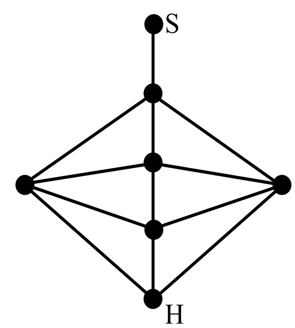
This is not a handout, but should be used as an opportunity for a quick review of Euler paths and circuits — that is, all participants should recognize quickly that the first and second graphs have neither Euler paths nor circuits and the third graph has an Euler circuit.

Point out the parallel between the two problems using TSP #8, and the major difference between the two problems using TSP #9 — both of which are condensed into Handout #3 — referring to how easy it was for participants to give a response to the road

inspector's problem for the three given graphs; from the solver's perspective, one problem allows for instant recognition of the solution, and the other results in definite puzzlement.

The contrast between the two types of problems should be emphasized; participants often have difficulty distinguishing between Euler and Hamilton paths, between the road inspector's problem and the traffic light inspector's problem, and where the letter carrier's problem fits into the picture. Activity #2 address the distinction kinesthetically.

Activity #2 — Going Home from School
(Allocated time = 10 minutes)



A. Moving to the graph on the floor (that has been prepared in advance), pose the following question: "Suppose you are a fourth grader and school (S) has just ended for the day. On your way home (H) you would like to stop by and visit five of your friends who live at the unlabeled vertices. What route would you take?" Ask for three volunteers, one at a time, to walk a variety of routes which will satisfy the criteria. As each volunteer walks the route, have another participant mark the appropriate edges using the large arrows. (At the same time, one of the lead teachers should record the route on the left side of TSP#10, which you can use for the subsequent discussion.) Once the three participants have "walked their walk" ask the group what was the same about the three walks (e.g., each walk visited every vertex exactly once) and what was different about the walks (e.g., each walk used a variety of edges and not every edge was necessarily traveled).

In the past, participants have found the distinction between Euler paths (and circuits) and Hamilton paths (and circuits) difficult to understand. This activity helps participants understand that Euler paths/circuits focus on travel along the "edges" and Hamilton paths/circuits focus on visits to the "vertices". In this first problem, we would like participants to recognize that a "visit to each vertex" is necessary, not a "travel over every edge". As a result, we're looking for a Hamilton path from S (school) to H (home). In the second problem (see below), travel over each edge is necessary. As a result, we're looking for an Euler path from school to home.

B. Then pose the following problem: "On the way home (H) from school (S), you want to stop by every house along every block so you can sell as many girl scout

cookies as possible. What route would you take?" Again, ask for three volunteers, one at a time, to walk a variety of routes which will satisfy the criteria and mark their routes using the arrows (and on the right side of TSP #10); once all three participants have "walked their walk" ask the group what was the same about the walks and what was different about the walks. (One thing they should notice is that every vertex, except for S, is visited twice in each of these walks.) When they return to their seats, show TSP #10 and review the activity with them; they should see that in the figures at the right all edges have been colored.

Activity #3 — Introducing the Traveling Salesperson Problem
(Allocated time = 35 minutes, including 10-15 minutes for each of parts A-C)

A. Use Hand-out #5 (=TSP #11), with the "Many Errands" problem, as an introduction to the Traveling Salesperson Problem. After explaining the problem, and noting its connection to Hamilton circuits, let them informally ("trial and error") find the best solution, and observe that it can be run in the opposite direction as well (but it's better to pick up the fish at the end of the trip!). Introduce the question of how you might verify that this is actually the best solution, and introduce the idea of a tree-diagram (left-to-right) to enumerate the possibilities; but only do two levels of the tree-diagram, just enough so that participants will be able to construct a tree-diagram for the next activity. Label the nodes in the tree-diagram, and include the distances on the tree-diagram, so that they will do the same with theirs.

Participants will later complete the tree-diagram for the "Many Errands" problem in their study groups and verify that "eighteen miles" is indeed the best answer. in the next activity, they will develop a tree diagram for a complete graph, which is easier.

B. Use Hand-out #6 (=TSP #12) involving the shortest route covering Chicago, St. Louis, Minneapolis, and Cleveland. Explain why this is called the "traveling salesperson problem". Note that TSP #12 introduces the notion of a weighted graph, and that yesterday's videotape showed an example of a weighted graph when they asked the question "Which route for the snowplows involves duplication of the smallest total distance?" Have participants use a tree-diagram to enumerate all the possibilities and find a shortest route. Review with them their solutions using TSP #13. Remind them that each branch of the tree diagram provides a Hamilton circuit in the graph; here the problem is not to find a Hamilton circuit — there are lots of them — but to find the cheapest one.

Remind participants to put the distances on the tree diagram. If they work together *with a partner, they can share the addition computations.*

Participants will of course find that there are six routes; they should also discover, because of the repetition of the totals, that each route is repeated (in the reverse direction) — so that only half of the routes have to be considered in determining which is the best.

Mention that two reasons for doing tree-diagrams are to convince ourselves that they give a good conceptual understanding for the situation, and to convince ourselves that they are not practical as a method of solution (except for small problems).

C. Using TSP about the various applications of the Traveling Salesman Problem (see TSP #14). A few obvious ones are UPS deliveries, ATM machine collections, and car pooling. Too less obvious ones are robots in drug warehouses, which are sent to retrieve a list of drugs and have to figure out (using a Traveling Salesperson Problem algorithm) in what order to retrieve them; and water sampling (where marine biologists have to collect samples of ocean water at hundreds of sites and need to know in which order to collect the samples). In each case, great savings would result from having an efficient way of solving the Traveling Salesperson Problem.

[Time for a 5-10 minute break]

D. Review the Multiplication Principle of Counting using TSP #15.

E. Observe from the tree diagram (TSP #13) that since there are three choices for the first stop, two choices for the following stop, and only one choice for the final stop before returning to Chicago; there are altogether 6 possible routes, corresponding to 3x2x1. Use TSP #16 to discuss the case where there are four cities besides Chicago, in which case, there would initially be four possibilities, then three, then two, then 1, so that altogether there would be 4x3x2x1 possibilities; in TSP #16 the 24 routes are partitioned according to first city visited so that they can also see the recursive version 4x6. When discussing that part of the tree in TSP #16 is actually the tree in TSP #13, it might be helpful to cover up part of TSP #16, leaving only the branch that looks like TSP #13 showing. Continue to generalize, introduce factorial notation, using explicitly both the explicit definition and recursive definition through 10! = 3,628,800 (see TSP #17. Then use TSP #18 to estimate how long it would take a computer to check the total length of all those routes, recalling the discussion at the beginning of the workshop of how long is a billion seconds. Provide them with Hand-out #7, which includes all the information on TSP #17 and TSP #18, at the beginning of this discussion.

This should be sufficient to convince participants that even for the Traveling Salesperson Problem on a modest number of cities, constructing a tree-diagram is a hopeless strategy. Moreover, the information on TSP #17 and TSP #18 should also convince participants that examining all possibilities is also a hopeless strategy, even for a computer. Even if computers became a billion times faster, adding 10 more cities would cancel out any progress made.

Activity #4 — Algorithms for the Traveling Salesperson Problem
(Allocated time = 40 minutes)

A. Show TSP #19 — but don't hand out Handout #8 yet — and note that "examining all the possibilities" using a tree-diagram is not an adequate strategy for the traveling salesperson problem even if there are only 9 cities to visit. Ask participants to try to generate some ideas about how we might solve such a problem; most likely, someone will make a suggestion that you can identify as the Nearest Neighbor Algorithm. Introduce the Nearest Neighbor Algorithm and use it to do the problem on TSP #20. Pause to explain the idea of an algorithm — a step-by-step procedure which can be applied to many different problems — and why the term "algorithm" can be applied to the method used to solving this problem.

B. Ask them to apply the NN algorithm to the example on Handout #9 (=TSP #19), the problem that they looked at a few minutes ago. After they present their solutions, ask the participants whether they think that the NN algorithm always provides the best solution to the traveling salesperson problem.

C. Return to the example in TSP #12 to show that the NN algorithm doesn't always give the best solution. (Indicate on TSP #12 the three possible routes using three colors.)

D. Review TSP #21 which has the NN solution for TSP #20, and discuss (overlaying TSPs) what solutions might be better; conclude that Chicago-Pittsburgh-Philadelphia-Cincinatti-St.Louis-Chicago seems to be the best (see TSP #22). The total is 1741 vs. 1846 for NN on TSP #21.

E. Discuss the concept of a "greedy algorithm", where you do what appears to be the best thing right now without paying any attention to the consequences of that decision; the Nearest Neighbor Algorithm is an example of a greedy algorithm. Ask them to find a better solution than the NN algorithm for the problem on TSP #19, and discuss their solutions using overlays of TSP #23. The NN solution is on TSP #24, and the optimal solution is on TSP #25. (They will try to apply the four-point switch to

their NN solution on TSP #21 – five-city problem – on the homework.)

F. How could we find these better routes? Show four-point switch on TSP #26 and apply that to the NN solution on TSP #24, using TSP #27 (a copy of TSP #23) overlaying the solution on TSP #24.

G. The four-point switch approach involves finding an initial solution using NN and then modifying it. We can also try different methods of finding initial solutions.

H. One way is to pretend that we're starting at another vertex and apply the NN algorithm to that vertex. Review TSP #28, discuss the Repeated Nearest Neighbor algorithm, and then distribute HO #9 (=TSP #29) with five copies of the diagram on TSP #28. Ask them to apply the NN algorithm to each of the five vertices. Discuss the results using an overlay of TSP #23.

I. Do these algorithms give the best solution? Go back to the four-city problem in TSP #12, and apply both algorithms to that situation. You'll find that RNN doesn't gives the best solution either, because the best solution is the one that avoids altogether the cheapest road, and all three pick it as soon as they see it! In this problem the best solution does not include the shortest edge, but by forgoing the shortest edge it avoids the longest edge.

J. In fact, no algorithm exists which always provides the best solution to the Traveling Salesperson Problem! (Stress the fact that this is a major unsolved problem in mathematics/computer science – see TSP #30.) A brief history of the Traveling Salesperson Problem is on the last page of the Resource Book.

Supplementary Notes

1. As a review of Hamilton and Euler circuits, consider the following "sewer problem": Read the top half of TSP #31 and ask participants to discuss which kind of problem this is; then read the bottom half of TSP #33 and ask participants to discuss which kind of problem this is. Then reinforce the difference between the different kinds of problems. (They will complete the problem on TSP #33 for homework.)

2. The shortest route for the 10 city Traveling Salesperson Problem in Exercise #7 appears to be circuit V to D to C to B to NY to A to M to H to LA to SF to V – for a total of 7847.

About how long is a million seconds?

Would a million seconds be measured in

- **hours?**
- **days?**
- **weeks?**
- **months?**
- **years?**
- **decades?**
- **centuries?**
- **millenia?**

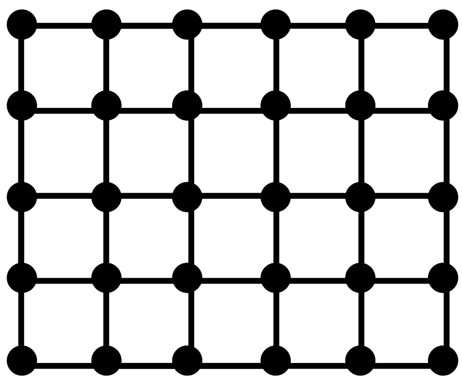
And about how long is a billion seconds?

Hand-out #1: Traffic Light Inspector's Problem

Can you inspect every intersection exactly once and end where you start?

Can you inspect every intersection exactly once if you don't have to end where you start?

1



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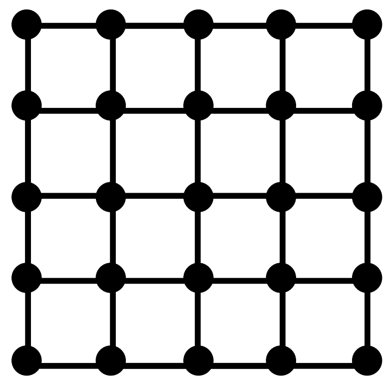

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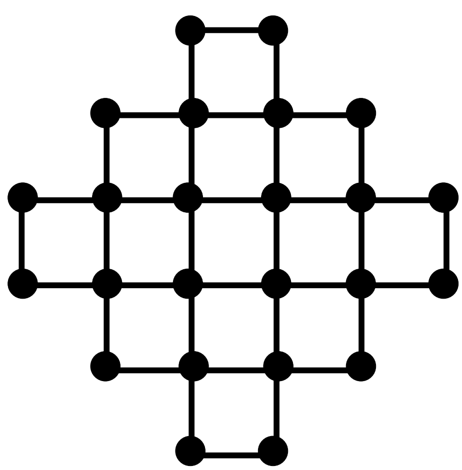


Hamilton path

A path in a graph which visits each vertex exactly once

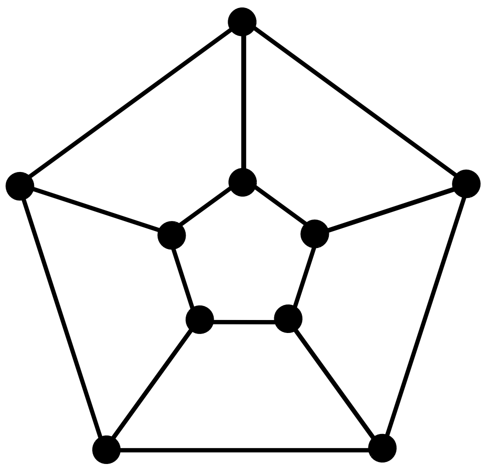
Hamilton circuit

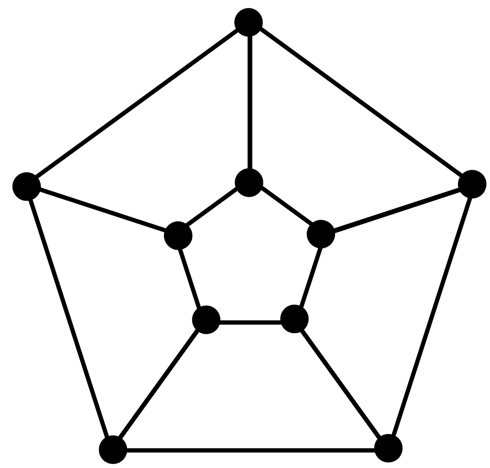
A Hamilton path which ends at the starting point

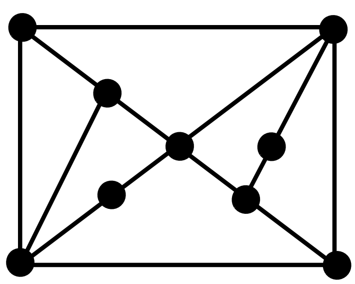
Sir William Rowan Hamilton (1805-1865) was a noted Irish mathematician and physicist who created (and marketed?) a puzzle that involved finding a Hamilton circuit for the vertices of a dodecahedron.

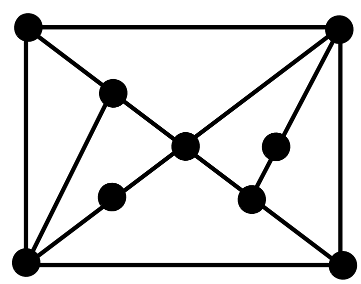
Hand-out #2: **Looking for a Hamilton circuit**

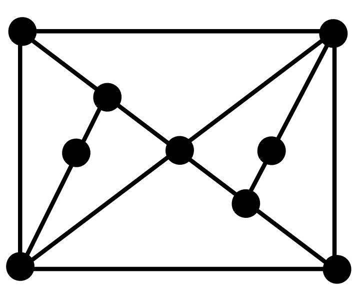
Can you find a Hamilton circuit for the following graphs (two copies of each are provided)? If you can't find one, can you explain why there isn't one?

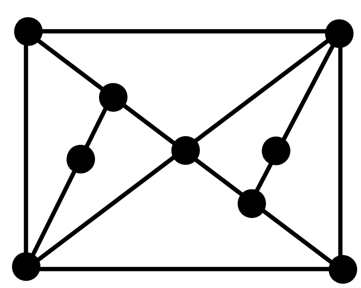






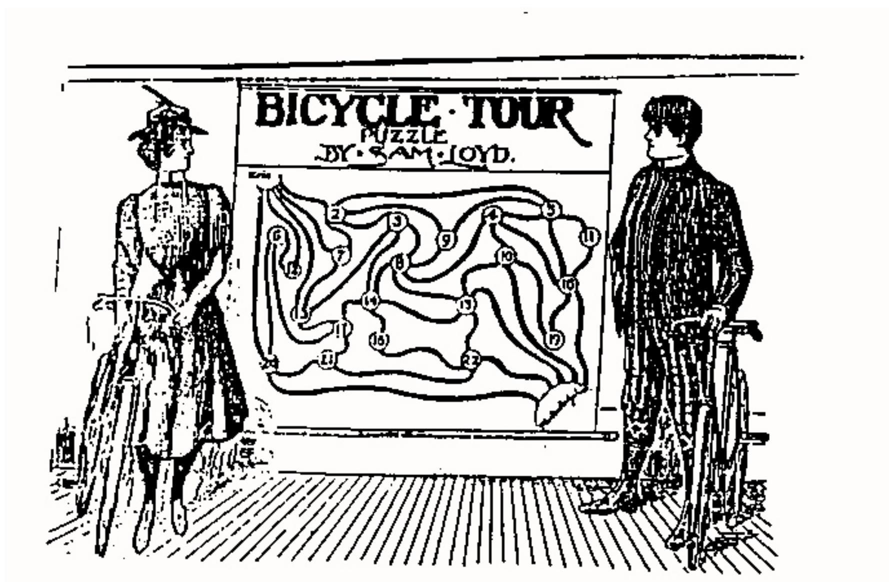


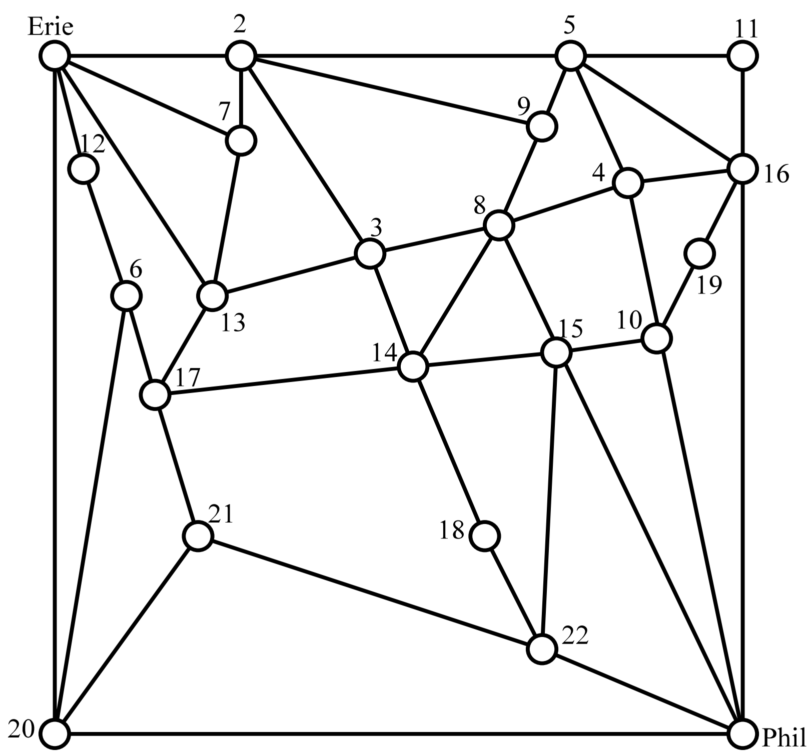




Sam Lloyd's Bicycle Tour

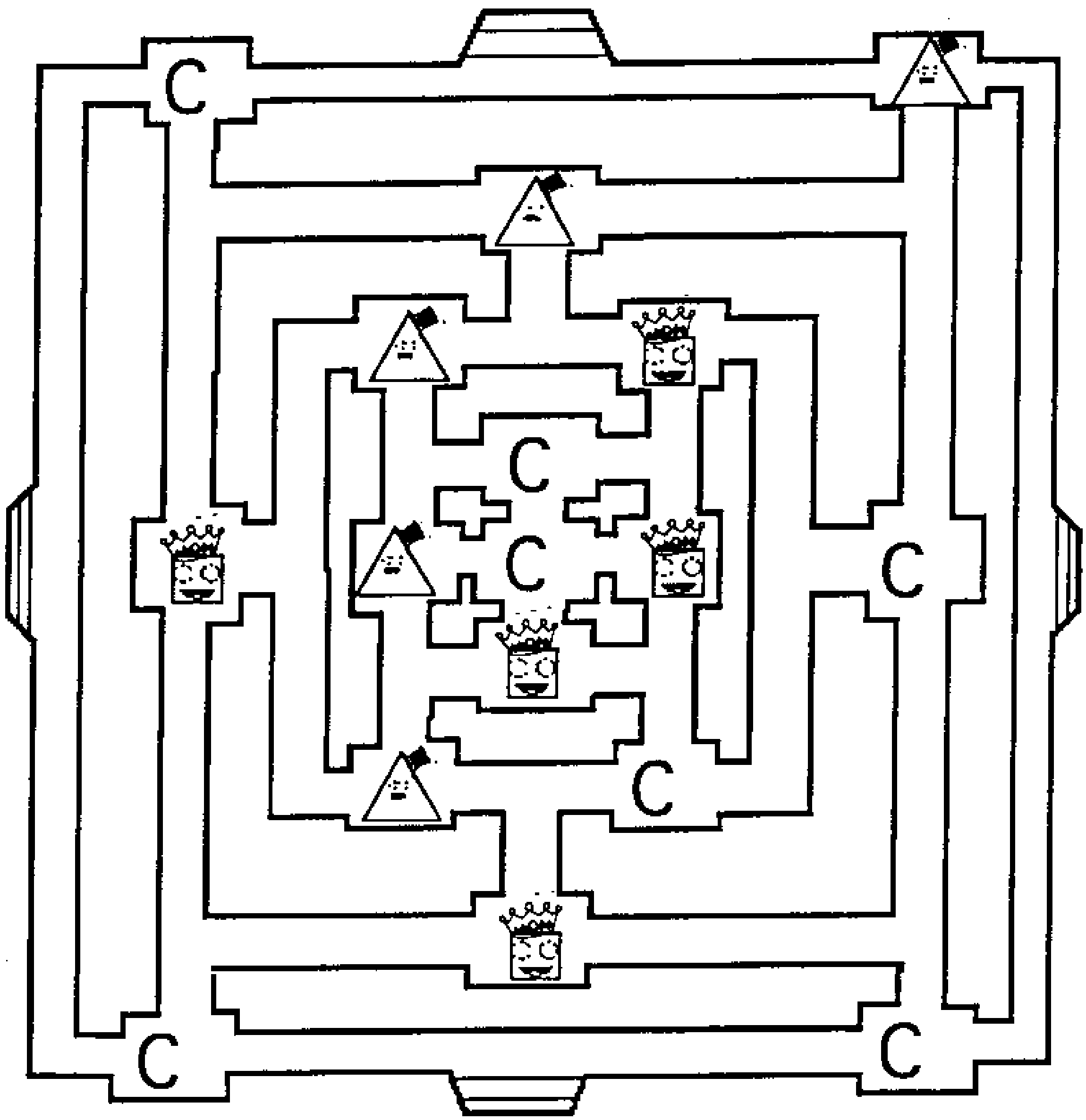
Find a route from Philadelphia to Erie, passing once through each town.





Temple of the Three Gods

Welcome, brave mortal, to our sacred temple. We are the gods Quadratus, Circulus and Triangulus, fated to watch over the treasures the goddess Geometria has scattered throughout the temple. Only the wisest of mortals can steal the treasures by entering the temple through any of the four staircases, visiting all 17 of our likenesses without retracing a path, and leaving by a different staircase. But beware — if you pass two likenesses of the same god in a row, that god will turn you to stone on the spot. Do you dare to enter the temple? (From GAMES Magazine — used with permission.)

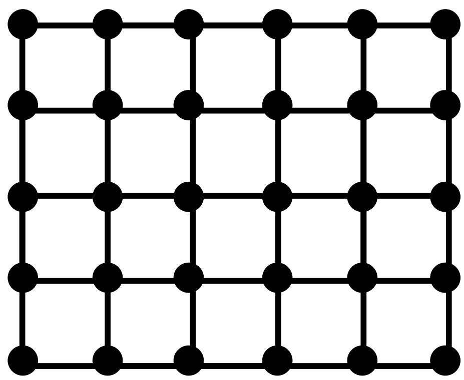


Road Inspector's Problem

Can you inspect every road exactly once and end where you start?

Can you inspect every road exactly once if you don't have to end where you start?

1



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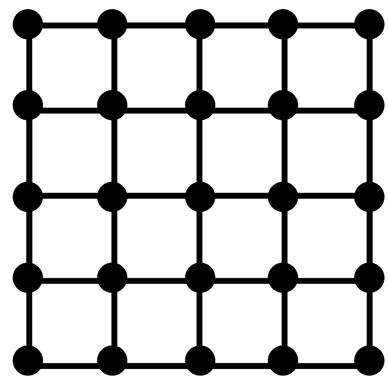

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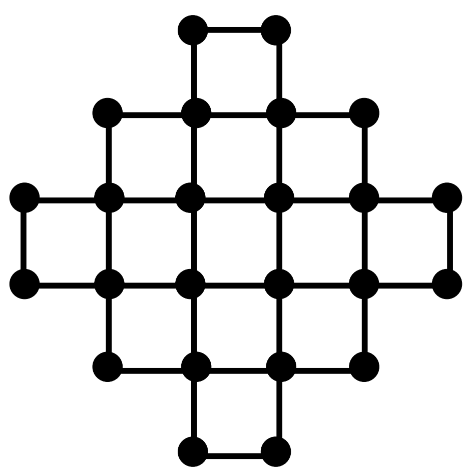


Two parallel problems.

Road Inspector's Problem

Can you inspect every road exactly once and end where you start?

Traffic Light Inspector's Problem

Can you inspect every intersection exactly once and end where you start?

Does a given graph have an Euler circuit?

... or does it have an Euler path?

Does a given graph have a Hamilton circuit?

... or does it have a Hamilton path?

Parallel problems with different answers

Euler circuits vs. Hamilton circuits

Does a given graph have an Euler circuit?

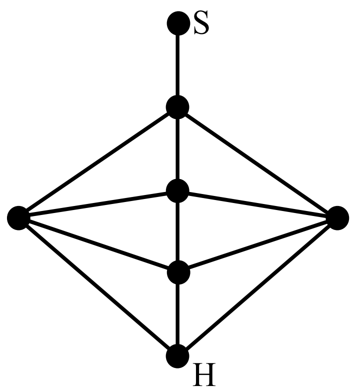
✓ *There is a simple and efficient way of telling whether or not a graph has an Euler circuit*

✓ *and, if there is an Euler circuit, it is easy to find*

Does a given graph have a Hamilton circuit?

✓ *There is no simple and efficient way of telling whether or not a graph has a Hamilton circuit*

✓ *and, even if you know that there is a Hamilton circuit, it may be hard to find*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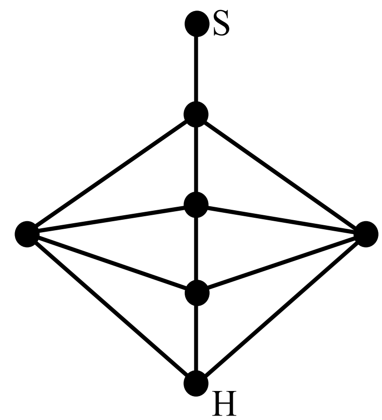
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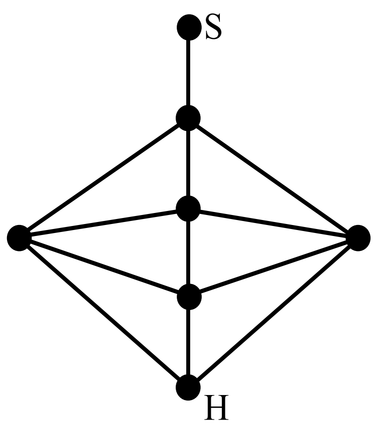
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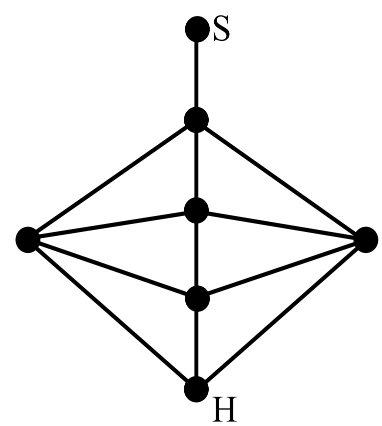
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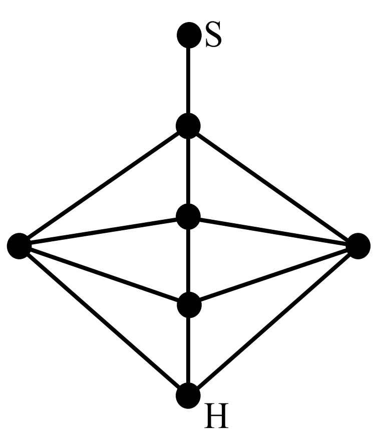
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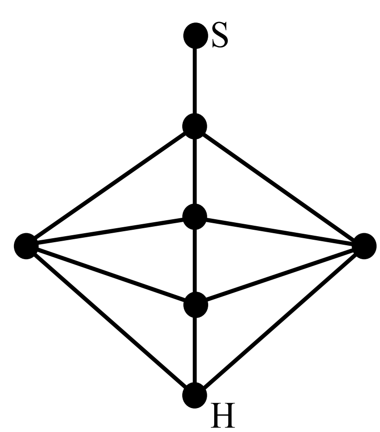
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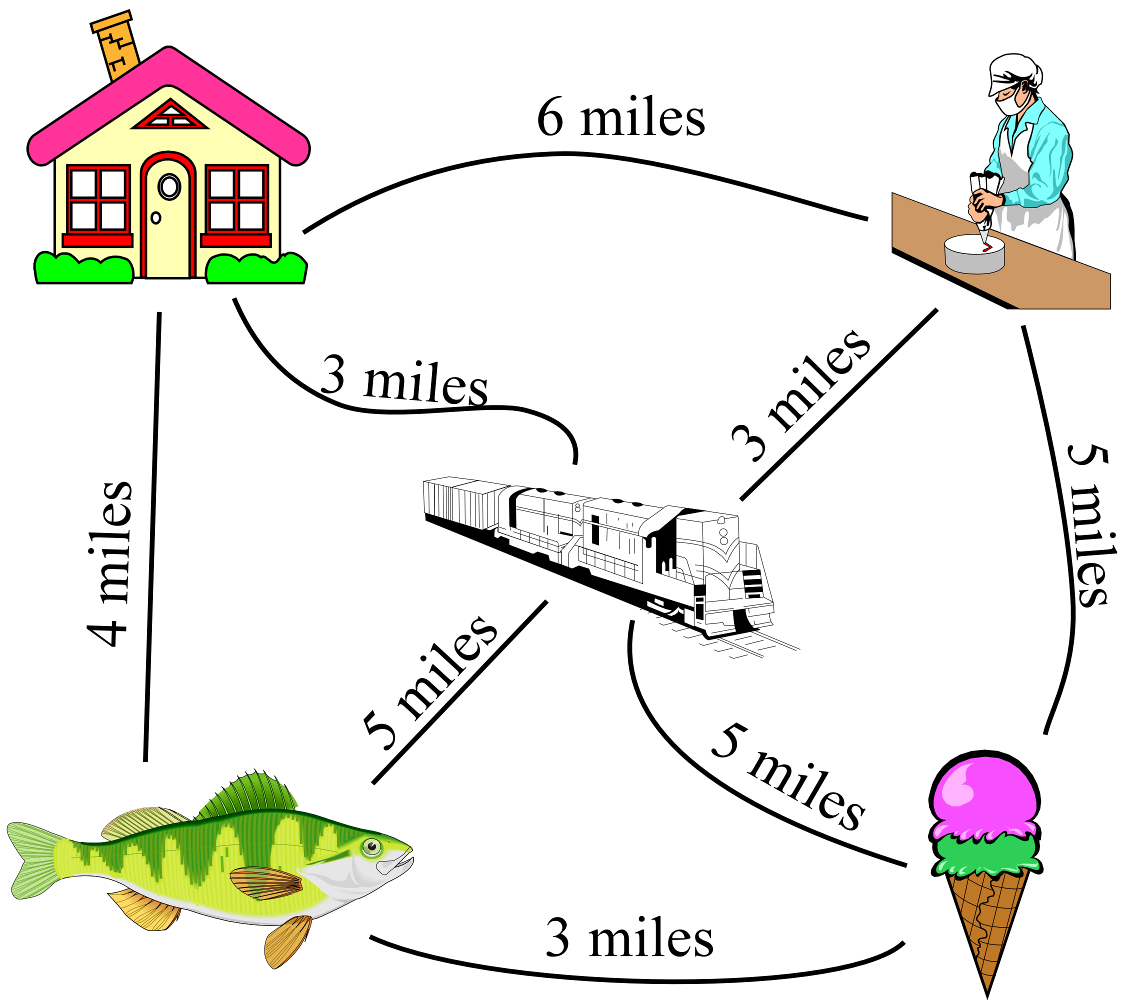
Hand-out #5 **Many errands**

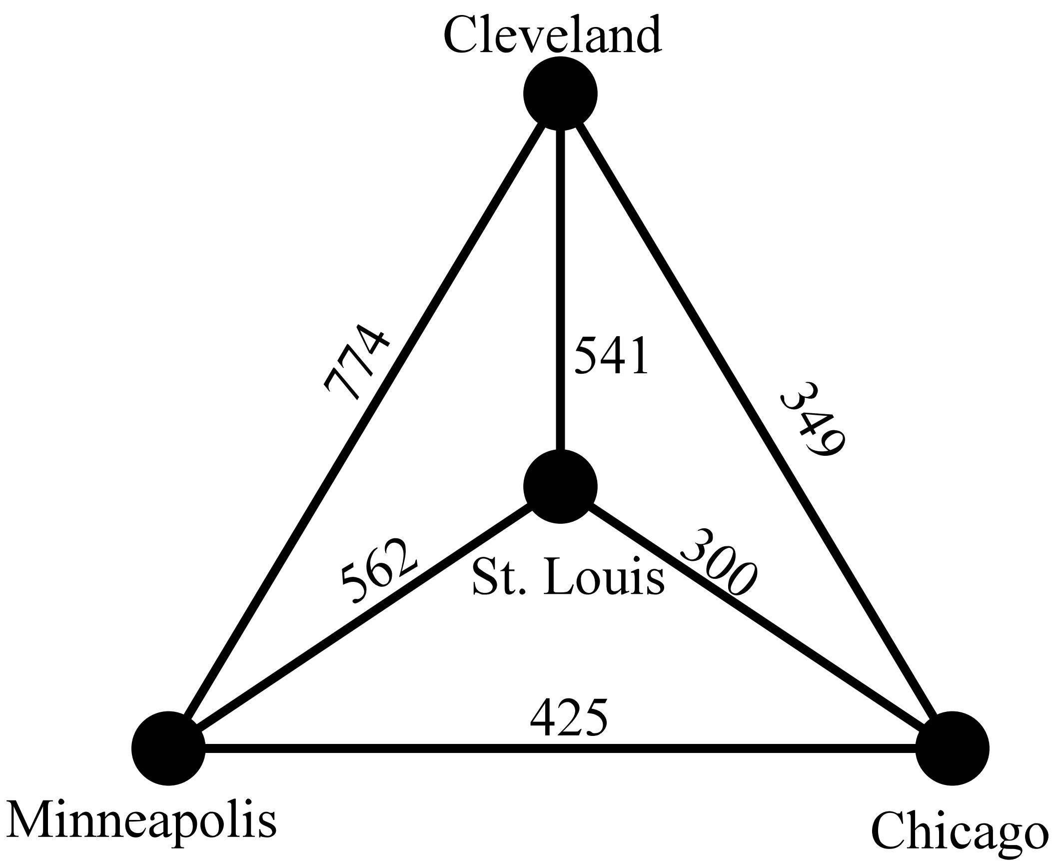
You have so many errands to run! Starting from your house, you have to go to the Fish Store, the Bakery, the Train Station, the Ice Cream Store, and then back home. You can go in any order, but you want to make the shortest trip you can. How many miles will you go?



Developed by Susan Picker, Leadership Program 1990

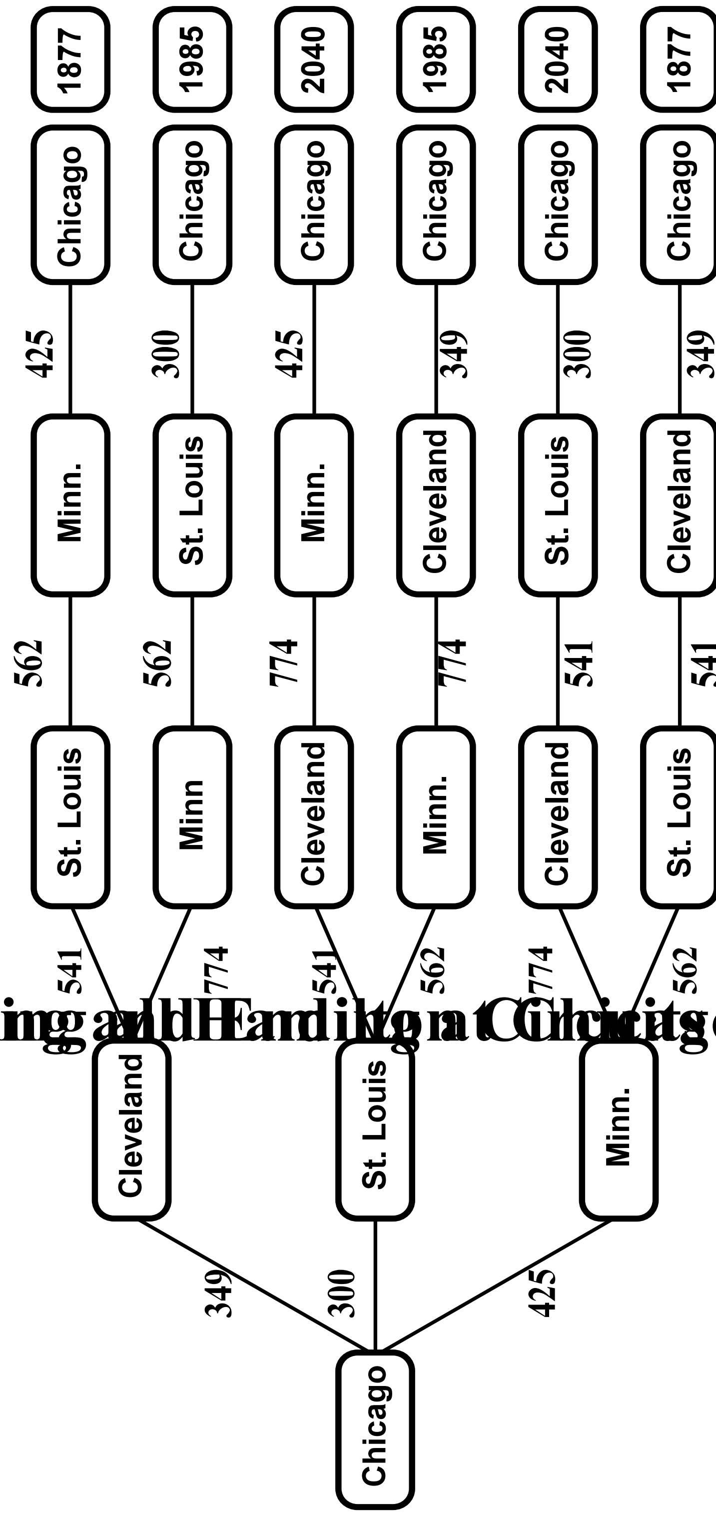
Hand-out #6 Traveling Salesperson Problem — for Four Cities

Use a tree diagram to enumerate all of the routes that begin in Chicago, visit the other three cities and return to Chicago. Which is the shortest route that a traveling salesperson should take?



This is an example of a weighted graph, where each edge is assigned a number, called the weight. In this example, the weight of an edge is the mileage between the two vertices.

Tree showing all Hamilton Circuits Beginning and Ending at Chicago



Applications of the Traveling Salesperson Problem

Everyday examples

- ☒ UPS deliveries
- ☒ ATM machine collections
- ☒ car pools

Industrial examples

- ☒ automated warehouses
- ☒ ocean sampling

Multiplication Principle of Counting

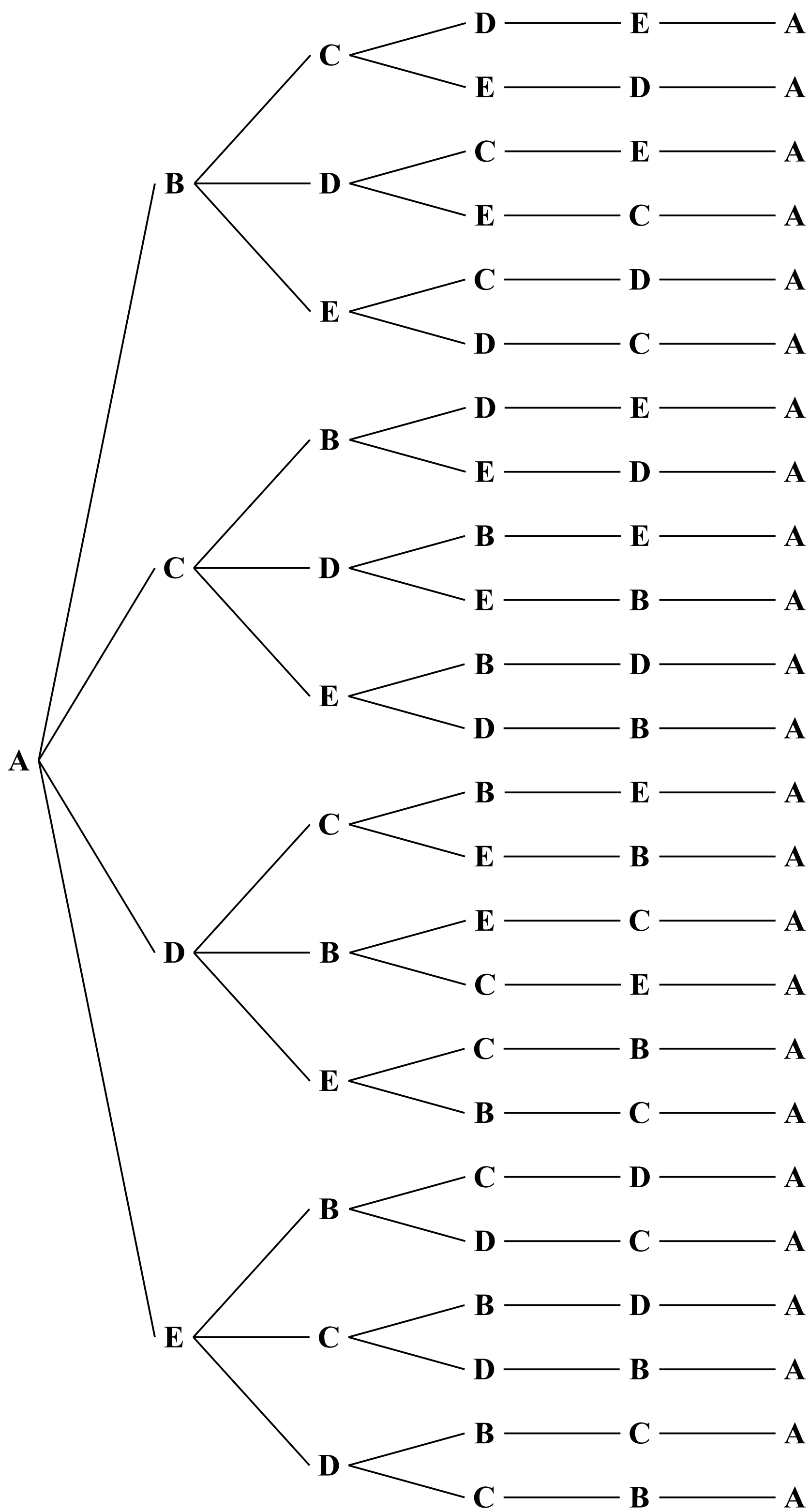
If a task consists of separate parts that are completed *consecutively and independently*, and the first part can be completed in A ways, the second part in B ways, the third part in C ways, etc., then the total number of ways of completing *all parts* of the task is AxBxCx…

- **The number of possible outcomes when you toss a die and then toss a second die is 6x6 = 36.**

- **The number of possible outcomes when you flip a coin four times is 2x2x2x2 = 16.**
- **The number of possible outcomes when you flip a coin and toss a die is 2x6=12.**

- **The number of possible ways of answering T/F to 10 questions is 2^{10}.**

The 24 Circuits on a Graph with 5 Cities



Factorials

$1! = 1$

$2! = 2\times1 = \quad 2 \times 1! = 2 \times 1 = 2$

$3! = 3\times2\times1 = \quad 3 \times 2! = 3 \times 2 = 6$

$4! = 4\times3\times2\times1 = \quad 4 \times 3! = 4 \times 6 = 24$

$5! = 5\times4\times3\times2\times1 = \quad 5 \times 4! = 5 \times 24 = 120$

$6! = 6\times5\times4\times3\times2\times1 = \quad 6 \times 5! = 6 \times 120 = 720$

$7! = 7\times6\times5\times4\times3\times2\times1 = \quad 7 \times 6! = 7 \times 720 = 5040$

$8! = 8\times7\times6\times5\times4\times3\times2\times1 = \quad 8 \times 7! = 8 \times 5040 = 40{,}320$

$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1 = 9 \times 8! =$
$9 \times 40{,}320 = 362{,}880$

$10! = 10\times9\times8\times7\times6\times5\times4\times3\times2\times1 = 10 \times 9! =$
$10 \times 362{,}880 = 3{,}628{,}800$

...

20! is about 2,000,000,000,000,000,000
or two billion billion

This is the total number of possible routes for a traveling salesperson who has to visit 21 cities.

How long would it take a computer to check two billion billion possible routes?

... if it checked one billion routes per second ...

... it would take two billion seconds ...

... and since one billion seconds is about 31.7 years ...

... it would take ...

63.4 years

How about if you had 30 cities?

It would be about a *billion times* as big as that!

Hand-out #8 Nearest Neighbor Algorithm

John lives in Gettysburg and wants to visit eight colleges (besides Gettysburg) before deciding to which ones he should apply. Use the Nearest Neighbor Algorithm to find a possible route that he might take. (Problem by Susan Simon.)

	Sc	Pi	Be	Ge	La	Wi	Re	Ph	Wa
Scranton		272	76	163	123	80	99	130	228
Pittsburgh	272		292	184	237	192	255	304	253
Bethlehem	76	292		127	80	121	49	54	178
Gettysburg	163	184	127		54	124	90	150	82
Lancaster	123	237	80	54		127	31	63	106
Williamsport	80	192	121	124	127		108	166	205
Reading	99	255	49	90	31	108		58	136
Philadelphia	130	304	54	150	63	166	58		136
Washington	228	253	178	82	106	205	136	136	

(The colleges John wants to visit in these cities are, respectively, the University of Scranton, Carnegie-Mellon, Lehigh, Gettysburg, Franklin and Marshall, Lycoming, Albright, University of Pennsylvania, and American University.)

Developed by Susan Simon,
Leadership Program 1989

Nearest Neighbor Algorithm

Nearest Neighbor Algorithm: At each step of the way, go to the nearest neighbor that has not yet been visited.

Use the Nearest Neighbor Algorithm to find a route for a traveling salesperson who begins his journey in Chicago and must visit four other cities before returning to Chicago.

	Chicago	St. Louis	Cincinnati	Philadelphia	Pittsburgh
Chicago		256	254	669	403
St. Louis	256		308	813	553
Cincinnati	254	308		507	256
Philadelphia	669	813	507		267
Pittsburgh	403	553	256	267	

Solution to the Five Cities Problem obtained by using the Nearest Neighbor Algorithm

Can you find a better solution?

Here's a better solution!

The total here is 1741 vs. 1846 (miles) for the Nearest Neighbor Solution!

Visiting eight colleges – which route is shortest?

Nearest Neighbor Algorithm solution

The optimal solution(?)

Applying the "Four Point Switch"

After finding an initial solution to the TSP using the nearest neighbor algorithm, try to improve on your solution using a four-point switch.

Four-point switch:

If A, B, C, and D are four consecutive vertices in the circuit, switch to A, then C, then B, then D if the sum of the weights on AC, CB, and BD is less than the sum of the weights on AB, BC, and CD.

... and here's how it works ...

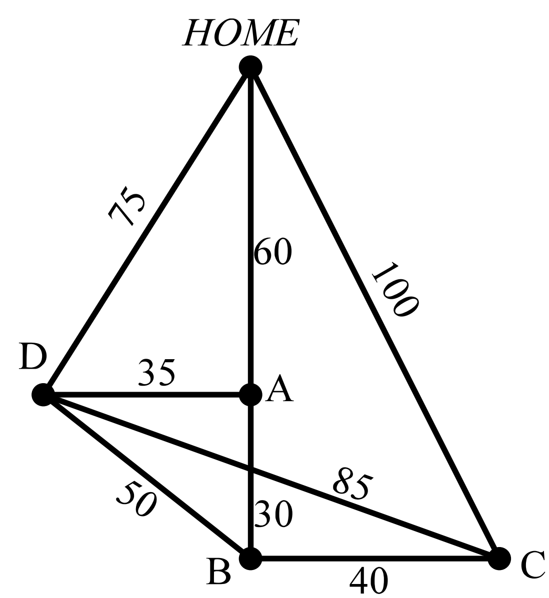
Algorithms for the Traveling Salesperson Problem

Nearest Neighbor Algorithm

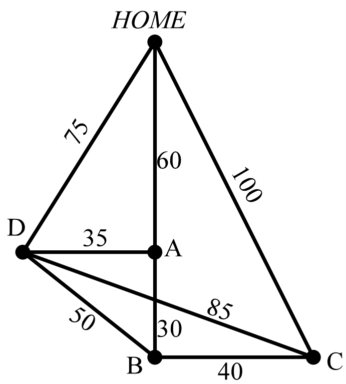
At each step along the way, go to the nearest neighbor which has not yet been visited.

Repeated Nearest Neighbor Algorithm

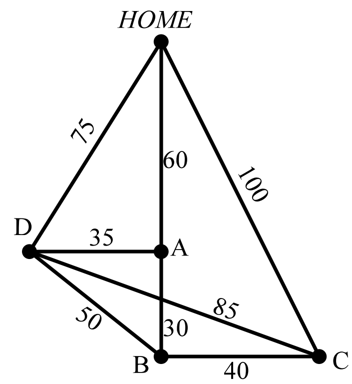
Repeat the Nearest Neighbor Algorithm, starting at each city, to obtain a number of different circuits. Use the circuit with the smallest total mileage, but start the tour at the original city.



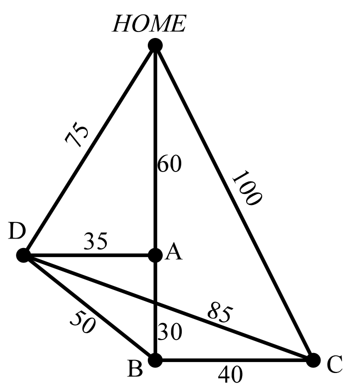
Handout #9: Repeated Nearest Neighbor Algorithm



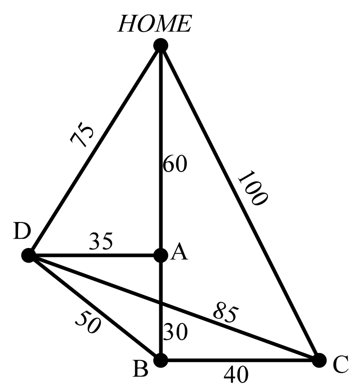
Nearest Neighbor – Start at "HOME"



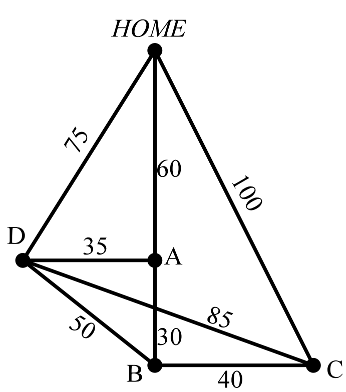
Nearest Neighbor – Start at "A"



Nearest Neighbor – Start at "B"



Nearest Neighbor – Start at "C"



Nearest Neighbor – Start at "D"

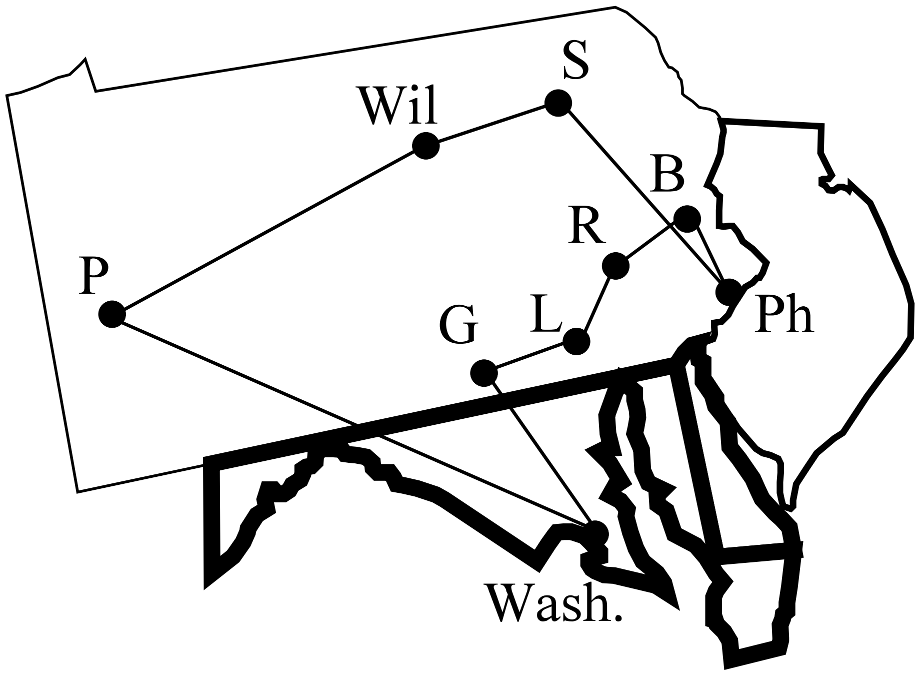
No efficient algorithm exists which always provides the best solution to the Traveling Salesperson Problem!

We could of course look at all possible solutions – the brute force method – but as we have seen that is not an efficient algorithm.

Not only is there no efficient algorithm, but it is not known whether such an algorithm is possible. But no one has proved that such an algorithm can't exist.

If anyone came up with an efficient algorithm that always works ... or showed that no such algorithm exists, they would be eligible for a $1,000,000 prize from the Clay Mathematics Institute – see
www.claymath.org/millennium/P_vs_NP/

The "P vs NP" problem is whether problems like the Traveling Salesperson Problem can be solved efficiently (P) or not (NP).



It would seem to make more sense to go

$$P \to G \to W \to L$$

than

$$P \to W \to G \to L.$$

Indeed, $P \to G \to W \to L$ is a total of

$$184 + 82 + 106 = 372 \text{ miles},$$

whereas, $P \to W \to G \to L$ is a total of

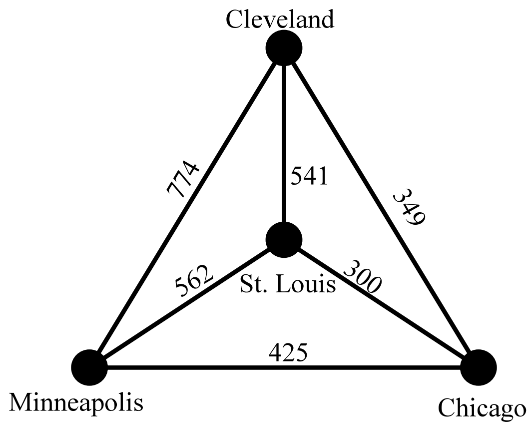
$$253 + 82 + 54 = 389 \text{ miles}.$$

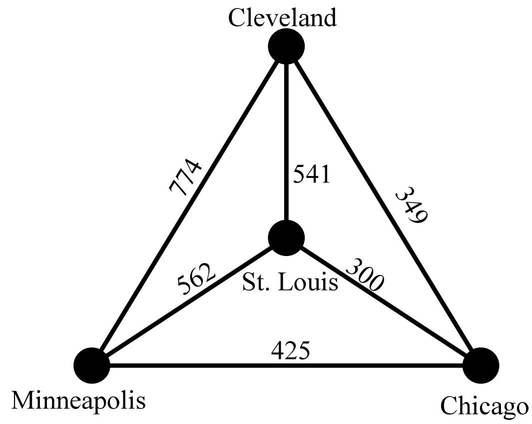
Similarly, it would make sense to go

$S \to B \to Ph \to R$, which is 188 miles,

rather than

$S \to Ph \to B \to R$, which is 233 miles.


Cleveland
774
541
349
562
St. Louis
300
425
Minneapolis
Chicago


Cleveland
774
541
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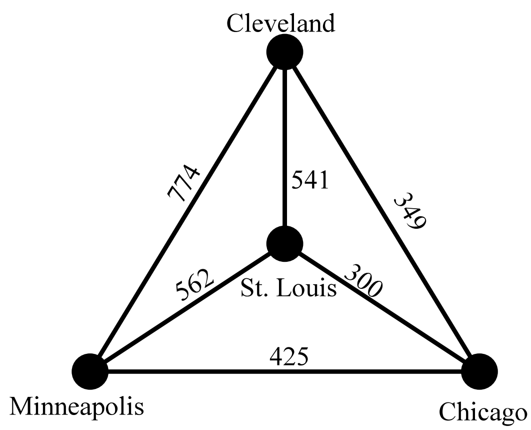
St. Louis
300
425
Minneapolis
Chicago


Cleveland
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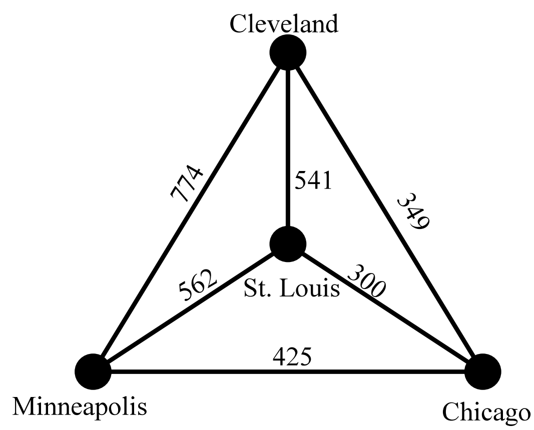
562
St. Louis
300
425
Minneapolis
Chicago


Cleveland
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St. Louis
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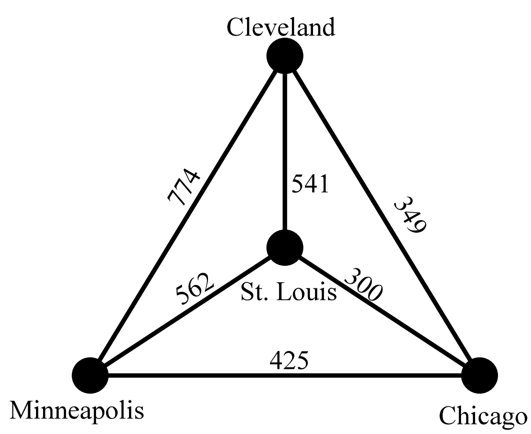
Minneapolis
Chicago


Cleveland
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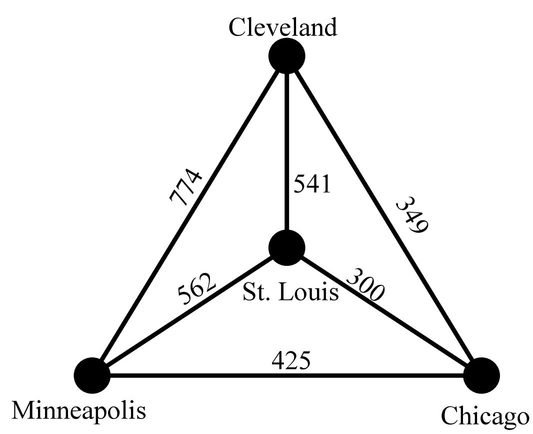
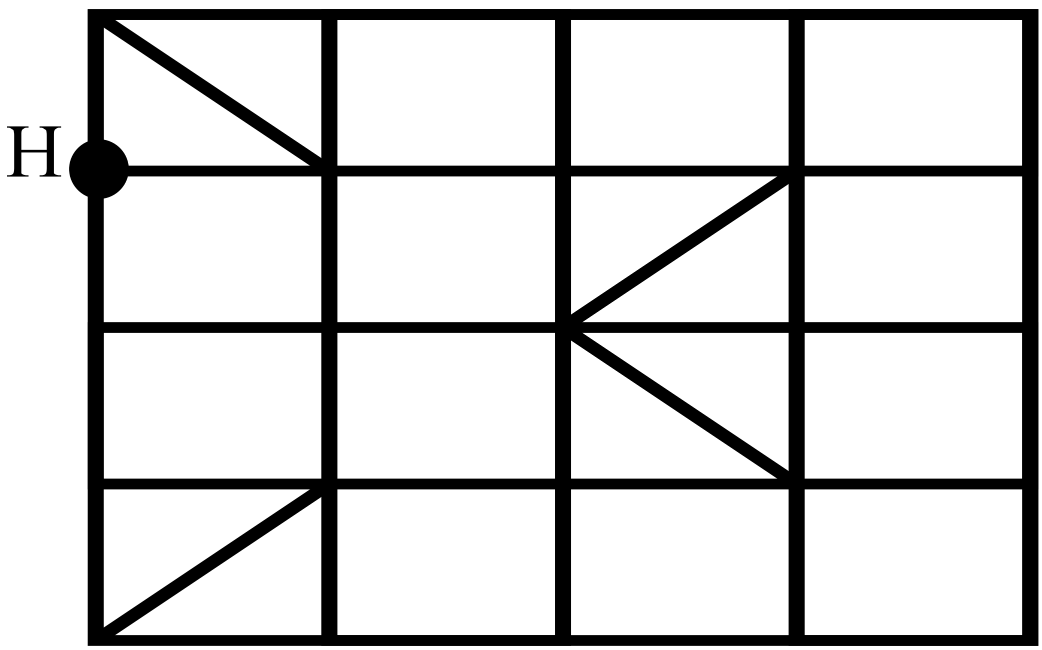

Cleveland
774
541
349
562
St. Louis
300
425
Minneapolis
Chicago

Does this problem involve finding an Euler circuit or a Hamilton circuit?

The following figure represents a town where there is a sewer located at each corner (where two or more streets meet). After every thunderstorm, the department of public works wishes to have a truck start at its headquarters (at vertex H) and make an inspection of sewer drains to be sure that leaves are not clogging them. Can a route start and end at H that visits each corner exactly once?



Assume that at equally spaced intervals along the blocks in this graph there are storm sewers that must be inspected after each thunderstorm to see if they are clogged. Is this a Hamilton circuit problem, an Euler circuit problem, or a Chinese postman problem?

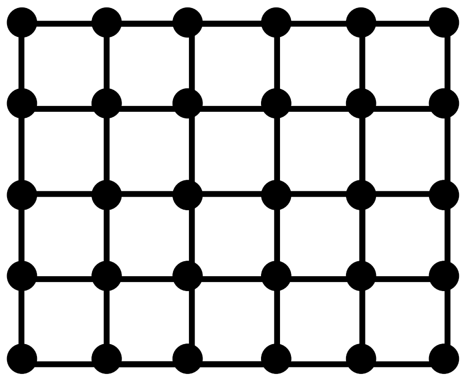
(This problem is from *For All Practical Purposes*, published by COMAP.)

Hand-out #1: Traffic Light Inspector's Problem

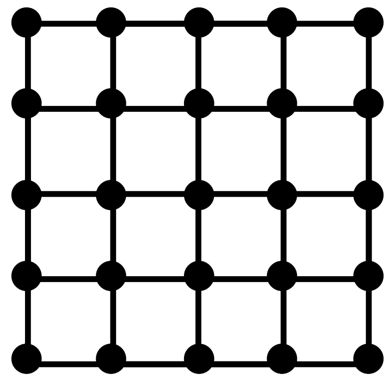
Can you inspect every intersection exactly once and end where you start?

Can you inspect every intersection exactly once if you don't have to end where you start?

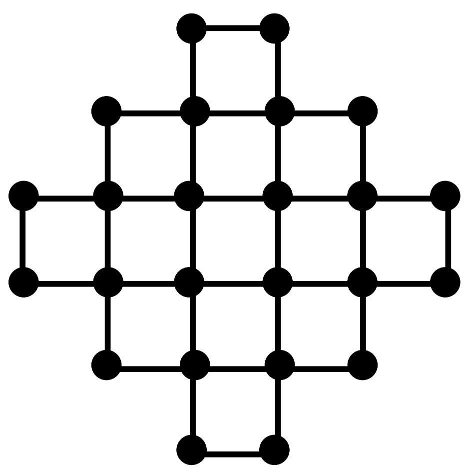
1



2

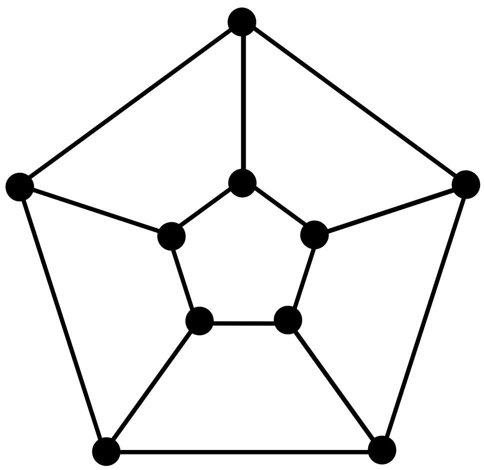


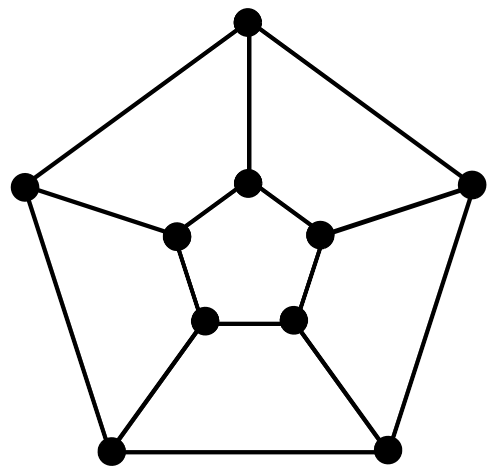
3

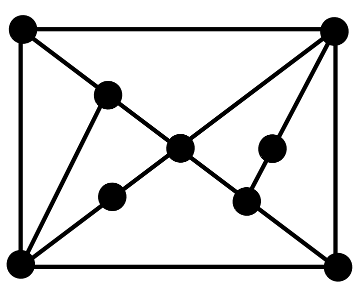


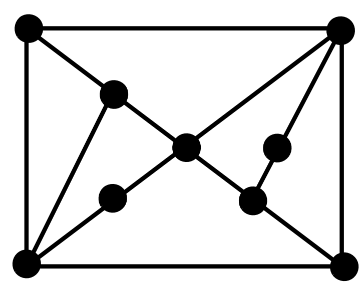
Hand-out #2: Looking for a Hamilton circuit

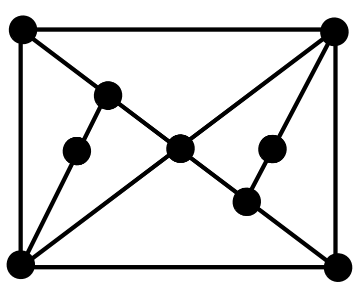
Can you find a Hamilton circuit for the following graphs (two copies of each are provided)? If you can't find one, can you explain why there isn't one?

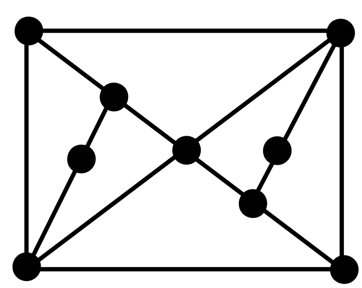






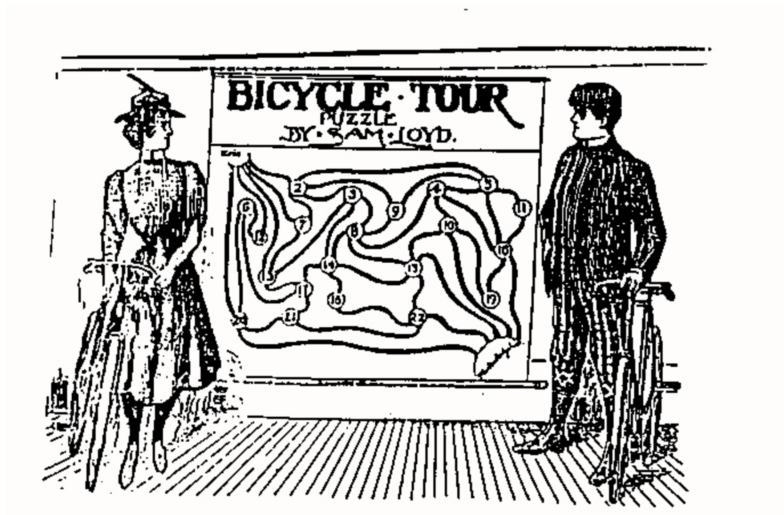


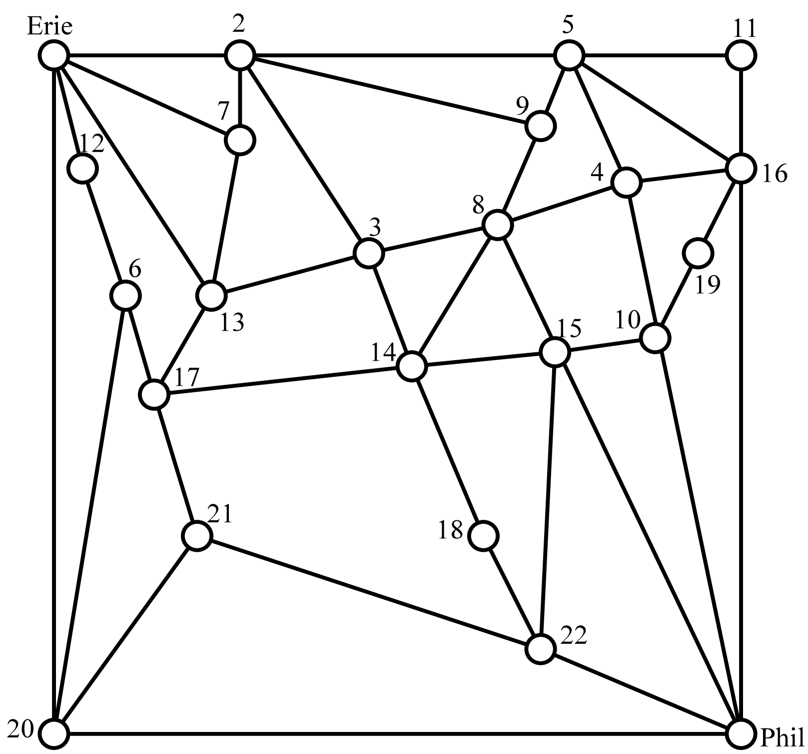




Handout #3 – Sam Lloyd's Bicycle Tour

Find a route from Philadelphia to Erie, passing once through each town.





Hand-out #4: Two Parallel Problems — Traffic Light and Road Inspector

A. Road Inspector's Problem:

Can you inspect every road exactly once and end where you start?

B. Traffic Light Inspector's Problem:

Can you inspect every intersection exactly once and end where you start?

Question A asks whether a given graph has an Euler circuit?

Question B asks whether a given graph has a Hamilton circuit?

These are parallel problems with different answers.

Does a given graph have an Euler circuit?

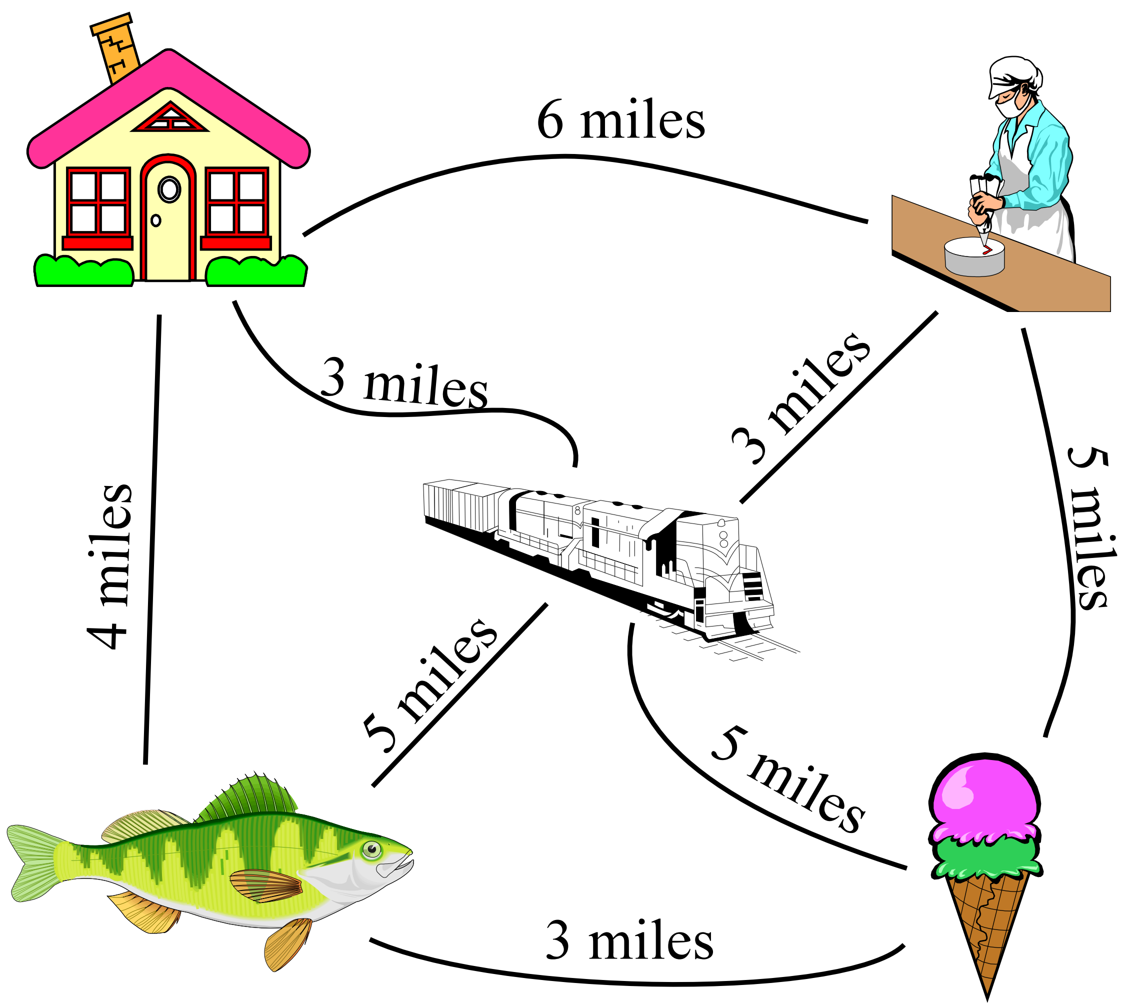
- ✓ ***There is a simple and efficient way of telling whether or not a graph has an Euler circuit***
- ✓ ***and, if there is an Euler circuit, it is easy to find***

Does a given graph have a Hamilton circuit?

- ✓ ***There is <u>no</u> simple and efficient way of telling whether or not a graph has a Hamilton circuit***
- ✓ ***and, even if you know that there is a Hamilton circuit, it may be hard to find***

Hand-out #5: Many errands

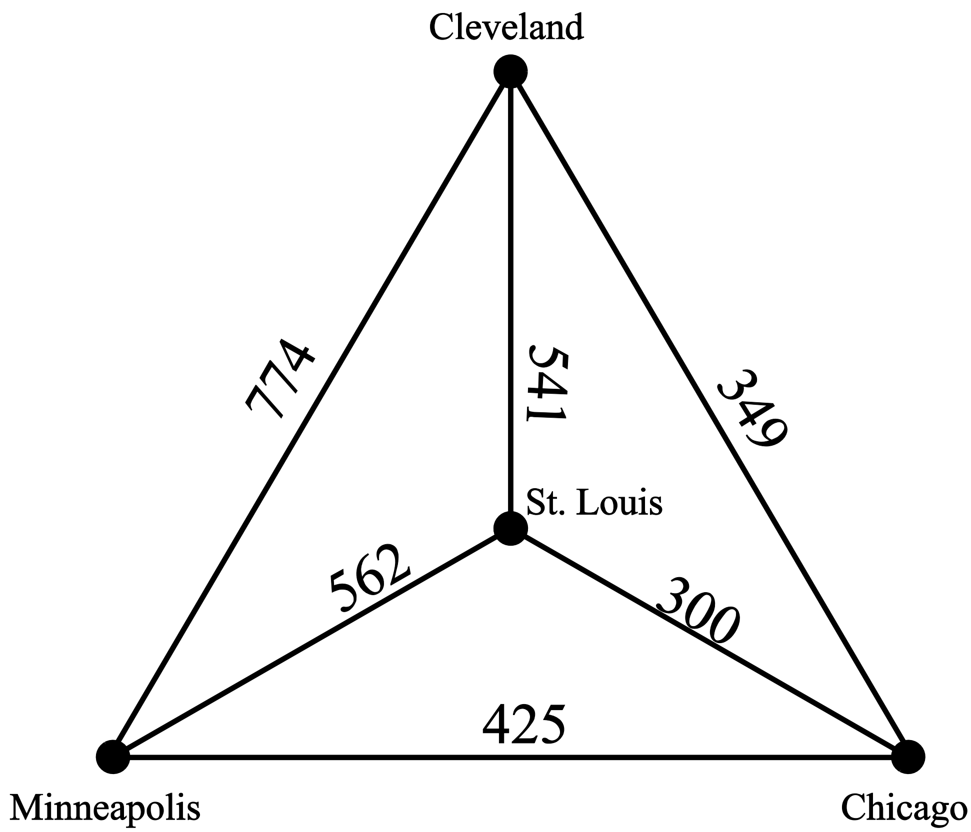
You have so many errands to run! Starting from your house, you have to go to the Fish Store, the Bakery, the Train Station, the Ice Cream Store, and then back home. You can go in any order, but you want to make the shortest trip you can. How many miles will you go?



Developed by Susan Picker
Leadership Program, 1990

Hand-out #6: Visiting three cities

Use a tree diagram to enumerate all of the routes that begin in Chicago, visit the other three cities and return to Chicago. Which is the shortest route that a traveling salesperson should take?



From *For All Practical Purposes,* Freeman, 1988, used with permission

Hand-out #7: Factorials

$1! = 1$	
$2! = 2\times1 =$	$2 \times 1! = 2 \times 1 = 2$
$3! = 3\times2\times1 =$	$3 \times 2! = 3 \times 2 = 6$
$4! = 4\times3\times2\times1 =$	$4 \times 3! = 4 \times 6 = 24$
$5! = 5\times4\times3\times2\times1 =$	$5 \times 4! = 5 \times 24 = 120$
$6! = 6\times5\times4\times3\times2\times1 =$	$6 \times 5! = 6 \times 120 = 720$
$7! = 7\times6\times5\times4\times3\times2\times1 =$	$7 \times 6! = 7 \times 720 = 5040$
$8! = 8\times7\times6\times5\times4\times3\times2\times1 =$	$8 \times 7! = 8 \times 5040 = 40{,}320$
$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1 =$	$9 \times 8! = 9 \times 40{,}320 = 362{,}880$
$10! = 10\times9\times8\times7\times6\times5\times4\times3\times2\times1 =$	$10 \times 9! = 10 \times 362{,}880 = 3{,}628{,}800$
...	

20! is about 2,000,000,000,000,000,000
or two billion billion

This is the total number of possible routes for a traveling salesperson who has to visit 20 cities.

How long would it take a computer to check two billion billion possible routes?

... if it checked one billion routes per second ...

... it would take two billion seconds ...

... which is about ...

63.4 years!

How about if you had 30 cities?

It would be about a *billion times* as big as that!

Hand-out #8: Nearest Neighbor Algorithm

John lives in Gettysburg and wants to visit eight colleges (besides Gettysburg) before deciding to which ones he should apply. Use the Nearest Neighbor Algorithm to find a possible route that he might take. (Problem by Susan Simon.)

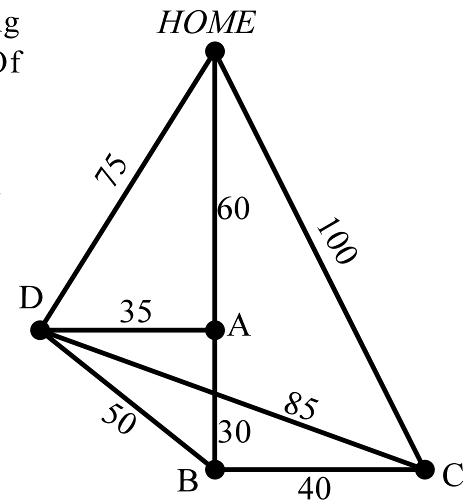
	Sc	Pi	Be	Ge	La	Wi	Re	Ph	Wa
Scranton		272	76	163	123	80	99	130	228
Pittsburgh	272		292	184	237	192	255	304	253
Bethlehem	76	292		127	80	121	49	54	178
Gettysburg	163	184	127		54	124	90	150	82
Lancaster	123	237	80	54		127	31	63	106
Williamsport	80	192	121	124	127		108	166	205
Reading	99	255	49	90	31	108		58	136
Philadelphia	130	304	54	150	63	166	58		136
Washington	228	253	178	82	106	205	136	136	

(The colleges John wants to visit in these cities are, respectively, the University of Scranton, Carnegie-Mellon, Lehigh, Gettysburg, Franklin and Marshall, Lycoming, Albright, University of Pennsylvania, and American University.)

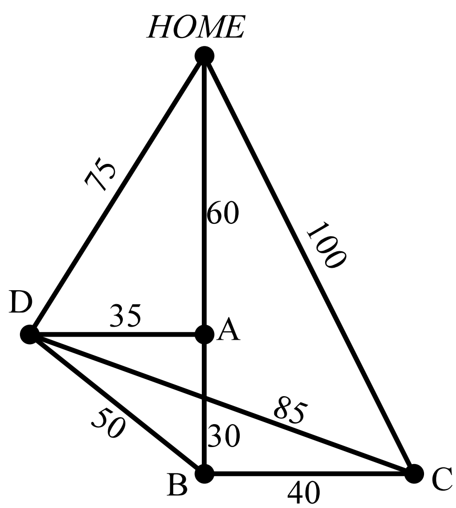
Developed by Susan Simon,
Leadership Program 1989

Hand-out #9: The Nearest Neighbor Algorithm, over and over.

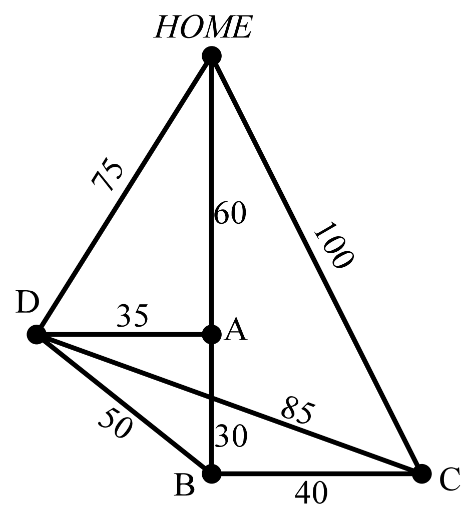
Apply the Nearest Algorithm to the given graph, pretending in each case that you are starting at the indicated vertex. Of course, once you find the best route, you would follow it starting from "home". The method of trying the Nearest Neighbor Algorithm over and over is called the "Repeated Nearest Neighbor Algorithm", since you repeat the algorithm starting at each vertex.



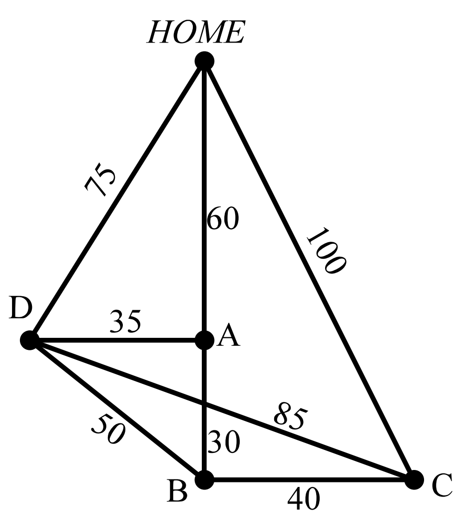
Nearest Neighbor — Start at "HOME"



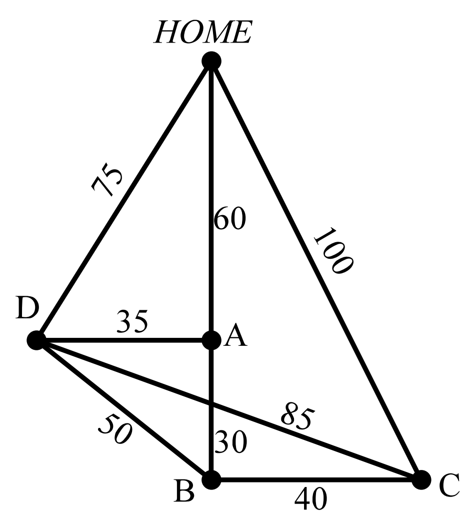
Nearest Neighbor — Start at "A"



Nearest Neighbor — Start at "B"



Nearest Neighbor — Start at "C"

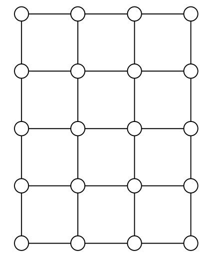


Nearest Neighbor — Start at "D"

Workshop 3 — Hamilton Circuits and the Traveling Salesperson Problem — Exercises

Practice problems:

1. Find a Hamilton circuit in the grid to the right.



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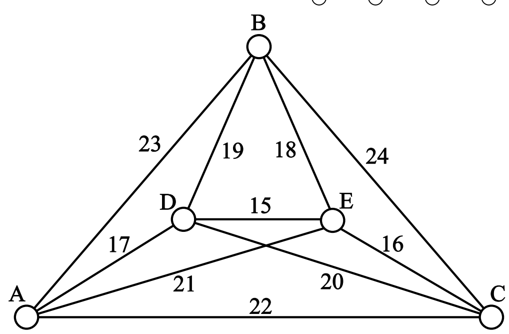
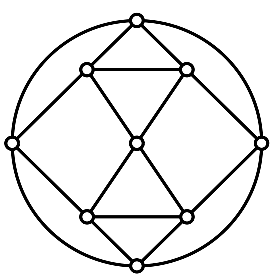
Study group problems:

2. Each person at the table picks a different one of the five vertices A, B, C, D, E. (See diagram at right.) Each person then uses the nearest neighbor algorithm to find a route starting and ending at his/her vertex and making stops at the other four vertices. Which of the Hamilton circuits is cheapest? Can you find a Hamilton circuit that is cheaper than all of the nearest neighbor routes.

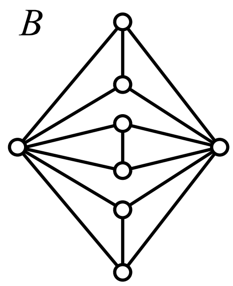


3. For each of the following graphs, find a Hamilton circuit or a Hamilton path. (Number the edges in the order used so that you can identify the path easily.) If you cannot find a Hamilton circuit, try to explain why none exists.

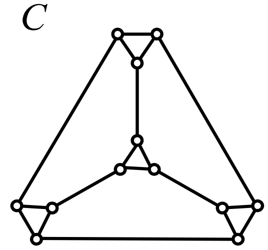
A



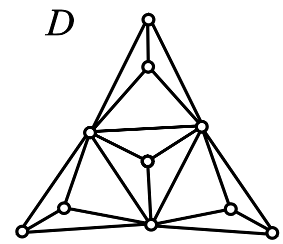
B



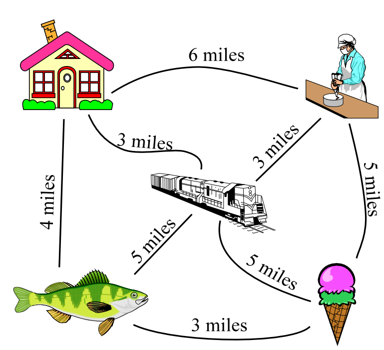
C



D



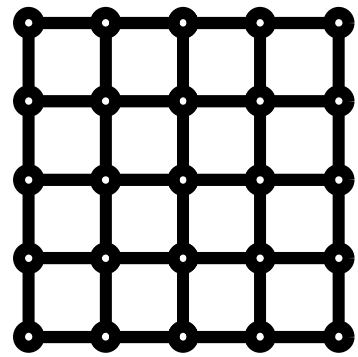
4. Find the best solution to the "Many Errands" problem in Hand-out #<u>4</u> (at right) by using a tree-diagram.



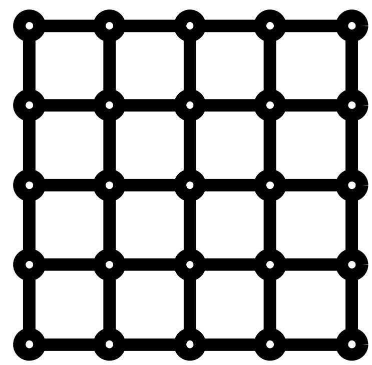
5. Each person at the table chooses a different starting point and uses the nearest neighbor algorithm to find a good route for a traveling salesperson who must visit the ten cities on the map below, starting and ending at his/her chosen point. Which of those routes is shortest? Starting with each route, can you find a shorter route by changing the order in which a few cities are visited?

	Atlanta	Boston	Chicago	Denver	Houston	Los Angeles	Miami	New York	San Francisco	Vancouver
Atlanta	0	946	606	1323	696	1942	597	755	2135	2228
Boston	946	0	851	1770	1606	2597	1256	190	2699	2516
Chicago	606	851	0	921	941	1746	1189	714	1859	1776
Denver	1323	1770	921	0	878	831	1726	1632	950	1052
Houston	696	1606	941	878	0	1374	969	1419	1645	2340
Los Angeles	1942	2597	1746	831	1374	0	2340	2452	347	1071
Miami	597	1256	1189	1726	969	2340	0	1089	2595	2803
New York	755	190	714	1632	1419	2452	1089	0	2572	2433
San Francisco	2135	2699	1859	950	1645	347	2595	2572	0	791
Vancouver	2228	2516	1776	1052	2340	1071	2803	2433	791	0

6. Explain why there is no Hamilton circuit in the diagram at the right (which appeared in Hand-out #1). (Suggestion: Construct a vertex coloring of this graph by alternating two colors, red and blue, and consider the answers to the following questions: If you walked along a path in this graph, what colors would you see? How many steps would there be in a Hamilton path in this graph? How many steps would there be in a Hamilton circuit? If you start at a red vertex and walk along a Hamilton circuit, what would be the color of the last vertex you reach before you came back to the start?)



7. The figure at the left represents a town where there is a sewer located at each corner, that is, where two or more streets meet. (Note the similarity between this graph and the graph in the preceding problem.) After every thunderstorm, the department of public works wishes to have a truck start at its headquarters (at vertex H) and make an inspection of sewer drains to be sure that leaves are not clogging them. (From *For All Practical Purposes* — COMAP.)



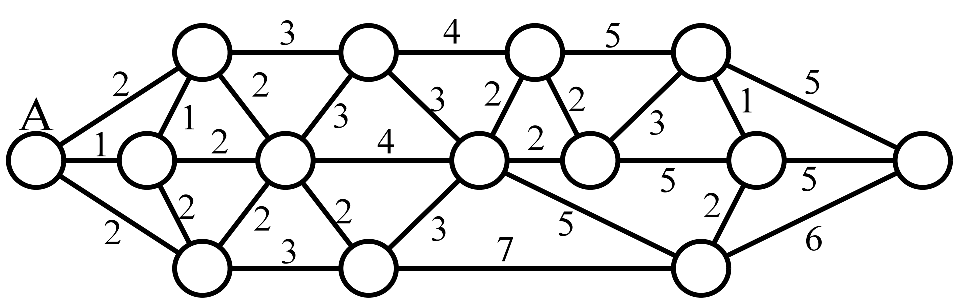
a. Can a route start and end at H that visits each corner exactly once?

b. Can you find two different routes each of which uses one of the two different diagonal streets?

d. Is there a route that uses both diagonal streets?

e. Is there a route that uses none of the diagonal streets?

8. A postman has to collect mail from public mailboxes located at each *corner* in the neighborhood pictured below. What is the shortest Hamilton circuit you can find?



9. Construct four different graphs, G, H, I and J, each with five vertices, so that:

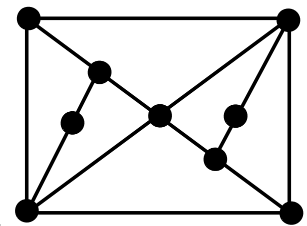
a. Graph G has neither a Hamilton circuit nor an Euler circuit

b. Graph H has a Hamilton circuit but not an Euler circuit

c. Graph I has no Hamilton circuit but does have an Euler circuit

d. Graph J has both a Hamilton circuit and an Euler circuit

10. Can you explain why the graph at the right has no Hamilton circuit?

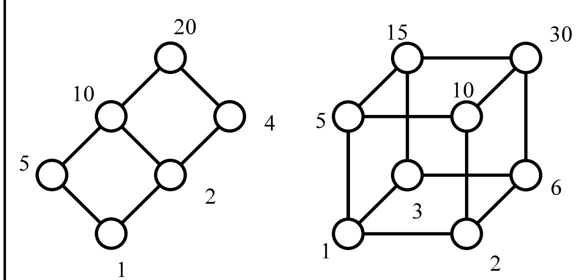


Extension problems:

11. For amusement, you might try the following problem of "The Homicidal Necrophobe": Think of a 4x4 grid as a cell-block. Each cell in the grid contains a prisoner; adjacent cells are connected by doors. One night all doors are accidentally left unlocked. This fact is discovered by the prisoner in the top left cell who happens to be a homicidal necrophobe, and who proceeds to kill all of the other prisoners in their own cells. Since he would never enter a room which contains a corpse, how can it happen that he ends up huddled in a corner in the bottom right cell?

12. Hand-out #1 discussed the Traffic Light Inspector's Problem (TLIP) for the case where the town was laid out on a grid involving four blocks in one direction and five blocks in the other direction. Try solving the TLIP for towns which are laid out on grids of various other sizes, such as 1x2, 2x2, 1x3, 2x3, 3x3, etc. Can you find a pattern for when the TLIP can be solved?

13. The number 20 has 6 factors: 1, 2, 4, 5, 10 and 20. If you let these factors be labels on the vertices of a graph, and draw an edge between two vertices if (and only if) their ratio is a prime number, then you get the graph shown to the right. As you can see, it has a Hamilton circuit. Similarly, if you draw the factor graph for 30, you get the other graph on the right, and it too has a Hamilton circuit.



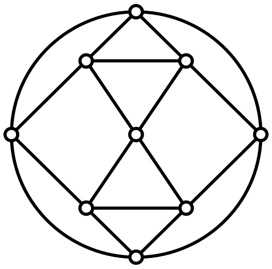
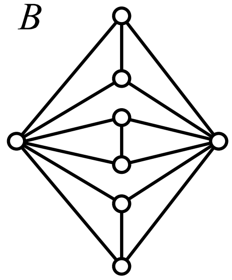
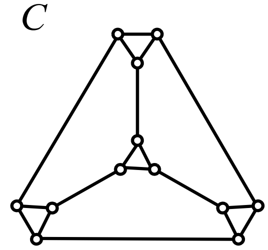
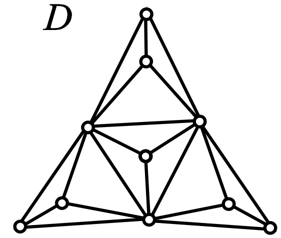
Factor graph for 20 Factor graph for 30

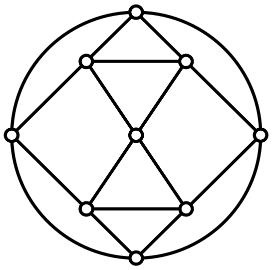
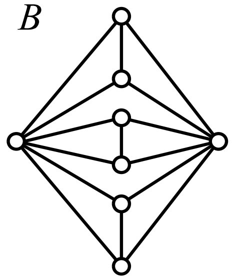
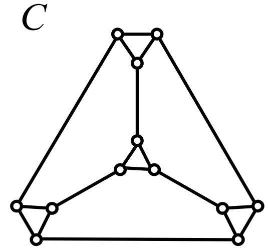
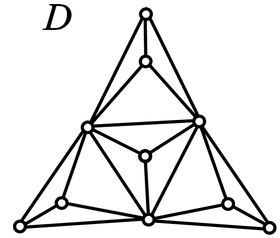
   a. Draw factor graphs for all of the numbers from 1 (just a single vertex) to 30
   b. Which of those graphs turn out to have Hamilton circuits?
   c. Can you make a conjecture regarding which factor graphs have Hamilton circuits?

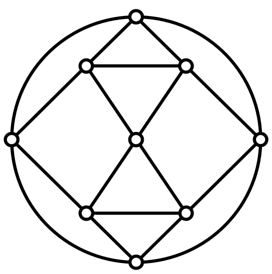
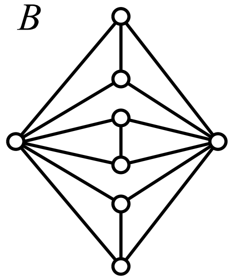
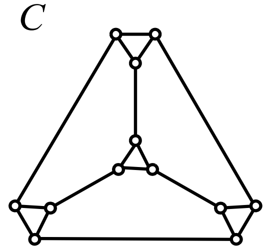
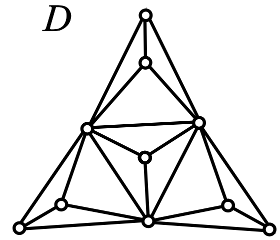
14. Can you come up with a rule based on problem 4b which says that if such-and-such happens in a graph, then it does not have a Hamilton circuit?

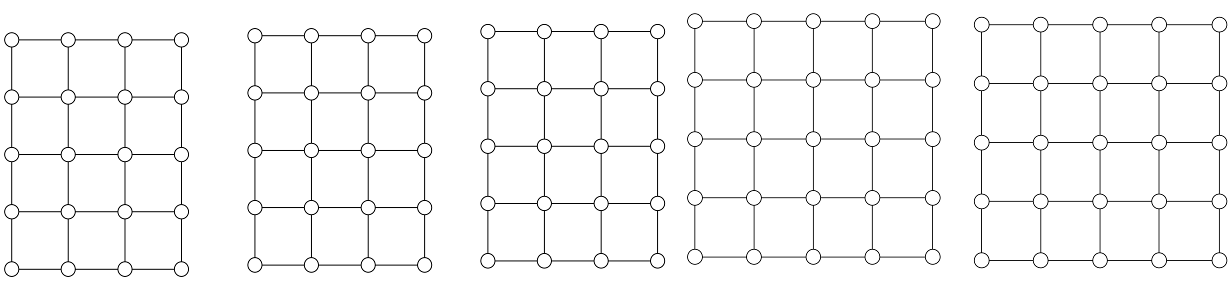
15. Use the four-point switch method to find a better solution to the Traveling Salesman Problem involving five cities than the solution generated by the Nearest Neighbor Algorithm. Use the four-point switch method a second time to find the optimal solution described in class.

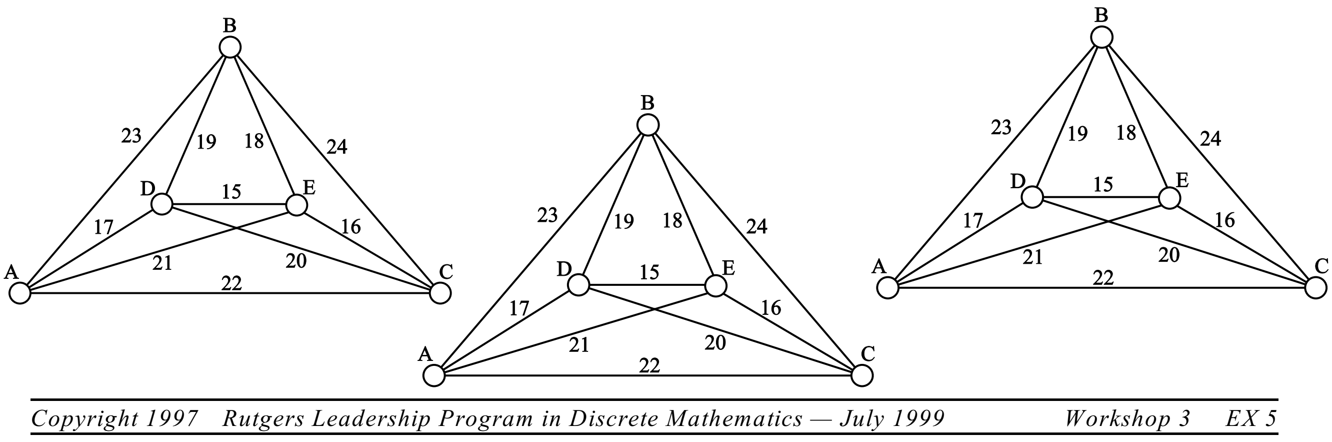
Four-point switch: If A, B, C, and D are four consecutive vertices in the circuit, switch to A, C, B, D if the sum of the weights on AC, CB, and BD is less than the sum of the weights on AB, BC, and CD.

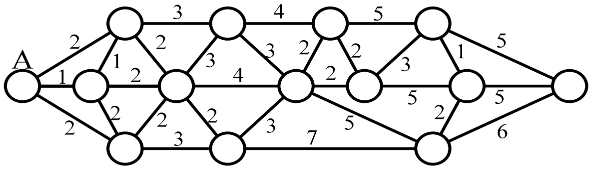
A  *B*  *C*  *D* 

A  *B*  *C*  *D* 

A  *B*  *C*  *D* 



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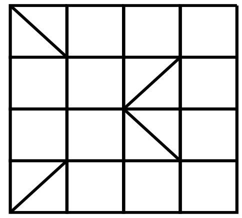

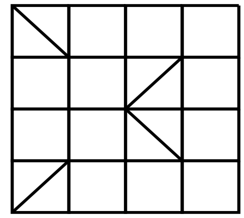

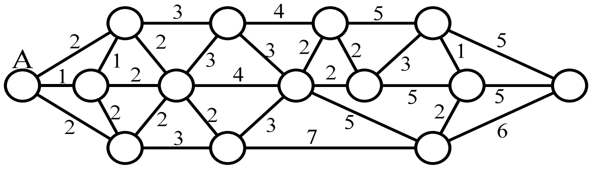
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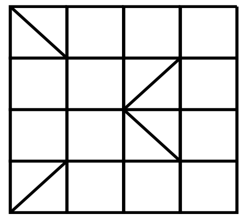

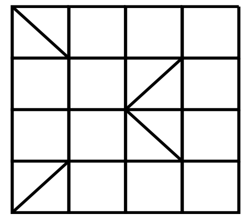

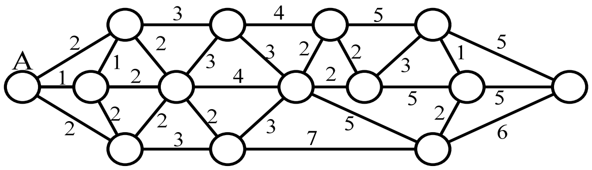
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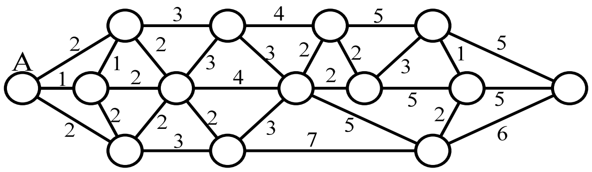
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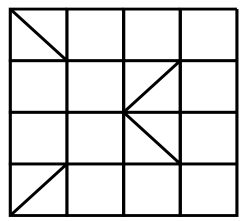

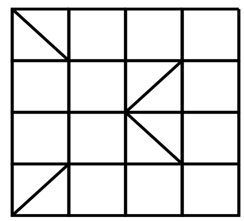

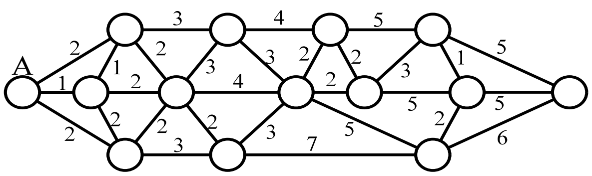
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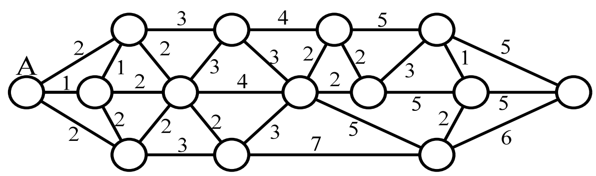
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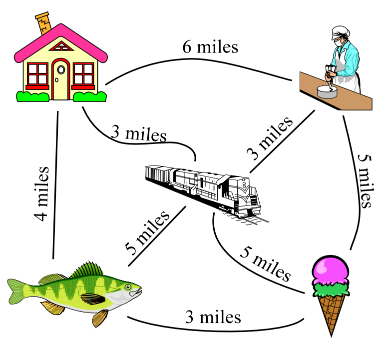
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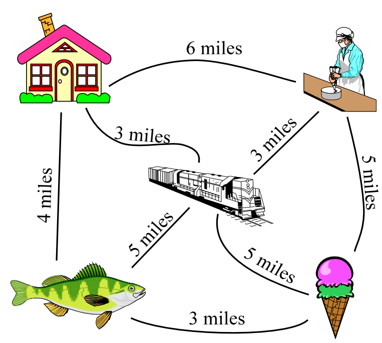
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Resource Book

Workshop 3: Hamilton Circuits and the Traveling Salesperson Problem

Table of Contents

The Resource Book contains activities that teachers can use in their classes in addition to those discussed in the institute workshop on the topics of Hamilton circuits and the Traveling Salesperson Problem.

Pages 1-3, in the section "Mathematical Background", contains the terminology introduced in this workshop. Pages 4-5 contain an outline of the Leadership Program's workshop on "Hamilton Circuits and the Traveling Salesperson Problem."

Pages 6-13 contain a variety of problems dealing with Hamilton paths and circuits, and a discussion (on page 9) of the parallels and differences between the search for Euler circuits and the search for Hamilton circuits.

Pages 14-32 focus on the Traveling Salesperson Problem — including a discussion of factorials on page 18, three activities (pages 26-31) where participants have to measure the distances between sites in the problem, and an historical account on page 32.

Mathematical Background

✓ **Traffic Light Inspectors' Problem:** Is it possible for a traffic light inspector to make an inspection tour of all intersections on a map, passing through each intersection exactly once, and ending back at the starting point?

✓ **Hamilton circuit** — A "**Hamilton circuit**" is a path which goes through each vertex exactly once and returns to the starting point.

✓ **Hamilton path** — A "**Hamilton path**" is a path which goes through each vertex exactly once but ends at a point other than the starting point.

✓ **Does a given graph have a Hamilton circuit?** There is <u>no</u> simple and efficient way of telling whether or not a graph has a Hamilton circuit; furthermore, even if you know that there is a Hamilton circuit, it may be hard to find.

✓ **Contrast with same question for Euler circuit —** There is a simple and efficient way of telling whether or not a graph has an Euler circuit (just check all of the vertices to see if any has odd degree); furthermore, one you know that the graph has an Euler circuit, it's easy to find one.

✓ **Road Inspector's Problem:** Is it possible for a road inspector to make an inspection tour of all roads on a map, traveling over each road exactly once and ending back at the starting point?

Mathematical Background (continued)

✓ **Contrast between Traffic Light Inspector's Problem and Road Inspector's Problem:** Although the two problems are started in almost identical ways, one problem is easy to solve (since it involves Euler circuits) and one is hard to solve (since it involves Hamilton circuits).

✓ **Traveling Salesperson Problem:** A traveling salesperson has to visit a number of cities in any order and return to the starting location; what route should be used so as to minimize the total distance (or cost)? Another way of expressing this problem is: In a complete graph where every edge has a cost attached (such as the mileage), find a Hamilton circuit for which the total cost of the edges is as little as possible.

✓ **Applications of the Traveling Salesperson Problem:** There are many daily situations which can be modeled by the traveling salesperson problem, ranging from UPS deliveries (in which order should the packages be delivered?) to ATM machine collections to car pooling (in which order should the kids be picked up?). There are also many less apparent applications, such as robots filling orders in an automated warehouse (in which order should the items be collected?), preparation of computer chips (in which order should the holes be drilled?), and collection of samples of ocean water (in which order should they be collected).

Resource: An interesting computer program which involves finding good solutions to the traveling salesperson program is the *Scavenger Hunt* on the Macintosh software *Turtle Math*, by Douglas H. Clements and Julie Sarama Meredith, Logo Computer Systems Inc. (LCSI), 1994.

✓ **Algorithm** — An algorithm is a mechanical method which can be applied to many problems of the same type; examples are the standard method for multiplying two numbers, the instructions for preparing a cake or programming a VCR, and computer programs.

✓ **Nearest Neighbor Algorithm:** The Nearest Neighbor Algorithm for the traveling salesperson problem determines a route for the salesperson by choosing at each step along the way to go to the closest city that has not yet been visited.

✓ **Cheapest Link Algorithm:** The Cheapest Link Algorithm for the traveling salesperson problem determines a route by choosing to travel along those edges which provide the cheapest links between two cities; more precisely, a path is created by choosing possibly disconnected edges, at each step adding the shortest edge which neither closes a loop (unless it includes all cities) nor ends up with three edges meeting at a city.

Resource Book

Workshop 3: Hamilton Circuits and the Traveling Salesperson Problem

Mathematical Background (continued)

✓ **Repeated Nearest Neighbor Algorithm:** The Repeated Nearest Neighbor Algorithm for the traveling salesperson problem determines a route by first repeating the Nearest Neighbor Algorithm, starting at each city, to obtain a number of different plausible routes, and then selects from those the route with the smallest total mileage. Although the best route selected by the Repeated Nearest Neighbor Algorithm may be based at a different city than where the traveling salesperson is based, that is where the salesperson begins the route.

✓ **Greedy algorithm:** A greedy algorithm is one where you do what seems best at the moment without paying attention to the consequences of your decisions. The Nearest Neighbor Algorithm is an example of a greedy algorithm. The Cheapest Link Algorithm is less greedy than the Nearest Neighbor Algorithm because it takes into consideration all edges of the graph. The Repeated Nearest Neighbor Algorithm is also less greedy than the Nearest Neighbor Algorithm because it also takes a more global view. However, all three are still relatively greedy because they will likely select a cheapest edge even though that may force the later selection of a very expensive edge.

✓ **Factorial notation n!:** The notation **n!** is used to represent the product of all the counting numbers from 1 to n; for example, 6! = 6x5x4x3x2x1. This is the explicit rule for factorials. The observation that 6! = 6 x 5! leads to the recursive rule for factorials, that n! = n x (n-1)!, where each factorial is described in terms of the previous factorial.

✓ **Number of routes for the traveling salesperson:** If there are **n** cities, then the first city visited (after the home city) can be chosen in **n-1** ways, the next city in **n-2** ways, etc. The total number of tours is **(n-1)•(n-2)•(n-3)•...•3•2•1** which is **(n-1)!**. Another way of saying this is that the complete graph with **n** vertices has **(n-1)!** Hamilton circuits. Of these, half are repeats of the other half, traveled backward. There are thus altogether **(n-1)!/2** distinct Hamilton circuits.

Resource Book

Workshop 3: Hamilton Circuits and the Traveling Salesperson Problem

Workshop Outline

1. **Traffic Light Inspector's Problem**
   a. Determine whether there is a solution to this problem in several examples.
   b. Introduce the ideas of Hamilton path and Hamilton circuit, and determine whether several graphs have Hamilton paths or circuits.
   c. There is no rule for determining whether a graph has a Hamilton circuit or path. This is quite different from the problem discussed in the previous workshop where there was a simple rule for determining whether or not a graph has an Euler path or circuit.
   d. Introduce road inspector problem for which, despite the parallel to the traffic light inspector problem, solutions can be found easily.

2. **Traveling Salesperson Problem**
   a. Use "errand problem" as introduction to the Traveling Salesperson Problem.
   b. Discuss possible solution strategies — such as "trial and error" and "list all possible paths" (tree diagram) — and determine the solution.
   c. Solve four-cities Traveling Salesperson Problem using a tree diagram. Reminder that these are examples of weighted graphs, a term used in the snow video in the previous workshop.

3. **Exponential growth and combinatorial explosion.**
   a. Take opportunity to count the number 3! = 6 of possible solutions in four-city problem, and to note that since each route can be run backwards, there are really only three distinct possibilities.
   b. Note that if there are five cities, then there would be 4! = 24 possible solutions to consider and show a tree diagram for those 24 solutions. If there were six cities, there would be 5! = 120 possible solutions. Note that one reason we did several problems involving tree diagrams was to become convinced that we don't want to solve larger problems using tree diagrams.
   c. Proceed to introduce both the standard and recursive definitions of "factorial" and to discuss the size of 10! and 20! which is about 2 billion billion.
   d. If a computer could review a billion possible solutions each second and determine the cost of each of those paths, then it would still take two billion seconds for it to find the best path.
   e. How long is a billion seconds? A simpler problem — how long is a million seconds? Elicit the answer that a million seconds is about 12 days, so that two billion seconds is about 12x2000 days, or about 70 years. Even if computers were a thousand times faster, they would be stymied by a problem involving a few additional cities. There must be another way to deal with Traveling Salesperson Problems!

Resource Book

Workshop 3: Hamilton Circuits and the Traveling Salesperson Problem

4. **Algorithms for the Traveling Salesperson Problem**
   a. Introduce "nearest neighbor algorithm (NNA)". Place this in the context of the idea of an "algorithm" — a mechanical method which can be applied to many different but similar problems. Apply the NNA to the four cities problem.
   b. Show by example that the NNA does not always give the best solution. Discuss the notion of a "greedy algorithm" and explain why such an algorithm often fails.
   c. Apply the "cheapest link algorithm" and the "repeated nearest neighbor algorithm" to the eight colleges problem.
   d. No algorithm exists which always provides the best solution to the Traveling Salesperson Problem.

5. **Applications of the Traveling Salesperson Problem**
   a. Common applications are to UPS deliveries, ATM collections, and car pools
   b. Industrial applications are to automated warehouses (where robots use a Traveling Salesperson Problem algorithm to determine in which order to retrieve the drugs on a list), drilling of computer chips (in which the drills have to be tole the order in which to drill the thousands of holes they need to make), and water sampling (where marine biologists have to decide in which order to collect water samples from hundreds of sites).

Resource Book

Workshop 3: Hamilton Circuits and the Traveling Salesperson Problem

Traffic Light Inspector's Problem and Road Inspector's Problem

A. Traffic Light Inspector's Problem:

Can you inspect every intersection exactly once and end where you start?

Can you inspect every intersection exactly once if you don't have to end where you start?

B. Road Inspector's Problem:

Can you inspect every road exactly once and end where you start?

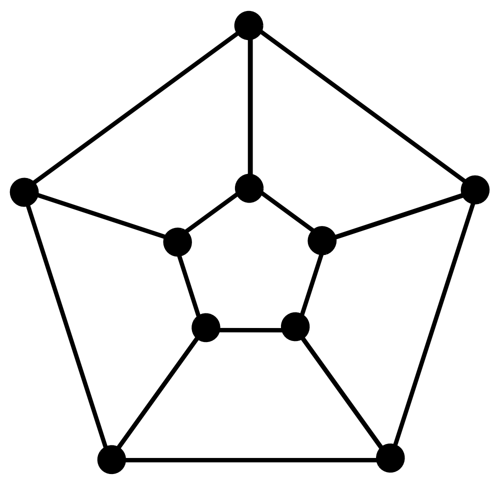
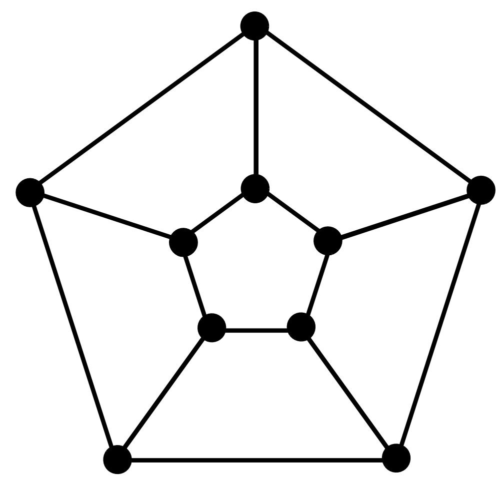
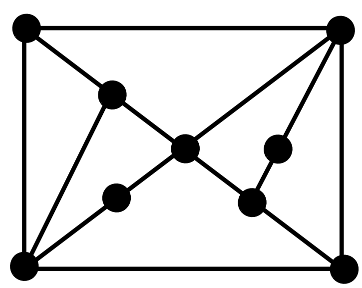
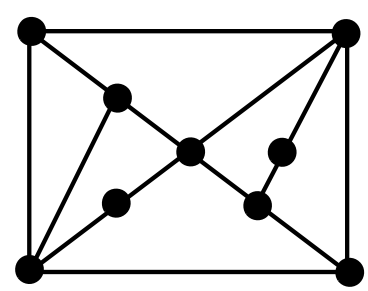
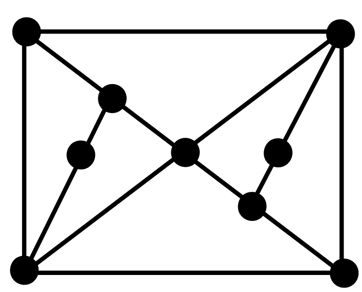
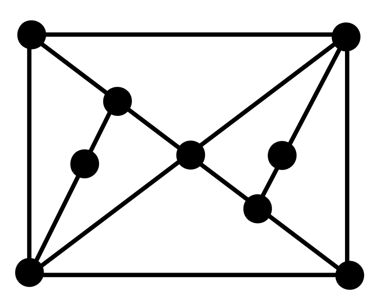
Can you inspect every road exactly once if you don't have to end where you start?

Resource Book

Workshop 3: Hamilton Circuits and the Traveling Salesperson Problem

Hand-out #2: Looking for a Hamilton circuit

Can you find a Hamilton circuit for the following graphs (two copies of each are provided)? If you can't find one, can you explain why there isn't one?

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A. Road Inspector's Problem:

Can you inspect every road exactly once and end where you start?

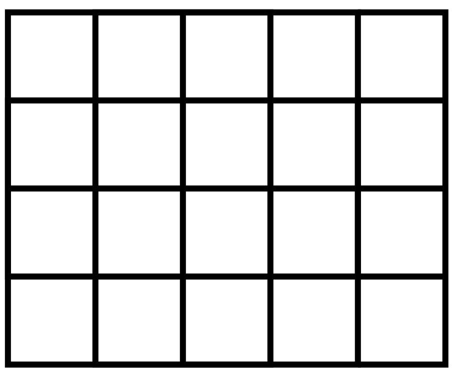
Can you inspect every road exactly once if you don't have to end where you start?

A. Traffic Light Inspector's Problem:

Can you inspect every intersection exactly once and end where you start?

Can you inspect every intersection exactly once if you don't have to end where you start?

1



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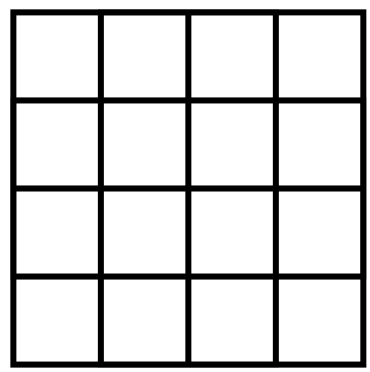

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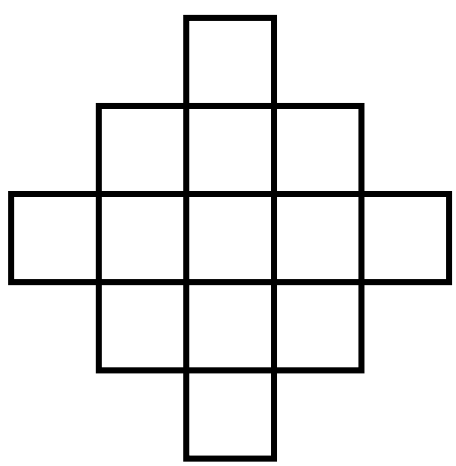

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Two Parallel Problems — Traffic Light and Road Inspector

A. Road Inspector's Problem:

Can you inspect every road exactly once and end where you start?

A. Traffic Light Inspector's Problem:

Can you inspect every intersection exactly once and end where you start?

Question A asks whether a given graph has an Euler circuit?

Question B asks whether a given graph has a Hamilton circuit?

These are parallel problems with different answers.

Does a given graph have an Euler circuit?

✓ *There is a simple and efficient way of telling whether or not a graph has an Euler circuit*

✓ *and, if there is an Euler circuit, it is easy to find*

Does a given graph have a Hamilton circuit?

✓ *There is <u>no</u> simple and efficient way of telling whether or not a graph has a Hamilton circuit*

✓ *and, even if you know that there is a Hamilton circuit, it may be hard to find*

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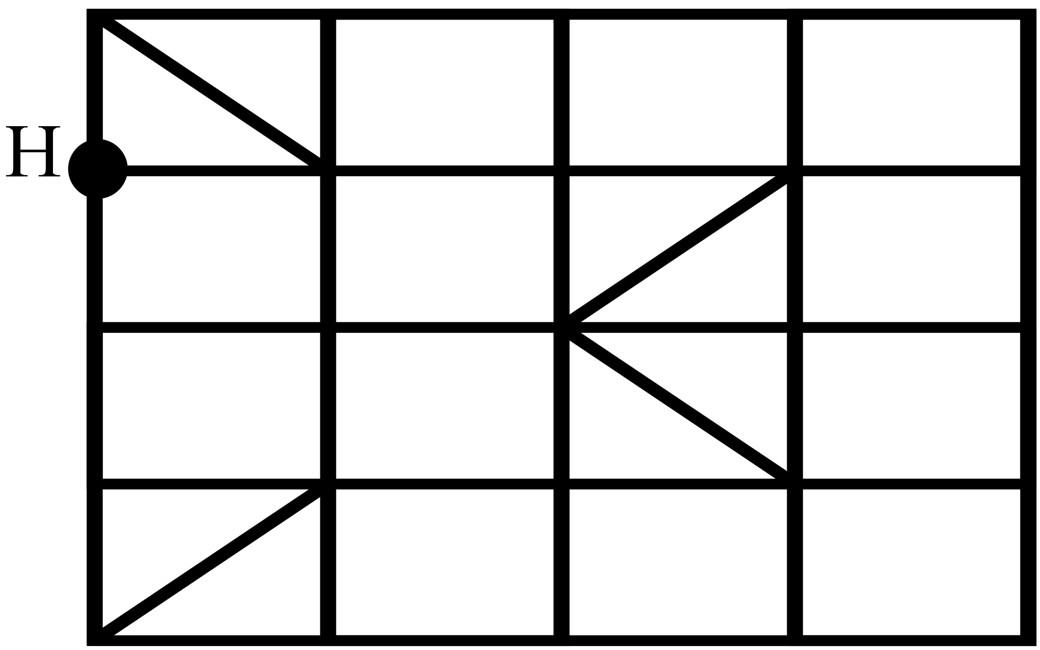
Does this problem involve finding an Euler circuit or a Hamilton circuit?

The following figure represents a town where there is a sewer located at each corner (where two or more streets meet). After every thunderstorm, the department of public works wishes to have a truck start at its headquarters (at vertex H) and make an inspection of sewer drains to be sure that leaves are not clogging them. Can a route start and end at H that visits each corner exactly once?



Assume that at equally spaced intervals along the blocks in this graph there are storm sewers that must be inspected after each thunderstorm to see if they are clogged. Is this a Hamilton circuit problem, an Euler circuit problem, or a Chinese postman problem?

(This problem is from *For All Practical Purposes*, published by COMAP.)

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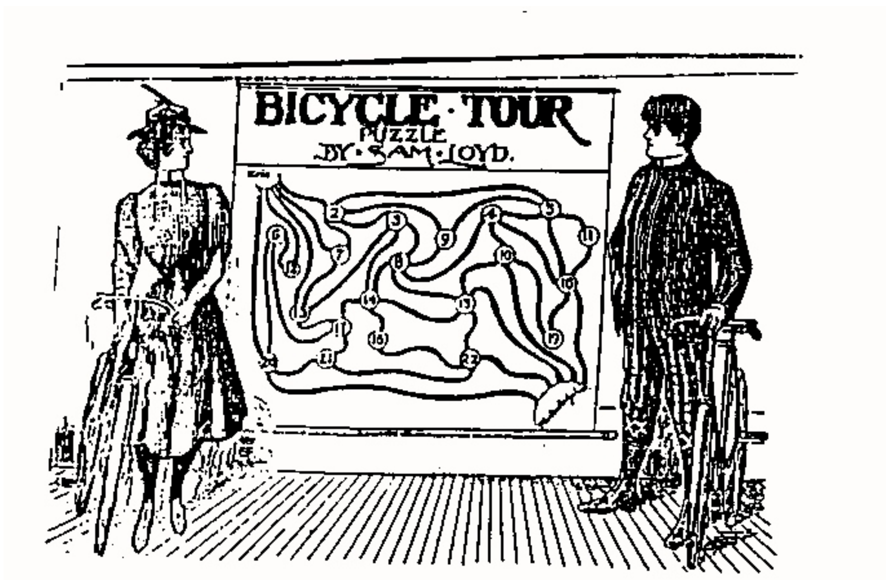
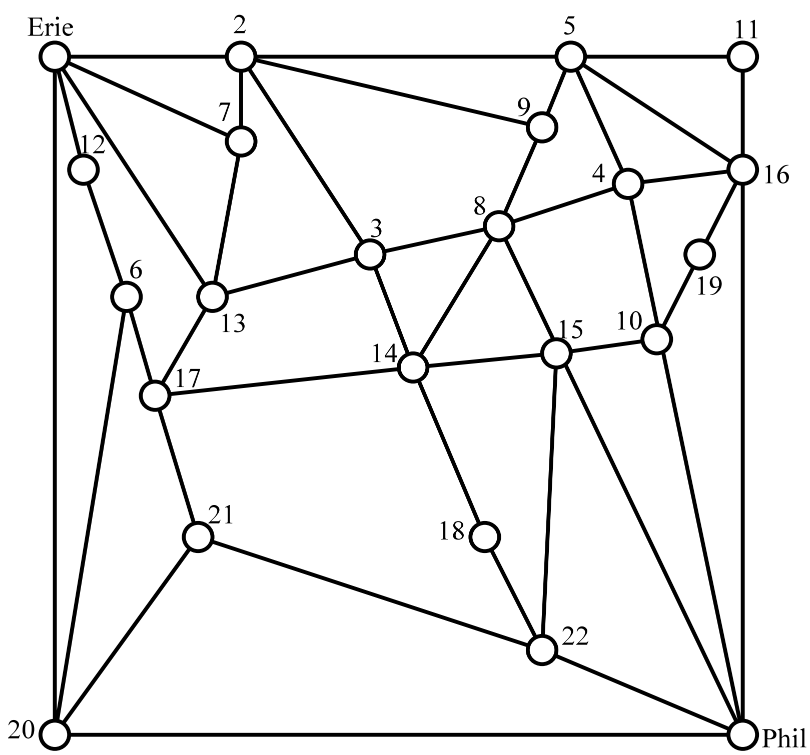
Sam Lloyd's Bicycle Tour

Find a route from Philadelphia to Erie, passing once through each town.





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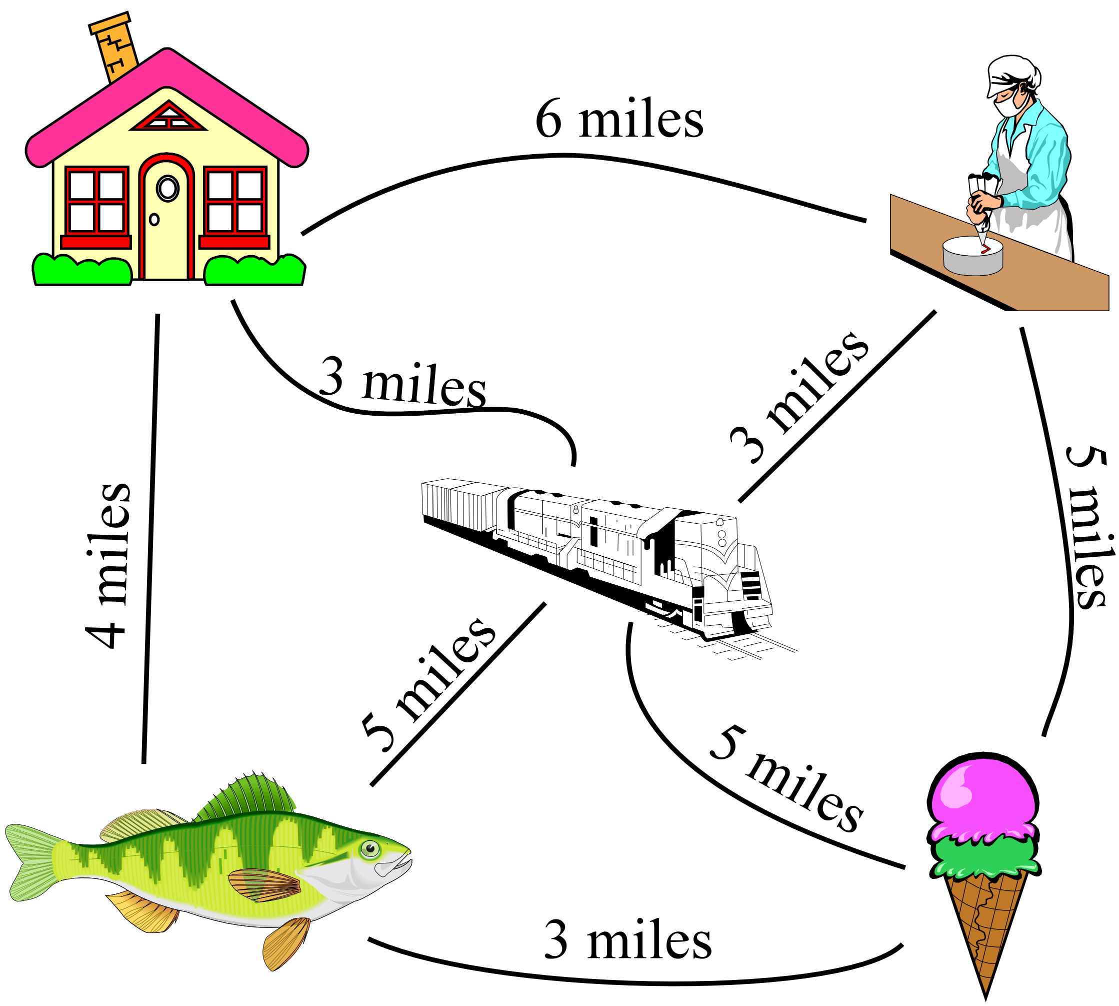
Many errands

You have so many errands to run! Starting from your house, you have to go to the Fish Store, the Bakery, the Train Station, the Ice Cream Store, and then back home. You can go in any order, but you want to make the shortest trip you can. How many miles



will you go?

Developed by Susan Picker
Leadership Program, 1990

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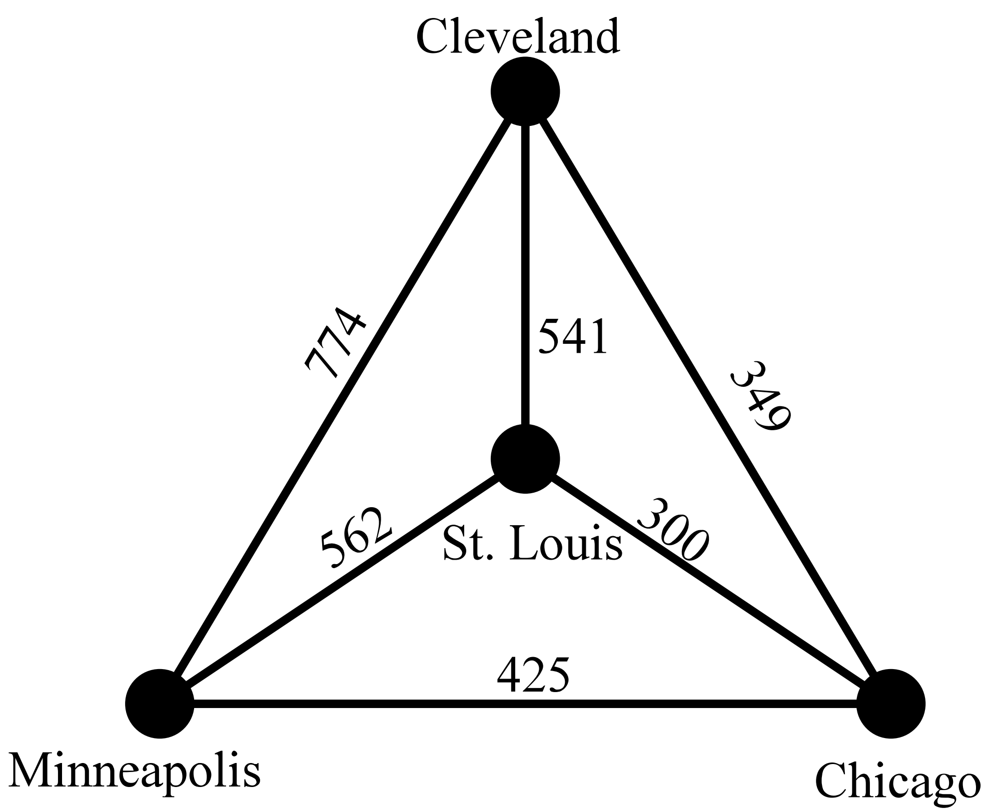
Visiting three cities

Use a tree diagram to enumerate all of the routes that begin in Chicago, visit the other three cities and return to Chicago. Which is the shortest route that a traveling salesperson should take?



From *For All Practical Purposes*,

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Freeman, 1988, used with permission

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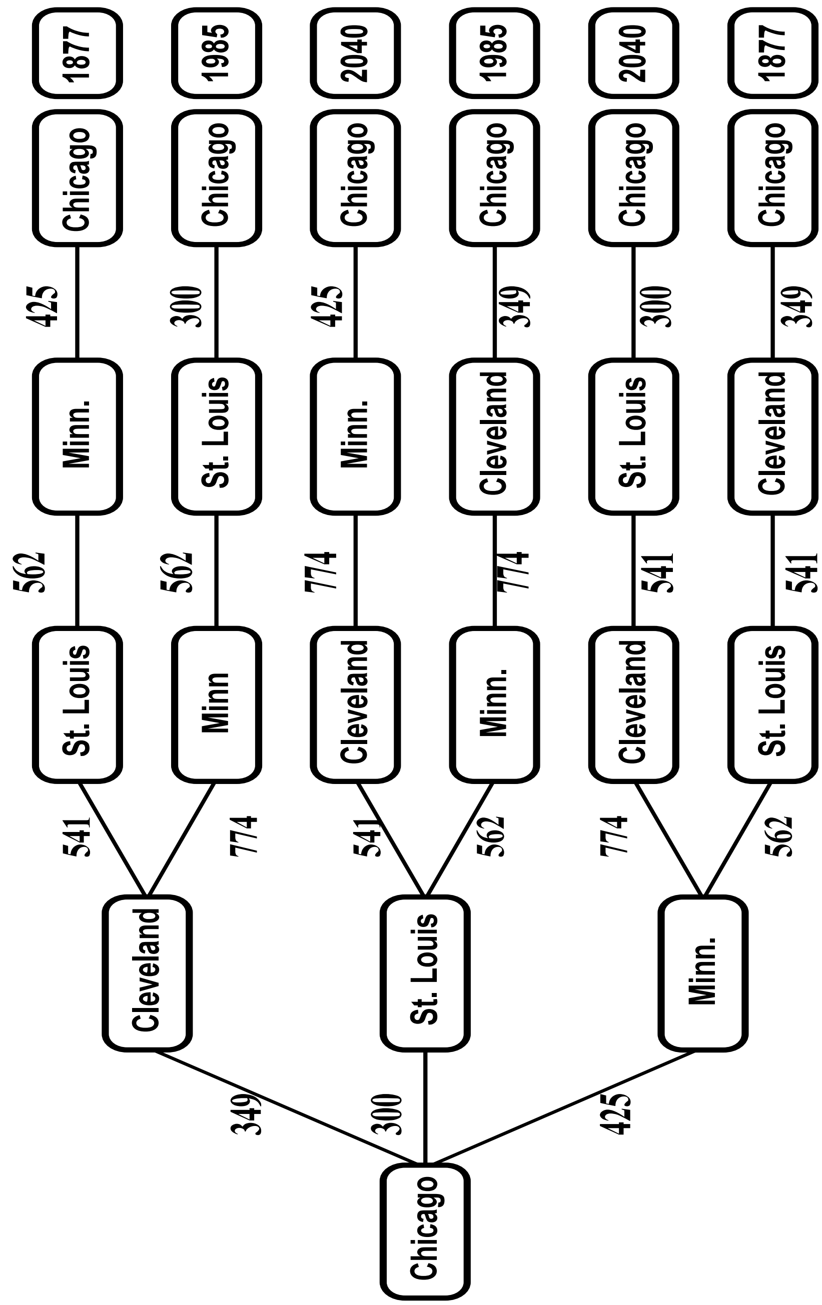

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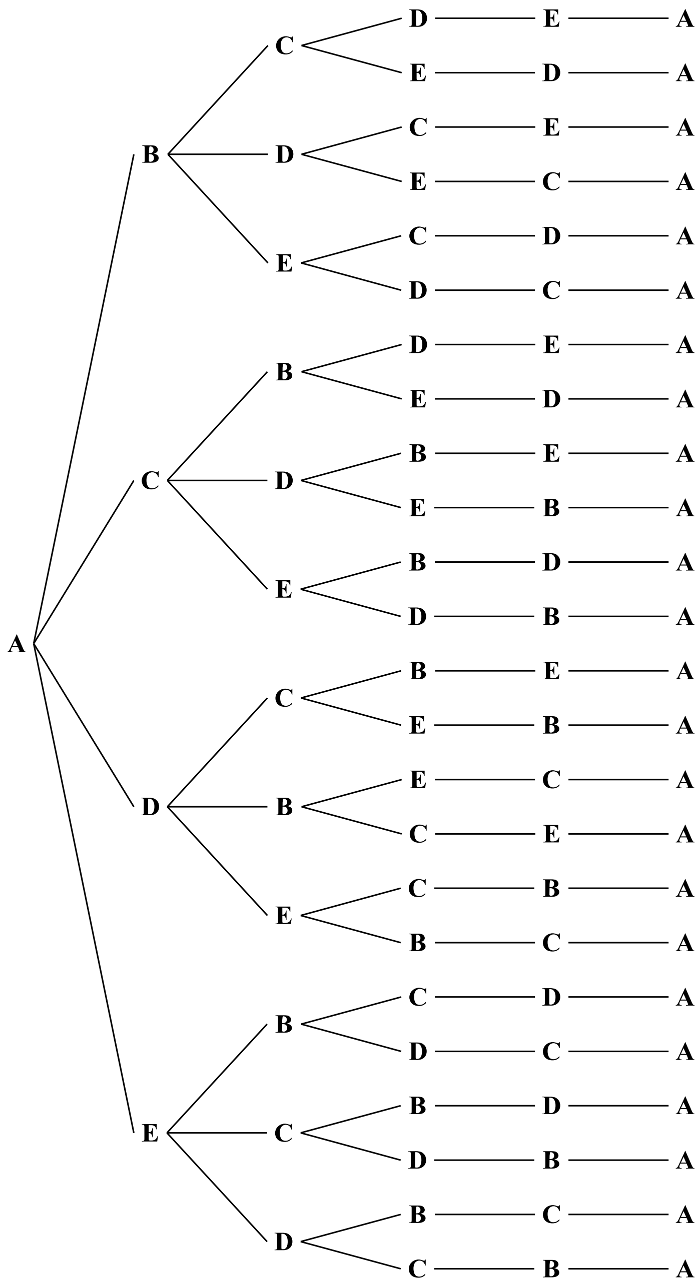

The 24 Circuits on a Graph with 5 Cities

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Factorials

$1! = 1$	
$2! = 2\times1 =$	$2 \times 1! = 2 \times 1 = 2$
$3! = 3\times2\times1 =$	$3 \times 2! = 3 \times 2 = 6$
$4! = 4\times3\times2\times1 =$	$4 \times 3! = 4 \times 6 = 24$
$5! = 5\times4\times3\times2\times1 =$	$5 \times 4! = 5 \times 24 = 120$
$6! = 6\times5\times4\times3\times2\times1 =$	$6 \times 5! = 6 \times 120 = 720$
$7! = 7\times6\times5\times4\times3\times2\times1 =$	$7 \times 6! = 7 \times 720 = 5040$
$8! = 8\times7\times6\times5\times4\times3\times2\times1 =$	$8 \times 7! = 8 \times 5040 = 40{,}320$
$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1 =$	$9 \times 8! = 9 \times 40{,}320 = 362{,}880$
$10! = 10\times9\times8\times7\times6\times5\times4\times3\times2\times1 =$	$10 \times 9! = 10 \times 362{,}880 = 3{,}628{,}800$

...

20! is about 2,000,000,000,000,000,000
or two billion billion

This is the total number of possible routes for a traveling salesperson who has to visit 20 cities.

How long would it take a computer to check two billion billion possible routes?

... if it checked one billion routes per second ...

1 Million seconds	About 11 days, 14 hours
2 Billion seconds	63.4 years

How about if you had 30 cities?

It would be about a *billion times* as big as that!

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Nearest Neighbor Algorithm

John lives in Gettysburg and wants to visit eight colleges (besides Gettysburg) before deciding to which ones he should apply. Use the Nearest Neighbor Algorithm to find a possible route that he might take. (Problem by Susan Simon.)

	Sc	Pi	Be	Ge	La	Wi	Re	Ph	Wa
Scranton		272	76	163	123	80	99	130	228
Pittsburgh	272		292	184	237	192	255	304	253
Bethlehem	76	292		127	80	121	49	54	178
Gettysburg	163	184	127		54	124	90	150	82
Lancaster	123	237	80	54		127	31	63	106
Williamsport	80	192	121	124	127		108	166	205
Reading	99	255	49	90	31	108		58	136
Philadelphia	130	304	54	150	63	166	58		136
Washington	228	253	178	82	106	205	136	136	

(The colleges John wants to visit in these cities are, respectively, the University of Scranton, Carnegie-Mellon, Lehigh, Gettysburg, Franklin and Marshall, Lycoming, Albright, University of Pennsylvania, and American University.)

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Developed by Susan Simon,
Leadership Program 1989

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A Ten City Problem

Use the nearest neighbor algorithm and the cheapest link algorithm to find good routes for a traveling salesperson who must visit the ten cities on the map below, one of which is both start and end of the

trip.

	Atlanta	Boston	Chicago	Denver	Houston	Los Angeles	Miami	New York	San Francisco	Vancouver
Atlanta	0	946	606	1323	696	1942	597	755	2135	2228
Boston	946	0	851	1770	1606	2597	1256	190	2699	2516
Chicago	606	851	0	921	941	1746	1189	714	1859	1776
Denver	1323	1770	921	0	878	831	1726	1632	950	1052
Houston	696	1606	941	878	0	1374	969	1419	1645	2340
Los Angeles	1942	2597	1746	831	1374	0	2340	2452	347	1071
Miami	597	1256	1189	1726	969	2340	0	1089	2595	2803
New York	755	190	714	1632	1419	2452	1089	0	2572	2433

San Francisco	2135	2699	1859	950	1645	347	2595	2572	0	791
Vancouver	2228	2516	1776	1052	2340	1071	2803	2433	791	0

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A Brief History of the Traveling Salesman Problem (TSP)

"The origins of the TSP are obscure." That's what it says in *Finding Cuts in the TSP*, a Technical Report of the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS), which sponsors the Leadership Program in Discrete Mathematics. This technical report, by David Applegate, Robert Bixby, Vasek Chvatal, and William Cook, begins with a historical introduction to the TSP (on which this page is based), which notes that although the problem was discussed in the 1920s, again in the 1930s, and in the 1940s, not much happened until the 1950s.

- ⊠ A breakthrough came in 1954, when George Dantzig, Ray Fulkerson, and Selmer Johnson published a method for solving the TSP (based on linear programming) and illustrated the power of this method by solving an instance with 49 cities, an impressive size at that time. The 49 cities consisted of one city from each of the then 48 states, together with Washington, D.C.

It is important to distinguish between "finding a good solution for a TSP" and "solving a TSP", that is, finding the best solution. In most applied situations, a good solution may very well be adequate, even if it not the best solution. The method of Dantzig, Fulkerson, and Johnson actually finds the best solution, but only works in certain kinds of TSP; for other problems, it does not work at all.

- ⊠ In 1972, Brian Kernahan and Shen Lin at AT&T Bell Labs, Murray Hill (NJ) devised a method for providing a solution to any TSP which is within 2% of the best solution; subsequent researchers have continued to refine their methods, so that finding nearly optimal routes even in fairly large TSP problems is a relatively easy task with the aid of computers.

- ⊠ Two notable examples of TSP problems where the optimal solution was actually obtained were an example of 532 cities (based on data from AT&T) and an example involving 2392 cities (solved in 1986 and 1988 respectively) by Manfred Padberg (Courant Institute of Mathematical Sciences of New York University) and Giovanni Rinaldi (Institute for Systems Analysis in Rome).

- ⊠ In 1991, Gerhard Reinelt collected together into a library called TSPLIB about 100 difficult examples of TSP arising from both industrial sources and artificially-defined-but-natural-examples like the 49-city example mentioned above. TSPLIB includes problems that have been solved, like the two problems (involving 532 and 2392 cities) mentioned above, and about 30 problems that had not been solved.

- ⊠ During the following four years, David Applegate (AT&T Bell Labs), Robert Bixby (Rice University), Vasek Chvatal (Rutgers University), and William Cook (Bellcore), using a network of over 50 workstations, and refining the methods of Dantzig's group, found optimal routes for 20 of the 30 unsolved problems in TSPLIB, including problems that involved 3038 cities (in 1992), 4461 cities (in 1993), and 7397 cities (in 1994).

It should be noted, however, that the number of cities alone is not what makes a problem difficult;

anyone can solve a TSP with 10,000 cities that are arranged on the circumference of a circle.

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