Master Document

Workshop 2 — Euler Paths and Circuits

Vertices need to be highlighted in Handout #1 and the corresponding TSP.

TSP #1 needs to be fixed – the vertical title has rotated to a horizontal position

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

revised July 28, 1999 (and, most recently, June 21, 2011)

Workshop 2 — Drawing pictures with one line

Materials and Pre-workshop Preparations — **Allocated time**

Activity #1 — The Case of the Stolen Diamonds . . . . **20 minutes**

- Props for "The Case of the Stolen Diamonds", a play which begins on page #9, 17, and copies of the script for the characters
- Participants should be assigned the various roles, and should be provided with scripts and props, early enough so that the troupe enters at the scheduled start time of the workshop; they should be coached to say their lines slowly and clearly so that the audience can follow the details of the script
- camera for taking cast picture

Activity #2 — Finding Euler paths on graphs . . . . **55 minutes**

- Each participant should be provided with a "communicator" into which they can insert a picture of a graph, and draw paths which are easily erasable.
- One lid full of sand (or sugar) for demonstration purposes (in case teachers don't have access to "communicators." (We used to provide lids full of sand, one per table for participants to "trace" paths in the sand, which were placed on the tables before the start of the workshop.)

Activity #3 — Finding Euler paths on maps . . . . **50 minutes**

- Video: COMAP's Video Geometry: New Tools for New Technologies — "Snowbound: Euler Circuits"

. . . . TOTAL WORKSHOP TIME: **125* minutes**

* In addition, ten minutes are allocated for a break in this 2 ¼ hour workshop.

Word Wall: Path, Euler Path, Circuit, Euler Circuit, Connected Graph, Disconnected Graph, Degree (of a vertex), Euler's Theorem, Balanced Graph, Multigraph

Activity #1 — The Case of the Stolen Diamonds
(Allocated time = 20 minutes, 10 for each part)

A. Participants perform Act I of "The Case of the Stolen Diamonds", a revised and revived version of a play often performed at Rutgers discrete mathematics institutes; a copy of the script is attached to the end of these Instructor's Notes. Before the beginning of the session, a "playbill" with the name of the play appears on the overhead projector (TSP #0). TSP #1 is a picture of the estate, with snow tracks indicated, and should be placed on the overhead (with light turned off) before the play begins. At the indicated time in the script, the light should be turned on so all can view the picture.

B. After Act I has been performed, ask each person to vote for who they think is the culprit using paper ballots, which are quickly collected, tallied, and reported. While the votes are being tallied, ask volunteers to explain why they voted the way they did. Most votes may be frivolous, but eventually someone who voted for the butler will explain the key idea — that each entry and departure from a house accounts for two edges at the corresponding vertex, so that we can tell from the tracks where the culprit must have begun and ended his or her route. As a result of this discussion, participants should be aware that the culprit has to be either the Butler or Lady Schmendrick.

You might ask the participants to trace a path that the burglar might have taken which would account for all the tracks in the snow. (Suggestion: trace the path by drawing in color on a blank transparency that overlays the transparency map of the grounds; then remove the bottom transparency so that the path becomes more visible.) Observe that by LaTour's reasoning any path would have to have started and ended at sites where there were three tracks; note that we can think of the buildings and the tracks as forming a graph (see TSP #2).

If anyone asks the question, respond that you can assume that no one repeated a path without making tracks, and that you are unable to tell the direction in which each part of the path is traversed. Some participants may recognize the path as the house that they tried to draw with one line when they were kids.

C. Participants perform Act II of the play, which should be followed by enthusiastic applause.

Activity #2 — Finding Euler paths on graphs
(Allocated time = 55 minutes; including 5 minutes for parts A, C & F; 10 minutes for parts B & E; and 15 minutes for part D. You may need to spend a bit more time on

parts A, D and E but be sure not to exceed 55 minutes altogether.)

A. View the above activity as an example of the problem of tracing a graph. That is, given a graph (thought of as a picture) can it be traced with one line without repeating any part of the picture? Use TSP #3 through TSP #7 as an introduction to this topic and to the terminology of paths, circuits, and Euler paths and circuits. (Historical remarks about Euler will be introduced later.)

Point out that vertices are sometimes represented by open circles, as we did in yesterday's workshop, sometimes by closed circles, as in TSP #3.

B. Ask participants to try their hand at drawing with one line the pictures on Hand-out #1 (which is TSP #8). Note that in this handout there should be a vertex (either open or closed circle) at each intersection of the picture.

Suggested format: "Work in pairs and determine which of the pictures in Hand-out #1 = TSP #8) can be drawn with one line." After a reasonable time, follow-up with: "When you're done, consult with the other pair at your table and see if you agree which drawings can be done and which can't."

Some participants may take the phrase "with one line" literally as "with one straight line", so the phrase "without lifting your pencil" should be used to clarify the meaning of "with one line".

Note that the point of #4 is for them to realize that for an Euler path to exist, the graph must be connected, so when that issue comes up in part C below, the terms "connected" and "disconnected" can be introduced.

Each participant should be provided with a "communicator." If they are not available, then each table should be provided with a shoebox lid containing sand or sugar for participants to use as needed for sketching graphs with one line; explain that this homemade "manipulative" is very useful in classrooms, especially when looking for Euler paths or circuits on more difficult graphs. For some participants, an appropriate alternative is to use tracing paper.

During this activity, participants may be asking how you can tell whether there is an Euler path vs. an Euler circuit; suggest that they hold that question for a while, since their discovering the difference will be the point of part D below.

C. On a blank transparency overlaying TSP #8, mark for each problem the

consensus of whether the picture could or could not be drawn. A number of participants might believe that they can draw #3 or #5 and they should be encouraged to come up and try to draw them. In connection with #4, use TSP #9 to introduce the notion of connectedness, and discuss the examples provided there. (Note that when a graph is depicted using open or closed circles, then the intersections where there are no circles are not regarded as vertices of the graph; as a result, graph D is disconnected, despite its similarity to graph C, which is connected.)

> *Note that we are expecting participants to realize that the pictures in Hand-out #1 can be thought of as graphs, not just as pictures. So it is worth pointing out to the participants that after one day in the program, they recognize a graph even if it is called a picture, and they are able to correctly introduce and use the "vertex" and "edge" terminology of graphs.*

D. Show TSP #10 and note that a helpful concept is the degree of a vertex. Explain this concept and, using a blank transparency over TSP #10, label each vertex on the graph with its degree. Then conclude that this graph has four vertices of degree 2, four vertices of degree 3, and one vertex of degree 4. Remind them that this graph is #5 on Handout #1. Do this example on the board (unless you have two overhead projectors) so that the information can be transferred to TSP #11 when you put that on the overhead projector. Do several other examples as well of calculating degrees of vertices in a graph.

E. Distribute Hand-out #2 (=TSP #11), the second part of Hand-out #1, and complete the row for graph #5; do this slowly since participants have trouble with the last two columns. Ask the participants to do the same for each of the seven graphs. Ask them to look for patterns in the chart which may provide clues as to whether a graph has an Euler circuit or Euler path. When participants have completed the chart, discuss why it is that graphs with 4 odd vertices cannot be drawn, and why graphs with 2 odd vertices may have an Euler path but can't have an Euler circuit.

> *Some participants will have problems with the notion of degree of a vertex, since this is the first time that the concept is actually used, so it will probably be useful to reinforce the idea that the degree of a vertex is the number of edges that meet at the vertex. Participants should be told that the terminology "odd vertex" is just an abbreviation for "vertex of odd degree".*

> *If the question arises, you might challenge participants to draw a graph which has 1 or 3 (or any other number) of vertices which have odd degree, and return to this a day or two later, at which time you can tell them that it's impossible and challenge them*

to find out why.

F. Provide participants with HO #3 which includes all of terminology that has already been introduced, as well as the content of TSP #12 and #13. Summarize the observations made above on TSP #12 State Euler's Theorem on TSP #13 but do not attempt to give an algorithm.

Note that the fact that there is a path (or a circuit) does not mean that it is easy to find; we will see an example of this shortly.

Summarize by reviewing the "Stolen Diamonds" graph in the context of Euler's Theorem.

You might note that unless you "paint yourself into a corner" (that is, end up with two disconnected pieces — as in #4 on Hand-out #1), you will always end up with an Euler path or circuit. It is also worthwhile noting the difference between TSP #12, which gives necessary conditions for the existence of Euler paths or circuits and TSP #13, which states that those necessary conditions are also sufficient; most participants, however, will not understand the difference, and so this difference should be treated lightly.

Another interesting activity you might do here is to ask the participants to draw a five-pointed star, and then ask them to draw a six-pointed star. Ask them how many drew the five-pointed star "with one line" — almost all will say that they did. Then ask them how many drew the six-pointed star with one line — almost all will say that they didn't. Both graphs in fact have all vertices of even degree — the five-pointed star has 5 vertices of degree 2 and 5 of degree 4, and the six-pointed star has 6 vertices of degree 2 and 6 of degree 4. But the simplest Euler circuit (go around the outside, then fill in the hexagon) is apparently more complicated than drawing two triangles with two lines.

F. Note that in many African cultures, the drawing of such graphs assumed great importance (see TSP #14). Use this transparency as a summarizing activity, eliciting from participants that both diagrams can be drawn, and where you need to start and finish for the second illustration.

This transparency is from a book called "Drawing Pictures with One Line"; much of the material in the book is taken from another book called "Ethnomathematics".

At the close of this activity, stress that the question "Does this graph have an Euler path?" can always be answered easily because there is a rule which tells you when there is and Euler path and when there isn't. Contrast this with the question "Does this graph have a coloring using three colors?" for which there is no such rule. Show the TSP from the previous day which discusses this question.

[Time for a 5-10 minute break]

Activity #3 — Finding Euler paths on maps

(Allocated time = 50 minutes, including 15 minutes for parts A-B, 15 minutes for part C, and 20 minutes for parts D-E; part C must be completed on time in order to allow 20 minutes for the videotape in D.)

A. Who is Euler? Tell the story of Euler and the Konigsberg bridges using TSP #15 and TSP #16. To lay the groundwork for the next activity, discuss explicitly the conversion from the map to the graph by placing TSP #17 over TSP #15. Alternatively, use a blank transparency overlaid on TSP#15, and draw the associated graph. Afterwards, remove the map so that everyone can see that on the graph there are many vertices of odd degree. Use the reasoning in previous discussions to conclude that because there are vertices of odd degree there is no solution to the Konigsberg bridge problem.

It should also be explained to participants that in this context we are allowing two vertices to be connected by more than one edge, since two sites in this map are connected by more than one bridge; the notion of "graph" is sufficiently flexible to allow multiple edges joining two vertices — such a graph is sometimes called a "multigraph".

B. Provide participants with Hand-out #4 (= TSP #18) which has on it the Manhattan bridges and tunnels and ask them to try to solve the problems given there. Review their responses to the problem in the context of previous discussions.

If participants are having difficulties with this problem, lead teachers should ask if they see any similarities between this problem and the Konigsberg bridge problem. Further suggestions might include drawing a graph as was done in part A above. Some participants may not see the land as a vertex and the bridge as an edge until this is suggested explicitly. Participants should be encouraged explicitly to write what the edges and vertices will represent in any given problem. Getting participants into this habit will be helpful in eliminating confusion when discussing Hamilton paths and circuits in the next workshop.

C. Letter carrier's problem. Provide participants with Hand-out #5 (= TSP #19, containing one 1×2 grid and one 1×3 grid) and ask whether the letter carrier can deliver mail on every block without repetitions and return to the post office (P) in each situation. The answer of course is "no". However, in the real world, the fact that no solution is possible is not in itself an acceptable answer — the mail still has to be delivered. Ask them to do those problems at their seats, and elicit from them the response that they can do the first one by repeating only the central line, and the second one by retracing only two line segments in the center, either a pair of horizontal ones, or a pair of vertical ones. Trace the path (on an overlaid blank transparency) and, in the process, add an edge to the graph. Point out that all vertices in the new graph have even degree; this is called a "balanced version" of the original graph (as in the videotape to be viewed shortly) or an "Eulerization" of the original graph. Use TSP #20 to explain this.

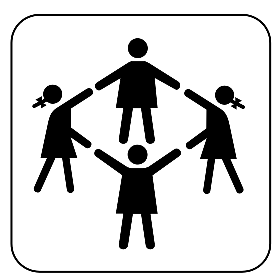
D. Play Unit V on the COMAP geometry videotape entitled "Snowbound: Euler Circuits."

Note that this video contains depictions of snow removal. In areas which do not experience regular snowfall, some words should be offered in advance of the showing of this video to reassure the viewers of the relevance of the techniques despite the absence of snow.

E. Discuss videotape and summarize today's workshop.

Supplementary Notes

1. After the homework review discussion of the Paris bridges on the following day, should mention that there is a "rule" which applies to one-way graphs, indicating whether or not it has a one-way Euler circuit, and suggest that participants may want to discover that rule for themselves.



2. On the afternoon of Day 2, or on a later day, a series of "human graphs" activities should be scheduled. One deals with graphs all of whose vertices have degree 2, one deals with graph coloring, and a third deals with Euler circuits.

The Case of the Stolen Diamonds
By Neil Goldstein
Revised by Deborah Franzblau and Susan Picker, 1996
Later revised by Joseph G. Rosenstein, 1997

Cast:	Suggested Props:
Narrator	
Rooster	
Lady Schmendrick	hat with plume, long pearl necklace, boa
Cook Cromwell	bakers hat, cooking, apron, wooden spoon
Chauffeur Hanson	chauffeur's hat
Maid Marian	fancy apron, duster
Butler Hornsby	serving tray with two glasses, tea towel
Inspector Oyl LaTour	magnifying glass, deerstalker hat

ACT I

Scene 1

Narr : It was a dark and snowy night at the Schmendrick Mansion ... Suddenly, just as dawn was breaking (*Rooster crows*), there was a loud cry from the upstairs bedroom.

Lady S : Aieee!!! My best diamonds ... I've been robbed!

Narr : The alarm was sounded and all the gates to the estate were locked ... the jewels must be somewhere on the grounds. But where?

(*Characters enter and step forward as they're introduced*)

Could they be in the Cook Cromwell's quarters? Or in Chauffeur Hanson's cottage? Did Butler Hornsby take the jewels ? Or perhaps it was Maid Marian.

Lady Schmendrick knew that she need the services of the one and only Inspector Oyl LaTour of Scotland Yard ...

(*lights down ... characters take seats in the "drawing room"*)

Scene 2

Narr : The prime suspects were gathered in the drawing room.

LaT : (*Aside to audience*) I knew something was up as soon as I entered the gates. I found fresh tracks in the snow—but couldn't tell which direction they pointed. The thief must have made extra tracks and dragged his boots ... Before the snow melted, I made a careful map of the grounds and the tracks.

(*Put map on overhead projector*)

I decided to question the suspects.

LaT : Lady Schmendrick, when did you last see the diamonds?

Lady S : Well, about midnight, I locked the diamonds in the safe as usual. I looked out the window, and all the lights in the servants' quarters were out, and it was just starting to snow. Then, around dawn I was awakened by the sound of running feet downstairs. When I got up, the safe was open and my diamonds were gone.

LaT : (*Aside*) Hmmm ... that means the tracks must have been made by the thief .
Thank you Lady Schmendrick. Now, Hanson, you're the chauffeur, what do you remember of last night?

Chauffeur : I was sleeping as usual, when I was awakened by my alarm clock. I suddenly heard someone run into my house and back out again. But I didn't get a good look at who it was.

LaT : Well, that sounds rather strange, but perhaps it could be true ...

Cook : It may sound strange Inspector, but that's exactly what happened to me. I couldn't sleep at all last night. I kept thinking about this recipe for Chunky Cherry Chiclet Puffs that I've been working on. They're really gooey and delicious, but there's just something missing ...
Anyway ... suddenly .. someone ... I couldn't tell you who it was ... ran right through my house and then out the other door. Then, a few minutes later, it happened again! I could see that the person was holding something but just couldn't tell what it was.

LaT : And you Hornsby, did you hear anything?

Butler (sounding somewhat suspicious): Well, I usually sleep very soundly, you know.

Something woke me briefly at some point, and I think I heard footsteps going in and out of the door a few times ... I was pretty groggy, so I can't really be sure.

Maid : Well you're not the only ones; I know someone ran in and out of my house too, at least twice.

LaT : My, my, this does get more and more interesting. The cook, the maid, and the butler all believe that someone ran in and out of their quarters two or more times, and was probably the thief. The Chauffeur says that someone ran in and out of his/her quarters once. From Lady Schmendrick's testimony, the thief must have run in and out of the mansion as well. That must be what woke her up.

This thief was clever enough to try to hide his tracks by running through all the buildings. However, I happen to know that one of you is lying. (*All gasp and look at one another*.)

Here is a map I made of the tracks in the snow this morning. Study it carefully. No one will leave this room until we get to the bottom of this.

Narr : We now take a brief leave from the Schmendrick drawing room, to give you, the audience, a chance to discover the truth. What did inspector LaTour see on her map that raised her suspicions? Can you guess who the culprit is and where the diamonds are hidden? If you can—don't give away the secret ... but write down your guess, and put it in this (box, hat, bucket...)

End of Act I

(During the intermission, participants record their suspicions of who the culprit is, the votes are tallied, and the workshop leader conducts a discussion of Act1.)

Act II

Narr : Now that you've had your chance to make your own deductions, let's return to the drawing room as Oyl LaTour exhibits the workings of her steel-trap mind.

LaT : Let's see how the map matches your stories. First of all, Lady Schmendrick could not have stolen her own diamonds: there are three tracks, so if she left the house with the diamonds, she'd have to go in and out again, and wouldn't have been in the house when she raised the alarm.

Lady S: Why Inspector, I can't believe you'd even think of suspecting *me, we*

Schmendricks are a very old and connected family, you know.

LaT : Sorry—but we have to consider all possibilities in this business, Lady Schmendrick.

The chauffeur has only two tracks leading to his house, which certainly fits his testimony. The cook and the maid both have four tracks to their houses, which means that their stories also check out.

Cook : I should say so, surely you don't think *I* have any use for diamonds!

LaT : That leaves only one more house ... the one with only three tracks going to it (*Everyone looks at the butler*). Hornsby ! You hid in the mansion last night, took the jewels, and ran around to confuse everyone, didn't you? I'll bet the jewels are hidden in your quarters. What do you have to say for yourself Hornsby?

Butler : Hey, wait a minute, why is everyone looking at me! You're not going to pin this on me so you can say "The butler did it!"

I don't only know how to serve martinis. I've been around. I know something about tracks in the snow myself.

You may be smart LaTour, but there's another scenario that you just didn't consider. Notice that there are also 3 tracks into Lady Schmendrick's mansion. I don't want to be too indiscreet, but I happen to know that she lost a lot of money recently investing in Naugahyde futures. Why , she can collect over a million from the insurance company on those diamonds—as long as we don't find them!

Hmmm ... I thought I smelled some familiar perfume in my living room last night ... (*Turning accusingly at Lady S*) Lady Schmendrick, you hid in my house last night and made the tracks in the snow just so that you could pin this on me. The diamonds are hidden somewhere in the mansion.

Lady S (*incensed*)**:** Not only do you steal my diamonds, but you also spy on me. You rat! You snoop! You loose-lipped louse!

LaT: The tracks show that the thief has to be either you, Hornsby, or you, Lady Schmendrick, and, since each of you was in your own house in the morning, the thief must have hidden in the other house some time before it stopped snowing. But which of you did that? Can either of you prove that you were home when it stopped snowing.

Butler: Yes, I can. Maid Marian spent the evening watching the late show on the telly with me, and she didn't leave until after the snow stopped. So I was home then. As I said, Lady Schmendrick is your thief!

LaT (*to Maid Marian***):** Is that true? Were you with the butler until after the snow stopped?

Maid: Yes, he couldn't have done it. It must have been Lady Schmendrick!

LaT (*to Maid Marian***):** Wait a minute! You said that someone ran in and out of her house, which is possible, but then you said "at least twice" and that is now impossible, since one of the tracks is your own. So you lied about that. Also if you walked home after it stopped snowing, there is no way of accounting for the three other tracks to your house. I don't believe that you were with Hornsby.

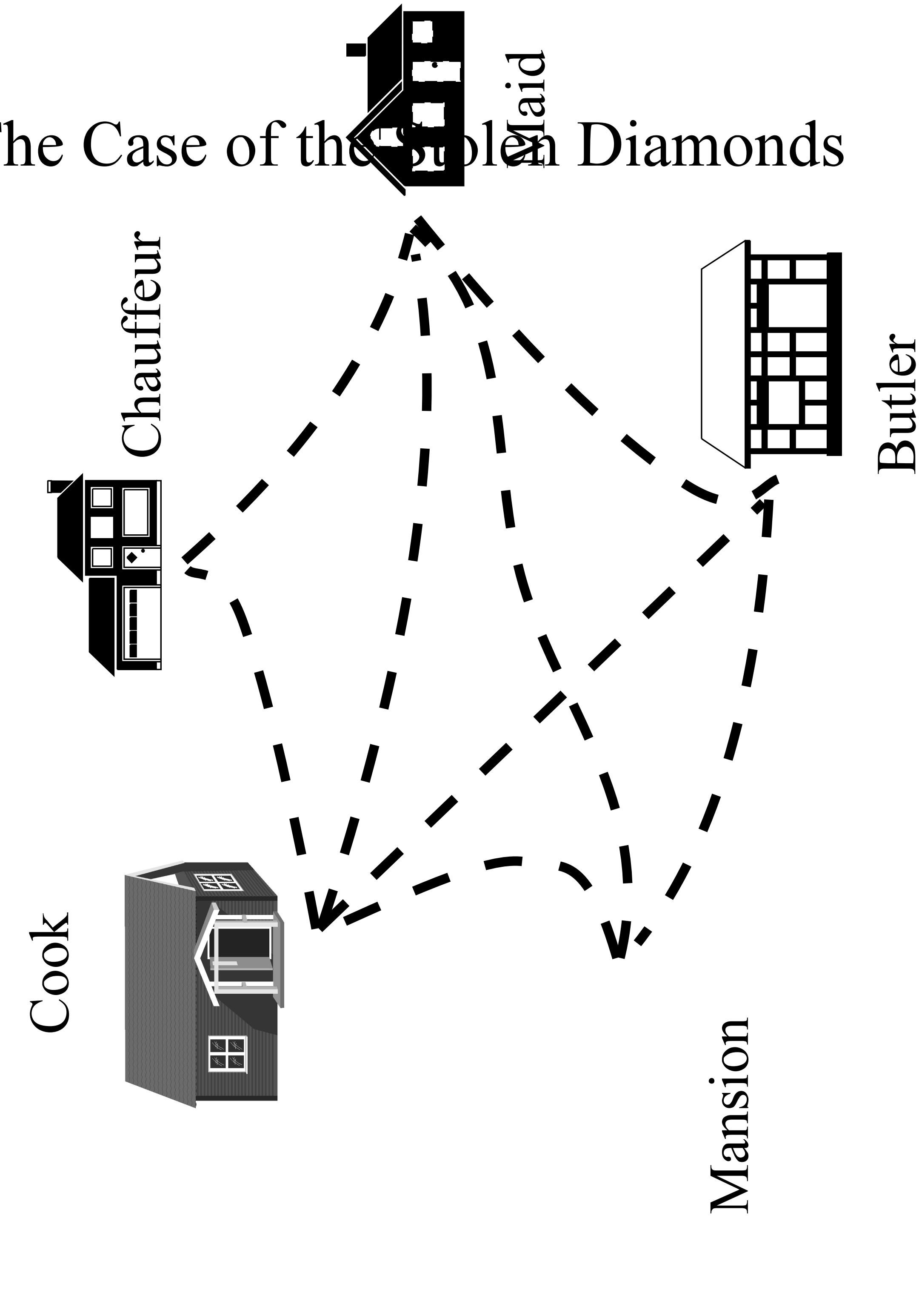
LaT (*turning to Butler***):** Sorry, Hornsby, you blew it. If you had kept quiet, I wouldn't have been able to figure out whether you or Lady Schmendrick was the thief. But you made up an alibi, and it doesn't hold water. So you're the one. I'm sure we'll find the jewels where you hid them. It's off to jail with you!

(*LaTour leads Butler away as lights fade*.)

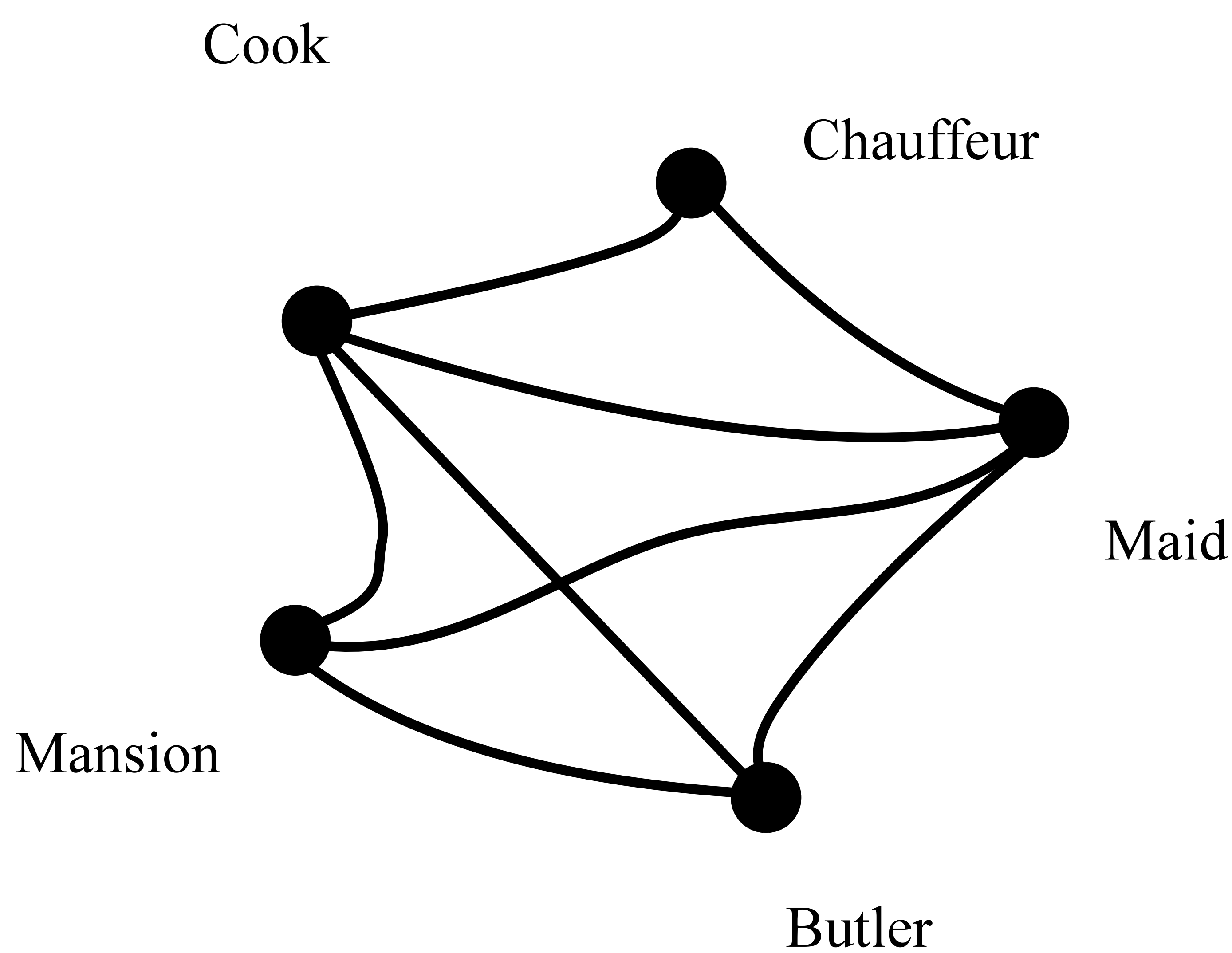
End of Act II

The End

The Case of the Stolen Diamonds

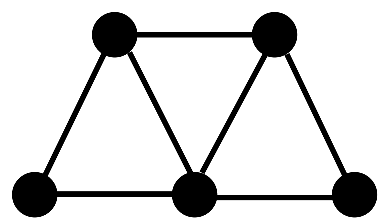


The Case of the Stolen Diamonds



Tracing Figures

Can you trace this picture (or graph) without removing your pencil from the paper and without repeating any part of the picture?

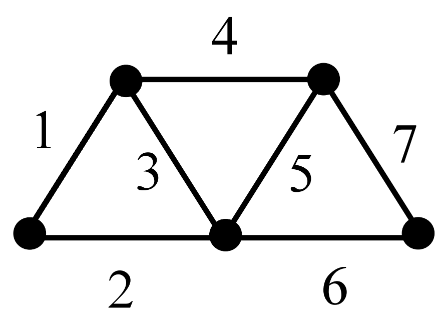


Terminology

Path A "path" is a sequence of edges in a graph, each of which begins where the previous one ends.

✓ The tracing is called an "Euler path".

Euler path A path in a graph which uses each edge exactly once.

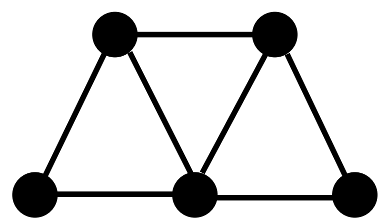


Note: Vertices may be revisited, but edges may not be repeated;

Note: "Euler" is pronounced "oiler"

Tracing Figures

Can you trace this picture (or graph) without removing your pencil from the paper and without repeating any part of the picture?



Can you trace this picture (or graph) without removing your pencil from the paper and without repeating any part of the picture ... **and end up where you began**?

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Terminology

Path A "path" is a sequence of edges in a graph, each of which begins where the previous one ends.

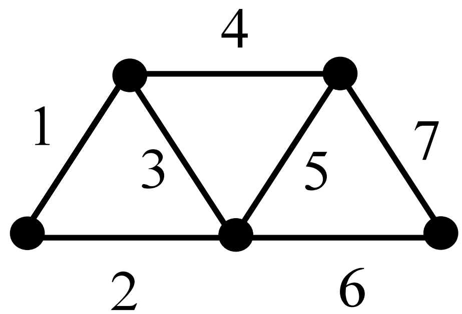
Circuit A "circuit" is a path which ends at the same vertex from which it starts.

What do you think...

Is every path a circuit???

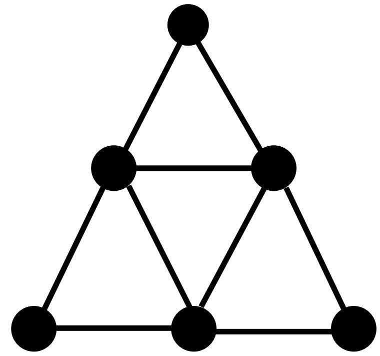
Is every circuit a path???

Euler path A path in a graph which uses each edge exactly once.



Euler circuit An Euler path which ends where it begins.

Does this graph have an Euler path? Does it have an Euler circuit?



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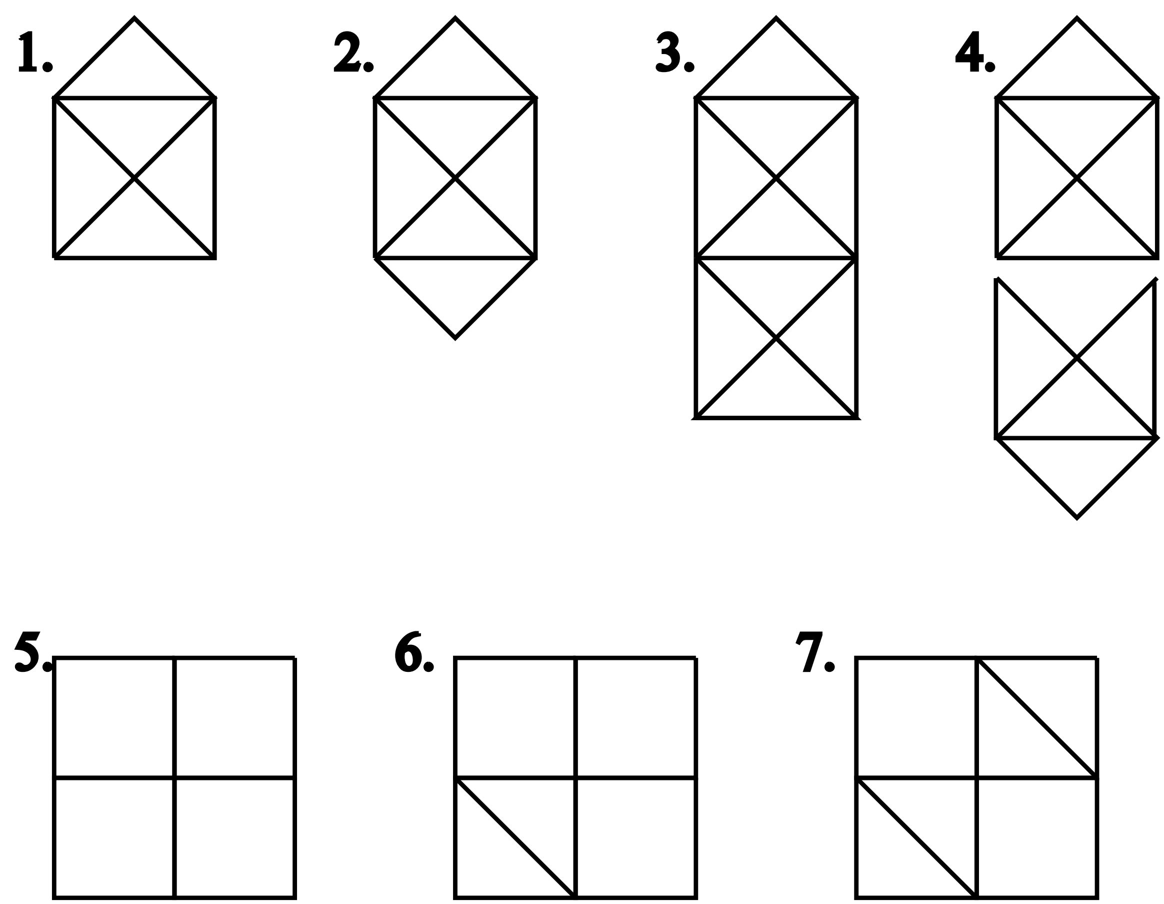
Hand-out #1:   Tracing pictures

Which of the following pictures can you trace without removing your pencil from the paper and without retracing any part of the picture? (Note that there is a vertex at each location where two or more line segments meet.)

1.



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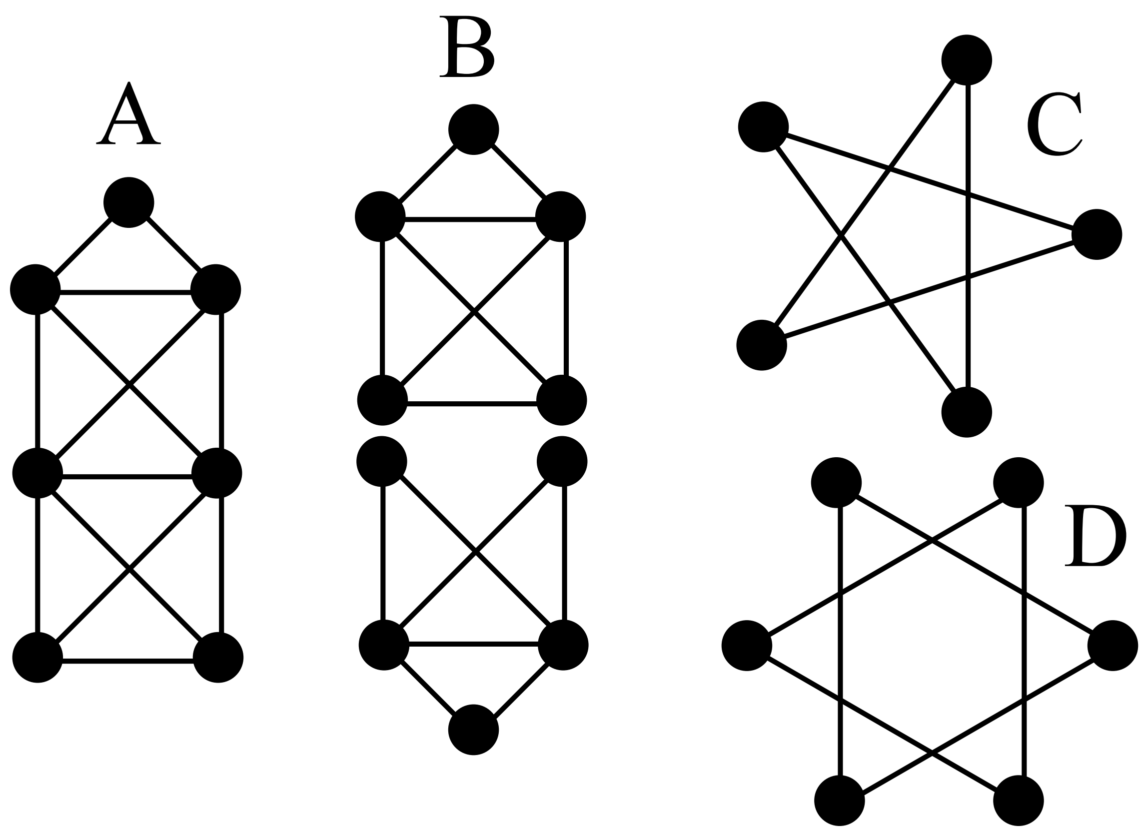
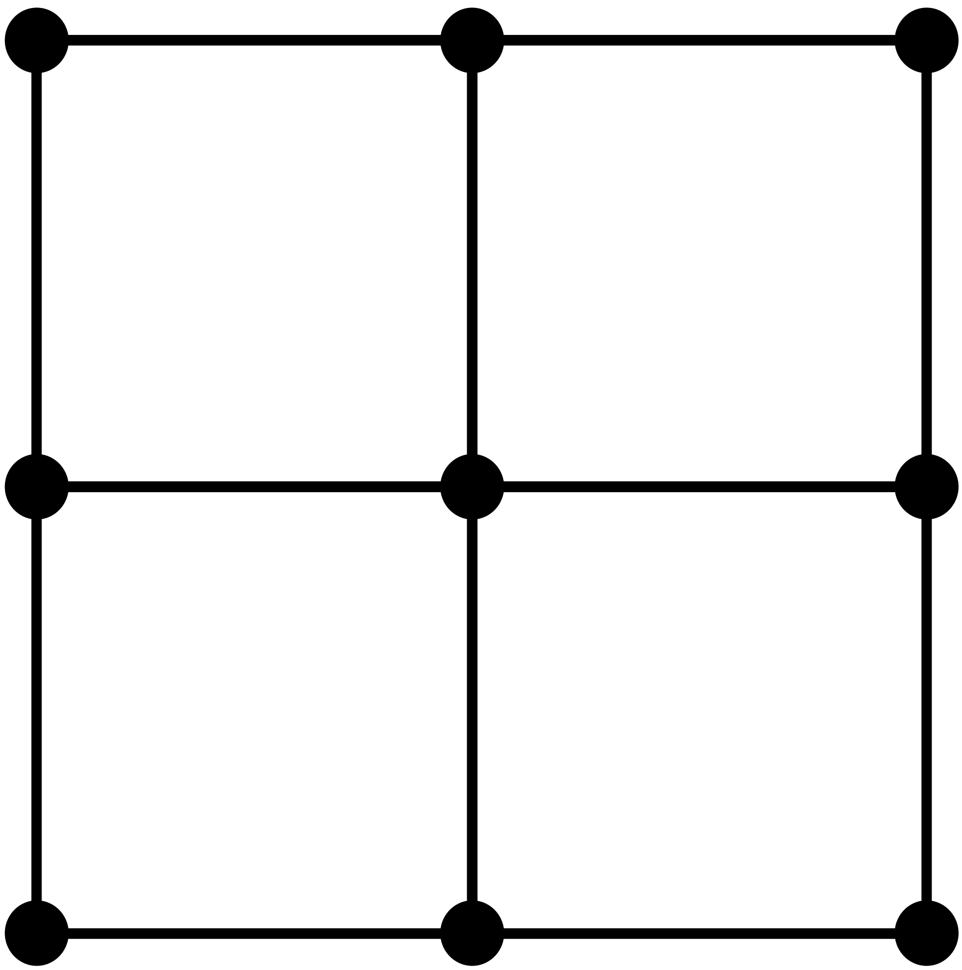
Connected A graph is connected if it is possible to get from any vertex to any other vertex along a path.

Disconnected A graph is disconnected if you can find two vertices with no path joining them.



The **"degree"** of a vertex is the number of edges that enter the vertex.



Hand-out #2: Tracing pictures — part 2

Complete the following chart. What conclusions can you draw and why?

Graph #	Euler path? (yes or no)	Euler circuit? (yes or no)	How many vertices of each degree?				Number of each type?	
			2	3	4	5	Even (2 or 4)	Odd (3 or 5)
1								
2								
3								
4								
5								
6								
7								

Requirements for an Euler path or circuit:

Euler circuit

A connected graph can only have an Euler circuit is there are <u>no</u> vertices of odd degree, that is, if every vertex has even degree.

Euler path

A connected graph can only have an Euler path if it has exactly <u>two</u> vertices of odd degree, that is, if all other vertices have even degree.

Euler path or Euler circuit

A connected graph can only have an Euler path or an Euler circuit if it has exactly <u>zero or two</u> vertices of odd degree.

Euler's Theorem

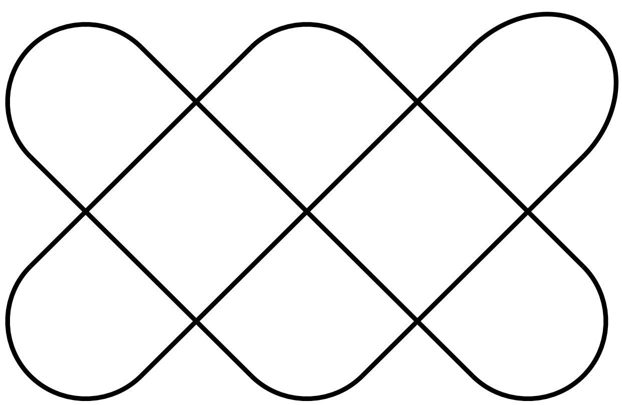
If the above requirements are met, then the graph actually has an Euler circuit or an Euler path. That is,

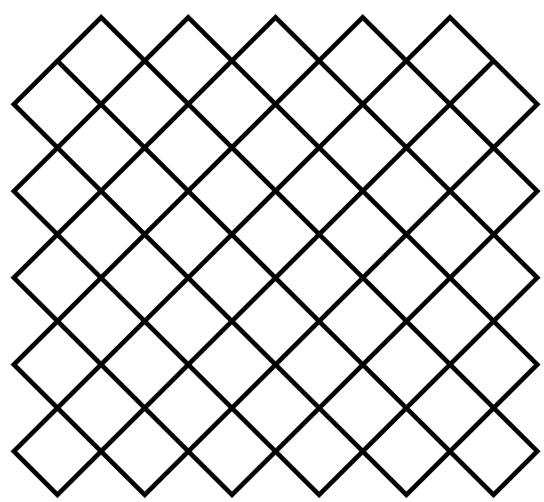
1. If a graph is connected, and has no vertices of odd degree, then the graph has an Euler circuit; an Euler circuit can begin and end at any vertex.

2. If a graph is connected, and has exactly two vertices of odd degree, then the graph has an Euler path; an Euler path must begin at one of the vertices of odd degree and end at the other vertex of odd degree.

The ability to draw pictures with one line plays an important role in some cultures.

The Bushong people of central Zaire are craft decorators for the 14 tribes that make up the Kuba chiefdom. The Bushong children, as part of learning the art of decoration, draw figures such as those below using a single continuous line. While you will find the first of these figures easy, the second is more challenging.

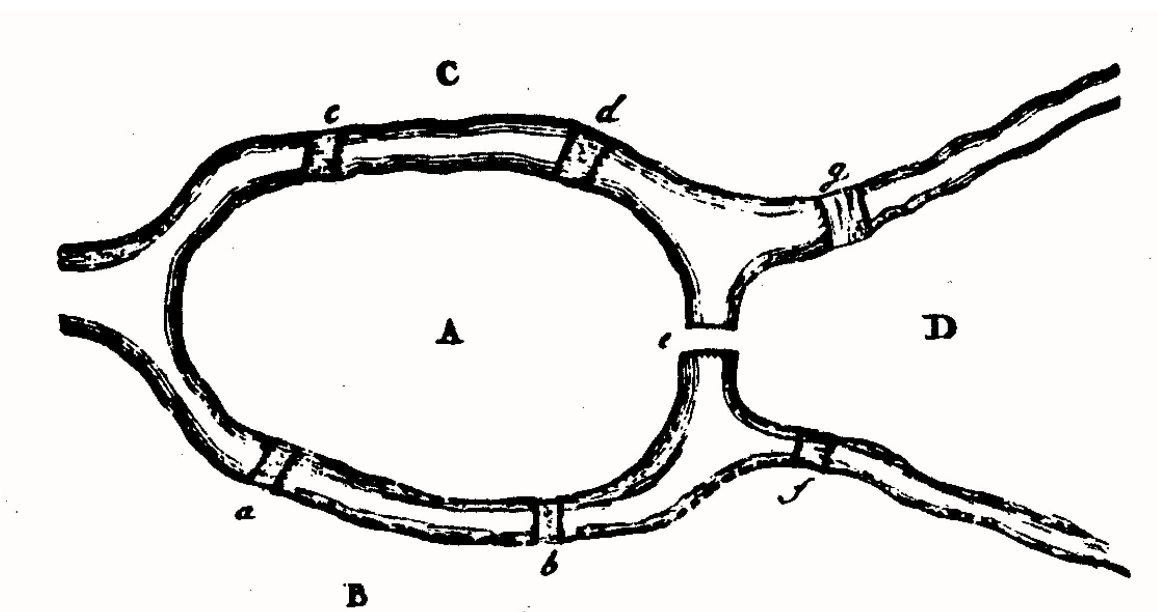




From COMAP's *Drawing Pictures with One Line* by Darrah Chavey

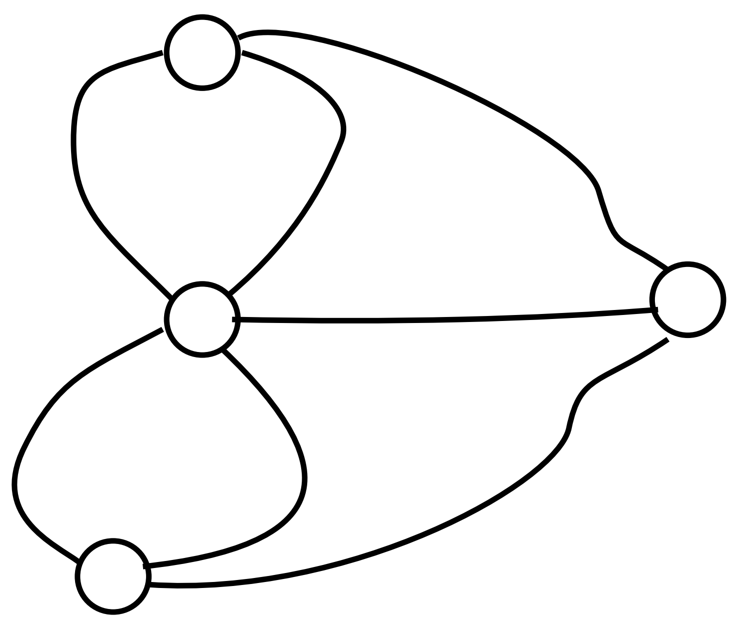
Who is Euler?

The Seven Bridges of Konigsberg — The city of Konigsberg in East Prussia (Kaliningrad on current maps) is on the banks of the River Pregel, and includes two islands Kneiphoff and "D". The parts of the city were connected in the 18th century by seven bridges. On Sundays the burghers would take a promenade around the town. A question often discussed was whether it was possible to plan the promenade so that one could walk across each bridge exactly once.



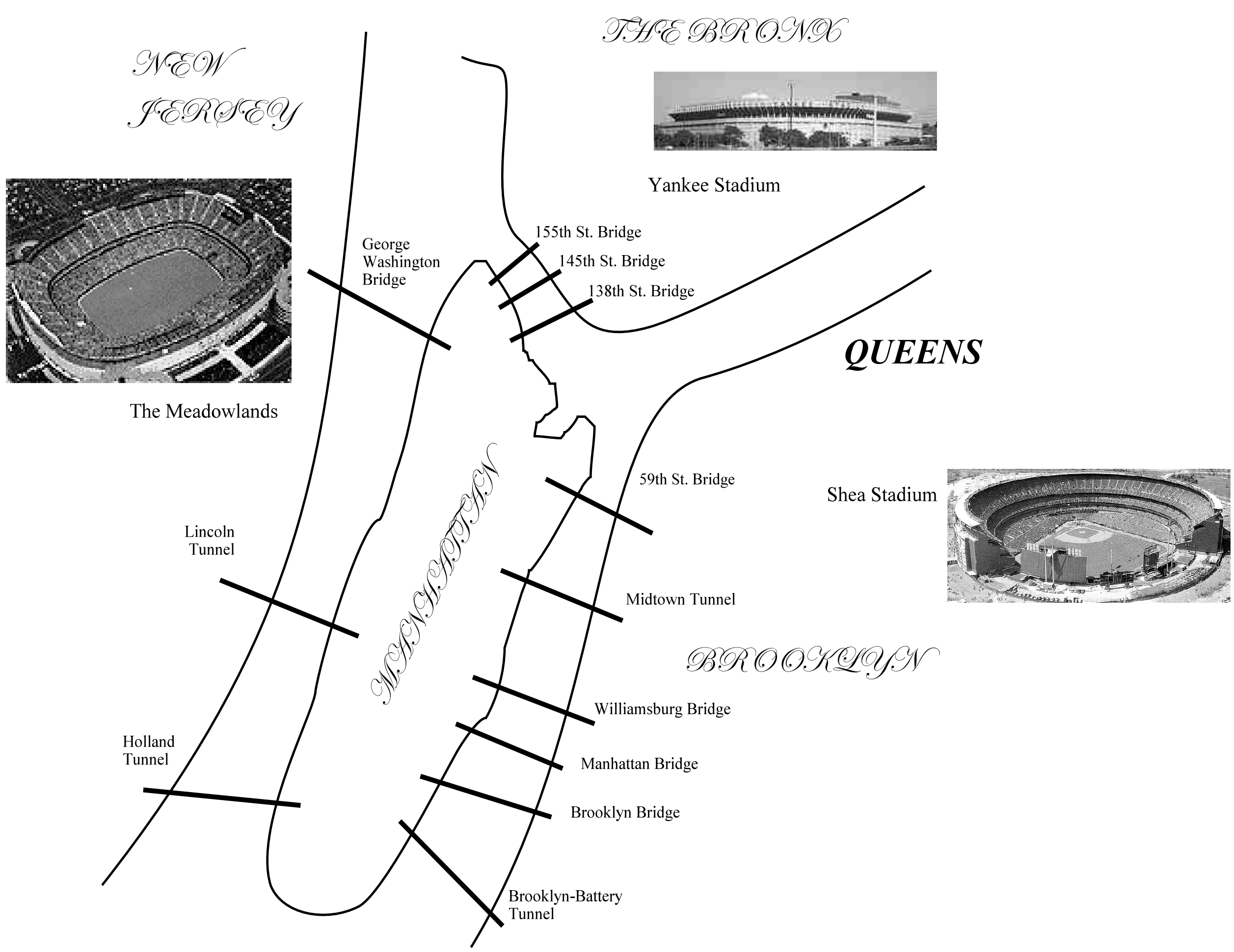
Leonhard Euler showed that this was impossible — and subsequently explained the circumstances when it was possible. For this reason, the theorem — the very first theorem in graph theory — is called Euler’s Theorem, and paths in graphs which use each edge exactly once are named after him.

The graph that corresponds to the Bridges of Konigsberg



Hand-out #4: Traveling around Manhattan

Is it possible to start at the Meadowlands, visit every bridge and tunnel on this map and end at Shea Stadium? (No bridge or tunnel may be visited more than once.) Is it possible to start at the Meadowlands and end at Yankee Stadium? Explain your answers. (Note that you are not asked to actually find a way of doing this – only to tell whether it is possible.)

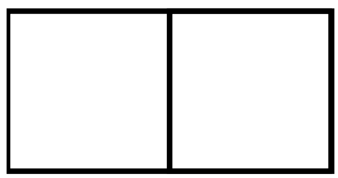


Developed by Susan Picker, LP '90

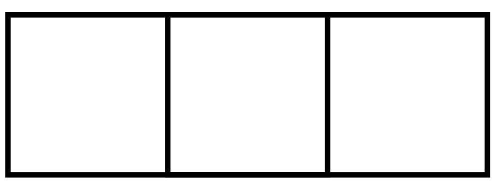
Hand-out #5: Letter Carrier's Problem

A letter carrier starts at the post office (located at the vertex marked P) and must deliver the mail to each block and return to the post office. Can this be done without repeating any blocks?

P



P



What is the least number of blocks the letter carrier must repeat?

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Chinese Postman Problem

The letter carrier's route is an Euler circuit in a new graph obtained from the original graph by duplicating edges. (Such a graph is called a "multigraph", because two vertices may be joined by multiple edges.)

The new graph is called an **"Eulerization"** of the original graph, because there will now be an Euler circuit.

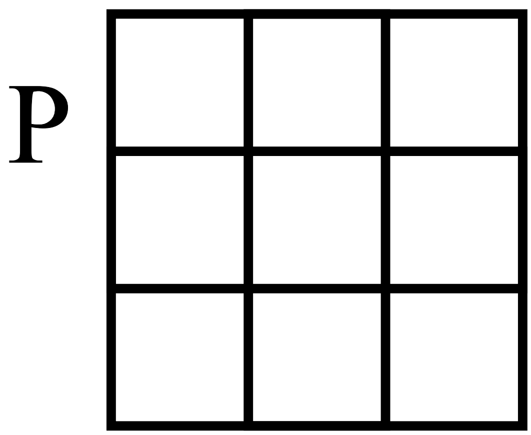
It is also called a **"balanced graph"** because all vertices have even degree.

This problem is called the Chinese Postman Problem after a Chinese mathematician/postman called Mei-ko Kwan who discussed this problem in 1962; it can be solved for any graph by finding a suitable Eulerization.

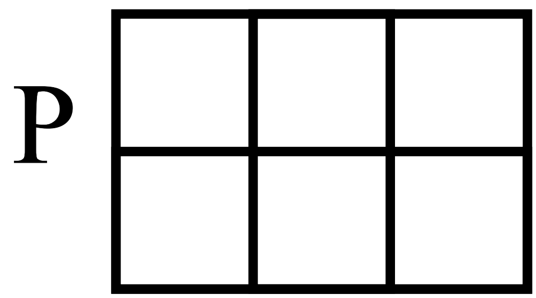
Another Letter Carrier's Problem

A letter carrier starts at the post office and must deliver the mail to each block and return to the post office. For each of the following examples, what is the least number of blocks the letter carrier must repeat?

1.



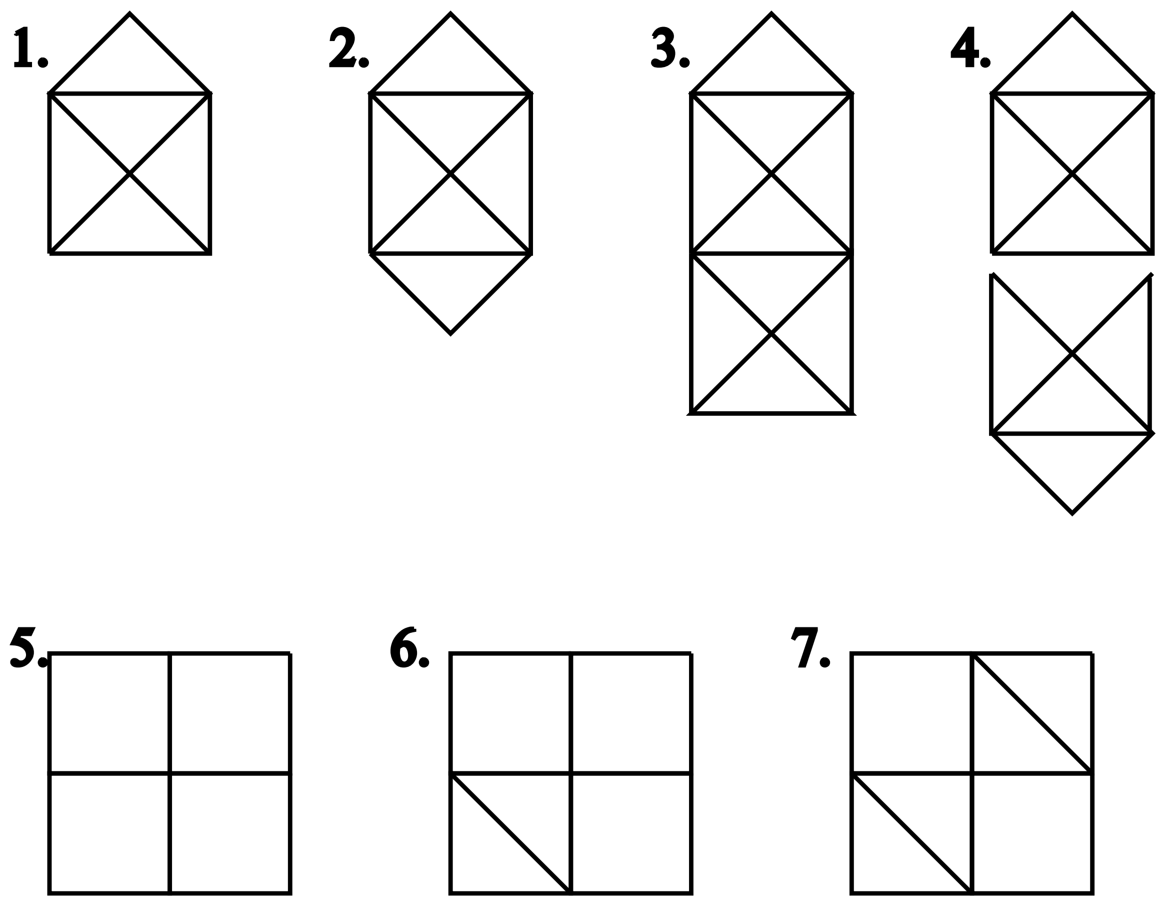
2.



Hand-out #1 — Tracing Pictures

Which of the following pictures can you trace without removing your pencil from the paper, and without retracing any part of the picture? (Note that there is a vertex at each location where two or more line segments meet.)

1.



Hand-out #2 — Tracing Pictures — part 2

Complete the following chart. What conclusions can you draw and why?

Graph #	Euler path? (yes or no)	Euler circuit? (yes or no)	How many vertices of each degree?				Number of each type?	
			2	3	4	5	Even (2 or 4)	Odd (3 or 5)
1								
2								
3								
4								
5								
6								
7								

(The degree of a vertex is the number of edges that enter the vertex.)

Hand-out #3 — Terminology

Path A sequence of edges in a graph, each of which begins where the previous one ends.

Circuit A path that ends at the same vertex from which it starts.

Euler path A path in a graph which uses each edge exactly once.

Note: Vertices may be repeated, but not edges;
"Euler" is pronounced "oiler".

Euler circuit An Euler path which ends where it begins.

Connected A graph is connected if it is possible to get from any vertex to any other vertex along a path.

Disconnected A graph is disconnected if you can find two vertices with no path joining them.

Connecting pictures with graphs: A graph has an Euler path if the corresponding picture can be traced without removing your pencil from the paper and without retracing any part of the picture.

Requirements for an Euler path or circuit: In order for a graph to have an Euler path, it must be connected and have

- ✓ either two vertices of odd degree
  - in which case there may be an Euler path which starts at one vertex of odd degree and ends at the other
- ✓ or no vertices of odd degree
  - in which case there may be an Euler circuit which starts at any vertex, and ends at the same vertex

Euler's Theorem

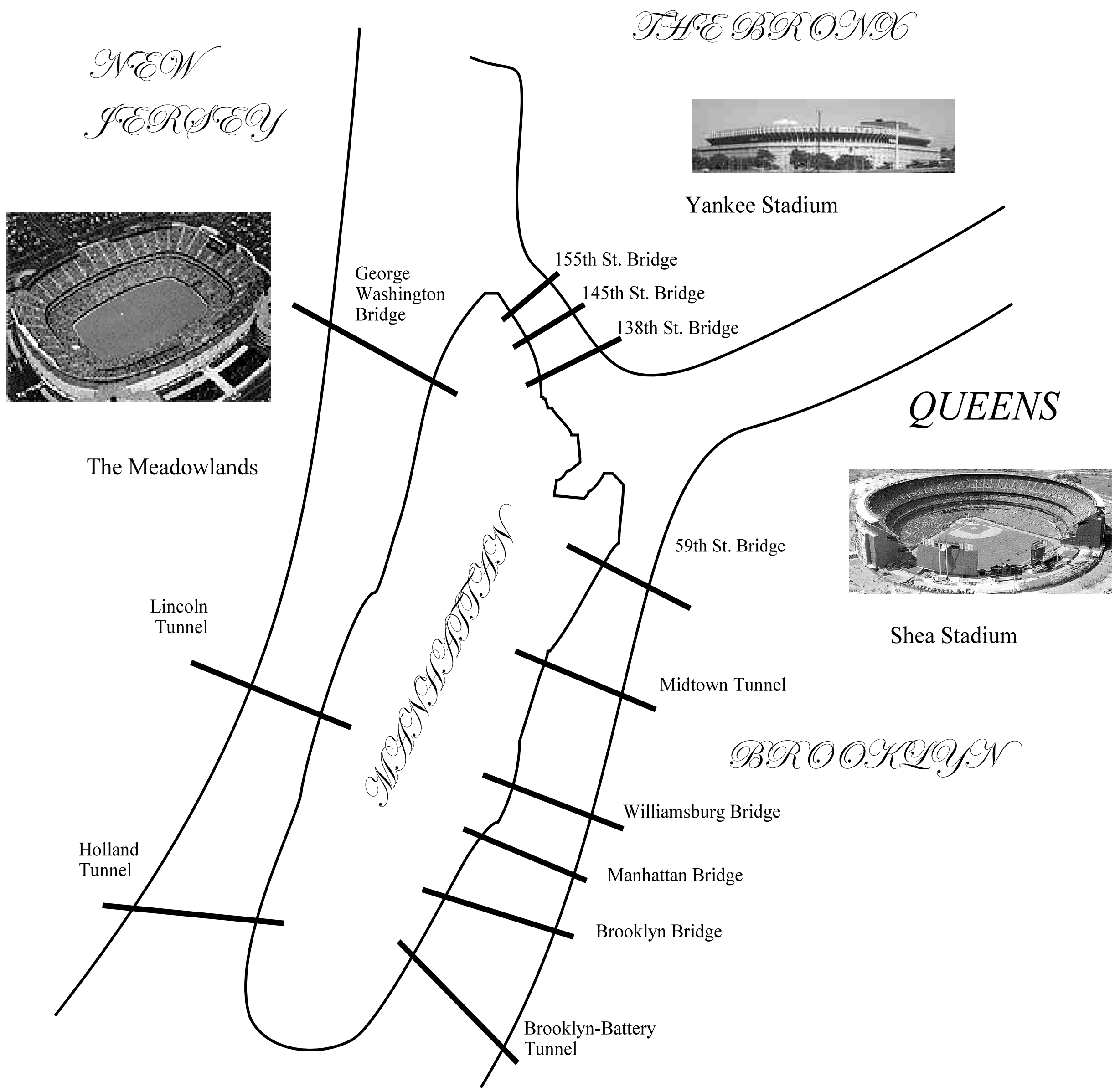
If the above requirements are met, then the graph actually has an Euler circuit or an Euler path. That is,

1. If a graph is connected, and has no vertices of odd degree, then the graph has an Euler circuit; an Euler circuit can begin and end at any vertex.

2. If a graph is connected, and has exactly two vertices of odd degree, then the graph has an Euler path; an Euler path must begin at one of the vertices of odd degree and end at the other vertex of odd degree.

Hand-out #4 — Traveling around Manhattan

Is it possible to start at the Meadowlands, visit every bridge and tunnel on this map and end at Shea Stadium? (No bridge or tunnel may be visited more than once.) Is it possible to start at the Meadowlands and end at Yankee Stadium? Explain your answers. (Note that you are not asked to actually find a way of doing this – only to tell whether it is possible.)

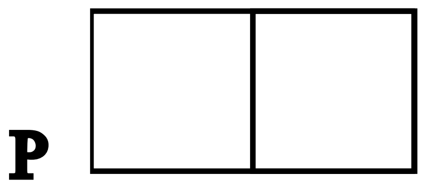


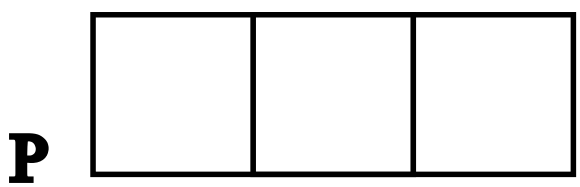
Developed by Susan Picker, LP '90

Hand-out #5: Letter Carrier's Problem

A letter carrier starts at the post office (located at the vertex marked P) and must deliver the mail to each block and return to the post office. Can this be done without repeating any blocks?

What is the least number of blocks the letter carrier must repeat?

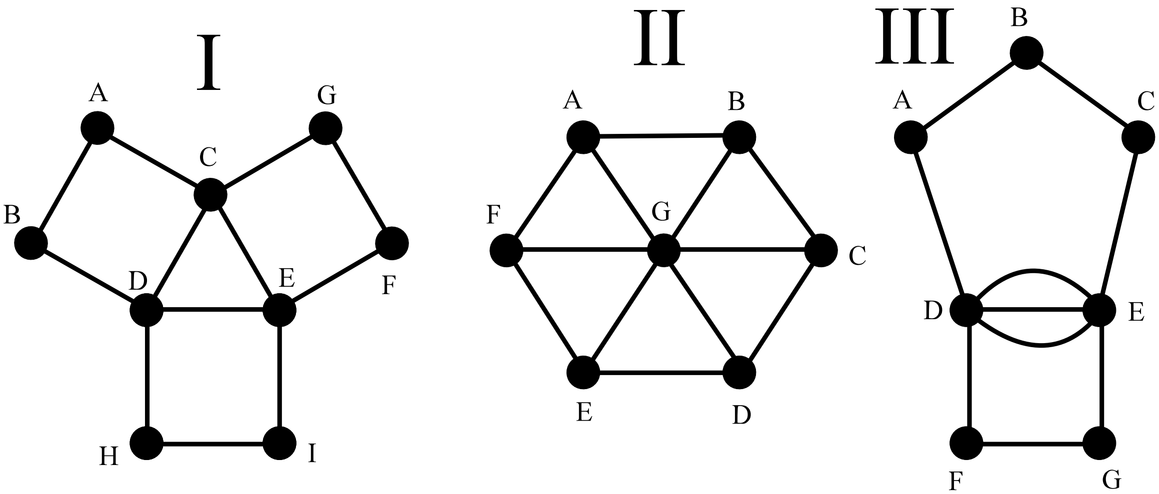




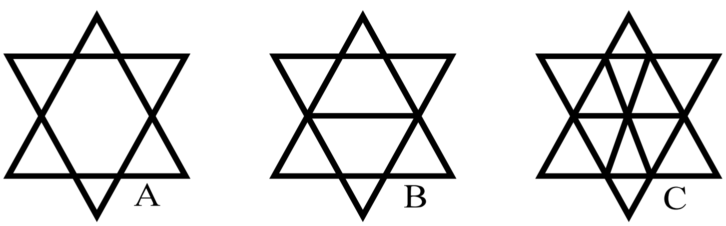
Workshop 2 — Drawing Pictures with One Line — Exercises

Practice Problems:

1. For each of the following graphs, determine whether it is possible to trace the graph without repeating any edges. (Helpful suggestion: To keep track of your path, number the edges in the order used. Extra copies of graphs may be found on EX #<u>6</u> - EX #<u>8</u>.)

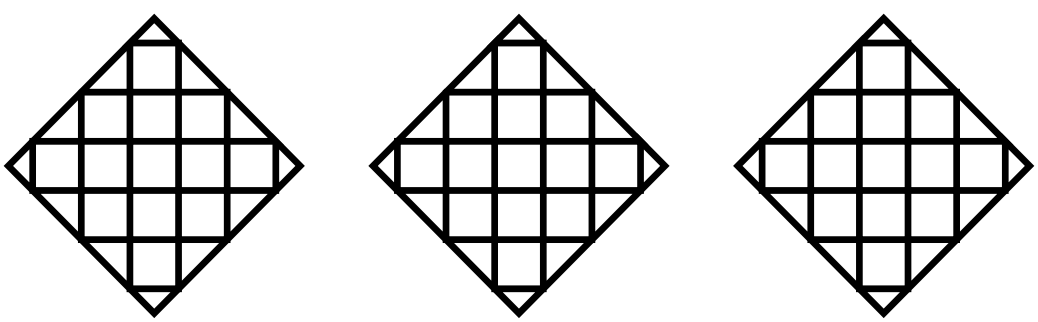


2. Which of the graphs below has an Euler path or circuit? (Note: In these graphs, there is a vertex wherever two line segments meet.)

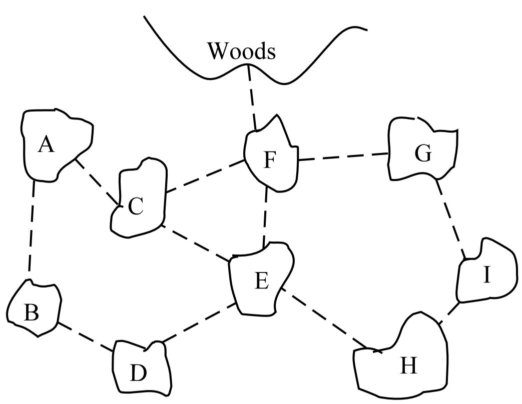


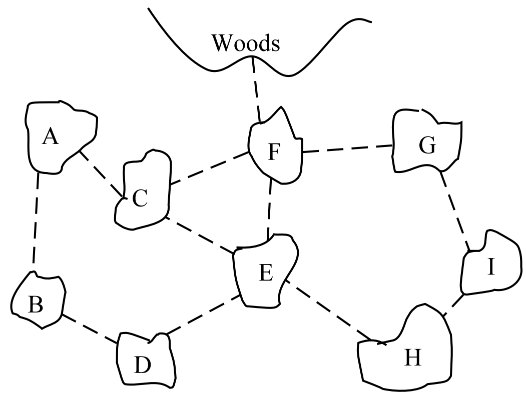
3. If a graph has 80 vertices, 30 of degree 6 and 50 of degree 8, how many edges does it have?

4. Find an Euler circuit in the picture below (three copies are provided).



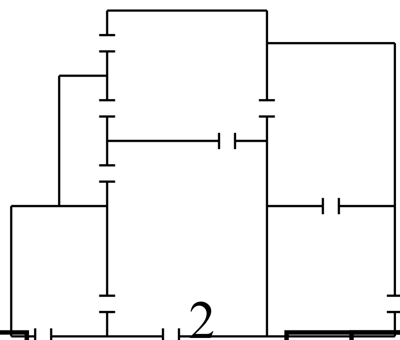
5. Walking through a snowy field one afternoon, you scare a rabbit from the woods at the top of the picture below. The rabbit scampers around the field, hiding under a succession of bushes. Where is the rabbit now? (From "Drawing Pictures with One Line", by Darrah Chavey, COMAP. Two copies are provided.)

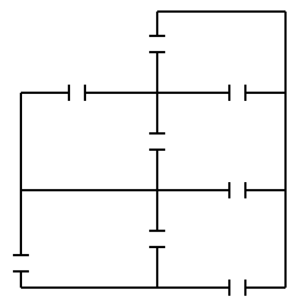




Study group problems:

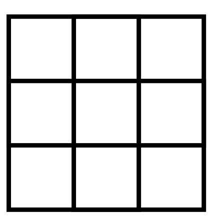
6. The two drawings to the right are floor plans for houses. In each case, is it possible to start somewhere, walk from room to room through the house (including going outside and then coming back into the house) and go through each door exactly once? (From "Drawing Pictures with One Line", by Darrah Chavey, COMAP.)



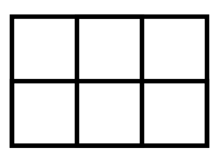


7. A letter carrier starts at the post office and must deliver the mail to each block and return to the post office (located at the intersection next to the P). For each of the following examples, what is the least number of blocks the letter carrier must repeat?

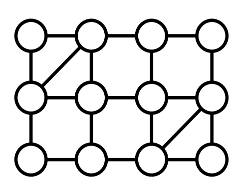
1 P



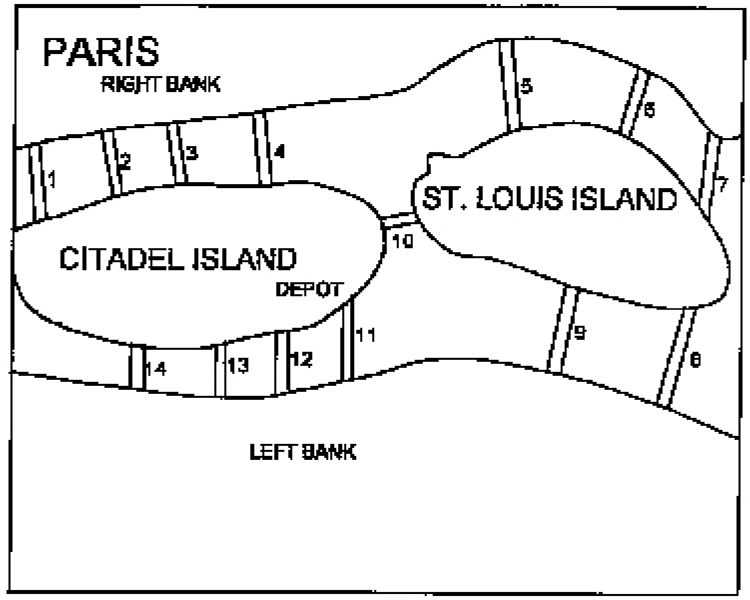
2 P



8. a. What is the least number of edges you can remove from the graph to the right so that the resulting graph has an Euler circuit? Which edges would you remove?
b. What is the least number of edges you can repeat in the graph so that the resulting graph has an Euler circuit? Which edges would you repeat?
c. What is the least number of edges you can add to the graph so that the resulting graph has an Euler circuit? Which edges would you add?



8. A huge shipment of potatoes has been spilled by farmers on Bridge 1 in protest against low prices. The bridge will have to be closed to traffic for the next 5 hours. The Paris bridge sweepers still must sweep all the other bridges. Use graphs to explain your answers to the following questions.

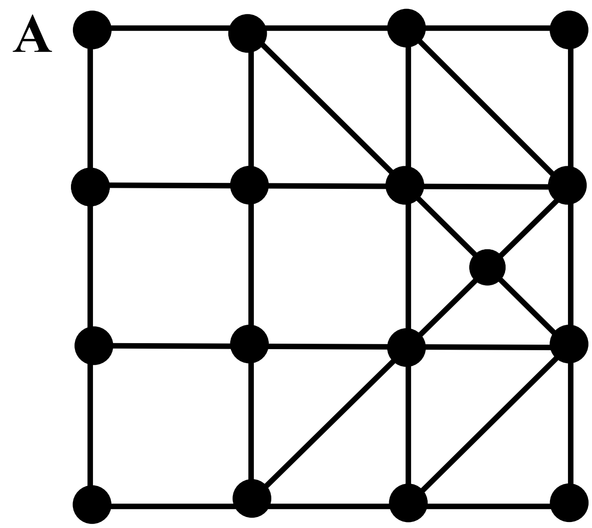


a. Is it possible for the sweep team to leave their depot on Citadel Island, sweep Bridges 2-14 and return to their depot without repeating a bridge?

b. Will it be possible for them to do this once Bridge 1 has been reopened?

c. How would your answers to these questions have been affected if all the bridges were one-way — for example, if all odd-numbered bridges were southbound only and all even-numbered bridges were northbound only (bridge 10 becomes eastbound only)?

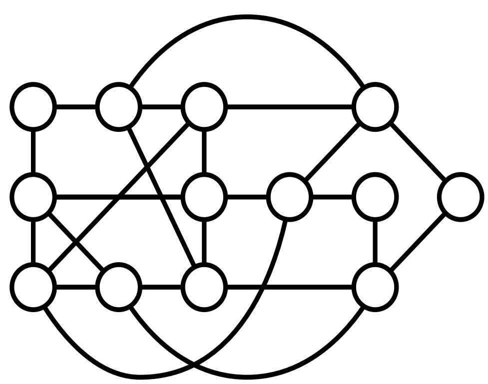
9. a. Given the street map at the right, is it possible to route a police car so that it starts and ends at A and travels each block exactly once?



b. What is the minimum number of blocks that have to be repeated if the police car starts and ends at A and travels each block at least once?

c. If it takes one minute to travel each block, what is the minimum total time that the police car will take to cover its beat?

10. Find an Euler circuit in the graph at the right. What is the least (positive) number of edges you can remove from this graph so that the resulting graph still has an Euler circuit?



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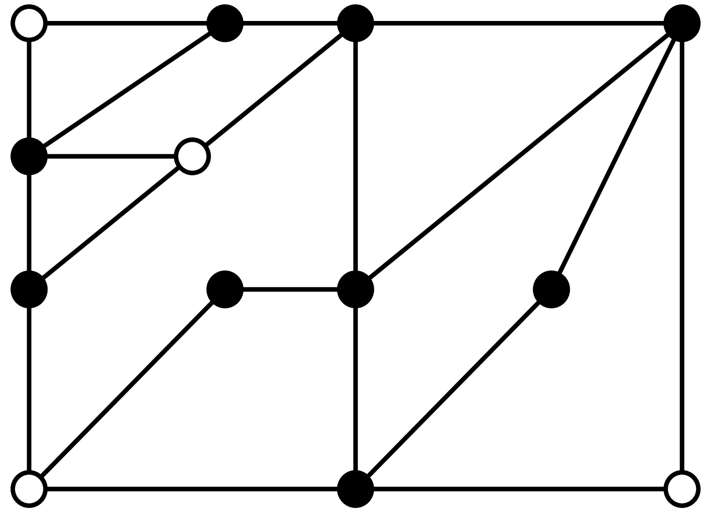
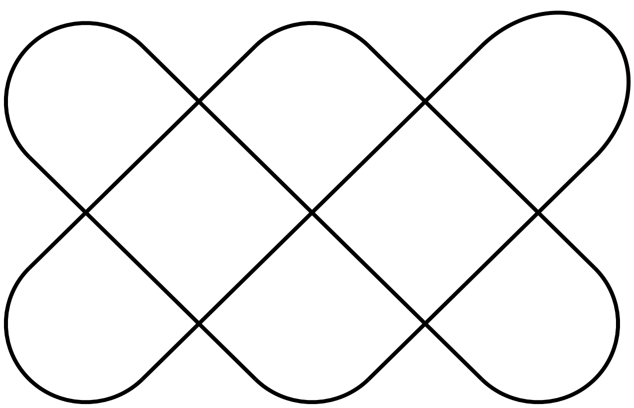
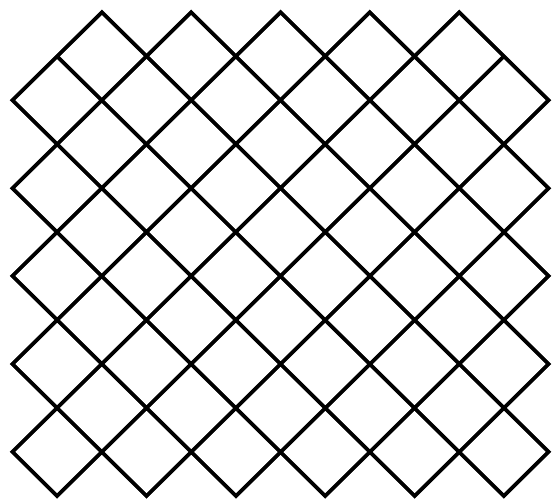
11. The Air Conditioning Inspector Problem. To locate the source of an obnoxious odor in an office building, an air-conditioning inspector is called in and asked to inspect the air-conditioning ducts thoroughly. Because the ducts are cramped, the inspector prefers not to go through any section more than once if it can be avoided. A scale map of the system is shown below with the access points shown as open circles. An access point can be used either as an entrance to the system or as an exit from it. (Note: The air conditioning inspector must crawl through the entire length of a duct in order to check it.)
a. Can the inspector crawl through all of the ducts without crawling through any of them more than once? Explain.
b. What is the best route for the inspector?



Extension Problems:

12. Can you find the "children's secret for drawing the networks". (See the book "Ethnomathematics" by Marcia Ascher for an extended discussion about how problems like this arise in various cultures.)

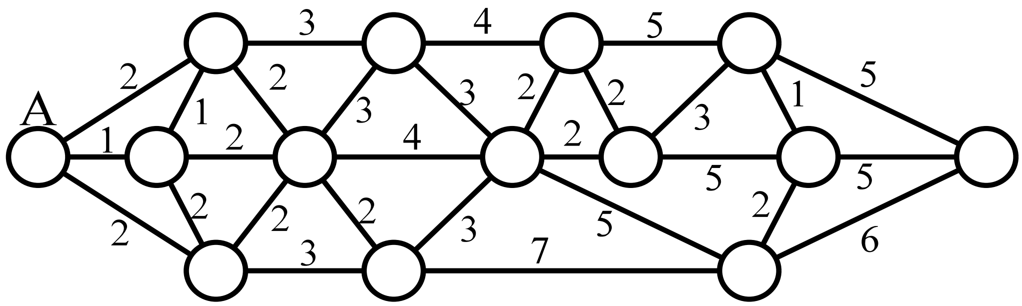


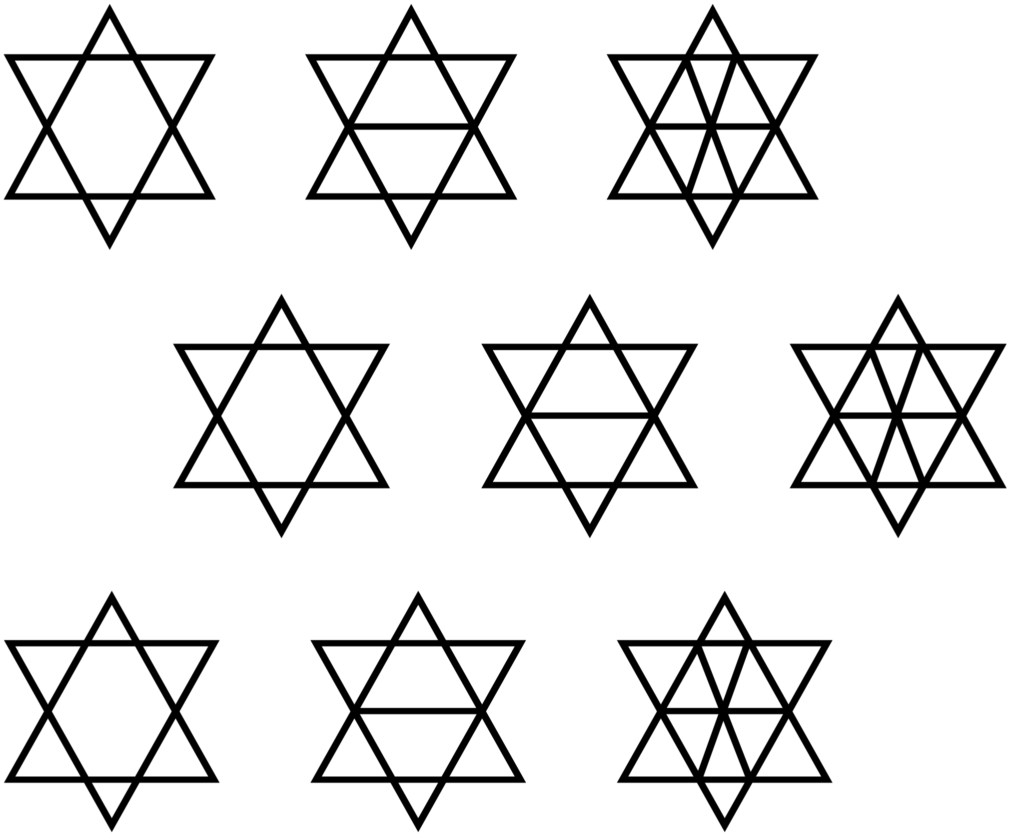


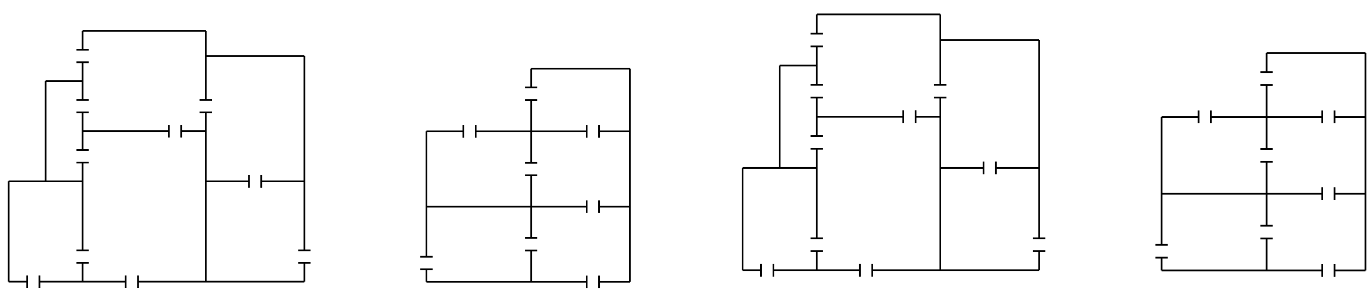
13. Which complete graphs have Euler circuits? Which "wheels" have Euler circuits?

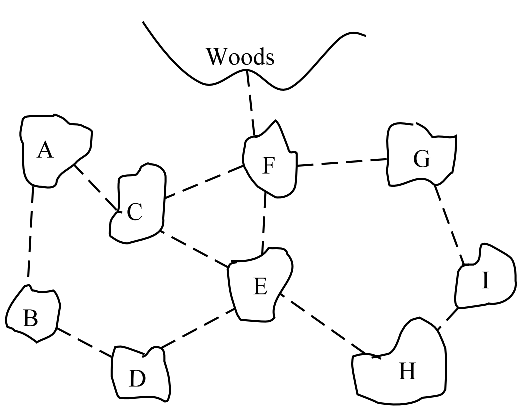
14. How many different graphs can you find with six vertices all of degree 2? with ten vertices all of degree 2? with two vertices of degree 2 and four vertices of degree 3?

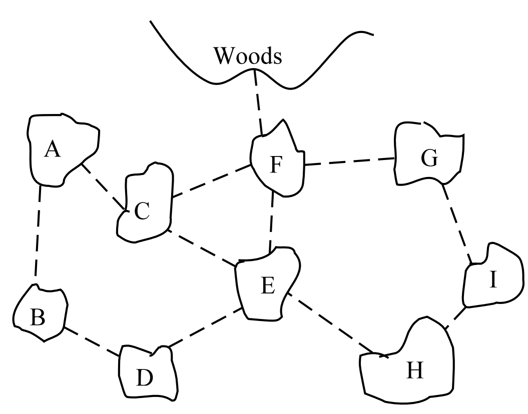
15. A postman must start at point A on the map below, traverse all the streets shown, and return to point A. The numbers on each edge show how many minutes it takes to deliver mail along that street. Since, as you can tell, this graph does not have an Euler Circuit, some edges will need to be repeated twice. What is the best route which you can find that will allow the postman to complete the mail delivery as quickly as possible, and how long will it take? What strategy did you use to find this route?











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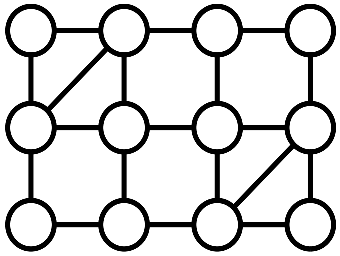
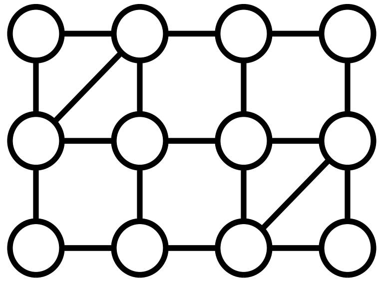
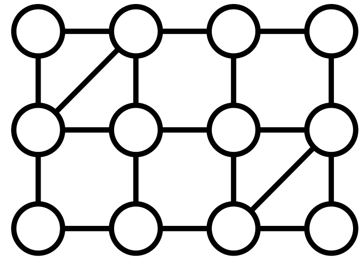
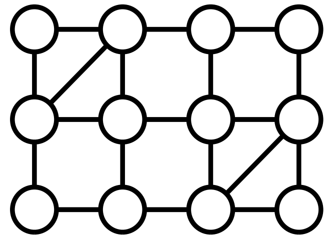
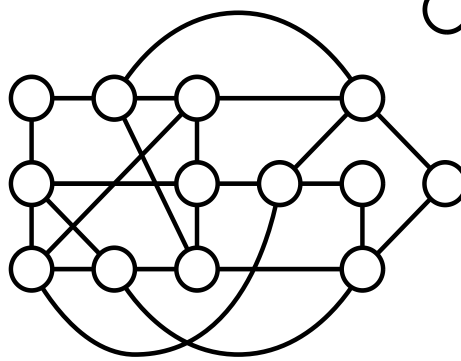
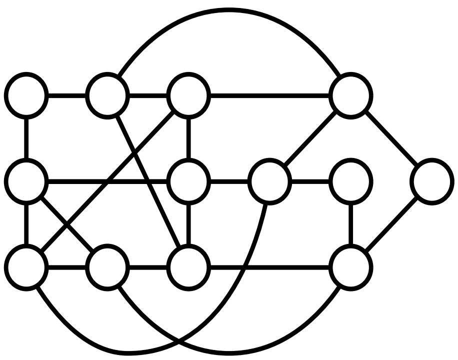


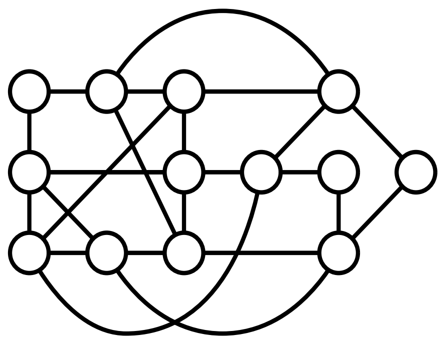


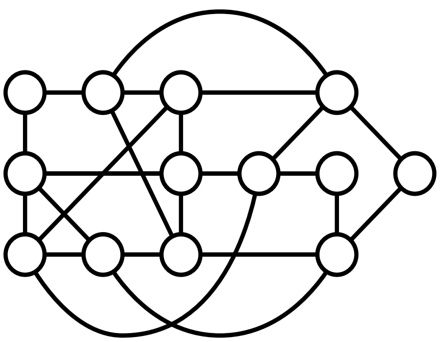


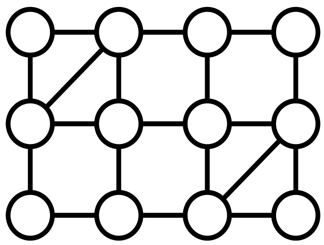


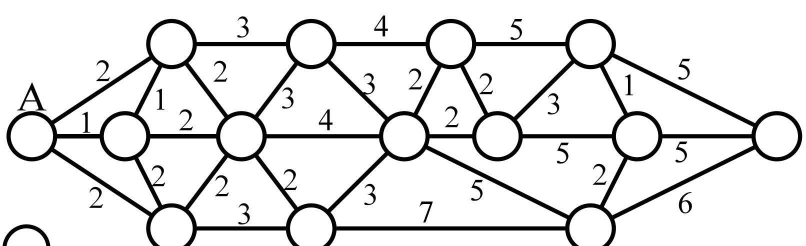


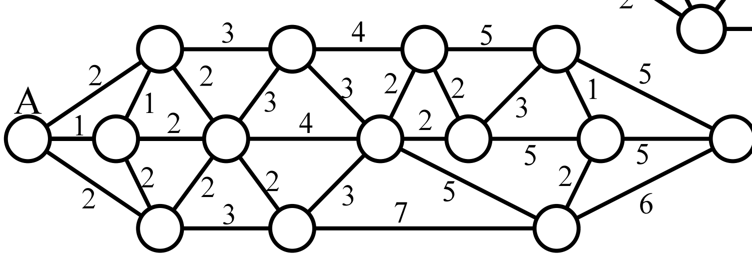


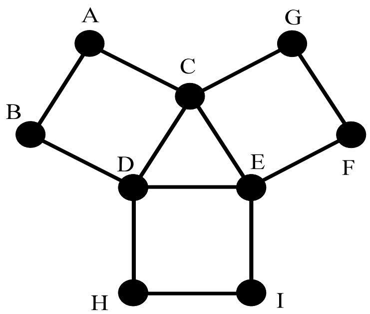


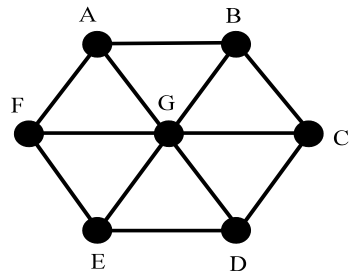


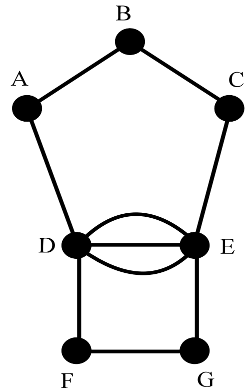


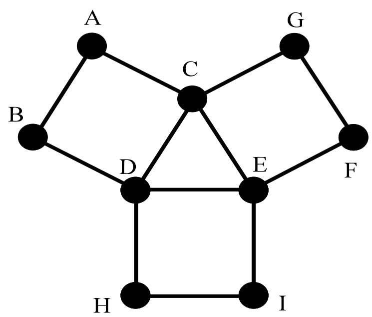

A
3 4 5
2 2 3 2 2 5
1 3 3 1
1 2 4 2 3
5 5
2 2 2 2
2 3 3 5
3 7 6

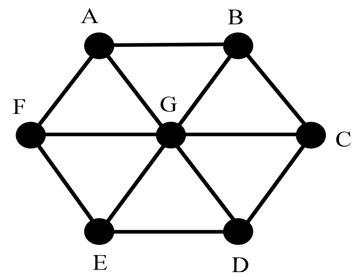

A
3 4 5
2 2 3 2 2 5
1 3 1
1 2 4 2 3
5 5
2 2 2 2
2 3 3 5
3 7 6

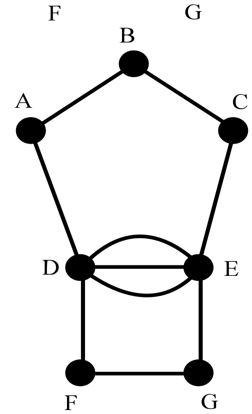


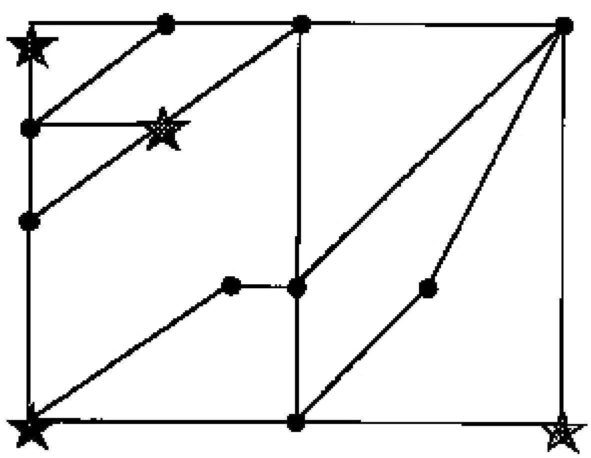


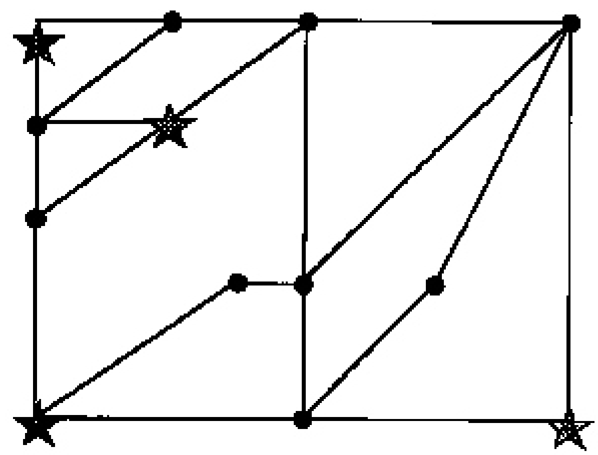


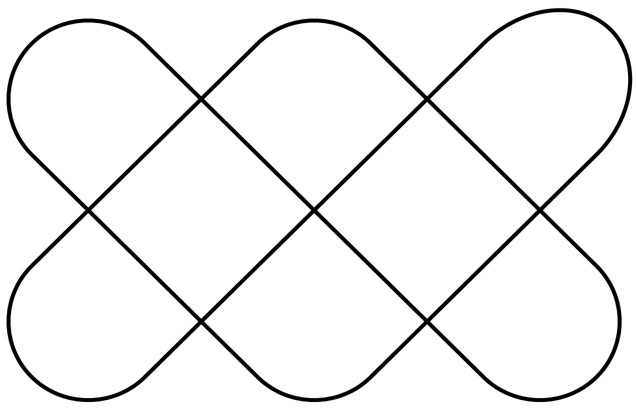


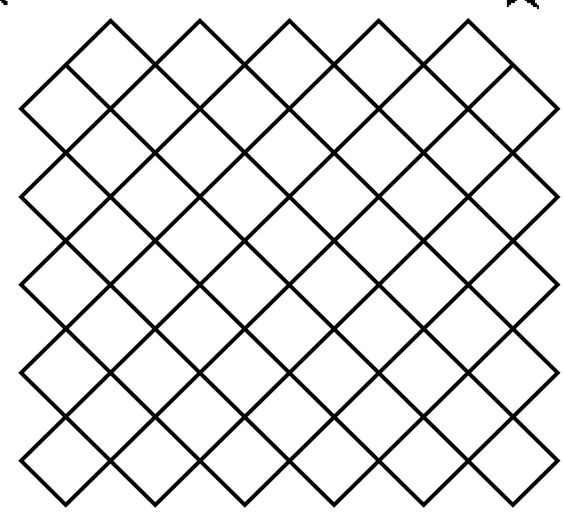












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Resource Book

Workshop 2: Drawing Graphs with One Line

Table of Contents

The Resource Book contains activities that teachers can use in their classes in addition to those discussed in the institute workshop on the drawing graphs with one line, and the applications of Euler paths and circuits.

Page 2, entitled "Mathematical Background", contains the terminology introduced in this workshop. Page 3 contains an outline of the Leadership Program workshop on “Drawing Graphs with One Line.”

Pages 4-9 focus on drawing graphs with one line; pages 4-7 contain simple example, and pages 8-9 contain examples of complicated patterns from other cultures.

Resources: Additional examples of interesting patterns that can be drawn with one line can be found in Darrah Chavey's *Drawing Pictures With One Line*, published by COMAP and the resources cited there, *Ethnomathematics* by Marcia Ascher and *Multicultural Mathematics: Interdisciplinary Cooperative Learning Activities* by Claudia Zaslasky.

Pages 10-17 provide examples of applications of Euler paths and circuits.

Pages 18-23 contains the script and map of the footprints for *The Case of the Stolen Diamonds*.

Resource Book

Workshop 2: Drawing Graphs with One Line

Mathematical Background

- ✓ A "**path**" is a sequence of edges in a graph, each of which begins where the previous one ends.
- ✓ An "**Euler path**" is a path in a graph which uses each edge exactly once. (Vertices may be repeated, but not edges; "Euler" is pronounced "oiler".)
- ✓ An "**Euler circuit**" is an Euler path which ends where it begins.
- ✓ A graph is "**connected**" if it is possible to get from any vertex to any other vertex along a path.
- ✓ A graph is "**disconnected**" if you can find two vertices with no path joining them.
- ✓ A "**multigraph**" is a graph where there may be more than one edge connecting two vertices; multigraphs are used to represent situations (as in the Konigsberg Problem) where there are multiple roads or bridges joining two sites.
- ✓ **Requirements for an Euler path or circuit:** In order for a graph to have an Euler path, it must be connected and have either (a) two vertices of odd degree (in which case there is an Euler path which starts at one vertex of odd degree and ends at the other), or (b) no vertices of odd degree (in which case there is an Euler circuit which starts at any vertex, and ends at the same vertex)
- ✓ **Euler's Theorem:** 1. If a graph is connected, and has no vertices of odd degree, then the graph has an Euler circuit; an Euler circuit can begin and end at any vertex. (2) If a graph is connected, and has exactly two vertices of odd degree, then the graph has an Euler path; an Euler path must begin at one of the vertices of odd degree and end at the other vertex of odd degree.
- ✓ **Road Inspector's Problem:** Is it possible for a road inspector to make an inspection tour of all roads on a map, traveling each road exactly once, and ending where he begins?
- ✓ **Letter Carrier's Problem:** A letter carrier starts at the post office and must deliver the mail to each block and return to the post office. What is the least number of blocks the letter carrier must walk?
- ✓ "**Eulerization**" is the process of repeating edges of a graph in order to ensure that the resulting graph has an Euler circuit; this process is used, for example, in solving the letter carrier's problem.
- ✓ A **"balanced graph"** is a graph each of whose vertices have even degree. The process of Eulerization results in a balanced graph.

Workshop 2: Drawing Graphs with One Line

Workshop Outline

1. **The Case of the Stolen Diamonds**
   a. Performance of this play followed by a discussion.

2. **Finding Euler paths on graphs**
   a. Can a graph (thought of as a picture) be traced with one line without repeating any part of the picture? Such a line is called an Euler (pronounced "oiler") path; if it ends where it begins it is called an Euler circuit.
   b. Determine which of the graphs on a handout can be traced with one line; boxes of sand are helpful for this activity.
   c. Calculate for each graph on the handout the number of vertices of each degree, and from patterns in the resulting information, determine when a graph has an Euler path or Euler circuit, using the reasoning in the discussion of the play.
   d. When does a graph have an Euler path or Euler circuit? The answer is given in Euler's Theorem. If it has no vertices of odd degree, then there is an Euler circuit. If there are exactly two vertices of odd degree, then there is an Euler path but no Euler circuit; if there are more vertices of odd degree, then there is no Euler path.
   e. Determine which decorations of the Bushong people can be drawn using a single continuous line.

3. **Finding Euler paths on maps**
   a. Discussion of Euler and the Konigsberg Bridge Problem.
   b. Problem of Manhattan bridges and tunnels.
   c. Letter carrier's problem. Can the letter carrier cover her entire route without repeating any streets? If not, what is the smallest number of streets that have to be repeated?
   d. Introduced the terminology "balanced graph" for one whose vertices all have even degree and the notion of "Eulerization" which by repeating edges of a graph results in a balanced graph.
   e. View and discuss the COMAP video "Snowbound: Euler Circuits" which shows how drawing Euler circuits and solving the letter carrier's problem help in removing snow efficiently.

Around the Office

1. Is it possible to enter the office complex below at A, go through each door exactly once, and exit at B?

   a. If so, what path would you use? (Indicate your path by listing in order the numbers of the rooms that you pass through.)

   b. If not, what path would you use in order to go through as many doors as possible without repetition? Which door or doors would be omitted?

2. How would the problem be different if the door between rooms 7 and 8 was sealed up?

Paris Potato Spill

A huge shipment of potatoes has been spilled by farmers on Bridge 1 in protest against low prices. The bridge will have to be closed to traffic for the next 5 hours. The Paris bridge sweepers still must sweep all the other bridges.

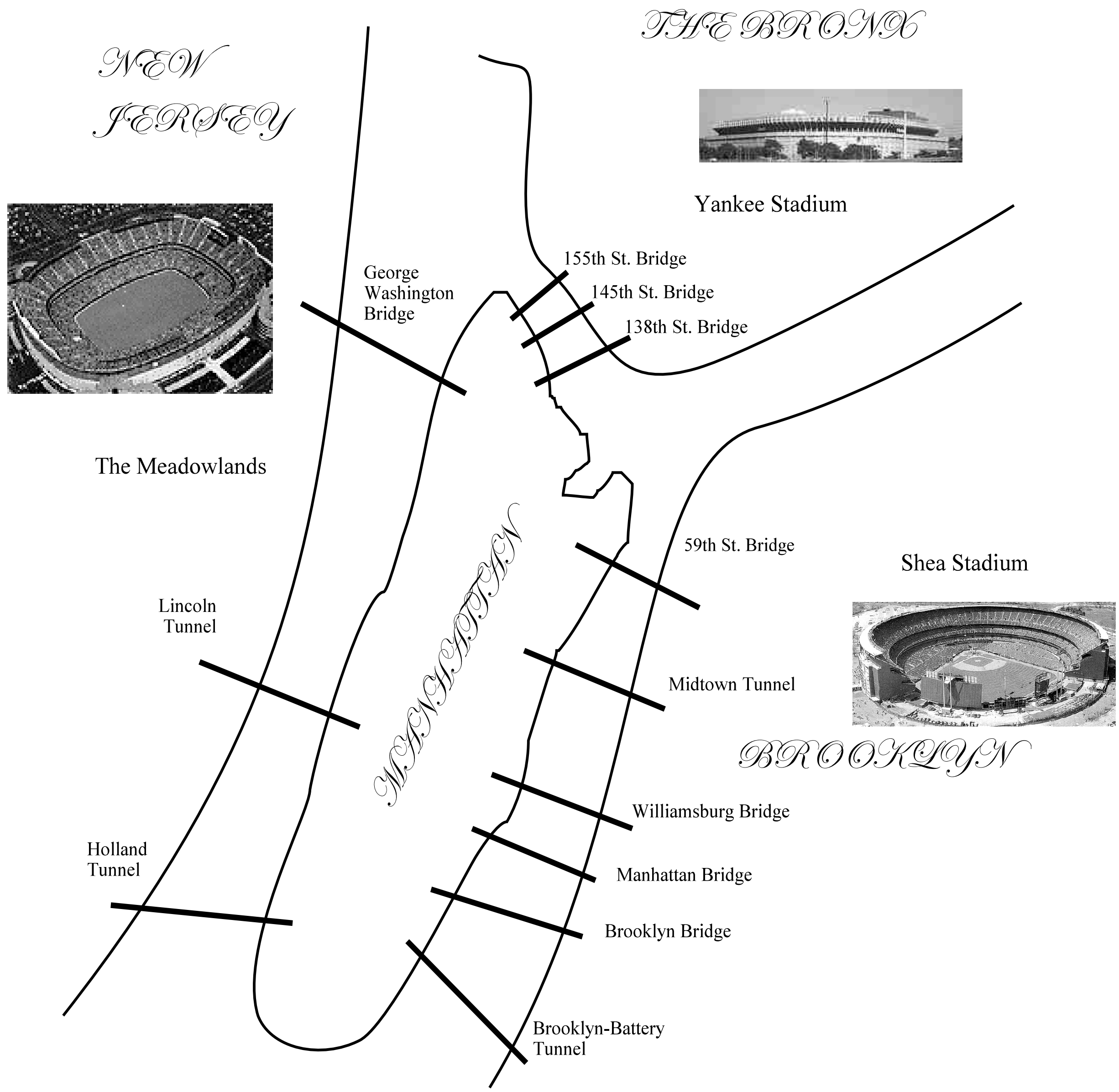
Is it possible for the sweep team to leave their depot on Citadel Island, sweep Bridges 2-14 and return to their depot without repeating a bridge? Will it be possible for them to do this once Bridge 1 has been reopened? Draw a graph and explain your answers.

Developed by Susan Picker, Leadership Program 1990

Workshop 2: Drawing Graphs with One Line

Traveling around Manhattan

Is it possible to start at the Meadowlands, visit every bridge and tunnel on this map and end at Shea Stadium? (No bridge or tunnel may be visited more than once.) Is it possible to start at the Meadowlands and end at Yankee Stadium? Explain your answers.



Developed by Susan Picker, LP '90

The Case of the Stolen Diamonds

By Neil Goldstein
Revised by Deborah Franzblau and Susan Picker, 1996
Later revised by Joseph G. Rosenstein, 1997

Cast:	Suggested Props:
Narrator	
Rooster	
Lady Schmendrick	hat with plume, long pearl necklace, boa
Cook Cromwell	bakers hat, cooking, apron, wooden spoon
Chauffeur Hanson	chauffeur's hat
Maid Marian	fancy apron, duster
Butler Hornsby	serving tray with two glasses, tea towel
Inspector Oyl LaTour	magnifying glass, deerstalker hat

ACT I

Scene 1

Narr : It was a dark and snowy night at the Schmendrick Mansion ... Suddenly, just as dawn was breaking (*Rooster crows*), there was a loud cry from the upstairs bedroom.

Lady S : Aieee!!! My best diamonds ... I've been robbed!

Narr : The alarm was sounded and all the gates to the estate were locked ... the jewels must be somewhere on the grounds. But where?

(*Characters enter and step forward as they're introduced*)

Could they be in the Cook Cromwell's quarters? Or in Chauffeur Hanson's cottage? Did Butler Hornsby take the jewels ? Or perhaps it was Maid Marian.

Lady Schmendrick knew that she need the services of the one and only Inspector Oyl LaTour of Scotland Yard ...

(*lights down ... characters take seats in the "drawing room"*)

Scene 2

Narr : The prime suspects were gathered in the drawing room.

LaT : (*Aside to audience*) I knew something was up as soon as I entered the gates. I found fresh tracks in the snow—but couldn't tell which direction they pointed. The thief must have made extra tracks and dragged his boots ... Before the snow melted, I made a careful map of the grounds and the tracks.

(*Put map on overhead projector*)

I decided to question the suspects.

LaT : Lady Schmendrick, when did you last see the diamonds?

Lady S : Well, about midnight, I locked the diamonds in the safe as usual. I looked out the window, and all the lights in the servants' quarters were out, and it was just starting to snow. Then, around dawn I was awakened by the sound of running feet downstairs. When I got up, the safe was open and my diamonds were gone.

LaT : (*Aside*) Hmmm ... that means the tracks must have been made by the thief .
Thank you Lady Schmendrick. Now, Hanson, you're the chauffeur, what do you remember of last night?

Chauffeur : I was sleeping as usual, when I was awakened by my alarm clock. I suddenly heard someone run into my house and back out again. But I didn't get a good look at who it was.

LaT : Well, that sounds rather strange, but perhaps it could be true ...

Cook : It may sound strange Inspector, but that's exactly what happened to me. I couldn't sleep at all last night. I kept thinking about this recipe for Chunky Cherry Chiclet Puffs that I've been working on. They're really gooey and delicious, but there's just something missing ...
Anyway, suddenly, someone I couldn't tell you who it was—ran right through my house and then out the other door. Then, a few minutes later, it happened again! I could see that the person was holding something but just couldn't tell what it was.

LaT : And you Hornsby, did you hear anything?

Butler (sounding somewhat suspicious): Well, I usually sleep very soundly, you know.

Something woke me briefly at some point, and I think I heard footsteps going in and out of the door a few times ... I was pretty groggy, so I can't really be sure.

Maid : Well you're not the only ones; I know someone ran in and out of my house too, at least twice.

LaT : My, my, this does get more and more interesting. The cook, the maid, and the butler all believe that someone ran in and out of their quarters two or more times, and was probably the thief. The Chauffeur says that someone ran in and out of his/her quarters once. From Lady Schmendrick's testimony, the thief must have run in and out of the mansion as well. That must be what woke her up.

This thief was clever enough to try to hide his tracks by running through all the buildings. However, I happen to know that one of you is lying. (*All gasp and look at one another*.)

Here is a map I made of the tracks in the snow this morning. Study it carefully. No one will leave this room until we get to the bottom of this.

Narr : We now take a brief leave from the Schmendrick drawing room, to give you, the audience, a chance to discover the truth. What did inspector LaTour see on her map that raised her suspicions? Can you guess who the culprit is and where the diamonds are hidden? If you can—don't give away the secret ... but write down your guess, and put it in this (box, hat, bucket...)

(*Intermission, with time for people to make guesses*)

(*Quickly count up votes for each character, and record them on a side board*)

Narr : Now that you've had your chance to make your own deductions, let's return to the drawing room as Oyl LaTour exhibits the workings of her steel-trap mind.

LaT : Let's see how the map matches your stories. First of all, Lady Schmendrick could not have stolen her own diamonds: there are three tracks, so if she left the house with the diamonds, she'd have to go in and out again, and wouldn't have been in the house when she raised the alarm.

Lady S: Why Inspector, I can't believe you'd even think of suspecting me, we Schmendricks are a very old and connected family, you know.

LaT : Sorry—but we have to consider all possibilities in this business, Lady Schmendrick.

The chauffeur has only two tracks leading to his house, which certainly fits his testimony. The cook and the maid both have four tracks to their houses, which means that their stories also check out.

Cook : I should say so, surely you don't think *I* have any use for diamonds!

LaT : That leaves only one more house ... the one with only three tracks going to it (*Everyone looks at the butler*). Hornsby ! You hid in the mansion last night, took the jewels, and ran around to confuse everyone, didn't you? I'll bet the jewels are hidden in your quarters. What do you have to say for yourself Hornsby?

Butler : Hey, wait a minute, why is everyone looking at me! You're not going to pin this on me so you can say "The butler did it!"

I don't only know how to serve martinis. I've been around. I know something about tracks in the snow myself.

You may be smart LaTour, but there's another scenario that you just didn't consider. Notice that there are also 3 tracks into Lady Schmendrick's mansion. I don't want to be too indiscreet, but I happen to know that she lost a lot of money recently investing in Naugahyde futures. Why , she can collect over a million from the insurance company on those diamonds—as long as we don't find them!

Hmmm ... I thought I smelled some familiar perfume in my living room last night ... (*Turning accusingly at Lady S*) Lady Schmendrick, you hid in my house last night and made the tracks in the snow just so that you could pin this on me. The diamonds are hidden somewhere in the mansion.

Lady S (*incensed*)**:** Not only do you steal my diamonds, but you also spy on me. You rat! You snoop! You loose-lipped louse!

LaT: The tracks show that the thief has to be either you, Hornsby, or you, Lady Schmendrick, and, since each of you was in your own house in the morning, the thief must have hidden in the other house some time before it stopped snowing. But which of you did that? Can either of you prove that you were home when it stopped snowing.

Butler: Yes, I can. Maid Marian spent the evening watching the late show on the telly with me, and she didn't leave until after the snow stopped. So I was home then. As I said, Lady Schmendrick is your thief!

LaT (*to Maid Marian***):** Is that true? Were you with the butler until after the snow stopped?

Maid: Yes, he couldn't have done it. It must have been Lady Schmendrick!

LaT (*to Maid Marian***):** Wait a minute! You said that someone ran in and out of her house, which is possible, but then you said "at least twice" and that is now impossible, since one of the tracks is your own. So you lied about that. Also if you walked home after it stopped snowing, there is no way of accounting for the three other tracks to your house. I don't believe that you were with Hornsby.

LaT (*turning to Butler***):** Sorry, Hornsby, you blew it. If you had kept quiet, I wouldn't have been able to figure out whether you or Lady Schmendrick was the thief. But you made up an alibi, and it doesn't hold water. So you're the one. I'm sure we'll find the jewels where you hid them. It's off to jail with you!

(*LaTour leads Butler away as lights fade*.)

The end