Master Document

Workshop 1 — Graph Coloring

Transparencies 7A, 8A, 9A are not in this file.

Changes in EX #2 and #6 are not in this file.

LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

Revised July 27, 1999 (and, most recently, June 21, 2011)

Workshop 1 — Coloring maps and resolving conflicts

Materials and Pre-Workshop Preparations — **Allocated Time**

Activity #1 — US Map Activity . . . . 25 minutes

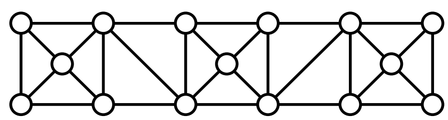
- Six large maps of the US (4' x 6') - distributed around the room
- Colored chips, cut from construction paper. Each group receives ten different colors, each color containing 20 chips, 10 large and 10 small.

Activity #2 — Map Coloring Challenge Activity . . . . 30 minutes

- Each participant should bring color markers or pencils to color maps
- Colored chalk for drawing maps on blackboard

Activity #3 — Graphs and Graph Colorings . . . . 35 minutes

- Large diagram on the floor of graph below with 15 vertices made of masking tape and the centers of paper plates (removing the rim permits the plates to lie flat) — the sides of each square should measure 4'



Activity #4 — Using Vertex Coloring to Resolve Conflicts . . . . 35 minutes

- Video: COMAP's Video Geometry: New Tools for New Technologies — "The Zoo Problem"

Activity #5 — Coloring Graphs with Three Colors . . . . 15 minutes

. . . . TOTAL WORKSHOP TIME: 140* minutes

* In addition, 10 minutes are allocated for a break in this 2 ½ hour workshop. On subsequent days, where the total time for the workshop is 2 ¼ hours, the time allocated for the activities is 125 minutes.

Word Wall – Map coloring, Chromatic Number, Four-Color Theorem, Vertex-Edge Graph, Vertex, Vertices, Edge, Adjacent Vertices, Graph Coloring, Cycle, Complete Graph, Wheel, Greedy Algorithm

Activity #1 — US Map activity
(Allocated time = 25 minutes)

A. Six large maps of the US (4' x 6') are distributed around the room, most on the floor, one or two on tables. (Alternatively, a smaller map (2' x 3') can be placed on each table.) Ask participants to cluster around the maps in groups, where each group includes approximately 1/6 of the participants. Give each group a pile of colored chips of ten or more different colors, and ask participants to color the map by placing a chip on each state (within the continental US), so that states with a common border have different colors.

You might tell the participants that: "You can imagine that the coloring of the states that you are constructing will be used to make a map in an atlas, and if two neighboring states are colored with the same color, you won't be able to tell where the border is."

Note that the instructions do not ask them to minimize the number of colors; indeed, they may start out by using all ten colors available.

During this time, participants should be encouraged to discuss (and even argue about) such questions as whether states which meet at a point (e.g., CO and AZ) have a common border — so that they see that there is a need to define the phrase 'common border". Lead teachers should be circulating among the groups, so that <u>once the participants recognize the need for a definition (but not before), the lead teachers can inform them that, for our purposes, two states which meet at a point do not have a common border</u>. Of course, we could have avoided this issue by announcing the definition in advance, but our perspective is that teachers can better understand the experience of mathematical problem solving if 1) they identify issues of "problem ambiguity" themselves, 2) discuss implications of the ambiguities, and 3) determine how to resolve such ambiguities.

Some groups may immediately report "four colors" because they have heard that "all maps can be colored with four colors". Yet when these same participants are asked, "Why?" or "Are you sure?" they are unable to support their immediate "four colors" claim. As a result, these groups should be encouraged to find out why three colors do not suffice. See section C below.

B. After the participants have colored the map, ask them if they can color the map using fewer colors, and indeed using as few colors as possible.

The instructor (not the lead teachers) should tell the participants that: "You can imagine that when your map is printed, each additional color complicates the production process, so that if you can use fewer colors, the total cost will be less." Some groups will move to this part of the activity without prompting; in any case, our experience is that all groups will be able, in a reasonably short amount of time, to color the map with four colors. Moreover, most groups will make good attempts to color the map with three colors.

C. At an appropriate point, after the lead teachers have indicated that all groups are done, ask each group whether they think that they have colored the map with the fewest number of colors possible, and verify that each group has done it with four colors. And now the hard part — ask them to discuss in their groups how they would justify that the number of colors they have used is in fact the smallest number needed, that they can't color the map with three colors.

You might tell the participants that: "You can imagine that the publisher, your employer, has asked you why you can't color the map with three colors, that it would mean a great savings in printing costs. How would you respond?"

Ask one person to report briefly from each group. Our experience has been that they will not be able to provide convincing justifications for why four colors are necessary — other than the vague statement that there were too many states bordering an individual state. If the US map had four countries which border each other, then we could expect them to find this situation and provide this justification for the need for four colors. However, that's not the case. The activities that follow are intended to make them aware of this possibility and of the possibility of a five-cycle surrounding a single vertex (referred to in the exercises as a "wheel"). We can't expect them to articulate this configuration at this point in the workshop, but some participants have in fact been able to do so. Moreover, many (even most) groups will identify the actual "trouble spots" — Nevada, Kentucky, West Virginia (the only three states which are the hubs of odd wheels) — although they may be unable to articulate precisely the problems that occur around those states.

Since we can't expect them to come up with a good reason for why a map can require four colors, we will instead ask them to try to ***construct*** *maps which* *require four colors — as in the activity below (which is similar to the Lewis Carroll Game in the Resource Book). See note below Activity #2A for a possible segue from this discussion to the next activity.*

At the end of this activity, participants return to their tables (if they are not already there).

Activity #2 — Map coloring challenge activity.
(Allocated time = 30 minutes)

A. Ask each participant to draw a map with a small number of countries (no more than 12) that requires four or more colors (show TSP #1 so that participants can also see the instructions); stress that this is an individual activity which everyone should complete on their own — otherwise, part B will not work well. Stress that they should draw their map on a separate piece of paper — since they will afterwards pass it to the person on their right — and that they should not color their map — that task is for the next person.

You might use the following as a segue from the previous discussion: "Many of you felt that four colors are needed for the US map; try to translate your intuition into a picture of a ***simpler*** *map which <u>requires</u> four or more colors".*

If in the previous discussion someone notes that four states which border on each other would require four colors, make sure (at that time) that everyone understands why that is correct; in that case, for this activity, ask each participant to create a map which requires four colors but which does not have four countries all of which border on each other.

B. Ask each participant to pass her/his map to the right (be specific about this), and to try to color the map they receive with three or fewer colors. Tell participants to make a colored mark in each country rather than filling in the whole map, so that they can easily change colors if they decide to try to do it differently.

C. Ask for six to ten volunteers to draw maps on the blackboard which seem to require four or more colors. (Ask them to use colored chalk, if available.)

You should use the blackboard rather than the projector so that all maps can be seen at the same time. Discuss with the group possible reasons why four colors are needed for these examples — hopefully they will come up with examples where *there are four countries bordering on each other (absent in the US map) or one country encircled by five bordering countries. Ask those who drew the maps why* ***they*** *thought they needed four colors. Try to get participants to discover the difference between wheels with an odd number of spokes and wheels with an even number of spokes (without using the terminology of "wheels").*

D. Assuming that they find such an example, hand out a copy of the US map (HO #1) and ask them whether they can find (in their groups) such an example there.

At an appropriate point, you can display TSP #2, which has the Nevada example. But you should also use TSP #3 (the US Map) to review the other real trouble spots that they noticed in the first activity (such as Kentucky and West Virginia) and potential trouble spots (such as Missouri and Tennessee, both of which are hubs of even wheels). (Use a blank transparency to overlay the US Map transparency and generate a coloring, on the blank TSP, to show whether each state suggested is a problem or not.)

It is important to consider each suggestion from a participant and determine whether it is valid or not. After completing this activity, point out to the participants that they also should be encouraging their students to make conjectures and test their conjectures in a variety of settings, that this pattern of behavior helps strengthen their problem solving abilities.

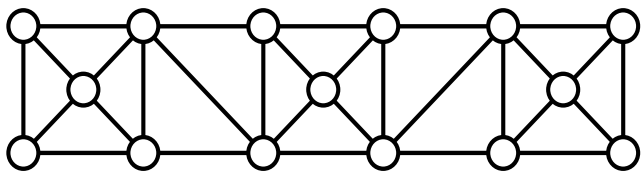
E. Observe that no one drew a map which had five countries all of which bordered on all the others. Ask them to try to draw five countries each two of which share a common border. After a brief time, ask them why they think that's impossible. You can point out that if four countries touch each other, then three of them must wrap around the other fourth one, preventing it from touching a fifth country. This observation leads into the next one.

F. Use TSP #4 to summarize the situation with regard to map-coloring, noting that: (a) it is known that every map can be colored with four colors and that this fact is known as the Four Color Theorem; (b) this was believed to be true for a long time, but was only proved in 1976 (this is discussed further in both the Resource Book and in a video that they will see later today); and (c) some maps can be colored with three colors, but some actually require four colors — and how to tell whether a particular map can be colored with three colors is sometimes difficult. (This issue will be discussed later — see Activity #5 and problem 12 of the exercises.)

Activity #3 — Graphs and graph colorings

(Allocated time = 35 minutes. Build in a 5-10 minute break after part A, before participants return to their seats.)

A. Have participants gather around a large diagram of a graph with 15 vertices that has been drawn (in advance) on the floor or a tarpaulin. (The edges and vertices are created using masking tape and paper plates — the centers of the paper plates representing the vertices, and single lines representing edges. Make sure that this graph is large enough; the squares should have sides of length at least 4 feet.) The graph used appears below and on TSP #6.



You might tell them that: "This is an example of what we call a graph or network — each paper plate is called a vertex and each line joining two vertices is called an edge. Two vertices are adjacent if there is an edge joining them. We want to color the vertices of the graph in such a way that adjacent vertices have different colors. Does this sound familiar? We'll talk more about the connection later. Here's how we're going to color it. Each of you will stand on a vertex, and notice which vertices are your neighbors — those are the ones you are connected to by an edge. (Model this by standing on a vertex and asking, "Which vertices are my neighbors?") When I say "go", you're going to color your vertex red, yellow, or green; then you're going to look at your neighbors and if necessary one of you is going to have to change colors (perhaps after a quick consultation). And that may have to happen again, and again, until we get a coloring of the vertices which works. Ready, set, and ... Oh, I forgot to tell you how you color your vertex. Remember that at a traffic light red is at the top, yellow is in the middle, and green is at the bottom — so you will do "hands-up" for red, "crouch-down" for green, and "stand-normal" for yellow. Ready, set, and go."

This activity works very well with this particular graph. What happens is that each of the three major "neighborhoods" is quickly colored, but their colorings are incompatible. The participants quickly realize that this is a lesson on cooperation, and that some neighborhoods have to abandon their colorings and start over. With this graph, participants discover (possibly with some guidance from the workshop leader) that you can color the entire graph by starting at one *end and propagating the coloring to the other end. (When originally introduced, the graph used was more complicated, and neither the participants nor the leader could easily determine where to begin and which vertices needed to change colors.)*

After carrying out this activity once, possibly giving participants guidance, try it again with another group of participants, this time without guidance. And if they are able to get it quickly (e.g., in less than a minute), have another combination of participants try it, but this time without communicating verbally during the activity.
You should point out that this activity is not only an exercise in coloring graphs, but also a lesson in cooperation, and an activity which will particularly benefit kinesthetic learners (children or adults who learn best by carrying out physical actions). Since some students might find confusing the extra step of identifying "red, yellow and green" with "top, middle and bottom," they might instead each be given three colored sheets of paper to hold up or three colored hats to wear.

[Time for a 5-10 minute break]

B. Use the above activity and TSP #5 to introduce the notions of a graph (or network). Overlay a blank transparency on TSP #5. **As you draw in the edges, reinforce the idea that edges can be curved lines, or jagged lines, as well as straight lines. Draw some other simple graphs on the board, or on a blank transparency, taking care to use small circles for vertices at the outset. Ask each participant to make his or her own graphs, using about six vertices and ten edges; ask if they have any questions about what graphs can look like. Note also that this simple mathematical model of "graph" is used to represent a variety of situations.**

C. Review TSP #6, pointing out that "vertex" is the singular of "vertices" not "vertice" and that some books use the terms "arc" instead of "edge" and "node" instead of "vertex". Introduce the notion of a vertex coloring of a graph, and give several examples (using the graphs you drew on the board earlier). Point out that the graph on TSP #6 is the same as the graph of Activity #3A above, and that that activity involved finding a vertex coloring of the graph on TSP #6.

Some participants will be thinking of coloring in terms of filling in the regions formed by the edges of the graph (as was done with map coloring in the previous activity), so you will need to make it clear that in this activity the colors are placed in the circles that we are calling vertices — hence the name "vertex coloring".

D. Ask participants to work in pairs to find vertex colorings of the graphs on Hand-out #2 (= TSP #7) — using as few colors as possible. (Since it is possible to view all eight graphs on this page as one disconnected graph, you should tell participants that there are eight graphs on this page, although the two graphs at the top of the columns are identical.)

In reviewing TSP #7, you might use K_3, add a vertex and an edge, add another edge, and note that three colors are still enough; add another edge and a fourth color is needed! Add another vertex and an edge, and another ... , and note that four colors is still enough. But when all five vertices are joined, then five colors are needed.

E. After they have found the number of colors needed for each of the graphs, they should find and then explain the pattern. This discussion should lead naturally to the notions of "cycle" and "complete graph" on TSP #8. At the conclusion of this discussion, introduce the notion of "chromatic number" on TSP #9, discuss the examples given there, in reference to TSP #7, and then determine the chromatic number of the given graphs, noting that, as before, adding the two missing edges results in increasing the chromatic number to five.

F. *(new)* Ask participants to work in pairs to find vertex colorings of the graphs on Hand-out #2A (= TSP #7A) – using as few colors as possible. After they have found the number of colors needed for each of the graphs, they should find and then explain the pattern. This discussion should lead naturally to the notions of "wheel" on TSP #8A. Show examples of graphs that are not wheels ... by deleting either a spoke or a edge in the cycle.

G. *(new)* Using TSP #9A, discuss question of how you know how many colors are needed for a graph.

H. Make the connection between vertex coloring of graphs and map coloring by overlaying TSP #2 (Nevada and its neighbors) with the graph of that map on TSP #10. (Or overlay a blank transparency on TSP #2 and construct the graph found on TSP #10 by hand.) Note that countries are represented by vertices, and when two countries share a common border we represent that by drawing an edge joining their associated vertices so that they become adjacent. Why do we use graphs to represent maps? As we will see shortly, graphs can be used to represent a variety of situations which are like map coloring.

Activity #4 — Using vertex coloring to resolve conflicts
(Allocated time = 35 minutes)

A. Have participants work on the problem of scheduling student projects — Hand-out #3 and TSP #11. Point out, using TSP #13, that this is similar to the map-coloring problem, except that here you have to assign days to projects rather than colors to states. Elicit from them what the vertices should be, and when edges should be drawn; draw a few edges and then have the participants complete the activity.

In this problem, many participants often have difficulty in determining that a vertex should represent a project. They feel intuitively that a vertex should represent a student and that vertices should be joined by an edge if they appear on the same project; this is natural because vertices are the basic elements in a graph and students are the basic element in the problem. Encourage participants to write answers to the following three questions before beginning any conflict resolution problem: 1) What things are in conflict? "In this problem, the projects are in conflict, so they should be vertices." 2) How can you tell whether two things are in conflict? "In this problem, two projects are in conflict if there is a student involved in both of them; in that case, the projects should be joined by an edge." 3) How can the conflict be resolved? "In this problem, the conflicts are resolved by assigning adjacent vertices (conflicting projects) different colors (different days)." Summarizing, the vertices are the projects, edges represent conflicts between projects, and colors represent assignment of days to projects. (This summary appears on TSP #13A.) Having participants get into the habit of thinking through "vertex", "edge" and "color" may assist in minimizing misconceptions and frustrations.

Another difficult spot for participants is the idea that "things" (i.e., two vertices) get "connected" (i.e., joined by an edge) if they "have a conflict". In the elementary school curriculum, teachers often have students connect two "like" objects by a line; they never connect two "unlike" objects by a line and hence initially find such a thing counterintuitive. Once participants get used to this idea, they are better able to solve a variety of conflict resolution problems.

B. Discuss participants' solutions, arriving at the graph-like diagram of TSP #12, and then stress the parallel between this problem and the map-coloring problem (see TSP #13).

C. Show and discuss the first part of Unit 2 (the zoo problem) from COMAP's video Geometry: New Tools for New Technologies. (This portion of the video takes 7 minutes.) Stop the video after they have found the first coloring of the zoo graph. This is just before the narrator says "Wait a minute!" (When we have shown the entire video, participants get focused on applying Brooks' Theorem; instead of staying in problem-solving mode, they try to use Brooks' Theorem for all the homework problems.)

D. Use TSP #14 to summarize Activity #4, showing how all three concrete problems (map, projects, zoo) are interpreted as graph-coloring problems — revisiting TSP #11 for projects, and using TSP #2 and TSP #10 for map coloring.

Activity #5 — Coloring graphs with three colors
(Allocated time = 15 minutes)

A. Use Handout #4 = TSP #15 to discuss the question of how many colors are needed for a graph. This activity involves three graphs, each of which is repeated. The top graph is a truncated (and flipped) version of the graph on TSP #6 (which they colored kinesthetically using three colors in Activity #3), and the other two graphs each has one additional edge. Encourage participants to notice the differences between the top and middle graph and between the top and bottom graph. They should use this information to make conjectures and help solve the problem of how many colors are needed to color the middle and bottom graphs.

> ***You can introduce this activity by asking the question "How do you know that a certain number of habitats are enough? Perhaps you can get by with one fewer habitat?" So it is important to know when you are indeed using the minimum number of colors. Try to color the three graphs on this hand-out using a minimum number of colors.***

Once they have worked on this problem in their groups, and found that the bottom graph requires four colors, although the others require only three colors, show them TSP #16 which says that there is no efficient method which will always tell you whether a graph can or cannot be colored with three colors! Tell them that this is a theme we will come back to several times in the coming days.

> ***The experience that many K-8 teachers have had with mathematics is the following: give a rule, follow the rule, get an answer. They are often surprised to learn that mathematics has "open problems" and that sometimes "there is no***

efficient method" available to answer some relatively simple questions. Participants often find the fact that there is no efficient method which will always tell you whether a graph can or cannot be colored with three colors disturbing. For many, this is the first time they have been exposed to the idea of "ambiguity in mathematics".

In the next workshop, you will contrast the answer to the general question of whether a graph is three-colorable with the answer to the general question of whether a graph is Eulerian. Participants seem to have a lot of difficulty accepting the idea that there is no simple way of answering a question like "How many colors does it take?" They really like Euler's Theorem (in the next workshop) because it tells them exactly when there is an Euler circuit, but they do not like the parallel situation with Hamilton circuits. When Euler's Theorem is discussed tomorrow, the contrast should be drawn, and TSP #16 should again be shown.

Supplementary Notes

1. **If you have time, you might show and discuss the remaining transparencies. TSP #17 shows 5 countries, where the outer and inner countries don't touch. Overlay TSP#18, which in effect relocates the map onto the surface of a torus. Now the inner and outer countries do touch. (Imagine that the ellipse which contains the map as being dran around the torus, and that the inner country goes through the hole and borders the outer country at that ellipse.) Thus on the torus you can have five countries each of which border all of the others. Finally, you can overlay TSP#19 adding a 6th country, touching each of the other 5, and leaving the other 5 still touching each other. If you wish, you can mention that it is possible to have a map on a torus with 7 countries, in which each country touches each of the other 6 countries along a border. These observations could be reinforced with an actual torus (e.g., a donut) on which the six countries depicted in the transparencies are actually shown.**

2. **It should be mentioned in the homework review session that your success in coloring a graph often depends on where and how you begin. If for example in problem #11 you are stingy in allocating colors, then you will need 5 colors, whereas if you are more generous, you can color it with 4 colors. It should also be noted that this example was generated by a LP participant who thought an odd wheel around an odd wheel would result in a map which required 5 colors.**

Map Coloring Challenge Activity

Now that you have discussed what makes it hard to color a map with a small number of colors ...

1. Draw your own map (involving no more than 12 countries) which you think <u>will require</u> four or more colors.

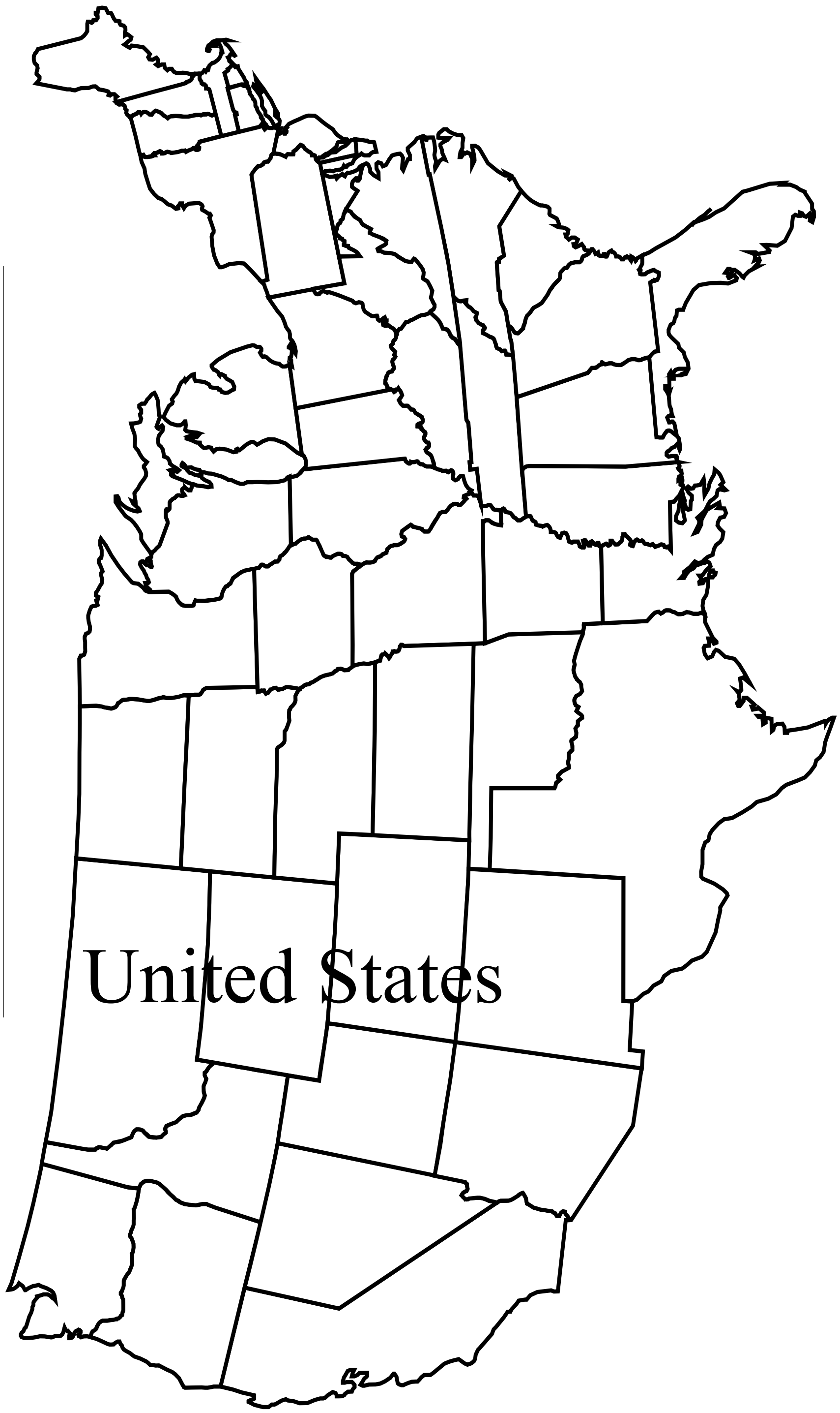
**** on a separate piece of paper**

**** don't color your map**

2. Pass your map to the right.

3. Try to color the map you receive with three colors.


OR
ID
NV
UT
CA
AZ


United States

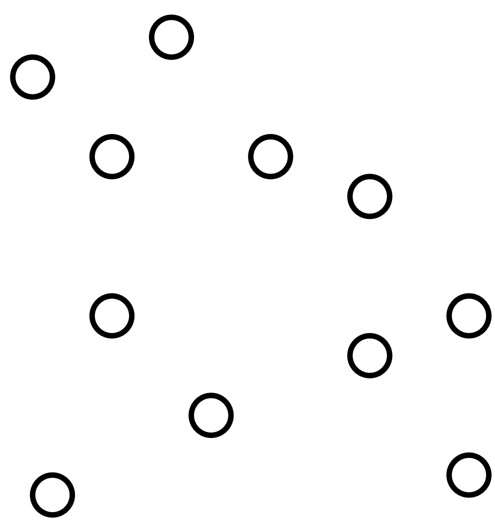
Map Coloring — Summary

- ✓ Every map can be colored with four colors!
- ✓ This fact is now known as the Four Color Theorem.
- ✓ It was believed to be true for a long time, but was only proved in 1976.
- ✓ Some maps can be colored with two or three colors, but some actually require four colors.
- ✓ How to tell whether or not a map can be colored with three colors is in general a difficult problem.

What is a vertex-edge graph or network?

- ✓ A vertex-edge graph or network is a collection of points some of which are joined by lines.

- ✓ The points are called vertices and are shown as small circles.



- ✓ The lines are called edges and are shown as lines which can be straight, curved, or wiggly.

- ✓ We often abbreviate to “graph.”

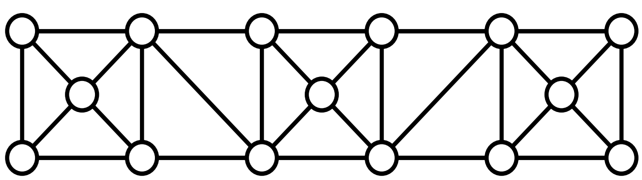
Basic information about graphs

- ✓ Vertices and edges are sometimes called "nodes" and "arcs"; the singular of "vertices" is "vertex" <u>not</u> "vertice".

- ✓ Each edge joins two different vertices.

- ✓ A given pair of vertices may or may not be joined by an edge.

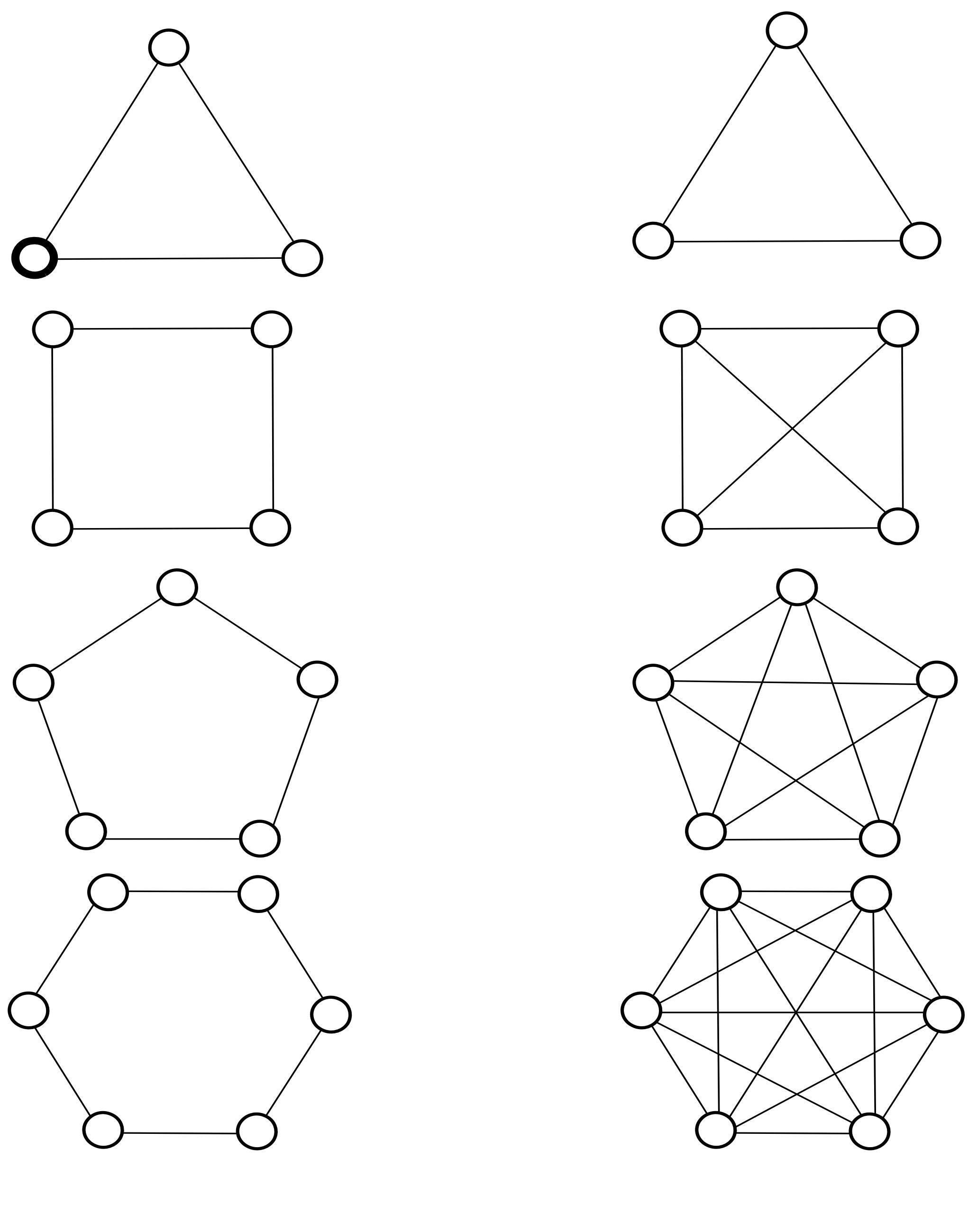
- ✓ Vertices joined by an edge are called "adjacent" or "neighbors".



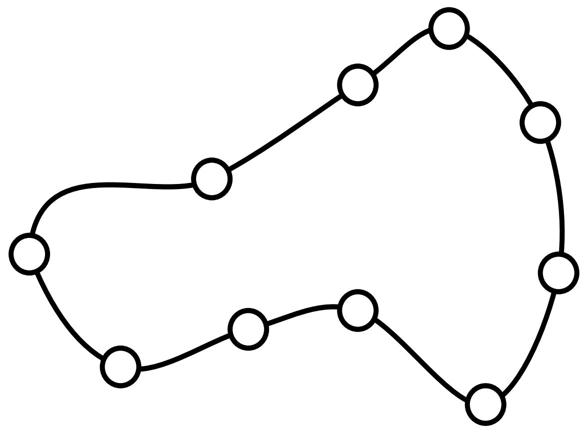
- ✓ A vertex coloring of a graph involves assigning colors to the vertices of a graph so that adjacent vertices are assigned different colors.

Hand-out #2: Vertex coloring of graphs

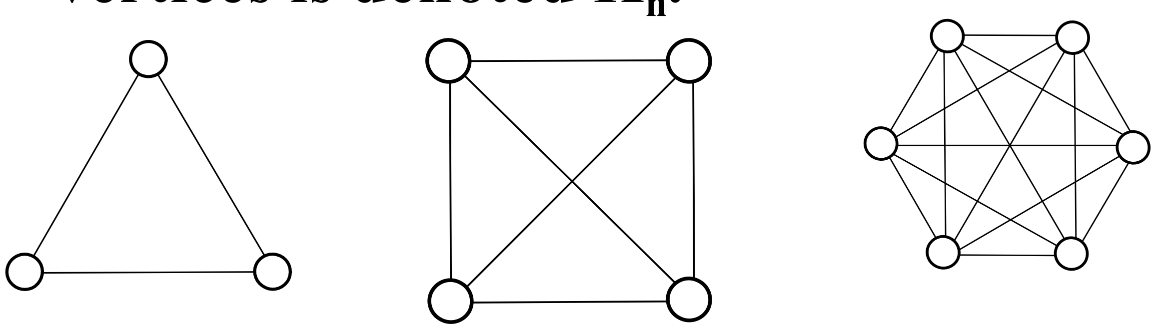
How many colors are needed for a vertex coloring of each of the following graphs? Do you see any patterns in your solutions?



✓ **A "cycle" is a graph where the vertices can be arranged in a circular fashion so that each vertex is adjacent to the two vertices which come before and after it in the circle. The cycle with n vertices is denoted C_n.**

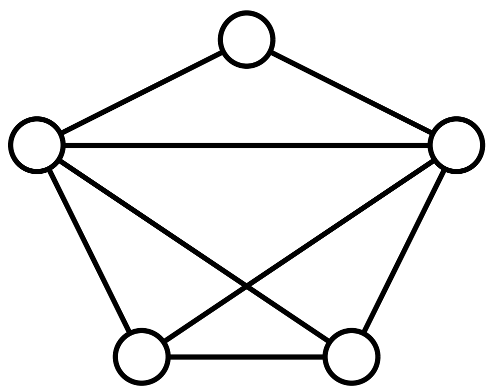


✓ **A "complete graph" is a graph in which every vertex is adjacent to every other vertex. The complete graph with n vertices is denoted K_n.**



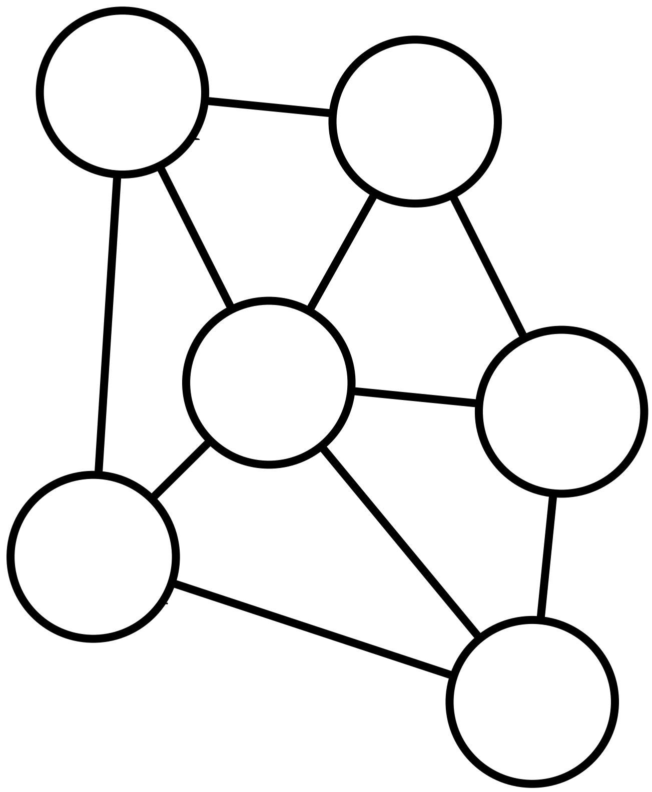
Chromatic Number of Graphs

✓ **The "chromatic number of a graph" is the smallest number of colors that can be used for a coloring of the graph. If the graph is called G, then the chromatic number of G is often written as $\chi(G)$, read "chi (kye) of G".**



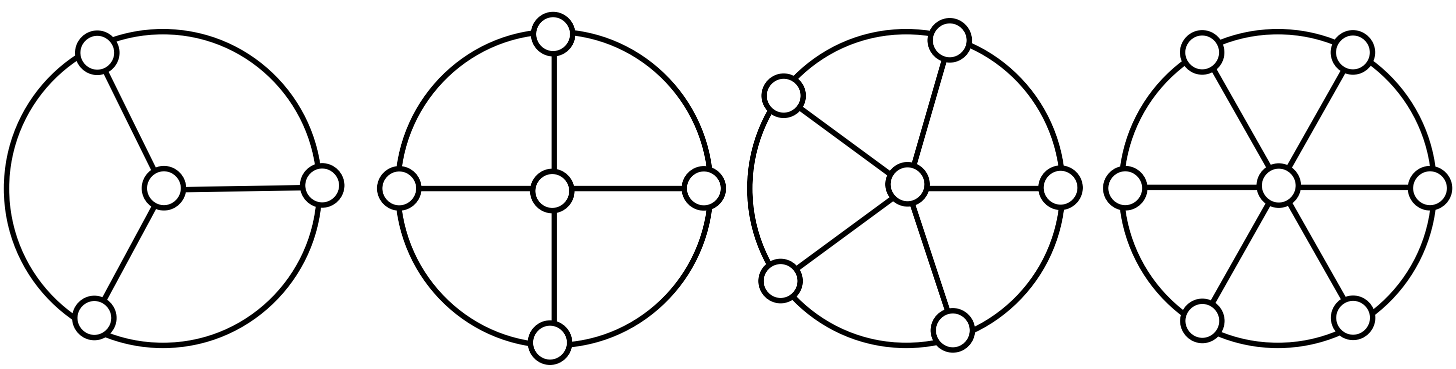
Examples:

$\chi(C_4) = 2$	$\chi(K_4) = 4$
$\chi(C_5) = 3$	$\chi(K_5) = 5$
$\chi(C_6) = 2$	$\chi(K_6) = 6$
$\chi(C_7) = 3$	$\chi(K_7) = 7$
etc.	etc.

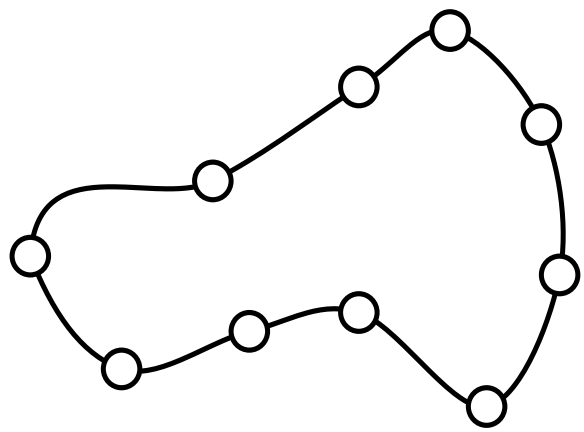


Hand-out #2A: Vertex coloring of graphs

How many colors are needed for a vertex coloring of each of the following graphs? Do you see any patterns in your solutions?



✓ **A "wheel" is a graph that consists of a cycle and one extra vertex (called the "hub" of the wheel) that is adjacent to all of the vertices in the cycle. A wheel with n vertices in the cycle is denoted W_n.**



Note that the wheel W_n has $n+1$ vertices. What is the chromatic number of a wheel?

$\chi(W_3) = 4$

$\chi(W_4) = 3$

$\chi(W_5) = 4$

$\chi(W_6) = 3$

$\chi(W_7) = 4$

An "odd wheel" has chromatic number 4.

How do you show that the chromatic number of a graph is 3 (or 4)?

1. Find a coloring with 3 (or 4) colors.

2. Show that the graph cannot be colored with fewer colors.

How do you show that a graph <u>cannot</u> be colored with 2 colors?

Find a portion of the graph that requires 3 colors – for example, an odd cycle.

How do you show that a graph <u>cannot</u> be colored with 3 colors?

Find a portion of the graph that requires 4 colors – for example, K_4 or an odd wheel.

Hand-out #3: Scheduling Class Projects

Eight students in your class are working on six projects, but they are having a hard time arranging for times to meet.

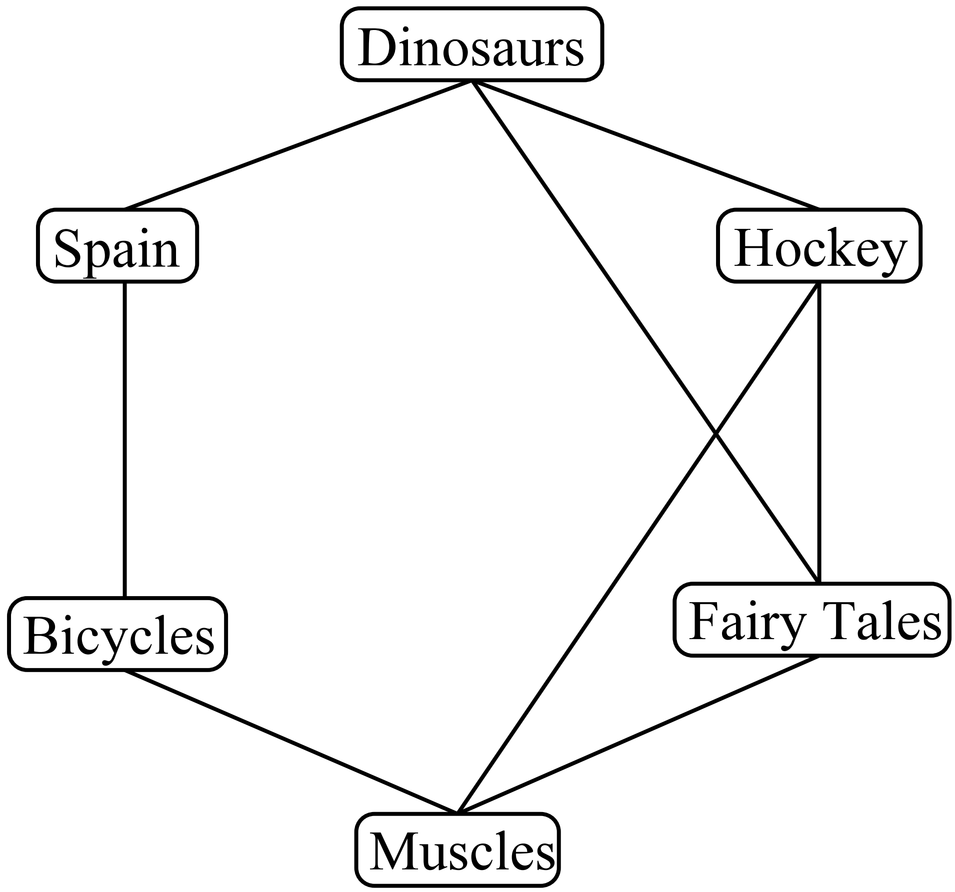
Each meeting will need a full 45-minute class period during seventh period on Monday, Tuesday, Wednesday, Thursday, or Friday, but you want to use as few class periods as necessary.

Can you help them out?

Projects	Students	Day assigned
Dinosaurs	Sarita, Barbara, Ravi	
Spain	Sarita, Roberto	
Bicycles	Roberto, Maimuna	
Muscles	Maimuna, Boris, Christie	
Fairy tales	Barbara, Boris, Jason	
Hockey	Ravi, Christie, Jason	

Comment: This problem is similar to the map-coloring problem — except that here you have to assign days to projects instead of colors to states.

Conflicts between class projects



Coloring Maps—Scheduling Class Projects

- ✓ **In a map coloring situation, two countries are "in conflict" if they share a border; we resolve the conflict by assigning conflicting countries different colors.**

- ✓ **In the project-scheduling situation, two groups are "in conflict" if they share a member; we resolve the conflict by assigning conflicting groups different meeting times.**

Vertex colorings of graphs can be used to solve a variety of problems which involve “conflict”.

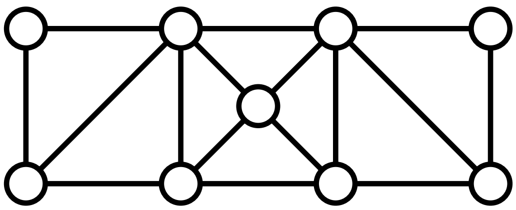
- ✓ **In a map coloring problem, two countries are “in conflict” if they share a border; we resolve the conflict by assigning conflicting countries different colors.**

- ✓ **In the project-scheduling problem, two projects are “in conflict” if they share a member; we resolve the conflict by assigning conflicting groups different meeting times.**

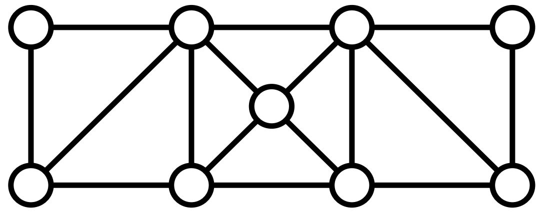
- ✓ **In the zoo problem, two animals are “in conflict” if one poses a danger to the other; we resolve the conflict by assigning conflicting animals different habitats.**

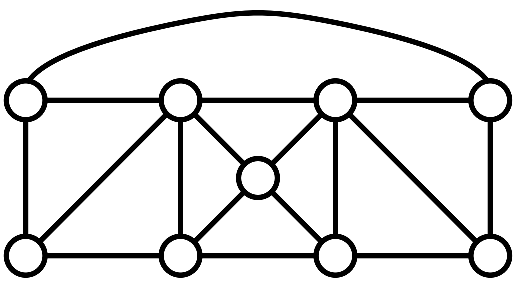
Hand-out #4: Vertex coloring of graphs

How many colors are needed for a vertex coloring of each of the following three graphs? (Two copies of each are provided.)

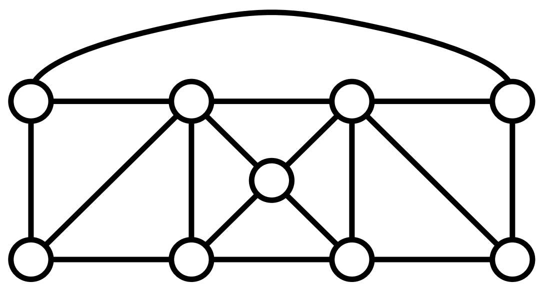


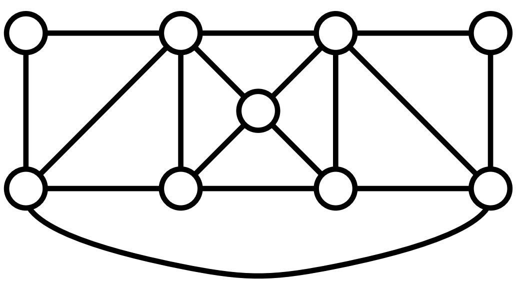
A



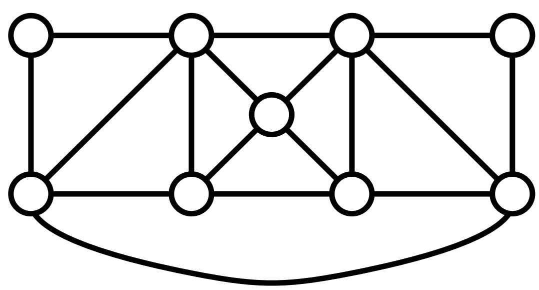


B





C



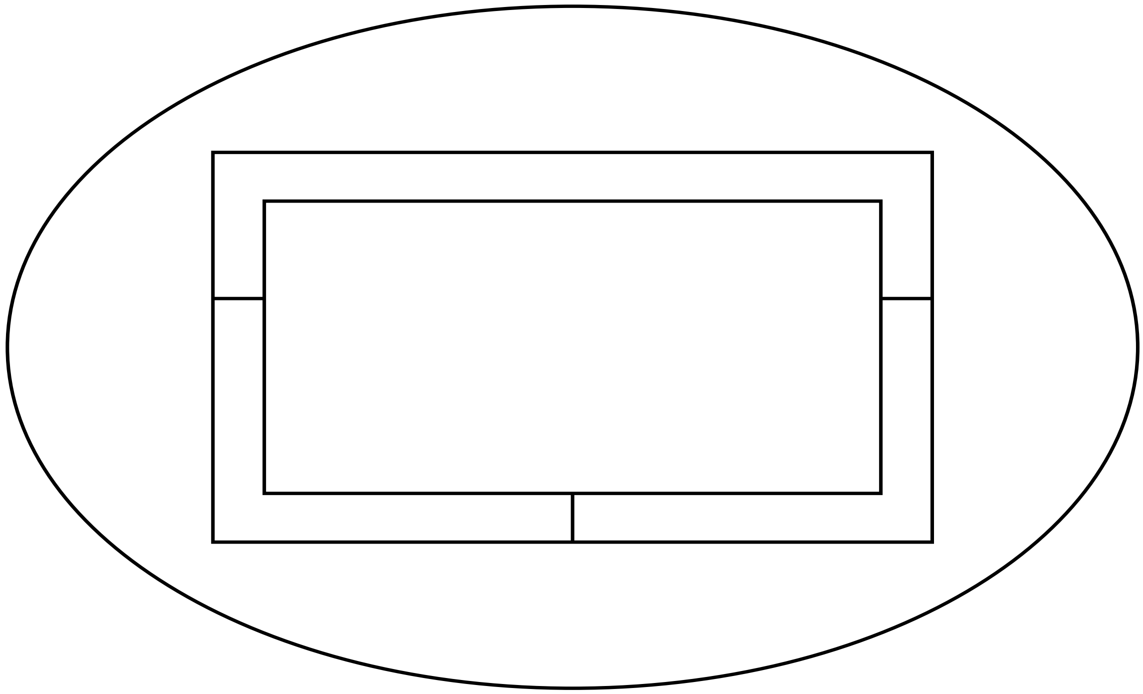
Question:

How can you tell if a graph can be colored with three colors or whether four colors are necessary?

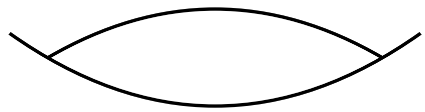
Answer:

There is no efficient way to tell in general whether or not a graph can be colored with three colors!

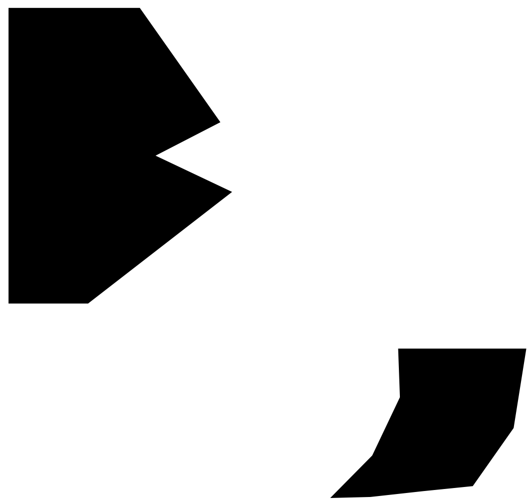
That means that every graph coloring problem is different; the methods that worked on the previous graphs you've examined may not work on this one.



Five regions on a map. Each region shares a border with each other region, except for the innermost and outermost.

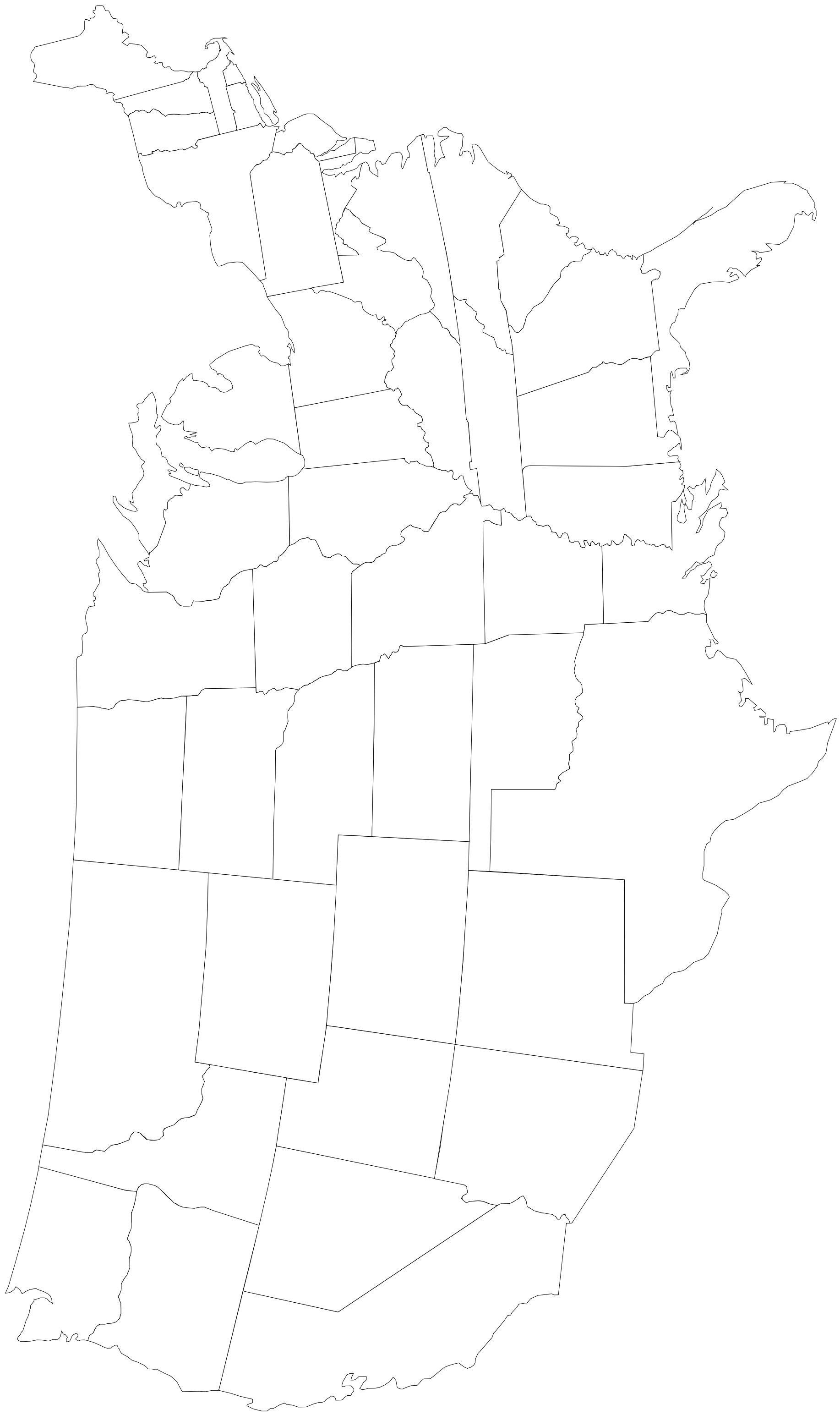


However, on a torus (donut) we *can* have the inside and outside regions also share a border.



In fact, we can also add a sixth region, sharing a border with each of the other five!

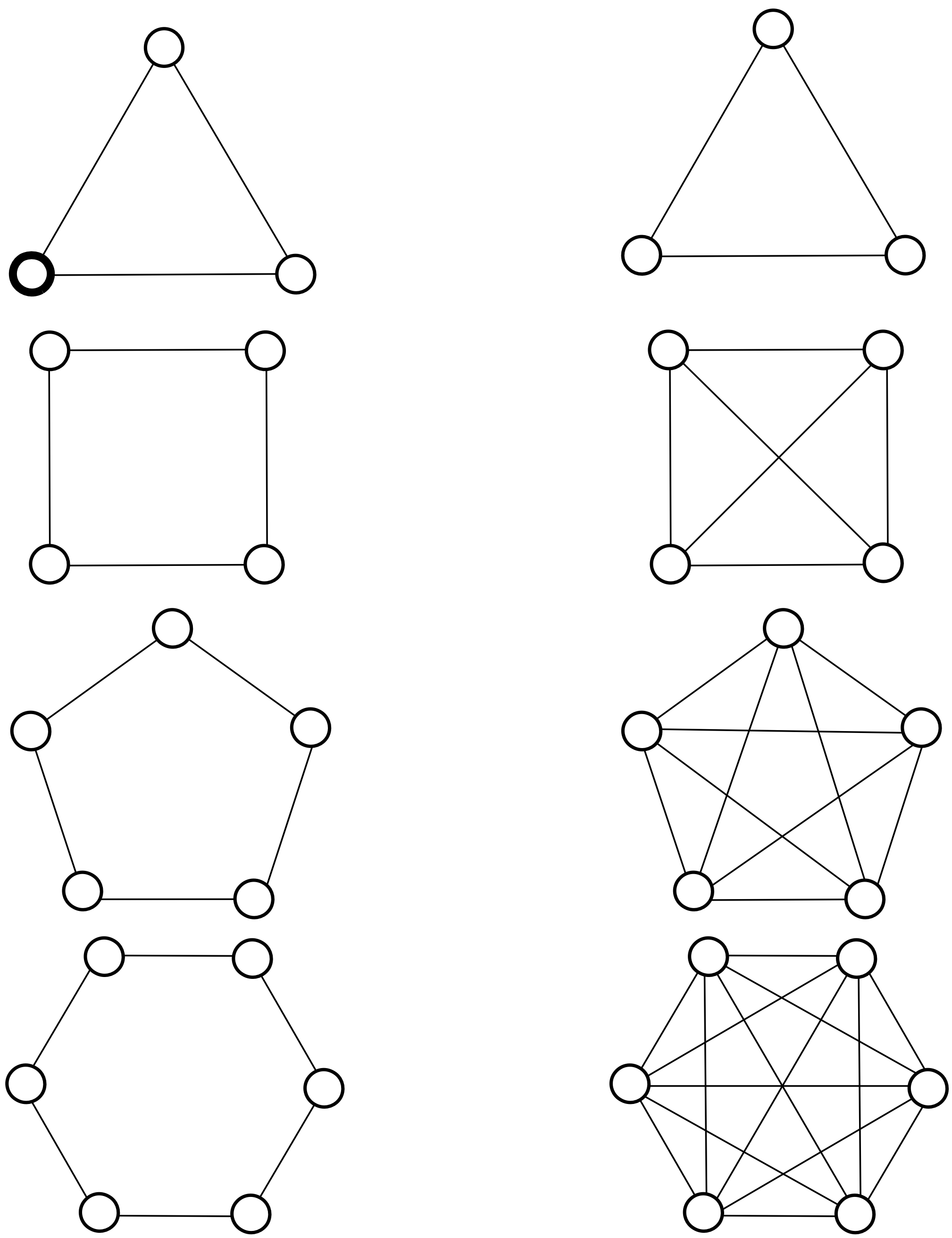
Hand-out #1:   US Map



Hand-out #2: Vertex coloring of graphs

How many colors are needed for a vertex coloring of each of the following graphs?

Do you see any patterns in your solutions?



Hand-out #3: Scheduling Class Projects

Eight students in your class are working on six projects, but they are having a hard time arranging for times to meet.

Each meeting will need a full 45-minute class period during seventh period on Monday, Tuesday, Wednesday, Thursday, or Friday, but you want to use as few class periods as necessary.

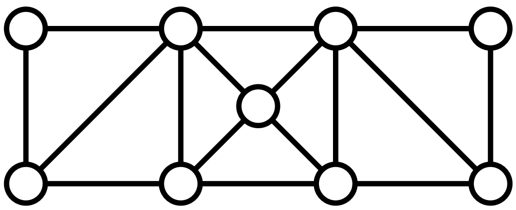
Can you help them out?

Projects	Students	Day assigned
Dinosaurs	Sarita, Barbara, Ravi	
Spain	Sarita, Roberto	
Bicycles	Roberto, Maimuna	
Muscles	Maimuna, Boris, Christie	
Fairy tales	Barbara, Boris, Jason	
Hockey	Ravi, Christie, Jason	

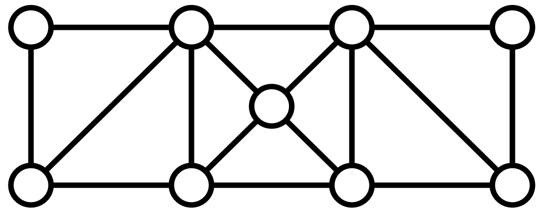
Comment: This problem is similar to the map-coloring problem — except that here you have to assign days to projects instead of colors to states.

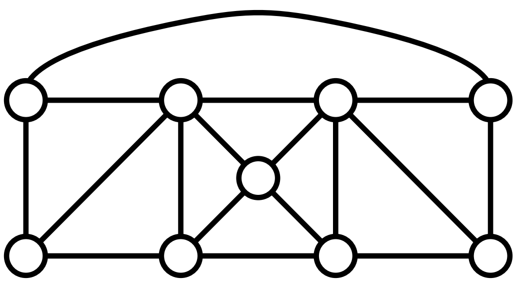
Hand-out #4: Vertex coloring of graphs

How many colors are needed for a vertex coloring of each of the following three graphs? (Two copies of each are provided.)

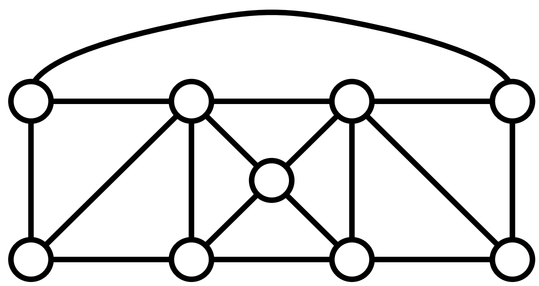


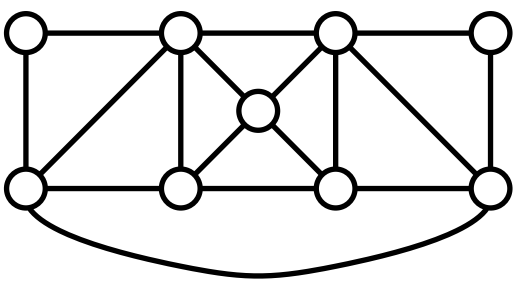
A



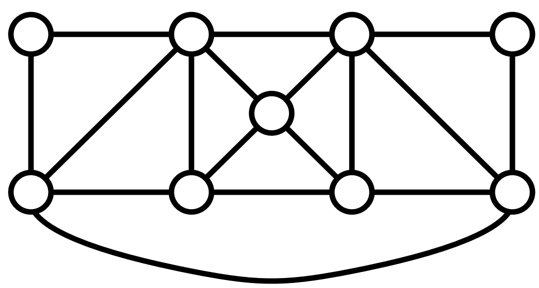


B





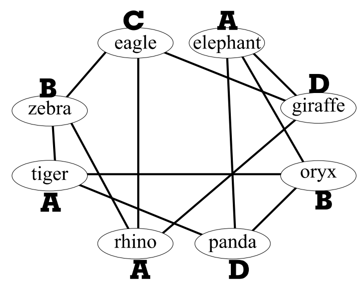
C

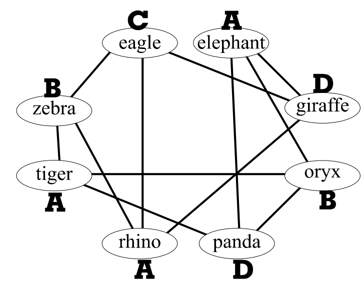


Workshop 1 — Coloring Maps and Resolving Conflicts — Exercises

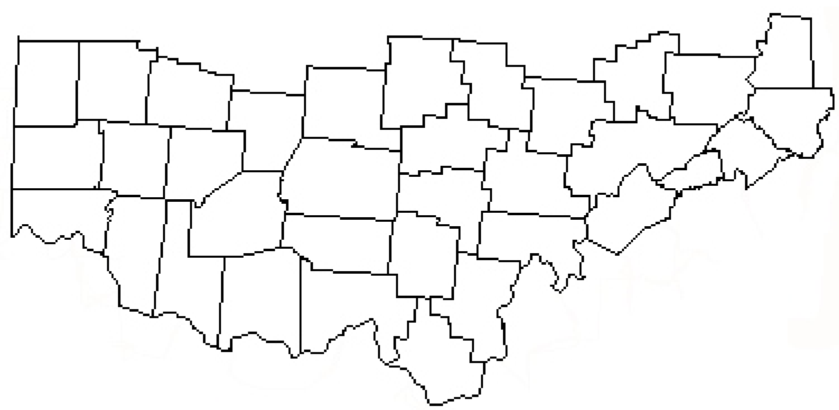
Practice problems:

1. **In the zoo problem, the initial solution (at left below) involved four habitats, labeled A,B,C,D. Can you find a solution which requires only three habitats?**

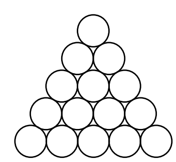




2. **Color the portion of the Ohio Counties map below using at most four colors. Can you color this map with three colors? Why or why not?**



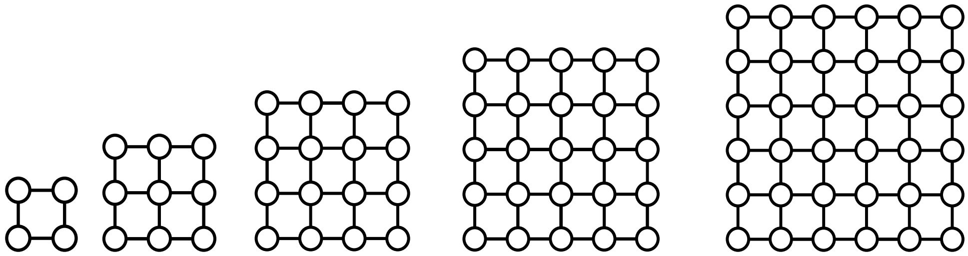
3. **How many colors do you need to color the 15 billiard balls so that touching balls have different colors?**



4. **B.B. (for "biology buff") joins your class of eight students (see below) and signs up for the dinosaur and muscle projects. Find a new schedule for the class projects.**

Projects	**Students**	**Day Assigned**
Dinosaurs:	Sarita, Barbara, Ravi	
Spain:	Sarita, Roberto	
Bicycles:	Roberto, Maimuna	
Muscles:	Maimuna, Boris, Christie	
Fairy Tales:	Barbara, Boris, Jason	
Hockey:	Ravi, Christie, Jason	

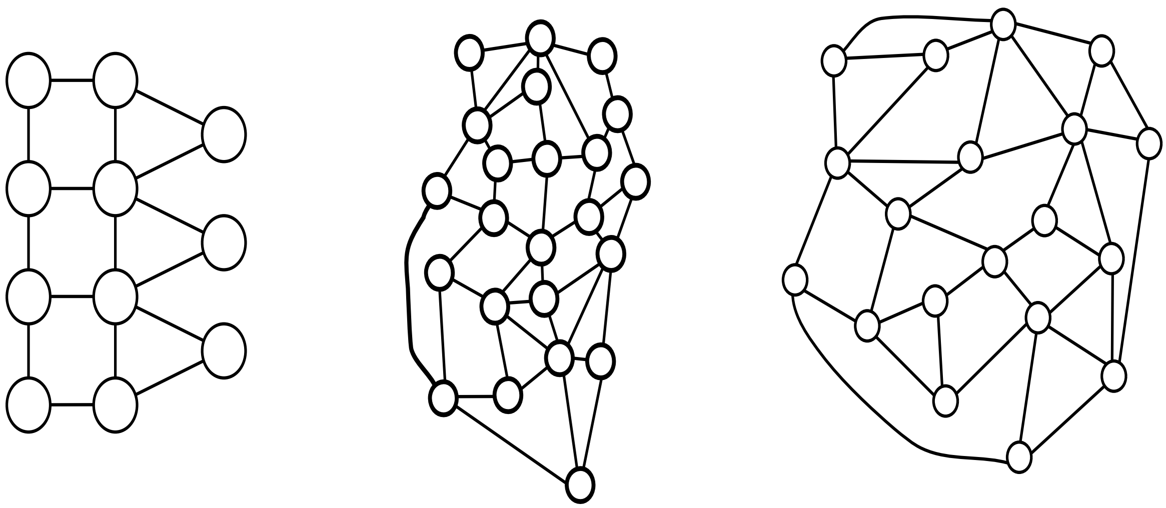
5. How many colors do you need for each of the five grids? Justify your answers. Do you see a pattern? Can you describe it?



Study group problems:

6. How many colors do you need to color each of the graphs below? How can you be sure that it can't be done with fewer colors? (Extra graphs to practice on can be found on page 6.)

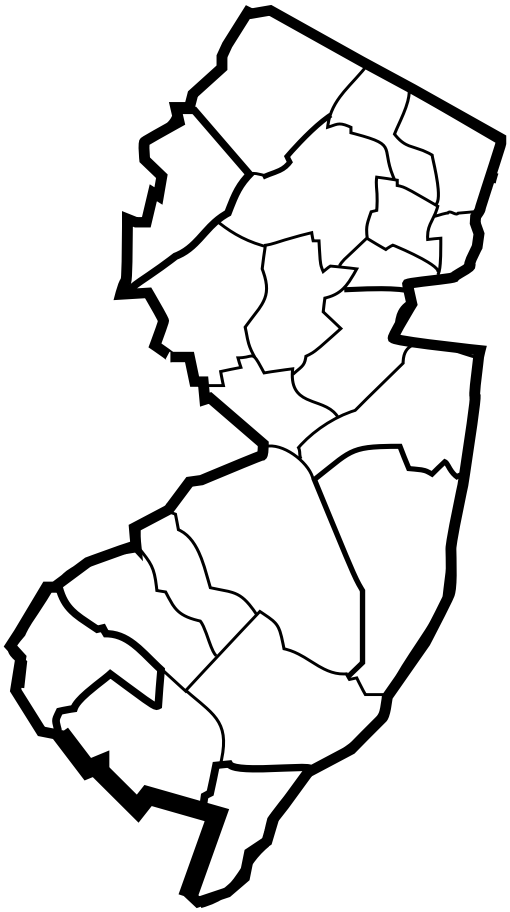
(A) (B) (C)

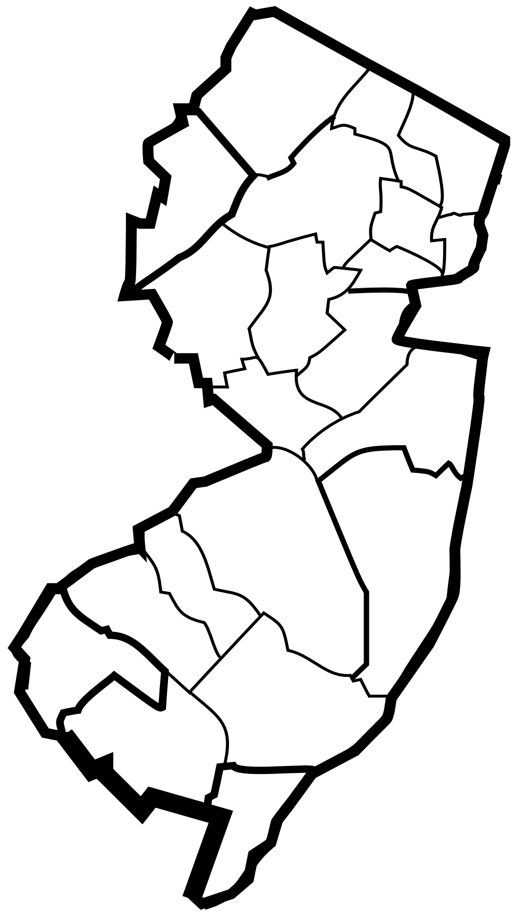


7. How many colors do you need to color each of the graphs below? How can you be sure that it can't be done with fewer colors? (Extra graphs to practice on can be found on page 6.)

(A) (B) (C)

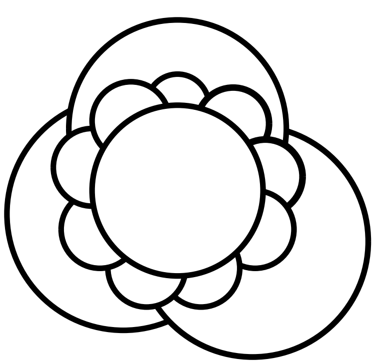
8. Color New Jersey's county map below using at most four colors. Can you color this map with three colors? Why or why not? (The Resource Book contains county maps for a few other states; you can get your state's county map from http://www.lib.utexas.edu/Libs/PCL/Map_collection/united_states.html)

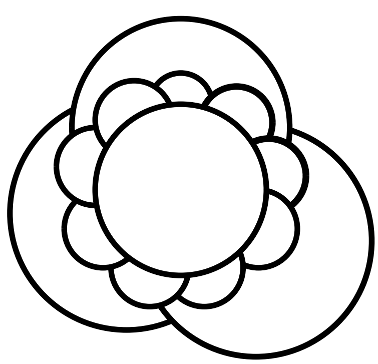


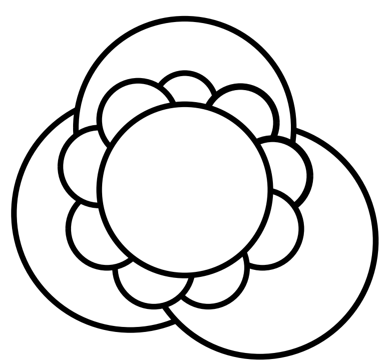


9. Do the problem labeled "Storing Chemicals" on page 8.

10. Can you color the map below with four colors? Three copies are provided.





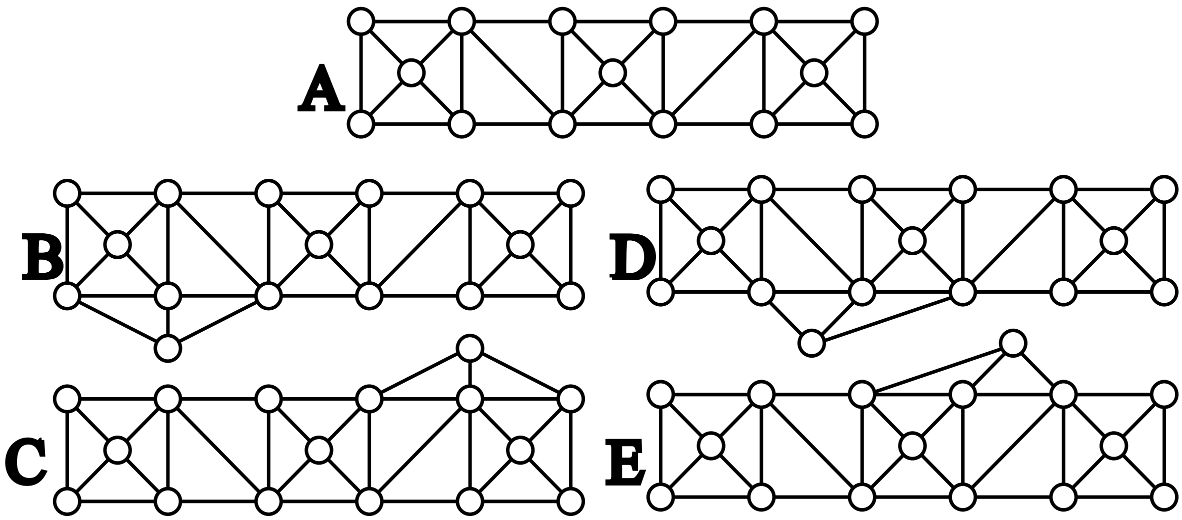


11. a. How many states can you find on the US map (see page 7) which are the hubs of wheels with five or seven spokes (as in problem 5.a. above); each of these provide a reason why four colors are <u>necessary</u> to color the US map. (What states are the hubs of wheels with an even number of spokes?)

b. In view of your answer to part a, if you wanted to color the states red, white, and blue, what would you guess is the smallest number of states that would have to be colored black (where contiguous states have different colors)? Now color the 48 states, trying to keep the number colored black to a minimum.

.

12. How many colors are required in a coloring of each of these graphs? How can your answer for A help you obtain your answers for the others?



13. Mr. Kling is inviting seven members of his (very) extended family to his wedding. Unfortunately, bitter rivalries through the year prevent him from seating certain family members at the same table. For example, Alan and David will not talk to either Barbara or Cathy, and Barbara and Cathy refuse to talk to each other. David and Goldie refuse to be seen with Esther or Frank, and Esther and Frank refuse to break bread with one another. Horrible arguments erupt when Alan talks with Goldie. What is the least number of tables that Mr. Kling needs for his 7 argumentative guests?

14. This problem concerns the map of 9 southeastern states shown to the right.
a) Draw the graph corresponding to this map.
b) What is the least number of colors required for a coloring of this graph?
c) Suppose that it costs $1 for each state that is colored red, $2 for each state colored blue, and $3 for each state colored yellow. How much does it cost to color the graph?
d) If you were allowed to use a fourth color, green, and if it cost $4 for each state which was colored green, might it have been possible to find a cheaper coloring than that of part c)?

Extension Problems:

15. The following routes of garbage trucks in New York city are being considered.
    - ✓ Route 1 picks up garbage at the Empire State Building (ESB), then Madison Square Garden (MSG), and then Pier 42 on the Hudson River.
    - ✓ Route 2 visits Greenwich Village, Pier 42, ESB, and the Metropolitan Opera House.
    - ✓ Route 3 visits Shea Stadium, the Bronx Zoo, and the Brooklyn Botanical Garden.
    - ✓ Route 4 goes to the Statue of Liberty and Pier 42.
    - ✓ Route 5 to the Statue of Liberty, the New York Stock Exchange (NYSE), and the ESB.
    - ✓ Route 6 to Shea Stadium, Yankee Stadium, and the Bronx Zoo.
    - ✓ Route 7 to the NYSE, Columbia University, and the Bronx Zoo.

Assume that no site may be visited more than once on any given day. In this problem we will try to determine the least number of days needed so that all 7 of the routes may be completed.

a) There seem to be two choices for what the vertices of our graph should represent: routes or sites. For each case, decide what the edges (conflicts) would be. Which seems to be a better model?

b) Generate a graph corresponding to the better model in part a).

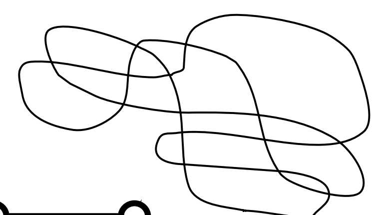
c) How many colors are needed in a proper coloring of the graph in part b)?

d) What is the least number of days required to complete all 7 routes?

16. Play a few games of Sprouts with a partner (see the Resource Book) where you start with two vertices. Which player do you think has an advantage in this game, if either?

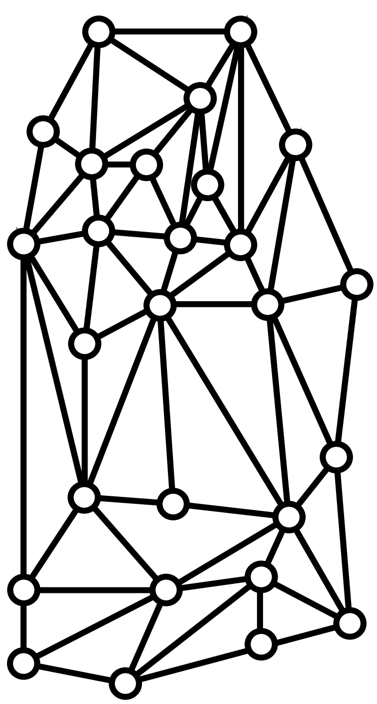
17. Make up one or more examples of conflict problems that would interest your students and that you could solve using graph coloring. (You might want first to look at some of the other problems in the Resource Book.)

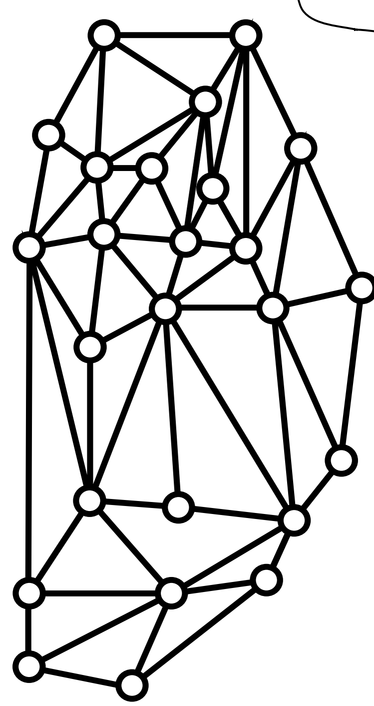
18. The diagram at the right is just a doodle created by drawing a single line which curved around and kept crossing itself. How many colors are needed to color the map that resulted. Try drawing similar doodles and figure out how many colors you need for your maps. Can you explain your conclusion?

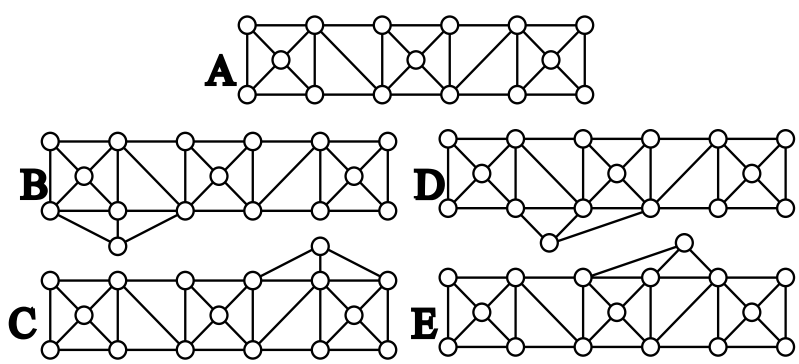


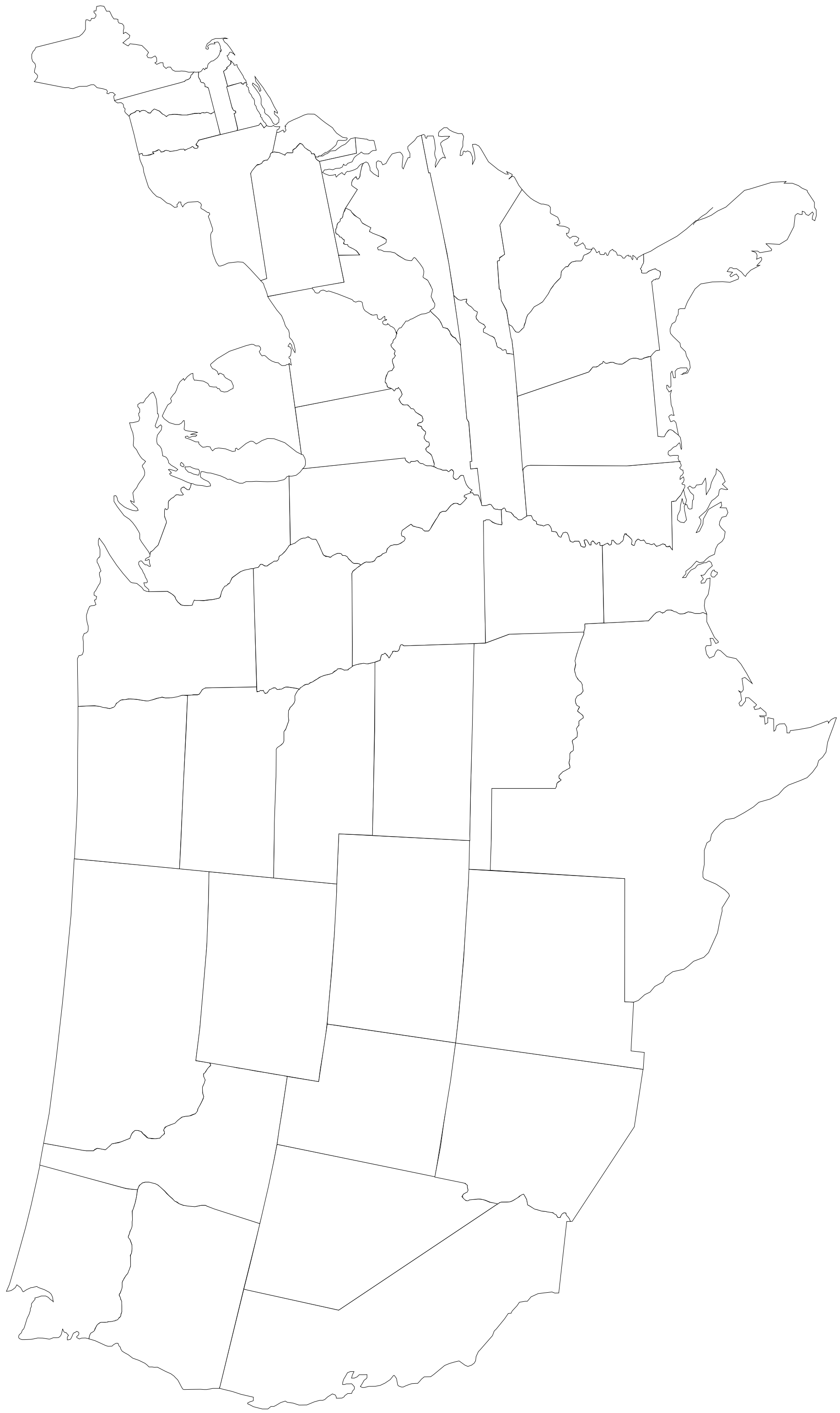
19. Can you draw a map with 6 countries on a donut so that each borders the others? with 7 countries?

20. How many colors do you need to color each of the graphs at the right? How can you be sure that it can't be done with fewer colors?






A
B
D
C
E



Storing Chemicals

A chemical manufacturer wishes to store some chemicals in a warehouse. Some chemicals react violently when in contact with each other, and the manufacturer decides to divide the warehouse into a number of rooms so as to separate dangerous pairs of chemicals. In the following table, an asterisk (*) indicates those pairs of chemicals which must be kept separate.

Draw the appropriate graph, and determine the smallest number of rooms needed to store these chemicals safely.

	A	B	C	D	E	F	G
A	-	*	*	*			*
B	*	-	*	*	*		*
C	*	*	-	*		*	
D	*	*	*	-		*	
E		*			-		
F			*	*		-	*
G	*	*				*	-

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

Table of Contents

Resources: Additional examples of maps can be found in *A Mathematician's Coloring Book* by R. Francis — COMAP Module #13. County maps from other states can be obtained through the Internet. The large (4' x 6') US Maps used in the program can be obtained (item #87105) at $12.49 plus shipping (in 1998) from J.L. Hammett, Company, PO Box 3106, 685 Liberty Avenue, Union, New Jersey 07083; the phone number is 1-800-333-4600.

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

Mathematical Background

- ✔ **A graph or network is a collection of points (or circles) some of which are joined by lines or curves. These are called "vertices" and "edges". (The singular of "vertices is "vertex", not "vertice".)**

- ✔ **The word "node" is sometimes used instead of "vertex"; the word "arc" is sometimes used in place of "edge".**

- ✔ **Each edge joins two different vertices.**

- ✔ **A given pair of vertices may or may not be joined by an edge.**

- ✔ **Vertices joined by an edge are called "adjacent" or "neighbors".**

- ✔ **A "coloring of a map" involves assigning colors to the countries of a map so that countries with a common border are assigned different colors.**

- ✔ **The "Four Color Conjecture" was the statement that "any map can be colored with four colors". This simple statement was thought to be true for over a hundred years before it was finally proved in 1976 by Kenneth Appel and Wolfgang Haken; then it became known as the "Four Color Theorem".**

- ✔ **A "coloring of a graph" involves assigning colors to the vertices of a graph so that adjacent vertices are assigned different colors.**

- ✔ **The "chromatic number of a graph"is the smallest number of colors that can be used for a coloring of the graph If the graph is called G, then the chromatic number of G is often written as $\chi(G)$, where χ is the Greek letter "chi" (read "kye").**

- ✔ **A "cycle" is a graph where the vertices can be arranged in a circular fashion so that each vertex is adjacent to the two vertices which come before and after it in the circle.**

- ✔ **A "complete graph" is a graph in which every vertex is adjacent to every other vertex.**

- ✔ **Vertex colorings of graphs can be used to solve a variety of problems which involve "conflict". In a situation involving maps, two countries are "in conflict" if they share a border; we resolve the conflict by assigning conflicting countries different colors. In a situation involving class projects, two project are "in conflict" if they share a member; we resolve the conflict by assigning conflicting projects different meeting times.**

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

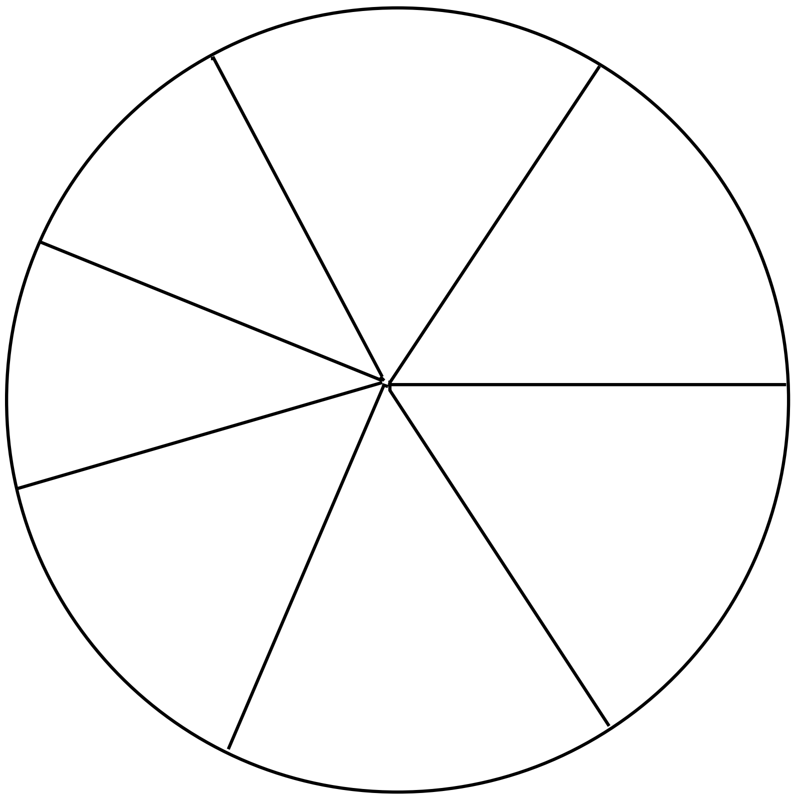
Workshop Outline

1. **US Map activity**
   a. **"Color" US maps by placing a colored circle on each state, so that states with a common border have different colors; groups discuss the meaning of "common border".**
   b. **Do the same using as few colors as possible.**
   c. **Explain why it can't be done with fewer colors, and identify "problem areas" in US map.**
2. **Map coloring challenge activity**
   a. **Draw a map which requires four or more colors.**
   b. **Try to color your neighbor's map with three colors.**
   c. **What kinds of maps require four colors? Major examples: four countries each of which border all of the others, or three or a larger odd number of countries encircling a central country. Can such maps be found in the US map?**
   d. **Summary of map coloring — every map on plane (or on sphere) can be colored with four (or fewer) colors; for a particular map, it may be hard to tell whether it can be colored with three colors.**
3. **Graph coloring activities**
   a. **Fifteen people stand on diagram on floor of a graph with 15 vertices. (A copy of the graph used appears on page 20.) They "color" the vertices "red", "yellow", or "green" by doing "hands-up", "stand normal", or "crouch down" (as with the colors on a traffic light). Adjacent vertices should be colored with different colors, so that if two adjacent vertices are colored with the same color, one of the two people must change color. Continue until a proper coloring has been obtained. The second group of volunteers should be able to complete the task more quickly; the third group should be able to complete it without any verbal communication.**
   b. **Introduce the notions of graphs, vertices (singular is vertex), and edges and provide a number of examples. Introduce the notion of vertex coloring of a graph.**
   c. **Find graph colorings of various graphs (and patterns of graphs) which use a minimal number of colors.**
   d. **Introduce the notions of cycles, complete graphs, and chromatic number, and find the chromatic number of cycles and complete graphs, and other graphs.**
   e. **Associate a graph with each map (one vertex for each state, an edge joins vertices associated with states that share a common border) so that any coloring of the map corresponds to a vertex coloring of the graph.**
4. **Using vertex coloring to resolve conflicts**
   a. **Work on handout involving scheduling student projects; create a graph which represents the situation and color the graph to find an appropriate schedule.**
   b. **Show first part of the "Zoo Problem" from COMAP's video "Geometry: New Tools for New Technologies", where graph coloring is used to assign habitats to the zoo animals.**
   c. **Parallel between different problems — in map coloring, two countries are "in conflict" if they share a border; in project scheduling, two class projects are "in conflict" if they share a member; in the zoo problem, two animals are "in conflict" if one poses a danger to the other. Graph coloring makes it possible to assign conflicting countries different colors, conflicting groups different meeting times, and conflicting animals different habitats.**
5. **Coloring graphs with three colors**
   a. **When are three colors sufficient? Similar graphs may require different number of colors.**

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

A Colorful Pie



Colo r the pie above so that no two pieces of the same color touch along a border.

1. **Did you need as many as seven different colors? Note that pieces of the same color may touch at a single point.**

2. **Could the coloring have been done with only three different colors?**

3. **Find a formula for coloring a pie with n pieces (above n =7). Can one formula work for n both even and odd?**

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

Kindergraph Theory

by Valerie A. DeBellis

For many K-2 classrooms, curricular topics in geometry include notions such as region, closed, open, inside, outside, next to, on top of, and under. The following kindergarten graph theory activity can be used to help reinforce such concepts.

Coloring books can be a good place to introduce young children to graph theory topics provided the pictures you use are *simple* enough. In this case, a simple coloring book picture refers to one with a few number of regions. For example, the teddy bear picture below contains fifteen regions. I have seen kindergarten children successfully complete this activity using coloring book pictures containing anywhere from eight to twenty regions.

insert Teddy Bear Figures

1. **Give each student a copy of the teddy bear pictured on the far left. It should be reproduced and enlarged so that one teddy bear fits on an 8 ½" x 11" page.**

2. **Ask your students "to place one dot inside each region" and show them an example of how to do this. See Figure B for completed version of this step. Of course, I am assuming that you have already introduced and reviewed the word "region" with your students. Most kindergarten children, even slower students, can successfully accomplish this step.**

3. **Then ask them to "connect the regions that touch each other by drawing a line from dot to dot." What you are asking your students to do is construct a conflict graph where two regions are "in conflict" if they share a border. This activity will allow you to visually assess the concept of "next to" by viewing what vertices each child has connected by an edge. Careful, you may be quite surprised at the work you'll receive!**

4. **Finally, you may want to use this conflict graph (see Figure D) to introduce the idea of a graph used in discrete mathematics. You could introduce vertex and edge terminology; ask the children to count the number of vertices and edges on a variety of graphs; ask them to describe what parts of the graph look the same as other parts of the graph or what parts of the graph look different; ask them to determine whether the graph is connected (together) or disconnected (in parts); or ask them to draw a picture and construct their own conflict graph. In addition, these conflict graphs can also be used to introduce vertex coloring, provided you use a circle to represent each vertex instead of a dot so that children can easily color inside them. After your students are able to construct conflict graphs, you may want to staple a piece of tracing paper on top of a simple coloring book picture and create the graphs on tracing paper; separate the pages and you're ready for a vertex coloring lesson!**

Hop into Map Coloring

not sure want to use this

Perrrrrfectly Colorful

not sure want to use this

RESOURCE BOOK

Color My World

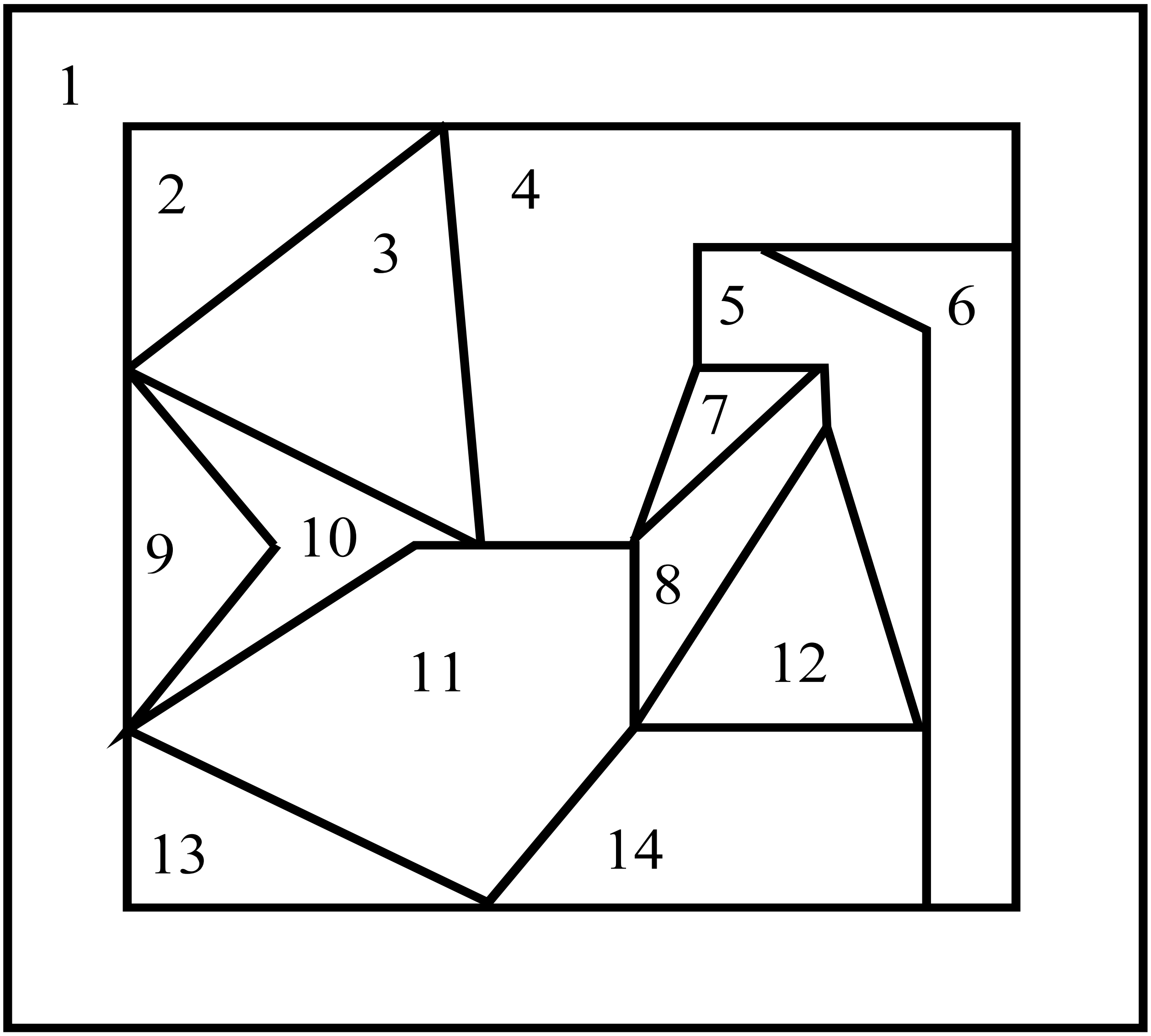
insert Judy Brown's Butterfly

RESOURCE BOOK

From Butterflies to Graphs

insert Judy Brown's page

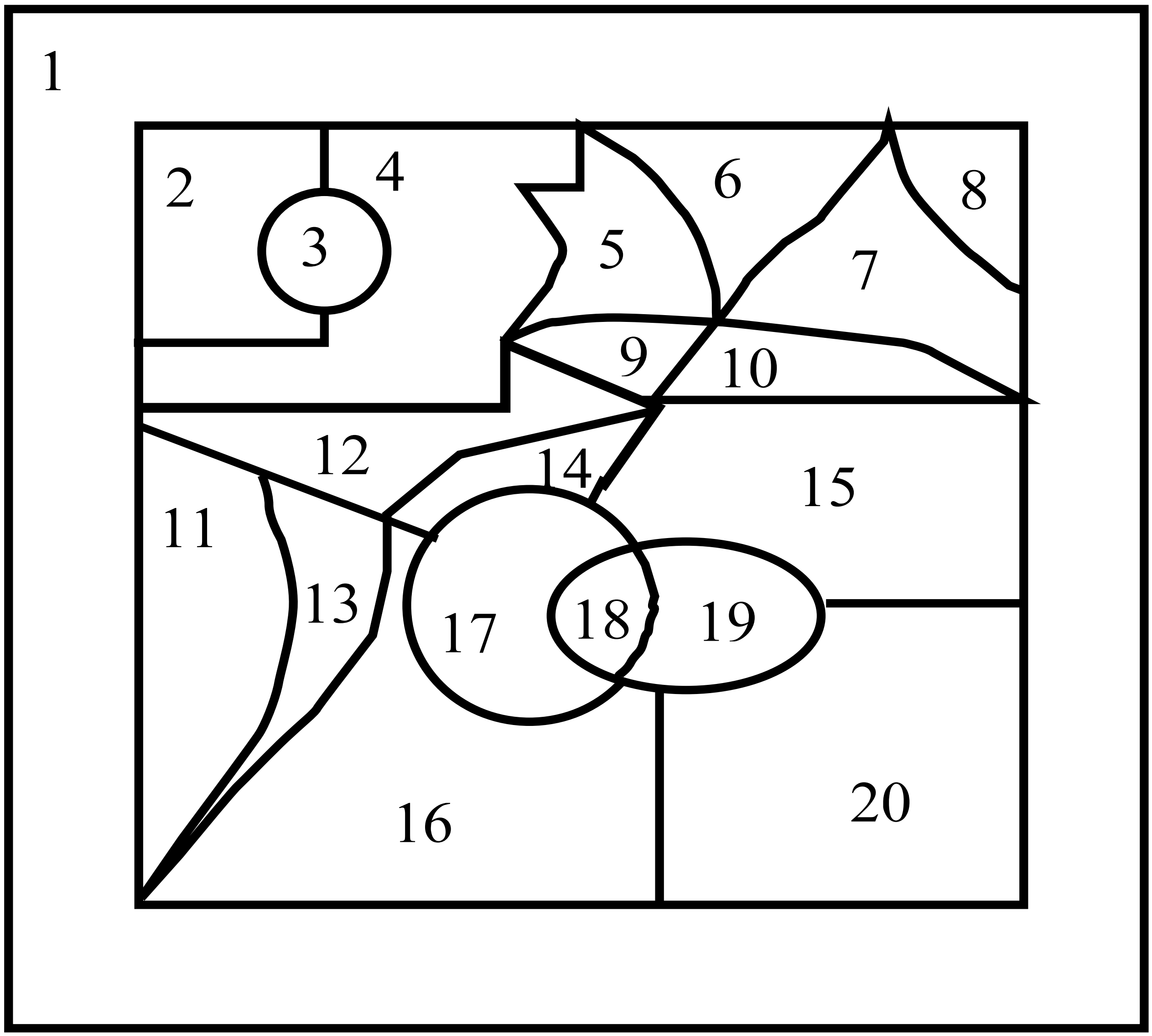
Workshop 1: Coloring Maps and Resolving Conflicts



From *Puzzles & Graphs* by John N. Fujii, copyright 1966 National Council of Teachers of Mathematics

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts



From *Puzzles & Graphs* by John N. Fujii, copyright 1966 National Council of Teachers of Mathematics

Lewis Carroll's Map Coloring Game

The Problem: A favorite puzzle of Lewis Carroll, the author of <u>Alice in Wonderland</u> and <u>Through the Looking Glass</u>, involved coloring a map with the fewest colors possible. It is a game for two players.

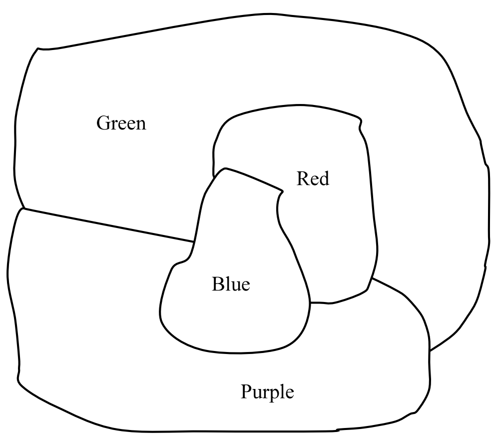
Suggested Materials: A blank sheet of paper for each student, colored pencils or colored markers.

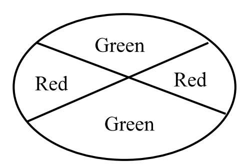
Prerequisite: None

Guided Exploration: The students need to be in pairs.
<u>Student A</u> is to draw a fictitious map, divided into "countries." Student B also draws a fictitious map. (I do not indicate the complexity of the drawings as I have discovered the more complex maps usually end up with the fewest number of colors needed to color it.)
<u>Student B</u> is to color the map drawn by student A using as few colors as possible. Two adjacent countries (touching along line) must have different colors. If the countries touch at a single point (like pieces of a pie), they may have the same color. At the same time, student A should be coloring the map student B drew.

<u>The Object</u> of the game is to force the opposing player to use as many colors as possible in coloring the map. The winner is the person using the fewest colors.

Examples:





Concluding the Exploration

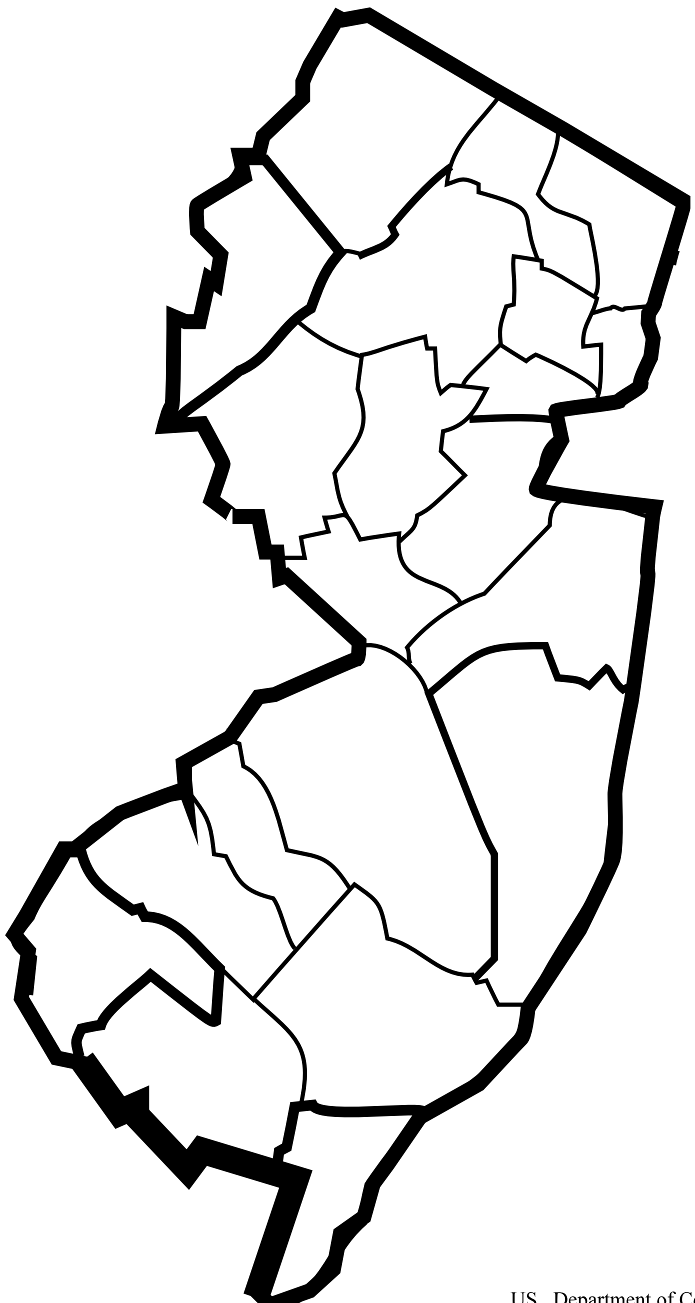
Have the students play the game two or three times. They do not see all the aspects of the game until they have played it once. Then share with the entire group the number of colors they actually needed. If they did it correctly, no one should use more than four colors. Do not yet tell them about the Four Color Map Theorem at this time, however.

Developed by Susan Simon, Leadership Program 1989

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

New Jersey County Map

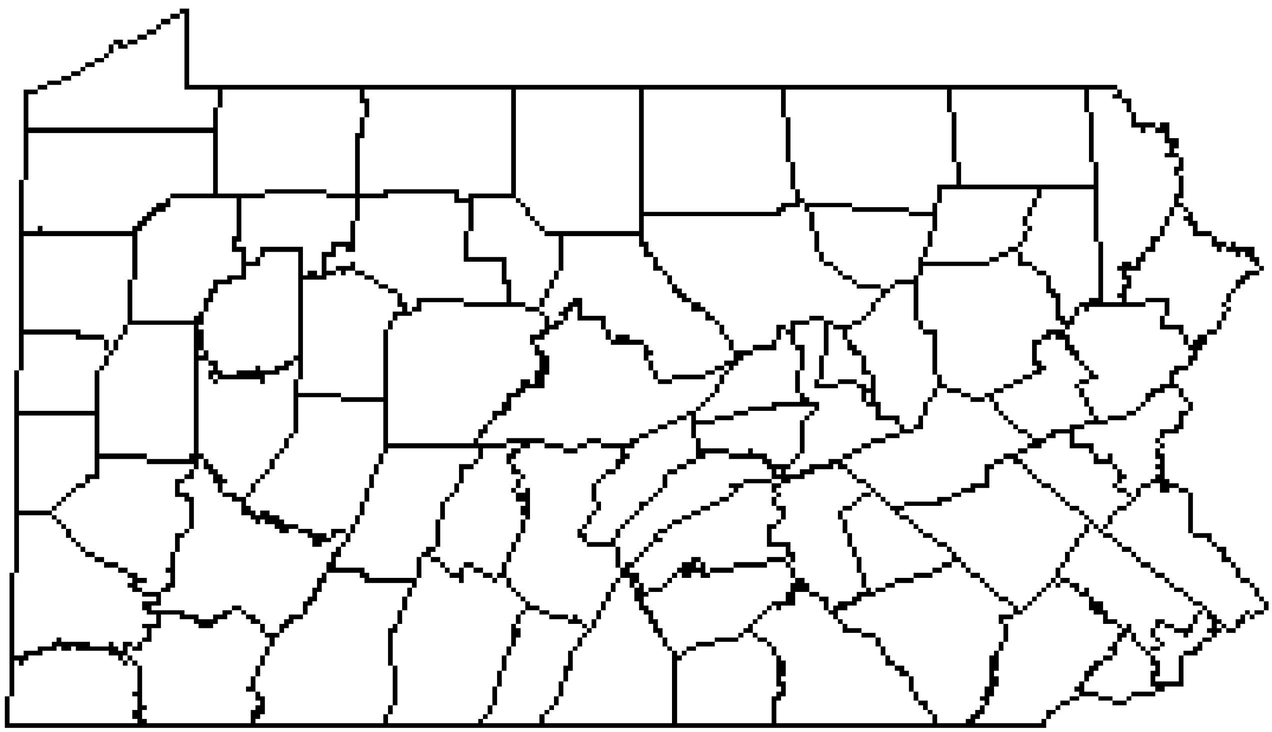


US. Department of Commerce Economics and
Statistics Administration Bureau of the Census

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

Pennsylvania Counties



This map (and those on the next four pages) were adapted from using the Internet site: http://tiger.census.gov

Click on either TMS Version 2.5 or TMS Version 1.3.1 (depending what your computer system will support) to begin your map creation.

US Department of Commerce Economics and Statistics Administration Bureau of the Census

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Workshop 1: Coloring Maps and Resolving Conflicts

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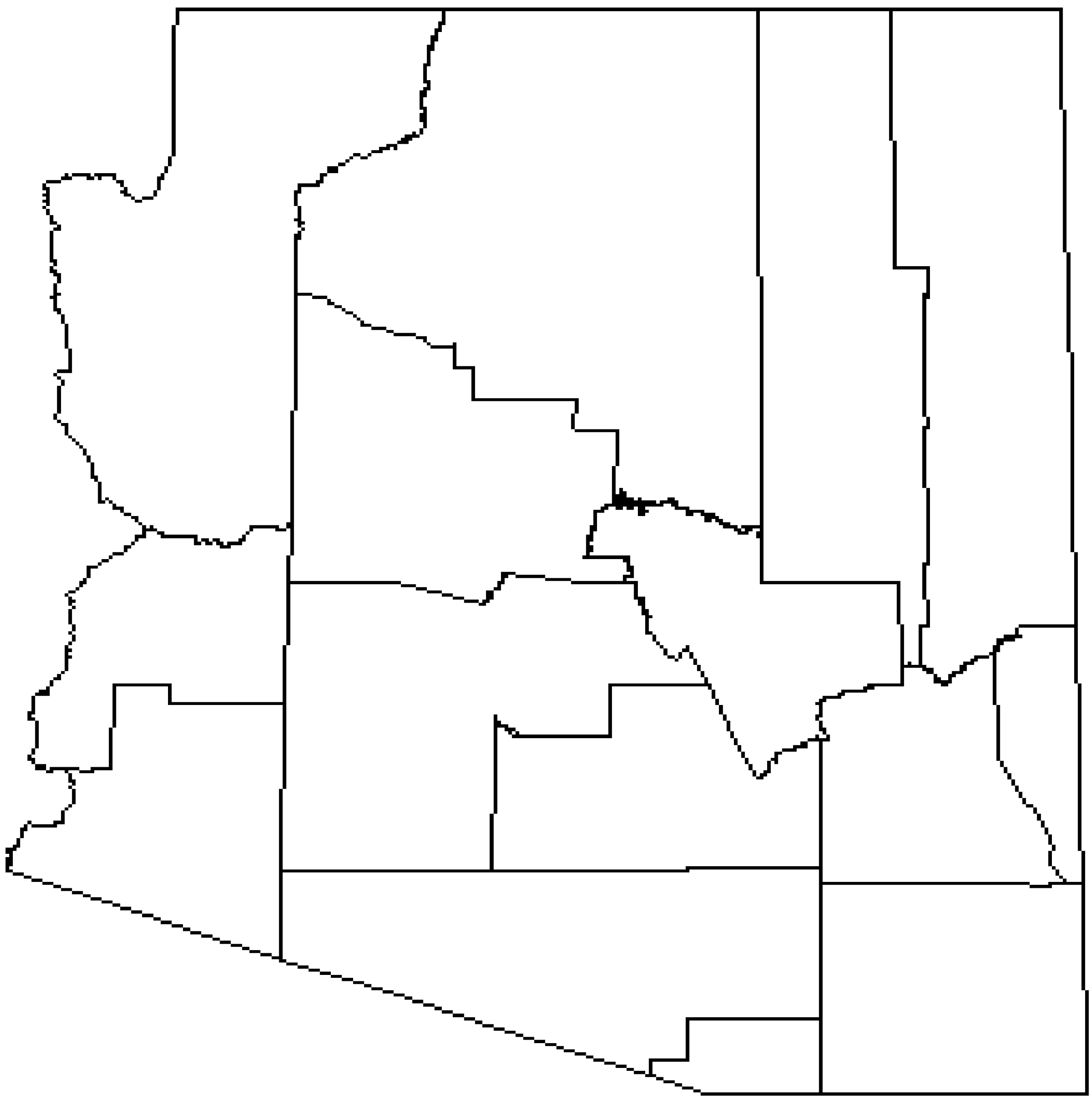
Workshop 1: Coloring Maps and Resolving Conflicts

Arizona County Map



Rhode Island, Massachusetts, and Connecticut County Maps

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

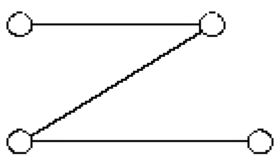
insert VA county map

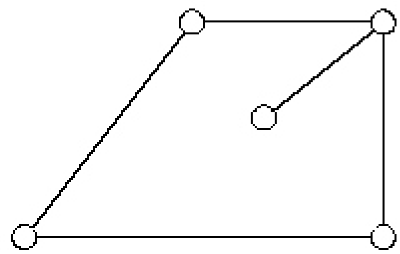
RESOURCE BOOK

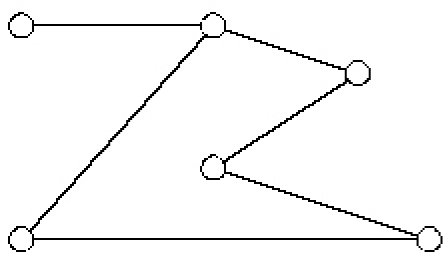
Workshop 1: Coloring Maps and Resolving Conflicts

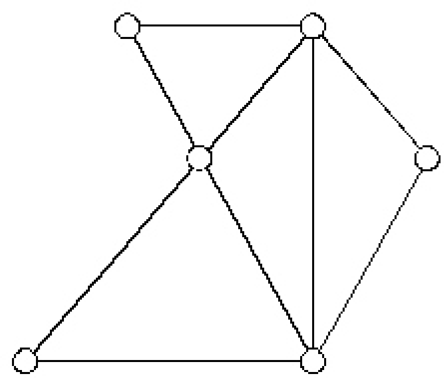
NAME: ______________________

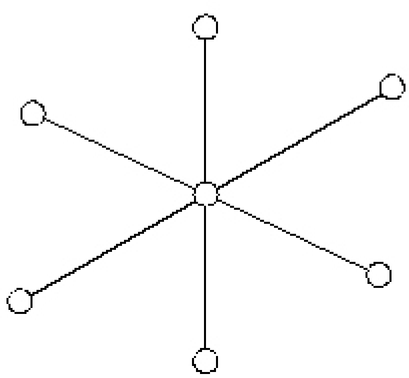
Color each graph so that no two adjacent vertices have the same color. The chromatic number of the graph, denoted by $\chi(G)$, is the smallest number of colors necessary to color the vertices of the graph.

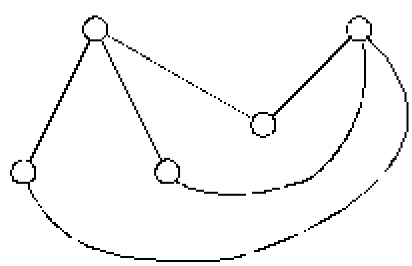


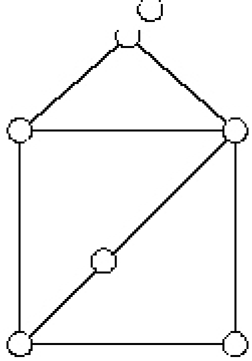


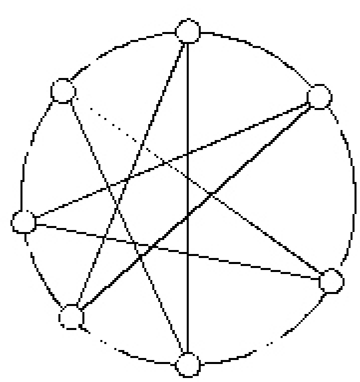






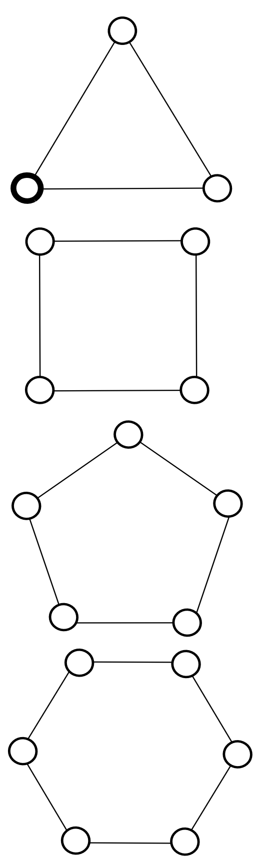


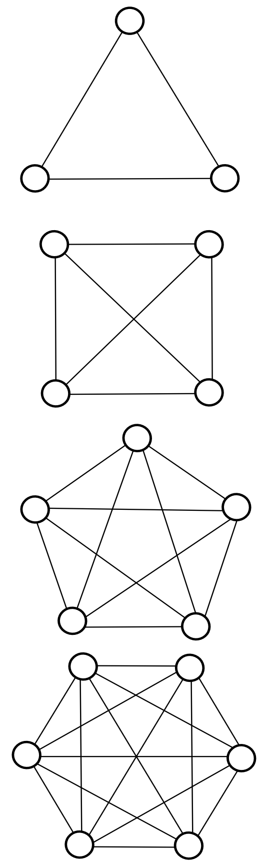




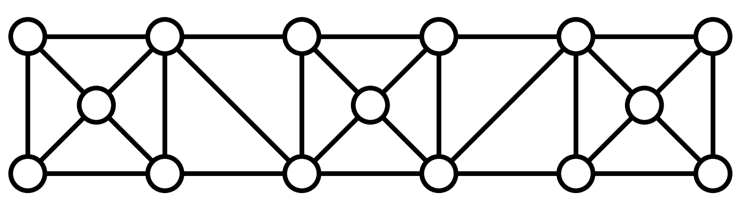
RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts





CYCLES **COMPLETE GRAPHS**



GRAPH USED FOR "Red, Green, Yellow" COLORING ACTIVITY

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

START HERE

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

RESOURCE BOOK

Coloring Applications

Vertex colorings of graphs can be used to solve a variety of problems which involve "conflict".

1. **In a map coloring problem, two countries are "in conflict" if they share a border; we reslove the conflict by assinging conflicting counrties different colors.**

2. **In the project-scheduling problem, two projects are "in conflict" if they share a member; we resolve the conflict by assigning conflicting groups different meeting times.**

3. **In the zoo problems, two animals are "in conflict" if one poses a danger to the other; we resolve the conflict by assigning conflicting animals different habitaits.**

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

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Workshop 1: Coloring Maps and Resolving Conflicts

THE ZOO PROBLEM

Taken from COMAP's - Geometry: New Tools for New Technologies

There are eight different animals in the zoo:

Amanda the Tiger
Candy the Oryx
Debbie the Zebra
George the Giraffe
Kanga the Elephant
Maribelle the Panda
Stefan the Rhino
Winifred the American Bald Eagle

One of the zoo planners thinks that he has found a way to place these 8 animals into 4 new habitats with no conflicts. With each new multi-species habitat costing as much as two million dollars, can the number of habitats be reduced?

The resident zoologist has provided the following list of the eight animals and their conflicts:

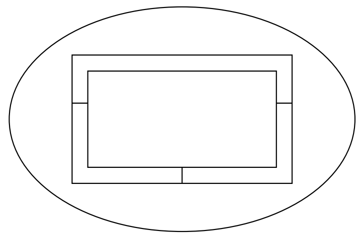
ANIMALS	CONFLICTS
Tiger	Oryx, Zebra, Panda
Oryx	Elephant, Panda, Tiger
Zebra	Tiger, Eagle, Rhino
Giraffe	Eagle, Rhino, Elephant
Elephant	Giraffe, Panda, Oryx
Panda	Tiger, Elephant, Oryx
Rhino	Giraffe, Zebra, Eagle
Eagle	Rhino, Giraffe, Zebra

RESOURCE BOOK

Workshop 1: Coloring Maps and Resolving Conflicts

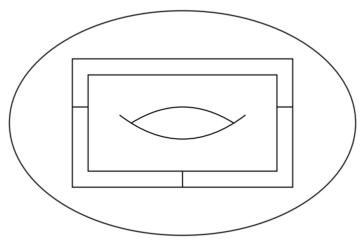
Each of the five regions in A shares a border with each other region, except for the innermost and outermost. It would be impossible for the inside region and the outside region to touch because we would then have five countries all touching each other!

A.



However, as can be seen from B, on a torus (donut) we *can* have the inside and outside regions also share a border, because the inside region continues through the hole to touch the outside region in the unseen part of the donut.

B.



In fact, we can also add a sixth region, sharing a border with each of the other five! (This region is shaded in C.) It is even possible to have senven countries in Donut-world, each sharing a border with all the others!

C.