LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

November 22, 1999

Follow-up: Variations on Coloring

Activity #1 — Refresher on vertex coloring of graphs
(Allocated time = ??????? minutes)

A. Show TSP # 1, which contains a map of a portion of the United States. Ask the participants to imagine that they are mapmakers, responsible for coloring this map. What can they say about coloring this map?

What you are looking for here is that adjacent regions must not get the same color, and that this map requires 4 colors. They should also be able to prove to you that this map takes 4 colors, since when a first color is chosen for Nevada, three additional colors must be used for the 5 states around it.

Ask the participants how we modeled such coloring activities. They should be able to tell you that a graph was used to model the map, with vertices representing regions and edges representing adjacencies. You can then put up TSP#2, overlaying it on TSP # 1, showing this graph. Redo the coloring activity just completed on the graph, with the map removed, to remind them that the model contains all the information from the map that we needed to do the coloring. Show TSP#3 To remind them of the rules of graph coloring and the definition of "chromatic number."

Also take this time to remind the participants of other conflict resolution problems which we solved by graph coloring. TSP#4 Shows several of these, taken from the first day's workshop. Explain, for each of the two problems, what the vertices and edges of the associated graphs would correspond to.

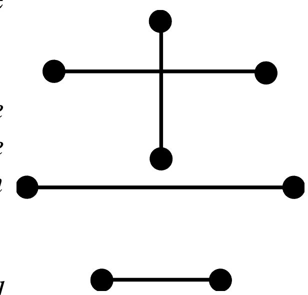
Activity #2 — Edge coloring of graphs
(Allocated time = ??????? minutes)

A. Put up the word problem shown on TSP#5, and read it with the participants.

The point of this slide here is not to actually solve the problem, but to discuss how we might model the problem. Take some time to discuss with the participants whether the people or meetings might represent the vertices, or edges. The next slide shows the corresponding "meeting graph."

When the time is right, put up the meeting graph on TSP#6. Continue the discussion with this visual aid, filling in "the blanks" on the slide. Now that they can see the model of the problem, try to lead them to discover the proper model for the solution, which is edge-coloring. This done, you can show TSP#7 which defines the notion of edge coloring and gives some terminology.

B. When they understand the goal of edge coloring, distribute HO #2, which asks them to find the chromatic indices of some classes of graphs with which they are already familiar — cycles, complete graphs and grids. These graphs can be found on TSPs #8 and 9.



The cycles and grids should be fairly easy, but the complete graphs will present a more significant challenge. For complete graphs with an even (for example, 8) number of vertices, you can show that the matching shown to the right can be one color class, and the rotates of those edges can form the other 6 color classes, yielding a legal 7-edge coloring of K_8. (This generalizes to all even n, showing that the chromatic index of K_n. is $n - 1$, for even n.) For odd complete graphs, the vertex in the center is a sort of "dummy vertex" which is "connected" to an unmatched vertex. This shows that the chromatic index of odd complete graphs is n.

It may not necessarily be prudent to actually show this construction to the participants, but if the instructor thinks it will be well-received, then go for it!

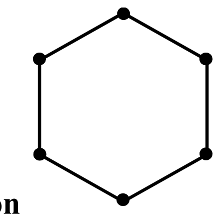
Mention to the participants that edge-coloring a complete graph corresponds to scheduling a round-robin tournament among some number of players.

It is probable that this is the application of edge-coloring that they are most likely to use in their classes, so it is important not to skip mentioning this.

C. Put up TSP #10 which has another word problem involving parent-teacher meetings.

This problem is intentionally difficult, and is there to show another application of edge coloring to scheduling, and to motivate the next slide which shows how to get the answer very quickly to this problem. During the first day of week 1, the participants were shown Brooks theorem about vertex colorings of graphs. The analogous theorem for edge *colorings is Vizing's theorem; both of these are shown on TSP #11, together with a bipartite version of Vizing's theorem. The interesting thing about the bipartite case is that it actually shows* equality*, and not just inequality. This makes the parent-teacher problem very easy!*

Finally, show TSP #11 with the three theorems mentioned above. If time permits, you may wish to note how the bipartite case of the theorem covers the grid graphs, which are bipartite. How do you know they are bipartite? Well, as they should have seen in the second summer of the program, 2-vertex-colorability implies bipartiteness.



Activity #3 — List Coloring of Graphs
(Allocated time = ??????? minutes)

A. The three graphs shown in this activity should all be laid out on the floor or on a tarp, made large enough so that participants can stand at the vertices without crowding one another. At each vertex should be two linker cubes — one red one and one blue one.

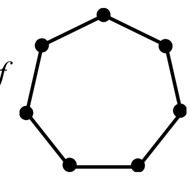
Have 6 participants stand at the vertices of the 6-cycle. Ask them to pick up the two linker cubes at their vertex. On the count of three, you want them to lay down one of the cubes on their vertex so that a legal coloring of the graph results. One - Two - Three! When they lay down their cubes, and finish adjusting, make sure that they have achieved a legal coloring. This shouldn't take too long.

They will probably have to discuss among themselves a little bit the idea that the colors have to alternate around the cycle, and this has to be coordinated. If such discussion doesn't seem to be taking place, you may wish to precipitate it by asking if there is a different coloring that will work.

That done, have the participants put their two cubes aside (perhaps you can quickly collect them) and then walk around the cycle with a bucket of separated linker cubes, and allow each participant to select two of them at random from the bucket. The only restriction is that no one should select two cubes of the same color. Now repeat the above activity: On the count of three, you want them to lay down one of the cubes on their vertex so that a legal coloring of the graph results. One - Two - Three! When they lay down their cubes, and finish adjusting, make sure that they have achieved a legal coloring. Ask them if this was easier or harder than the previous case.

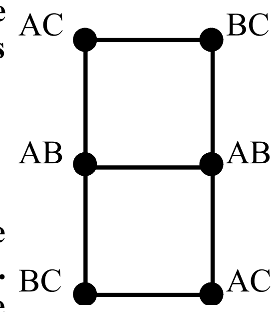
It is likely that they will say it was easier because there was more freedom because the same colors didn't appear at each vertex. This, of course, is not always true, as they will see in the case of the grid below. After the grid activity, we will return to this and the 7-

cycle, and ask the participants if they think it is possible to assign lists of colors to the vertices so that there is no legal coloring possible.



B. Repeat activity A. on the 7-cycle. There should again be two blocks sitting on each vertex, the same two colors at each vertex. Participants this time, on the count of three, will fail to find a legal coloring. Discuss it a bit, and ask them what they think would happen if you let them all pick randomly from the bucket again.

Collect their blocks and let them select two again from the bucket. Then see if they can, on the count of three, color the vertices of the graph.



Most likely they will be able to. See! It is easier!

C. Repeat the activity on the grid graph shown here. Since this graph is bipartite, they will have little trouble 2-coloring it. Collect their cubes, and again "randomly" hand out colors to the participants. This time, however, you will hand them their "random" colors from the bucket so that the people at the various vertices get assigned the lists shown in the figure. This will work best if you make it look random.

From these lists a legal coloring is not possible. For if the left center vertex is colored "A", then the vertex above must be "C," which forces the vertex on the top right to be "B," which forces the right center vertex to also be "A." But it's adjacent to our first "A" vertex. And if the left center vertex is colored "B," then the contradiction arises by chasing the colors around the bottom.

The participants will realize that this is not possible, but you should have them prove it, along the lines given above. But what about our earlier belief that "spreading out" the colors will make the problem of coloring easier? As they can see, this is not always the case.

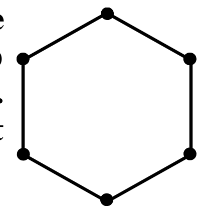
D. Return to the 6-cycle, and ask the participants if they think that "spreading out" the colors can make it impossible to color this graph, if each vertex is given two colors to choose from. Give them a few minutes to try this, and in the end, they will conclude that it is not possible. Probably.

In fact, if the lists (of size two) on the vertices of any cycle, odd or even, are not all the same, then there will always be a legal coloring. The proof that it is always possible goes like this: Select a vertex v *which has a different list than an adjacent vertex* w*. Color* v *with a color that is not on* w*'s list, and then move around the cycle, in the* *direction away from* w*, coloring each vertex with a color on its list different from the color used on the previous vertex. When you go all the way around the cycle, back to* w*, you will have no problem, because* v *is colored with a color not even on* w*'s list.*

It is not clear that the participants will be able to come up with this proof. If not, you can leave it for the handout, which follows, or you can show it to them now.

You can show TSP #12 which shows these graphs and the terminology of list coloring.

E. Distribute HO #3 = TSP #13 which contains several graphs. These graphs are all bipartite, and so are all 2-colorable. Their job on the top three graphs is to decide if they are colorable from the given list assignments. On the bottom graph, their job is to assign lists of size 2 to the graph so that there is no legal coloring. (12, 13, 23 on each side does it.)



The first graph has a good coloring, the second one does not (for the same "chasing colors" reason as the grid graph above), and the third one does not (because, for each of the 4 ways to color the two vertices on the left and right, there is a vertex in the middle column whose list has only those two colors, and thus can't be colored).

The bottom one contains the grid graph above as a proper subgraph, so you can already use those lists. Another way is that shown in the bold text above.

F. Finally, you may wish to show them that bipartite graphs, which are 2-colorable, can sometimes not be colored even if you have 20 colors available at each vertex! To do this, show TSPs #14 and 15, containing the complete bipartite graph $K_{3,27}$, and a list assignment of same. You can mention how this can generalize to having *k* vertices on one side, and k^k vertices on the other side, with lists of size *k* on each vertex, without admitting a coloring. It's a good counting problem.

Activity #3 — On-Line Coloring of Graphs
(Allocated time = ??????? minutes)

A. The t

1. On-line coloring
   a. Examples
      i. P(4) has chromatic number 2, but on-line, it can require 3 colors
      ii. Pair of graphs which need 2 colors each, but on-line they require 3.
      iii. However, as a pair, those two graphs can require 4 colors if the colorer

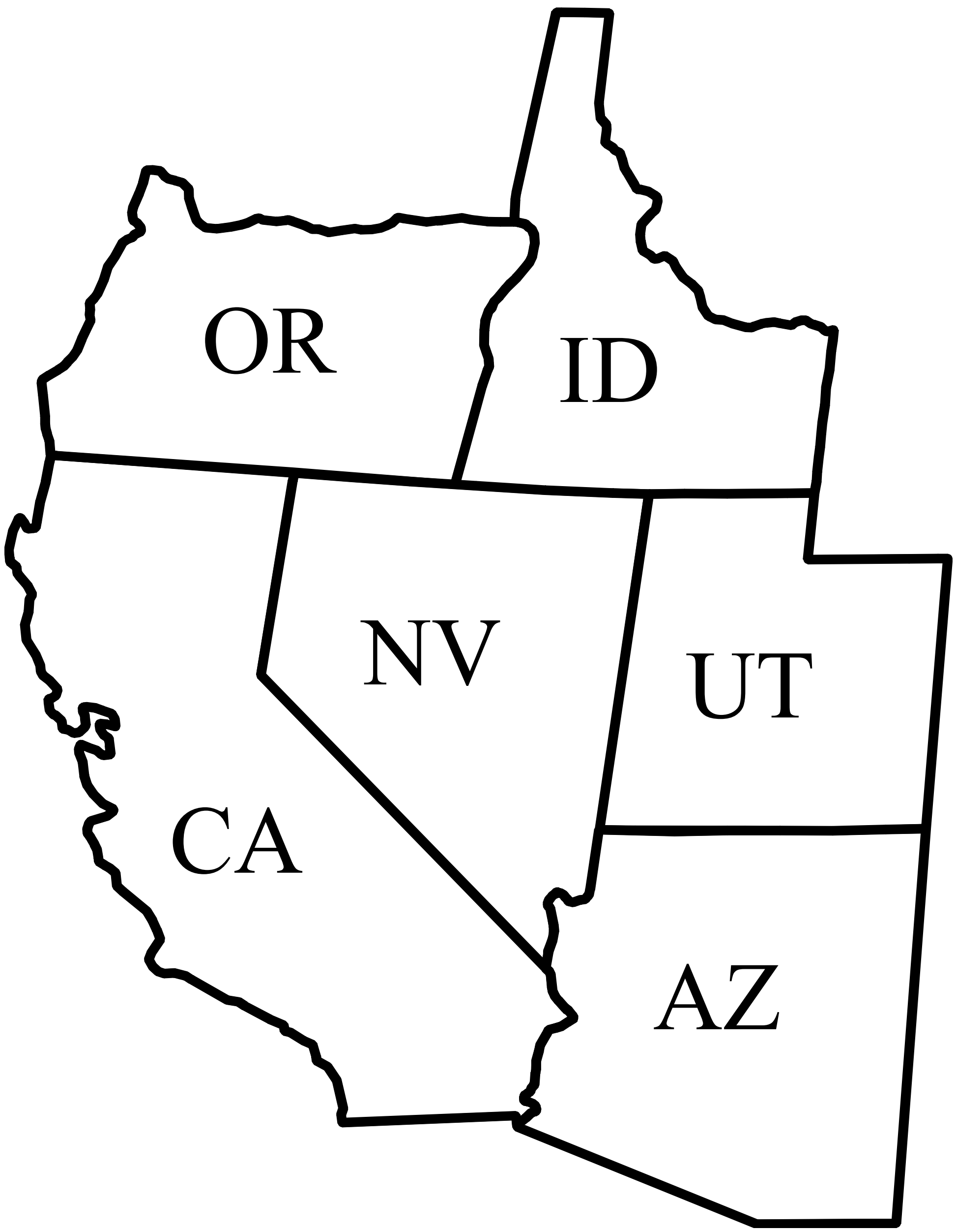
does not know which graph he is coloring!

b. Bipartite graphs can be arbitrarily high

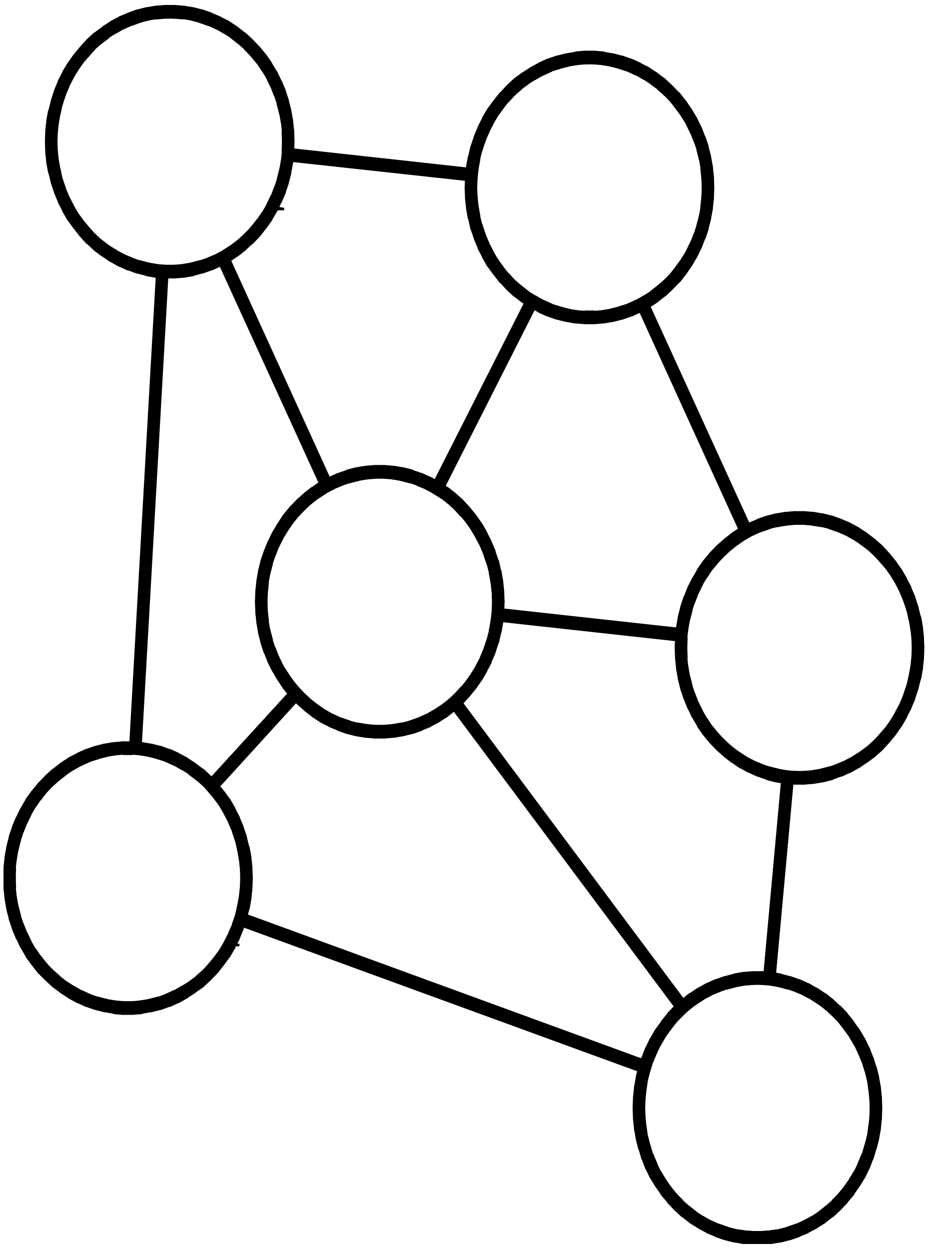
c. Even trees can!

2. Coloring with an uncooperative partner

A Portion of the United States Map



The Graph Corresponding to the Map

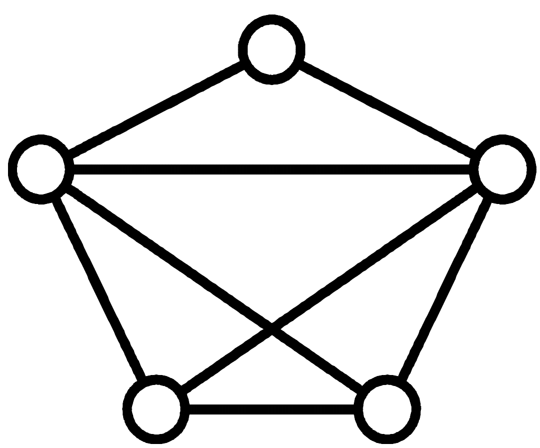


Rule for Graph Coloring

- Adjacent vertices must get different colors.
- Recall that two vertices are *adjacent* if they are connected by an edge.

Chromatic Number of a Graph

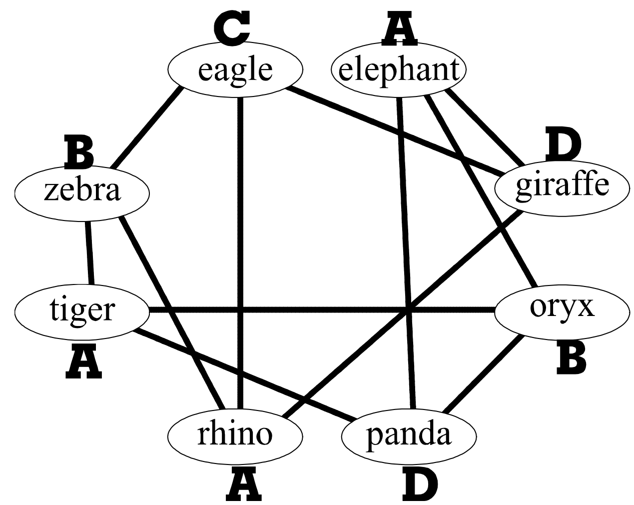
The "chromatic number of a graph" is the smallest number of colors that can be used for a coloring of the graph. If the graph is called G, then the chromatic number of G is often written as $\chi(G)$, read "chi (kye) of G".



For the graph shown above, $\chi(G) =$ _____

Some Scheduling Problems Solved by Vertex Coloring

1. In the zoo problem, the initial solution (at right) involved four habitats, labeled A, B, C, D. Can you find a solution which requires only three habitats?



2. Find a schedule for the class projects shown below.

Projects	Students	Day Assigned
Dinosaurs:	Sarita, Barbara, Ravi	
Spain:	Sarita, Roberto	
Bicycles:	Roberto, Maimuna	
Muscles:	Maimuna, Boris, Christie	
Fairy Tales:	Barbara, Boris, Jason	
Hockey:	Ravi, Christie, Jason	

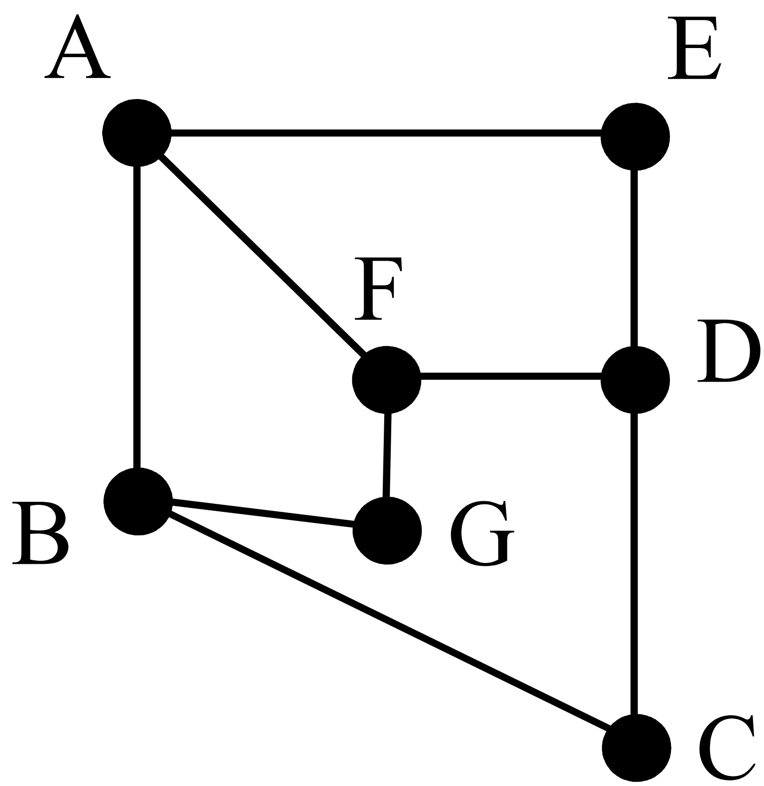
Morning Meetings

Art, Bob, Carol, Dee, Egbert, Fran and Geza are writers for a newspaper. Each morning, before doing any other work, they get together for one-on-one meetings to discuss assignments for the day. Below is a chart showing who needs to meet with whom on Monday morning:

Art	Fran
Bob	Art, Carol
Carol	Bob
Dee	Carol, Egbert, Fran
Egbert	Art, Dee
Fran	Art, Dan
Geza	Bob, Fran

The meetings last 15 minutes each, and several meetings may take place in each 15 minute block.

What schedule for these meetings will require the fewest blocks?



The Meeting Graph

How can this graph be used to schedule the meetings?

The people correspond to ________ in this graph

The meetings correspond to ________ in this graph

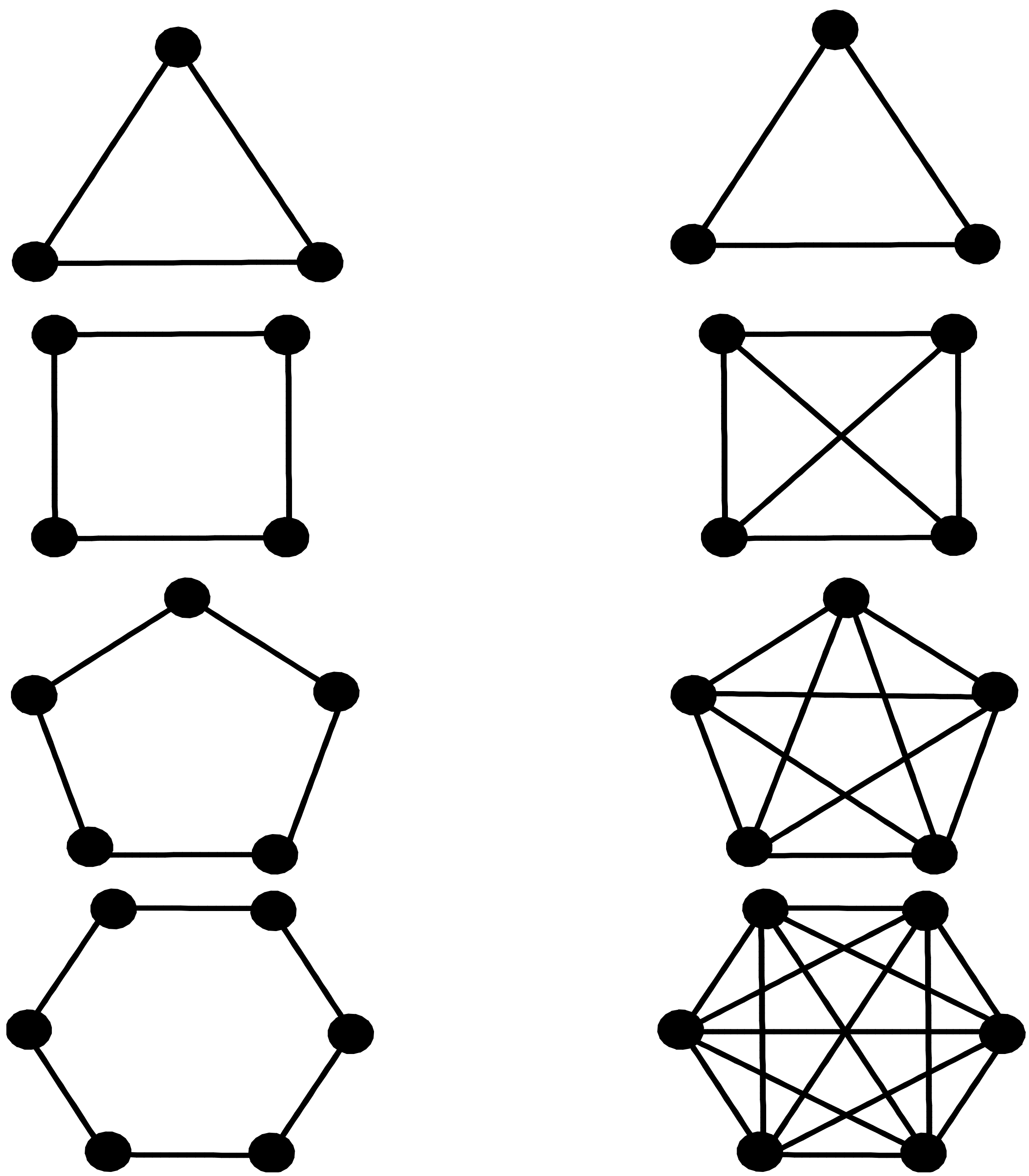
What should we color?

What is the coloring restriction?

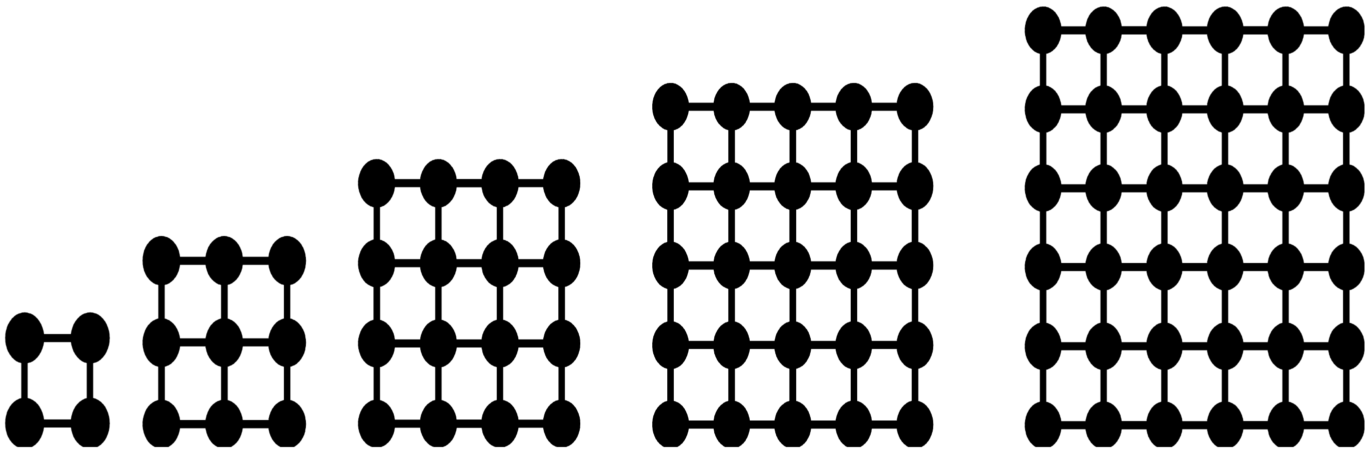
Edge Colorings of Graphs

- We color the edges of the graph, not the vertices
- Edges which meet at a vertex (incident edges) must get different colors
- The least number of colors needed to edge color a graph is called the *chromatic index* of the graph
- The chromatic index of G is denoted $\chi'(G)$, "chi-prime of G "

Chromatic Index of some Graphs



Chromatic Index of some Graphs

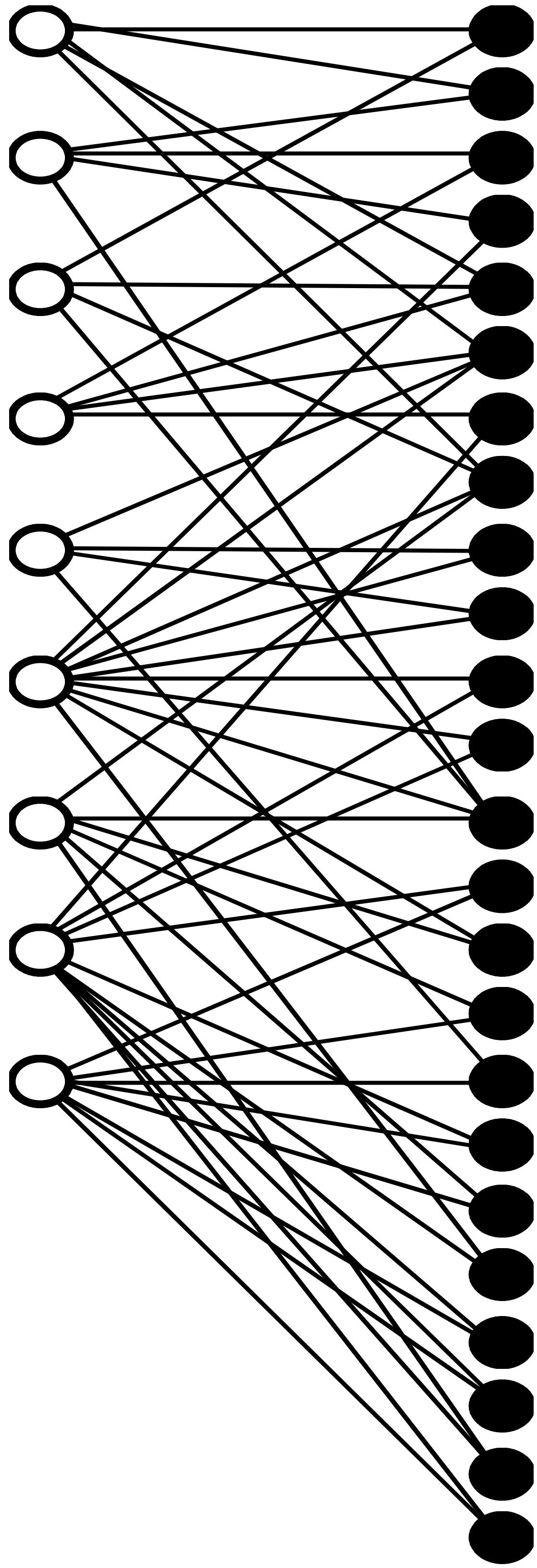


The chromatic index of a cycle on n vertices is ______

The chromatic index of a complete graph
on n vertices is _______

The chromatic index of a grid graph is _______

Parent-Teacher Meetings



Tonight is parent-teacher night. The graph to the right shows which parents (shaded) need to meet with the various teachers (unshaded).

If the meetings take place in 10-minute blocks, what is the fewest number of blocks needed to complete all the meetings?

Don't Panic!

We have a theorem...

Brooks' Theorem

If D is the largest degree in a graph G, then

$$\chi(G) \leq D$$

(unless G is an odd cycle or a complete graph).

Vizing's Theorem

If D is the largest degree in a graph G, then

$$\chi'(G) \leq D + 1$$

For Bipartite Graphs

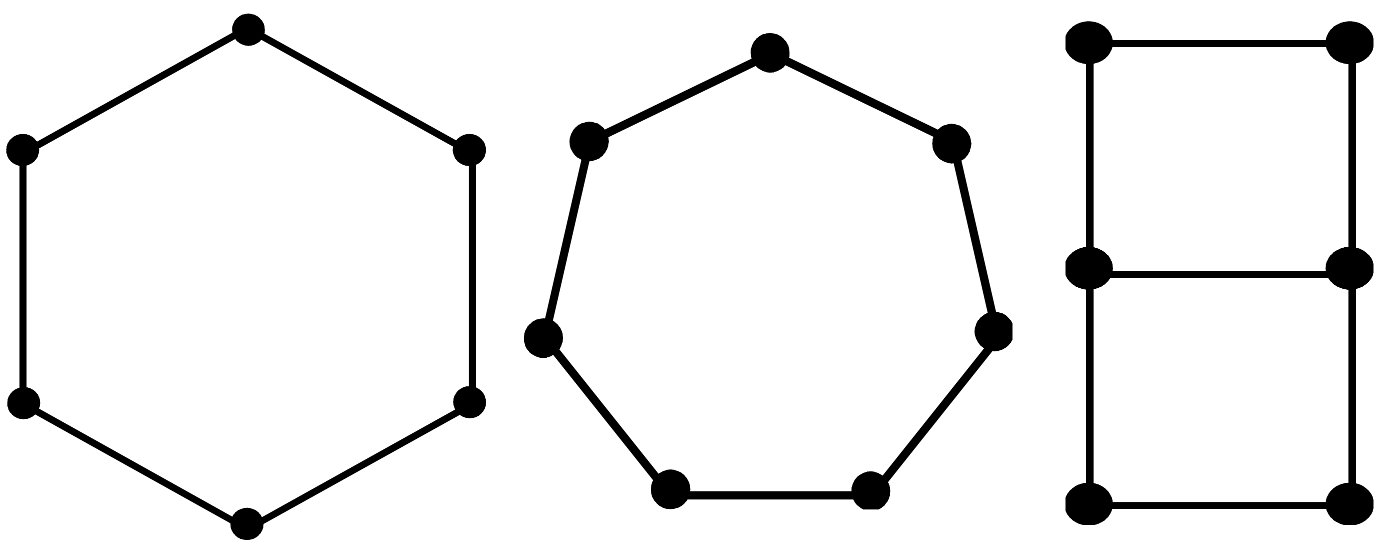
If D is the largest degree in a bipartite graph G, then

$$\chi'(G) = D$$

Note that this applies to grid graphs.

List Coloring

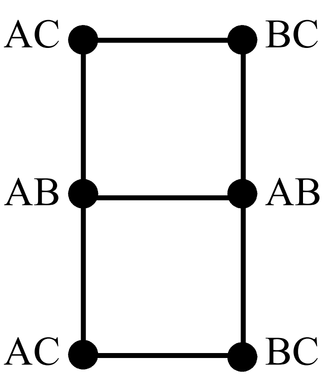
- The sets of colors at each vertex are called *lists*.
- Coloring the vertices from given lists at each vertex is called *list coloring*.
- A graph is called 2-choosable if it can always be colored from lists of size 2 at each vertex, no matter how "bad" the lists are. Same for 3-choosable, etc...
- The smallest list size that always yields a coloring is called the *choice number* of the graph.
- The choice number can be the same as the chromatic

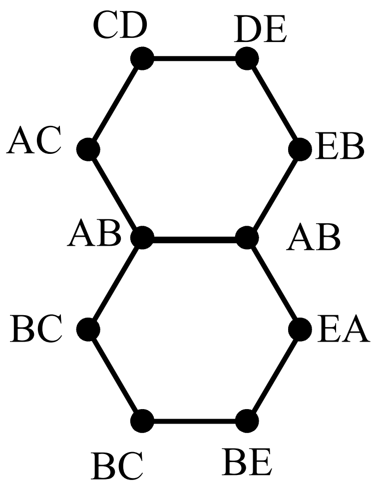


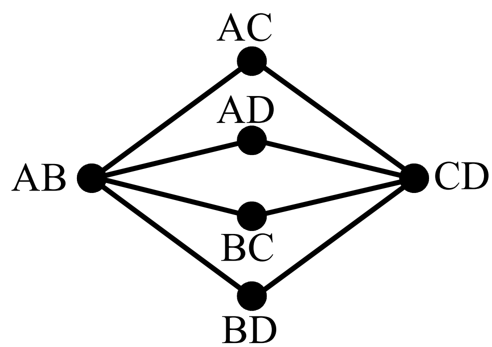
number, or it may be bigger, but it can never be smaller.

Handout #3 — List Colorings of Graphs

1. Which of the following graphs can be vertex-

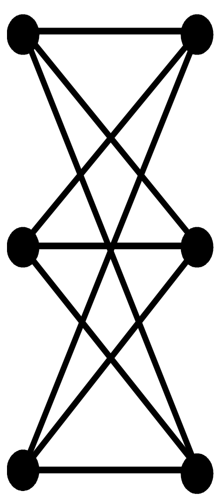
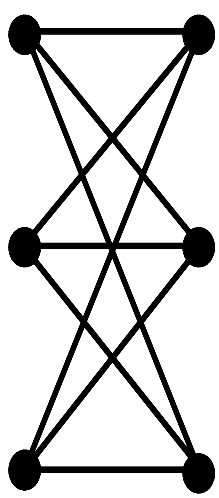
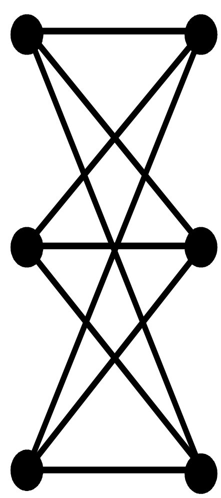
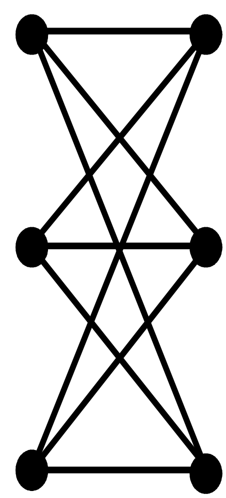




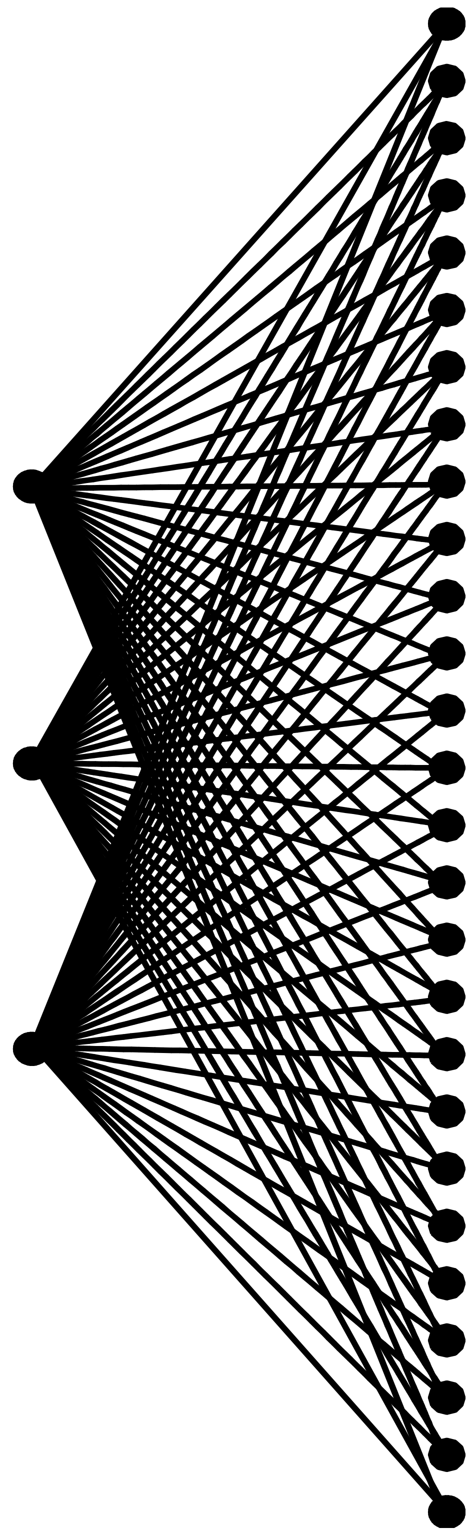


colored from the given list assignments?

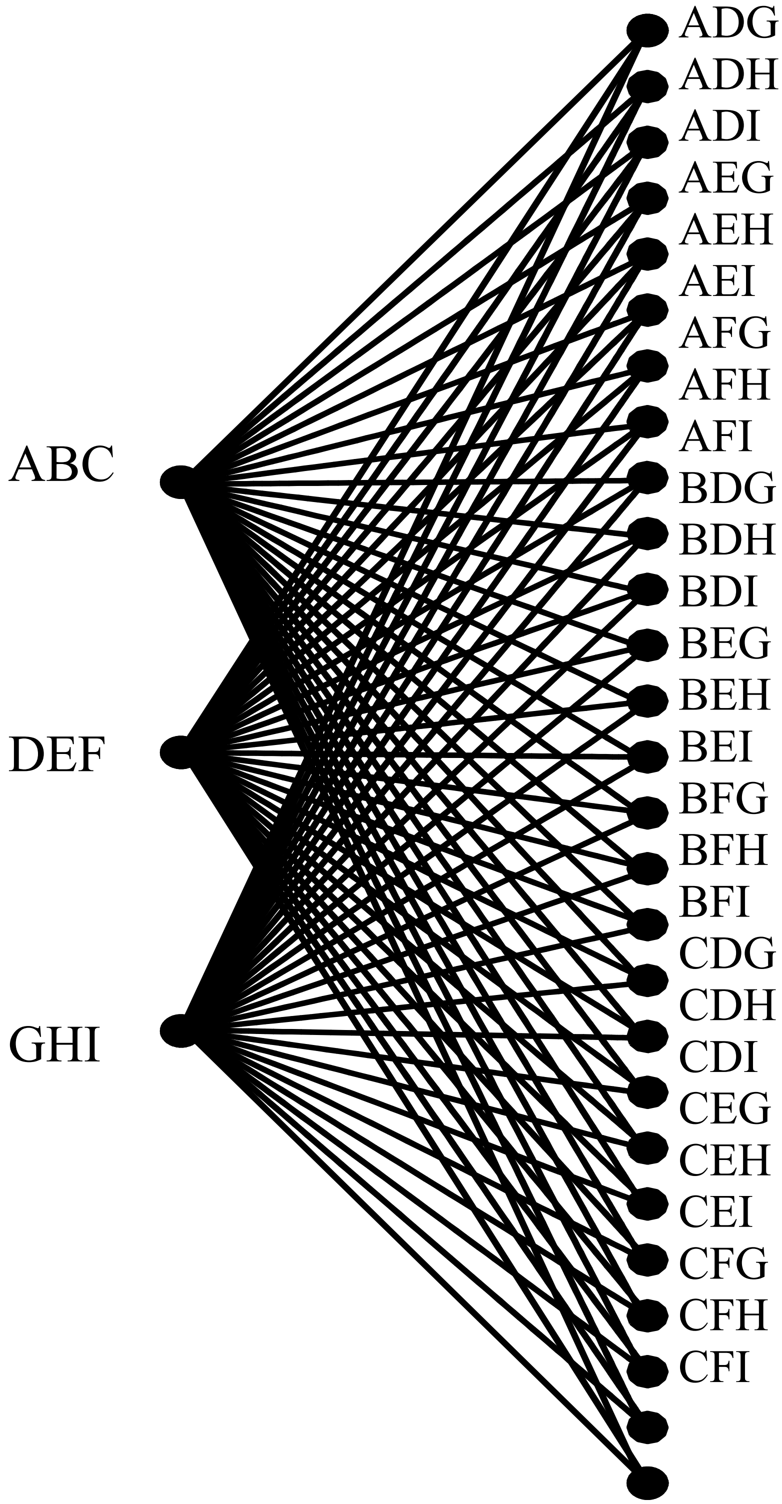
1. The following graph has a list assignment which does not yield a good coloring. Can you find it? (Four copies of the graph are provided.)

A Bipartite Graph which is not 3-Choosable



A Bipartite Graph which is not 3-Choosable



Some Facts about List Coloring

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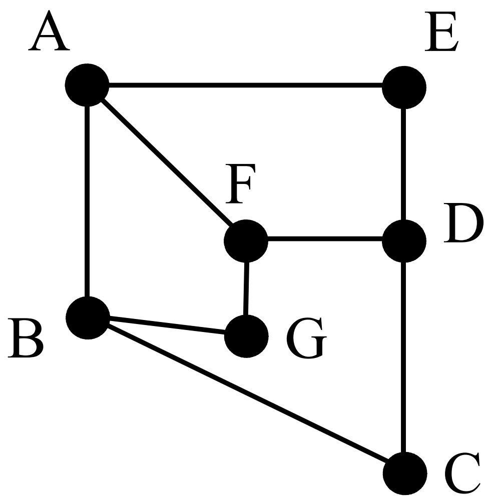
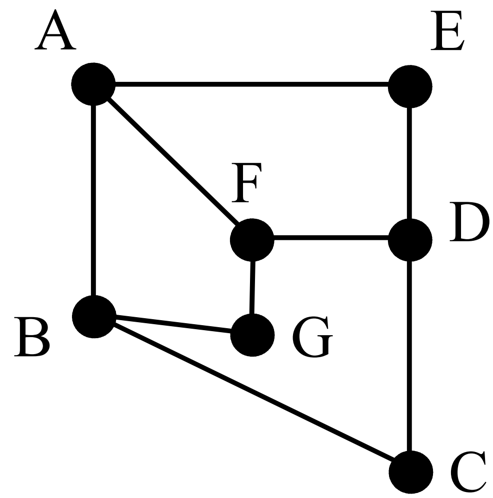
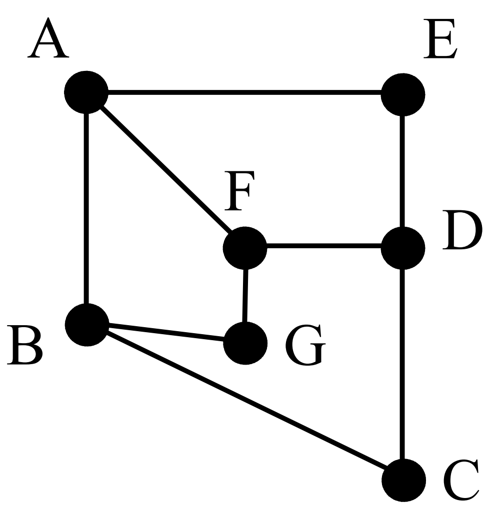
Handout #1 — Scheduling the Meetings

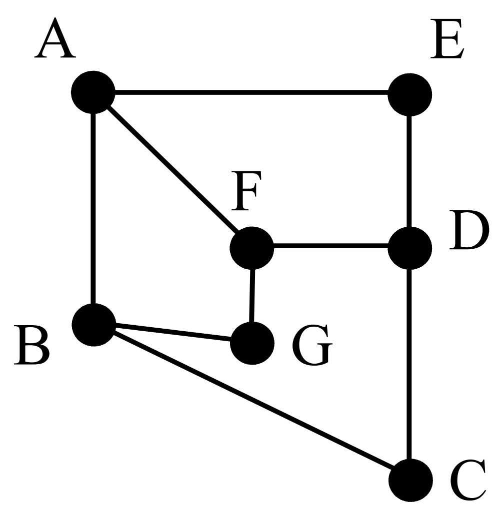
What is the fewest number of colors needed to edge-color this graph (4 copies are provided) so that incident



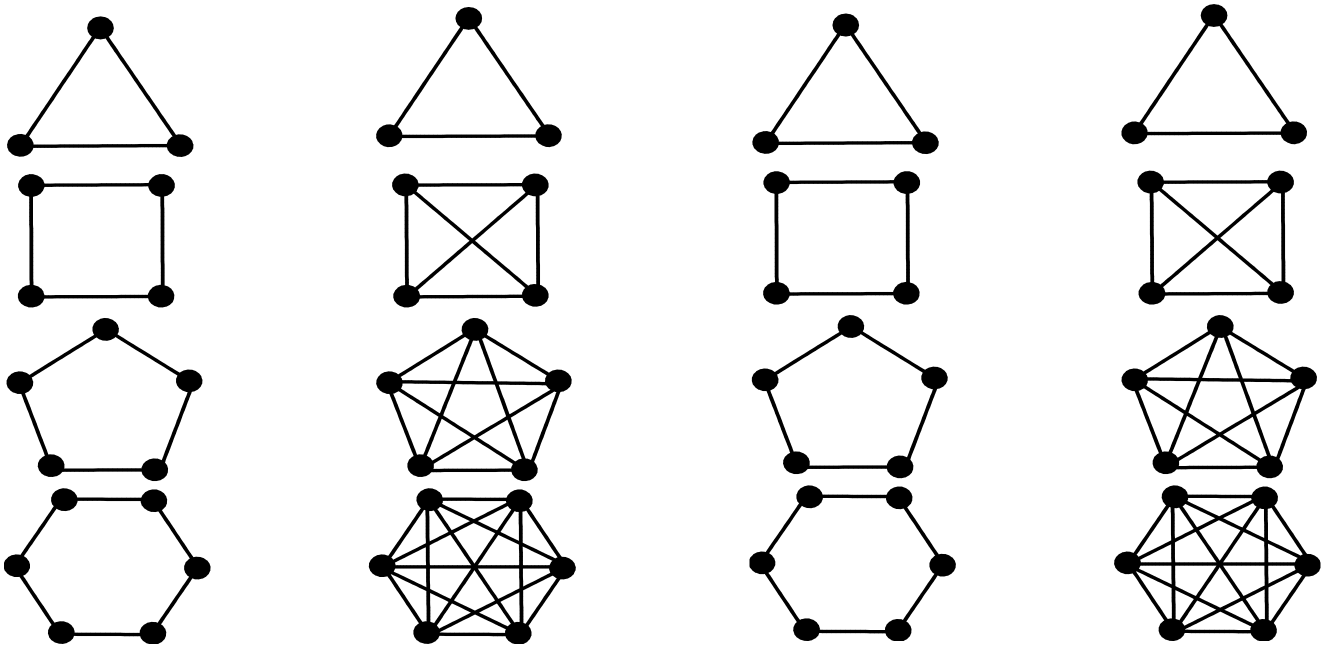


edges get different colors?

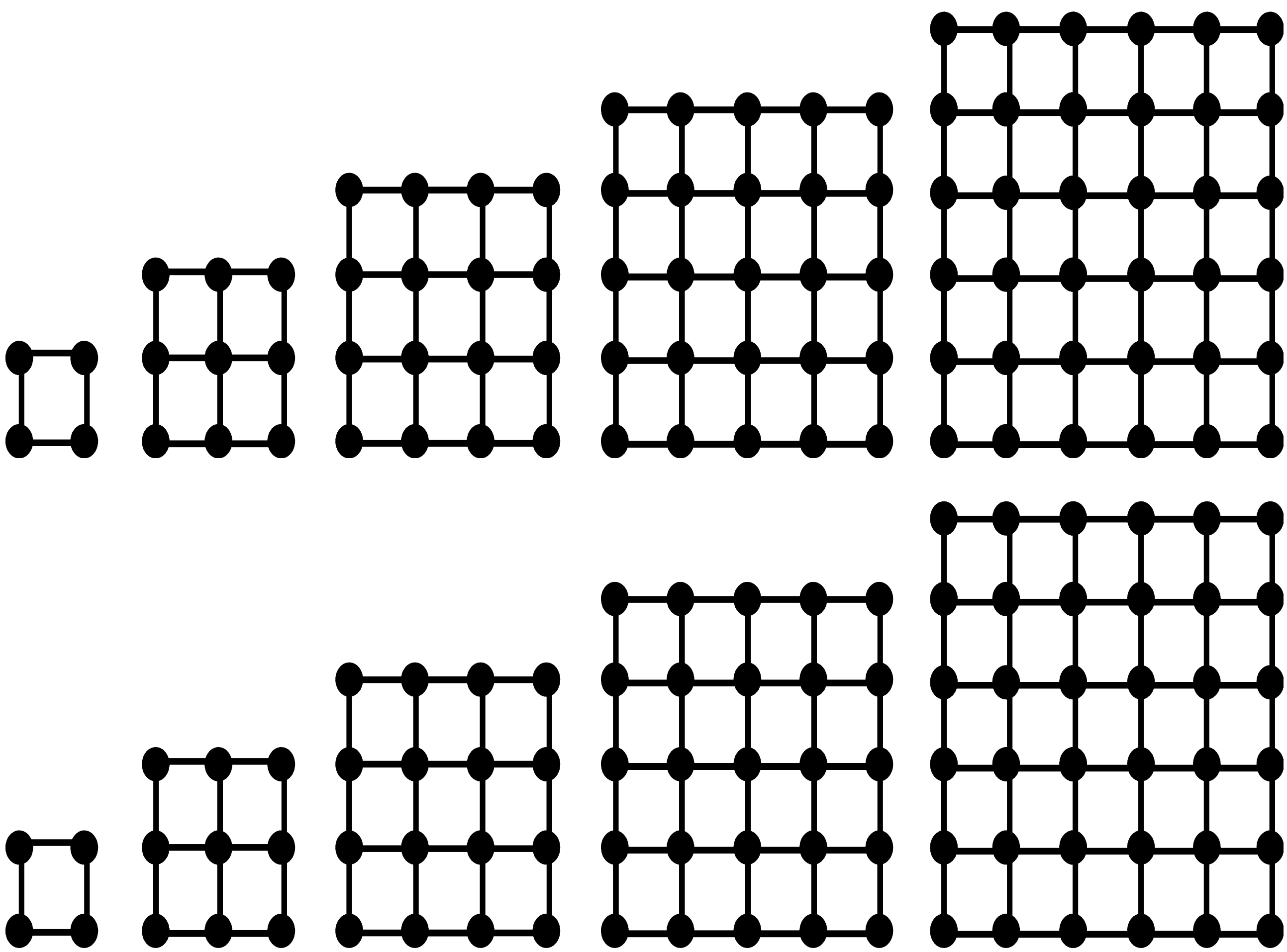




Handout #2 — Chromatic Index

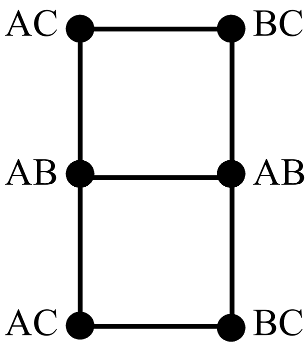


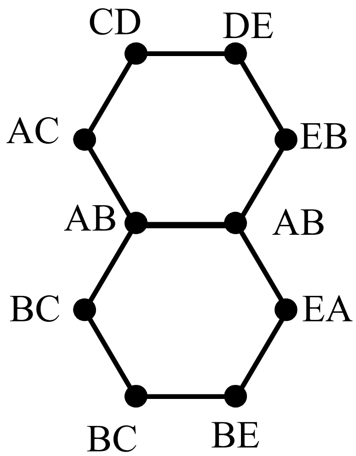
Find the chromatic index of each of the following graphs. (Two copies of each are provided.)

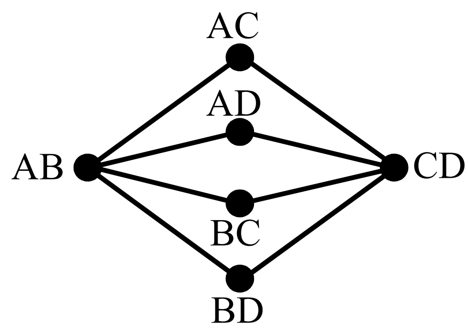


Handout #3 — List Colorings of Graphs

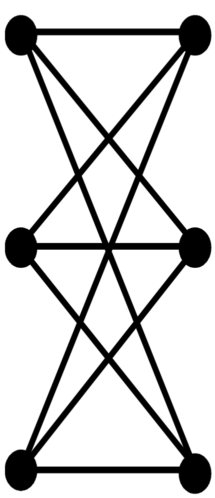
1. Which of the following graphs can be vertex-colored from the given list assignments?

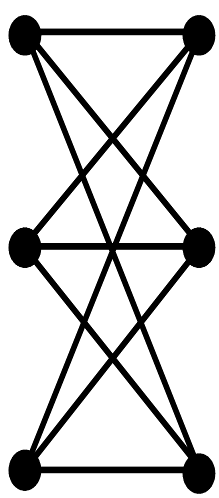


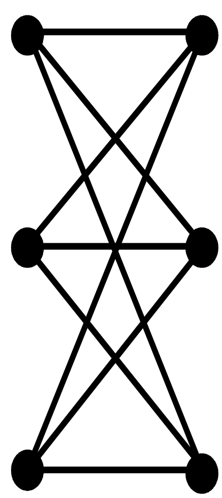


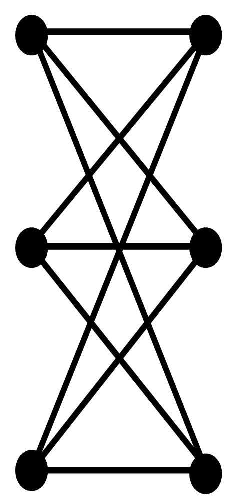


2. The following graph has a list assignment which does not yield a good coloring. Can you find it? (Four copies of the graph are provided.)

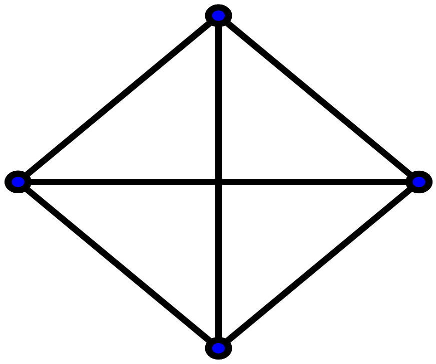




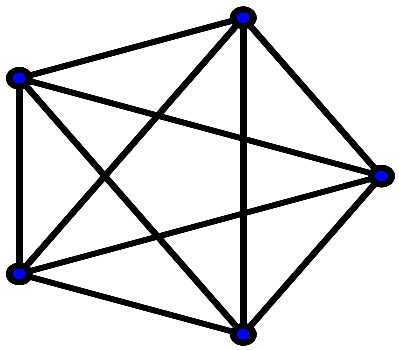


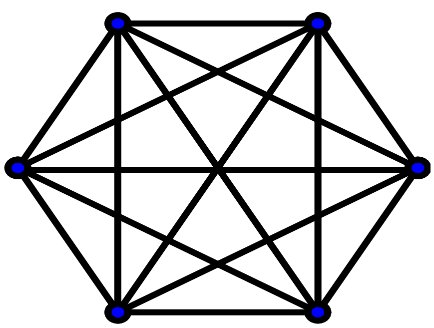


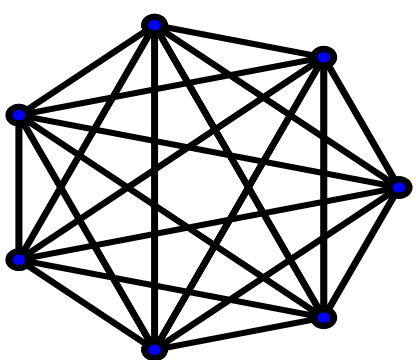
Handout #4 — Crossing Number of Graphs?



What is the minimum number of crossings with which each of these graphs can be drawn in the plane?





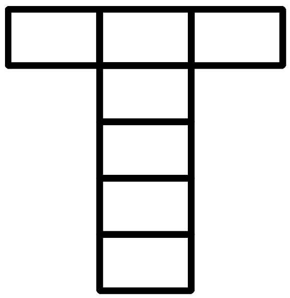


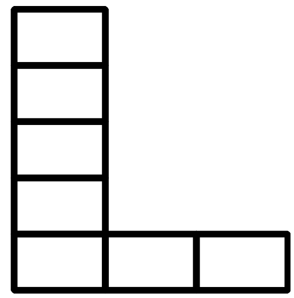
Drawing Graphs

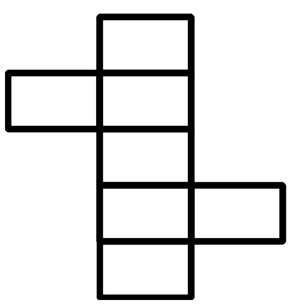
Follow-up session

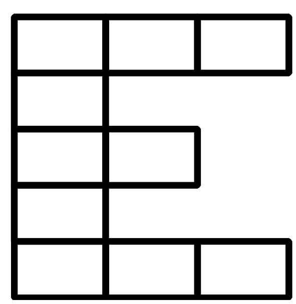
Winter/Spring 1999

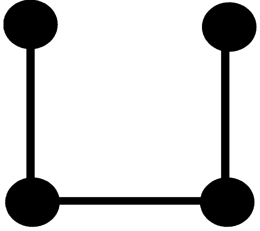
One of these things is not like the other











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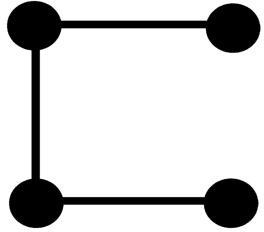

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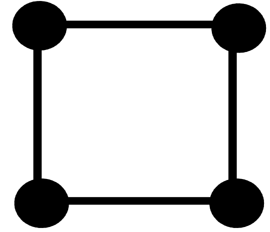

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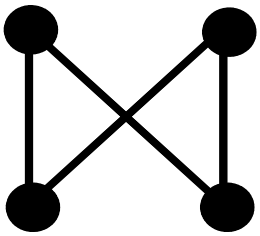

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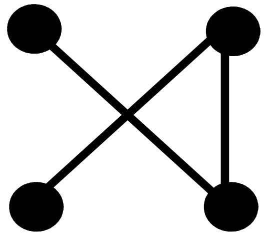

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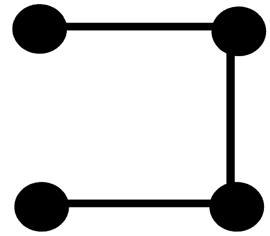

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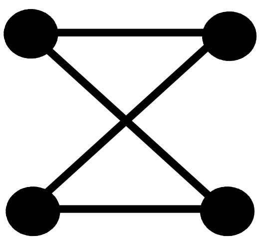

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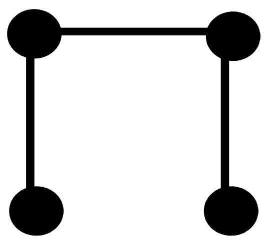

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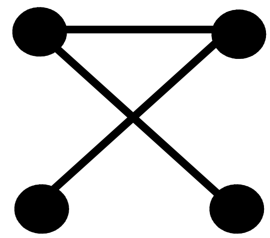

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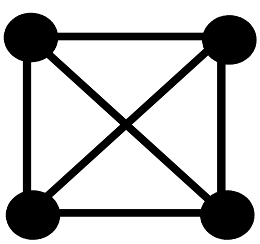

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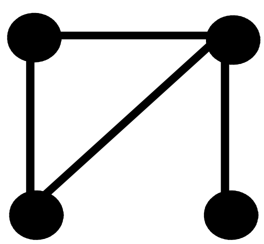

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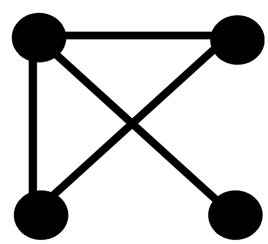
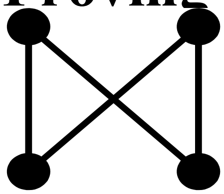
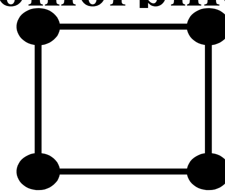
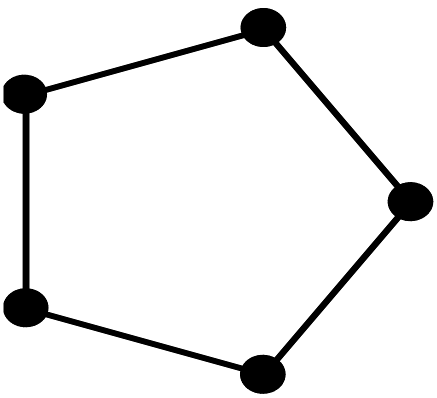


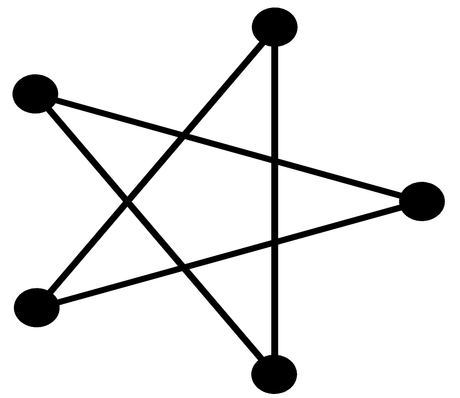
L

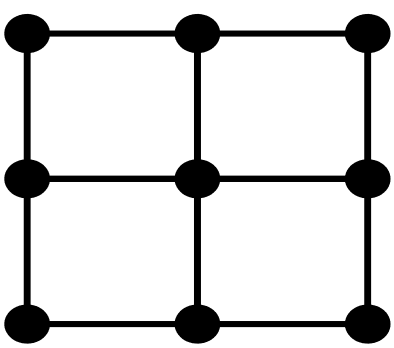
Proving that Two Graphs are Isomorphic

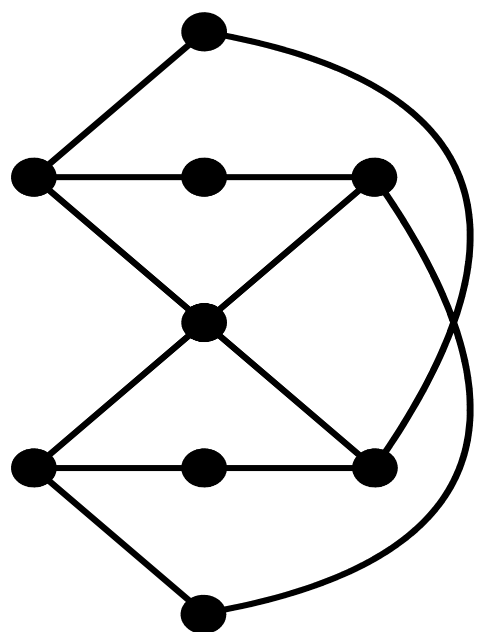




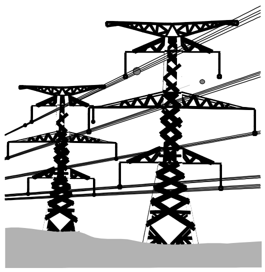


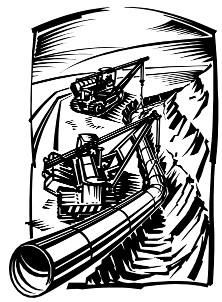


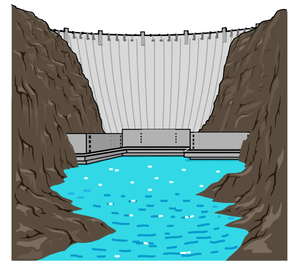


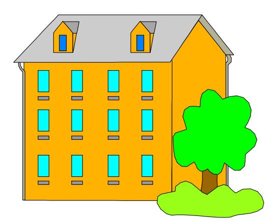


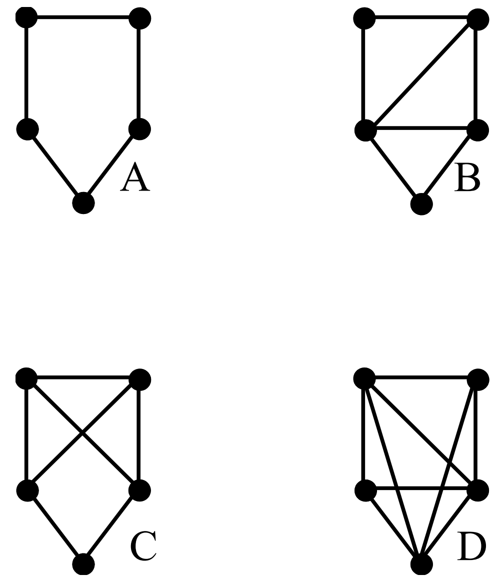
Is it possible to connect each house with each utility without crossing “lines?”











Graph Isomorphism

Two graphs are *isomorphic* to one another if, speaking loosely, they are actually the same graph, just drawn differently.

More precisely, suppose you number the vertices in one graph 1, 2, 3, ..., (up to the number of vertices in that graph) and then number the vertices in another graph 1, 2, 3, ..., (up to the same number of vertices). And suppose also that for every edge in the first graph there is an edge connecting the corresponding vertices in the second graph, and that for every edge in the second graph there is an edge connecting the corresponding vertices in the first graph. Then these graphs are *isomorphic.*

In particular, if two graphs are isomorphic, then:

- They must have the same number of vertices
- They must have the same number of edges
- They must have the same number of vertices of degree1
- They must have the same number of vertices of degree 2
- etc...
- If one graph has a triangle in it (a 3-cycle) then the other must also have a triangle
- In general, any structure which is found in one graph must also be present in any graph which is isomorphic to it

Thus, if you wish to prove that two graphs are *not* isomorphic to one another, all you need to do is find some structure or property which is present in one of the graphs, but is not present in the other graph. This property must not depend on the way the graphs are drawn, but only on the structural properties of the graphs.

Planar Graphs

A graph which can be drawn in the plane without any crossing edges is called *planar*.

Some examples of planar graphs are shown to the right:

Note that a graph is called planar if it *can* be drawn in the plane without crossings. Thus even though some of the graphs to the right are drawn with crossings, they *can* be drawn without crossings.

Crossing Numbers of Complete Graphs

This table shows all that is currently known about the crossing numbers of complete graphs:

# Vertices	Predicted Value	Known Value	Straight edges only
1	0	0	0
2	0	0	0

3	0	0	0
4	0	0	0
5	1	1	1
6	3	3	3
7	9	9	9

8	18	18	19
9	36	36	36
10	60	60	61 or 62
11	100		
12	150		

13	225		

Handout #? — Same or Different?

For each of the following pairs of graphs, classify them as either the same or different and provide a reason why.

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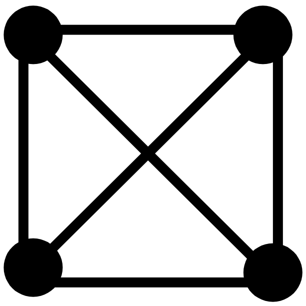
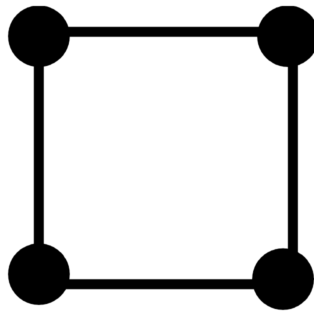
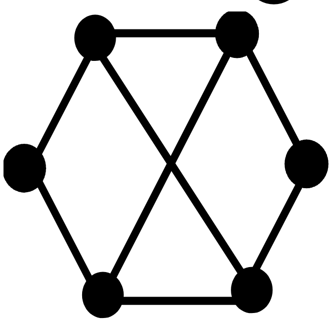
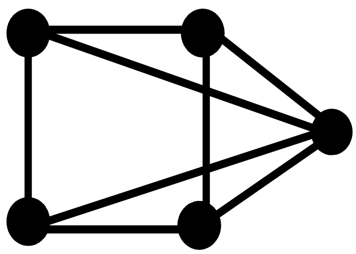
<td>  Why?  </td>
<td>SAME

or

DIFFERENT

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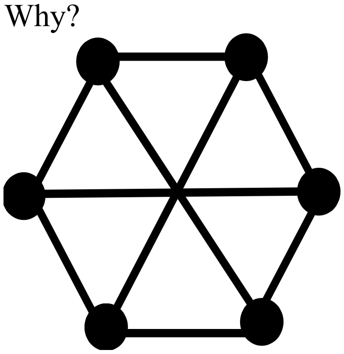 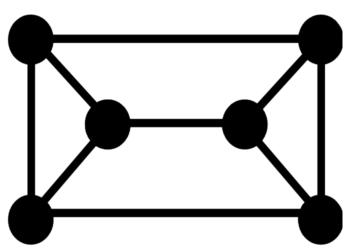</td>
<td>SAME

or

DIFFERENT

?</td>
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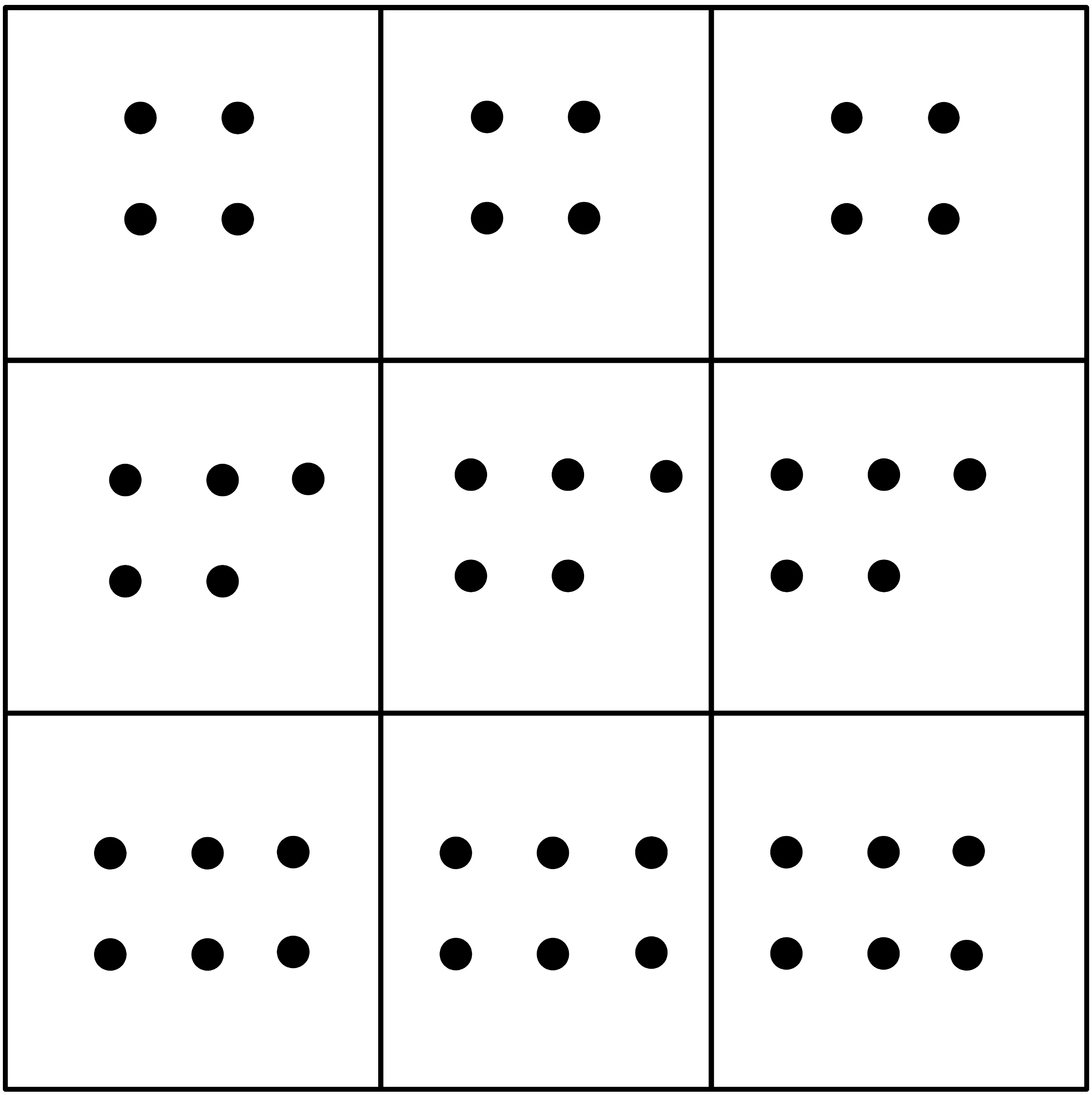
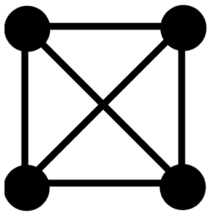
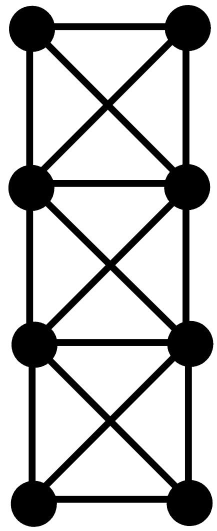
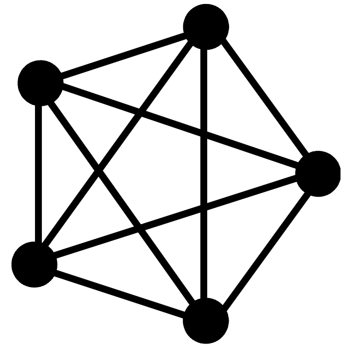
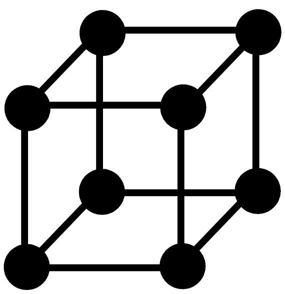
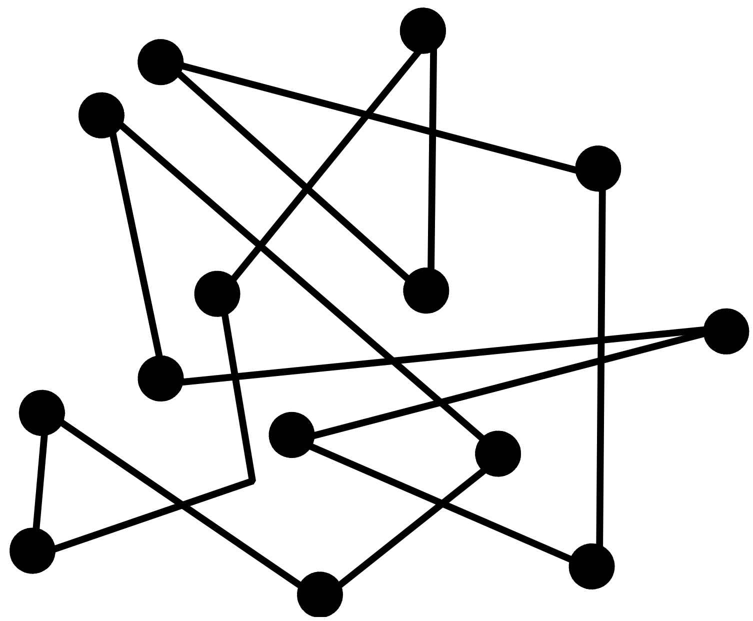
Why?	SAME or DIFFERENT ?

Handout #2 — Connect the Dots — Without Crossing!

Connect each dot in the top row with each dot in the bottom row.
For each case, three copies are provided.



Handout #3 — Which graphs are planar?

Determine which of the following graphs are planar, and which are not.

Handout #4 — Crossing Number of Graphs?

What is the minimum number of crossings with which each of these graphs can be drawn in the plane?