LEADERSHIP PROGRAM IN DISCRETE MATHEMATICS

Instructor's Notes

December 15, 1999

Follow-up: Blazing Pascal

Materials Needed | **Allocated time**

Activity #1 — Refresher of Pascal's Triangle . . . . 35 minutes
- Nothing special

Activity #2 — Pascal's Triangle under Different Moduli . . . . 45 minutes
- "Blazing Pascal's Modular Explorer" software
- Laptop computer and viewing system for class

Activity #3 — Using the mod 2 coloring to solve the Tower of Hanoi problem . . . 20 minutes
- 6 linker cubes
- 4 books of different sizes

Activity #4 — Coin Flipping and the Binomial Distribution . . . . 30 minutes
- Nothing special

Activity #5 — Optional — The Catalan Numbers . . . . Remaining Time
- Nothing special

. . . . TOTAL WORKSHOP TIME: 135* minutes

* In addition, ten minutes are allocated for a break in this 2½ hour workshop.

Activity #1 — Refresher on Pascal's Triangle
(Allocated time = 35 minutes)

A. Put up TSP #1 and distribute HO #1, showing a portion of Pascal's Triangle. Ask the participants to point out as many patterns as they can find or recall about the array of numbers shown.

Some of the things that they will likely point out, since they will recall them from the previous summer(s), or from their own experiences, are

1. *That each number is the sum of the two numbers above it*
2. *That the outside numbers are all 1*
3. *That the triangle is symmetric*
4. *That the first diagonal shows the counting numbers*
5. *That the sum of the rows gives the powers of 2*
6. *That each entry is an appropriate "choose number."*
7. *The Fibonacci numbers are in there along the diagonals.*

These are all shown on TSP #2

When participants mention the choose numbers, you can show TSP #3 which contains several applications of the choose numbers. Go through as many examples as necessary to remind them about how the choose numbers work.

When participants mention the Fibonacci numbers, you can show TSP #4 which uses color to show how they are contained in Pascal's Triangle. For an explanation of why they occur as they do, distribute HO #2 (= TSP #5) which looks into how the Fibonacci numbers give the number of ways to write a natural number as the sum of 1s and 2s.

This Fibonacci activity worked well when it was first done. There was some worry that it would be too theoretical, but it turned out very nicely, with many "ah-ha's."

Any of the above which were not mentioned by participants should be mentioned by the instructor, especially the choose numbers. For the powers of 11, it will probably be necessary to show the participants how, when the entries of Pascal's Triangle are more than one digit, you need to carry to find the powers of 11. It is probably worth it to go to row 9 also, to show what to do when you have a 3-digit number.

B. Using TSP #6, show the participants three "arithmetic" properties in Pascal's triangle. The graphics on this slide can be superimposed on TSP #1.

The "Hockey Stick" gives a way of adding the entries in a diagonal of Pascal's Triangle, as long as that diagonal begins along one side, at a "1." The sum of the numbers in the handle can be found on the bottom.

The "Funnel" gives a way to find the sum of the numbers in a rhomboid section of the Triangle, as long as it extends all the way to both sides. (It's the number in the circle, minus 1.) If time allows, it might be nice to show how the funnel rule follows from the hockey stick rule, and which also shows why that "minus 1" shows up.

The "Star of David" is made of two triangles. When the three entries at the corners of one triangle are multiplied, they give the same product as the three entries at the corners of the other triangle.

Activity #2 — Pascal's Triangle under Different Moduli
(Allocated time = 45 minutes)

A. Show TSP #<u>7</u>, which revisits an activity they did during their first summer. On this slide, the even numbers are shaded black and the odd numbers are unshaded. The resulting pattern is reminiscent of the Sierpinski Triangle.

A distinction that you may wish to make is that as you generate the Sierpinski Triangle, the triangle stays the same size, and a fractal emerges in the end. Here, however, as one considers more and more rows, the triangle grows, and what results (an "infinite size triangle") cannot properly be called a fractal. This is corrected by reducing the size of each "cell" so that more and more rows of Pascal's triangles are crowded into the same space, as shown on TSP #<u>8</u>.

Since the participants have already done this activity, all you need do here is remind them of this pattern which they have already seen. Also show them TSP #<u>8</u> so that they can see what would have resulted if they had colored up to 128 rows.

B. Distribute HO #<u>4</u>, containing 16 rows of blank cells. Ask the participants to do an activity similar to the one just shown, except that now we will color in only the multiples of 3. After a few minutes, tell them that you have a better idea, and would like to make the activity a little more interesting.

Distribute HO #<u>10</u>, which is the same as the previous handout, except that the title is different. On this handout, tell them that you would like to do the same thing, except that for entries which are *not* multiples of 3, they should find the remainder they get when the entry is divided by 3, and color according to that remainder. In other words, they should color the triangle "mod 3." You should do a few rows (say rows 0-4) for them by way of example. Then let the participants continue on their own.

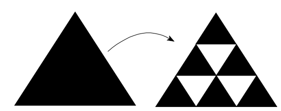
When the numbers get too unweildy, you can ask them to pause for a few seconds and look up front, for you have a better way to show them! Show TSP #<u>11</u> and spend a few minutes introducing the participants to the idea of adding remainders. When they seem to have this idea mastered, continue the idea by showing TSP #<u>12</u> where you complete the move from adding numbers and taking remainders, to adding remainders, to just adding colors. Finally, put up TSP #<u>13</u>, where you can use this "just add colors" idea to color, very quickly, the first several rows of Pascal's Triangle.

What they are actually playing with here is a "cellular automaton." Each hexagon in this grid is "programmed" to find out what the colors of the two cells above it are, and then to decide its own color based on those colors.

Put up TSP #14 showing up to 243 rows colored mod 3.

There are several interesting things to point out here.

- *One of the exercise activities they did during their first summer was called "Triangle Variation," in which they replaced each shaded triangle with six shaded and three unshaded triangles, as shown in the little graphic above. The resulting fractal is to Sierpinski's Triangle as the mod 3 coloring is to the mod 2 coloring, except that you have 2 colors making up the shaded spaces instead of 1.*
- *If you look at the 81-row diagram on TSP #14 as being made up of 6 copies of the 27-row figure, then you can see that 5 of those copies are all identical to the 27-row figure, but the 6^{th} copy (bottom center) has red and blue switched. This pattern continues forward and backward; in fact, the top 3 rows of the triangle already exhibit this pattern, even though it only contains 6 blocks altogether.*
- *As part of the "Triangle Variation" exercise set, the participants had proved that "in the end," the fractal would have shaded area 0. This corresponds to the fact that "almost all" numbers in Pascal's Triangle are multiples of 3.*



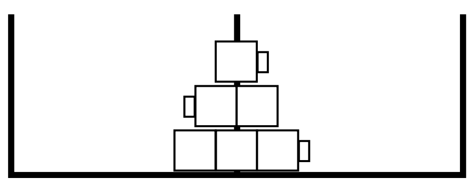
C. Now we will do a little quick exploration of Pascal's Triangle under different moduli. Put up TSP #15 showing a mod 4 coloring of Pascal's Triangle. In this figure, the multiples of 4 are black, and the entries congruent to 2 mod 4 are red. 1 mod 4 is blue, and 3 mod 4 is yellow, though these are not very distinguishable at the resolution shown. An interesting thing about this picture is that it could be created by taking a regular mod 2, sierpinski-like coloring, and placing a red mod 2 coloring within each black triangle!

Other pattens you can point out using the Modular Explorer include:

- *For a prime modulus, you get a generalization of the Sierpinski-like patterns we've seen already with 2 and 3. A nice way to emphasize this is to start with 100 as the modulus, 500 as the range and "divisible only" as the coloring scheme. Move the window so that the value in the range (100 at this point) is off screen, but the little up and down arrows are visible. Click "Draw" to generate the figure, and ask the participants if they think this is given by a prime modulus. Give feedback, so that they know this is not a prime modulus. Then increment the modulus to 101 (they still don't know the values) and ask if this figure is generated by a prime modulus. This time give them a little feedback so that they know it is prime. Continue in this fashion up to about 108. They should be able to clearly recognize the primes (101, 103, 107). After this little activity you can go back to some smaller primes (like 11 or 17, but still with range 500) so that they can see the self-similarity emerging.*
- *For moduli which are the square of a prime, you get the same figure as the prime, but with littler versions of the figure within the black triangular regions.*

- *For moduli which are the cube of a prime, you get triangles within regions within regions, like for squares of primes, but an extra layer deep. And the same goes for higher powers of primes.*
- *For a simple composite number, like 6, you can see a mod 2 and a mod 3 coloring at the same time. This looks even better if you select "remainder" for the coloring scheme. When you did the number 106 (= 2×53) above, the participants may have mentioned seeing something like this. And with the number 105 (=3×5×7) you can actually see all three layers, but not as easily. 1000 is a good range for 106.*
- *If you change the rule a little bit by playing around in the "Recursion Variants" section, you can get some very pretty patterns. Set the range to 250, and try a bunch of variants for various moduli. The following triples of (K, L, M) are particularly nice:*
    - *(2,1,1) mod 3*
    - *(2,2,0) mod 3 (with the "Lock Ones" box checked) The "Lock Ones" box causes all the entries in the leftmost diagonal to be "1," even if that is not what the recursion rule yields.*
    - *(2,2,1) mod 3*
    - *(2,2,1) mod 4*
    - *(2,1,1) mod 4*
    - *(2,1,1) mod 5*
    - *(a,b,0) mod p, with p prime and a and b both relatively prime to p. In this case you can get different color arrangements with the remainders, but with the "divisible only" coloring scheme selected, you cannot see any effect by changing a and b. This is a consequence of Fermat's Little theorem!*

Activity #3 — Using the mod 2 coloring to solve the Tower of Hanoi problem
(Allocated time = 20 minutes)



A. Put up TSP #16 showing the Tower of Hanoi problem, and remind the participants of the game and how it works. Tell the participants that a solution for the puzzle, that solves it in the least number of moves possible, is hidden in the mod 2 coloring of Pascal's Triangle. Put up TSP #17 and tell them that this is the solution to a Tower of Hanoi with 3 discs in it. (You can tell it's 3 discs by scanning down the center axis of the triangle and counting the number of shaded triangles you meet.) There are 3 sizes of black triangles here, small, medium and large. Now, reading across the bottom, ask the participants to read out with you what size triangles you see. On TSP #17 they come in the order: *small, medium, small, large, small, medium, small.*

When all participants can see how you got that sequence, put three rods made out of linker cubes on the overhead projector, atop the template shown on TSP #18 (as shown in the figure to the right). The sequence of sizes is shown at the top of that slide, indicating the order in which you should move the blocks. The tabs that stick out of the end of the rods indicate in which direction each rod moves. So the smallest rod is always moving to the right, and the

medium sized rod is always moving to the left, with wraparound when you get to an edge, of course. Like magic, this algorithm will move the pile to the right post.

B. Put up TSP #19 which shows the solution for a 6-high tower. Ask the class to read off with you the sequence that results when the black triangles are read off along the bottom row. (You can number the triangles, instead of coming up with 6 different adjectives!) The sequence is ***121312141213121512131214121312161213121412131215121312141213121*****. Explain to the class that you will use only a portion of it to solve a Tower of Hanoi problem of height 4 (and you can cover up all but the appropriate lower-left portion of the triangle.) Write the portion "*****121312141213121*****" on a slide.**

In order to demonstrate another way to use this idea in the classroom, take 4 books of different sizes and stack them on a desk where everybody can see them. Arrange them so that the spines alternate directions (left, right, left, right) indicating in which direction each book is to move, like the tabs did above.

Invite a participant to come up and read off the solution, while moving the books on the table in the proper directions.

This worked wonderfully the first time it was done. The participants were impressed, and seemed grateful for the excellent way to solve this puzzle.

Activity #4 — Coin Flipping and the Binomial Distribution
(Allocated time = 30 minutes)

A. Distribute HO #5 (= TSP #20) which asks the participants to determine the number of ways to get exactly *a* heads when you flip 5 coins, for *a* from 0 to 5. You can fill in the 0 and 5 columns with them, and probably the 1 column as well, so that they see what you're looking for. In the bottom row, they should indicate the total number of ways to get each number of heads.

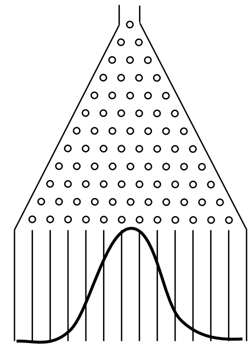
You will probably need to be quite vigilant as the participants do this activity. Experience has indicated that not all participants remember how to list systematically as well as they might have. Walking around and encouraging will probably be necessary.

Regroup and put the results on TSP #20. Go over the questions on the bottom of that slide with the group, showing first that these numbers are choose numbers, and are therefore in Pascal's Triangle, and second, that the probability can be gotten by dividing 10 by the sum of all the numbers (1+5+10+10+5+1). You can also use the fact that we had discussed earlier, that the sum of the entries in any row was a power of 2, thus making it easier to find the probabilities.

To reinforce these ideas, make up some more coin-tossing problems and ask the participants to solve them.

Some problems might be:

- *If you toss a coin 5 times, what is the probability that no more than 2 heads appear?*
- *If you toss a coin 8 times, what is the probability that you get 1, 2 or 3 heads?*
- *If you toss a coin 6 times, what is the probability of getting an odd number of heads?*



B. Put up TSP #21 and ask the participants to consider the question at the bottom of the slide. It is likely that someone will suggest the entries in the 12th row of Pascal's Triangle. To explain why this is indeed the answer, you may wish to point out that as a ball falls down the pegboard, it hits 12 pegs, and at each peg it has a 50-50 chance of going in each direction. This is just like flipping a coin 12 times. Dropping all those balls is like flipping 4096 coins 12 times.

In the Boston Museum of Science, there is a math section which has a huge pegboard along the lines shown here. Thousands of marbles are dropped into it, and you can just stand back and watch as they plunk their way down the board, and into the collectors. And sure enough, as if they didn't have a choice, they form a beautiful Bell curve. Of course, this doesn't ever have to happen, since it is conceivable that you could have many more balls fall to the right than to the left. But it never does!

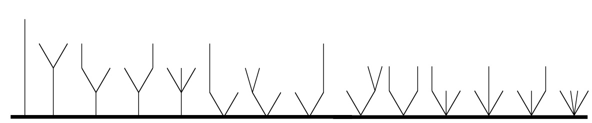
Plot the numbers in the 12th row of Pascal's Triangle using the collectors as a sort of "graph paper." They will probably be able to identify this as the famous "Bell Curve" or "Normal Distribution." It seems that these are words with which the participants are familiar, but about which they don't have anything mathematically concrete to say. So tell them that the bar graph that you draw with 13 collectors very roughly approximates this Normal distribution, but as you increase the number of collectors, the approximation gets better and better. That is, the rows of Pascal's Triangle, when graphed, give better and better approximations to the Bell Curve.

Activity #5 — The Catalan Numbers
(Optional — Use if there is remaining time)

Discuss the problems on TSP #?, and show how the answers can be gotten off of Pascal's Triangle, as shown on TSP #?.

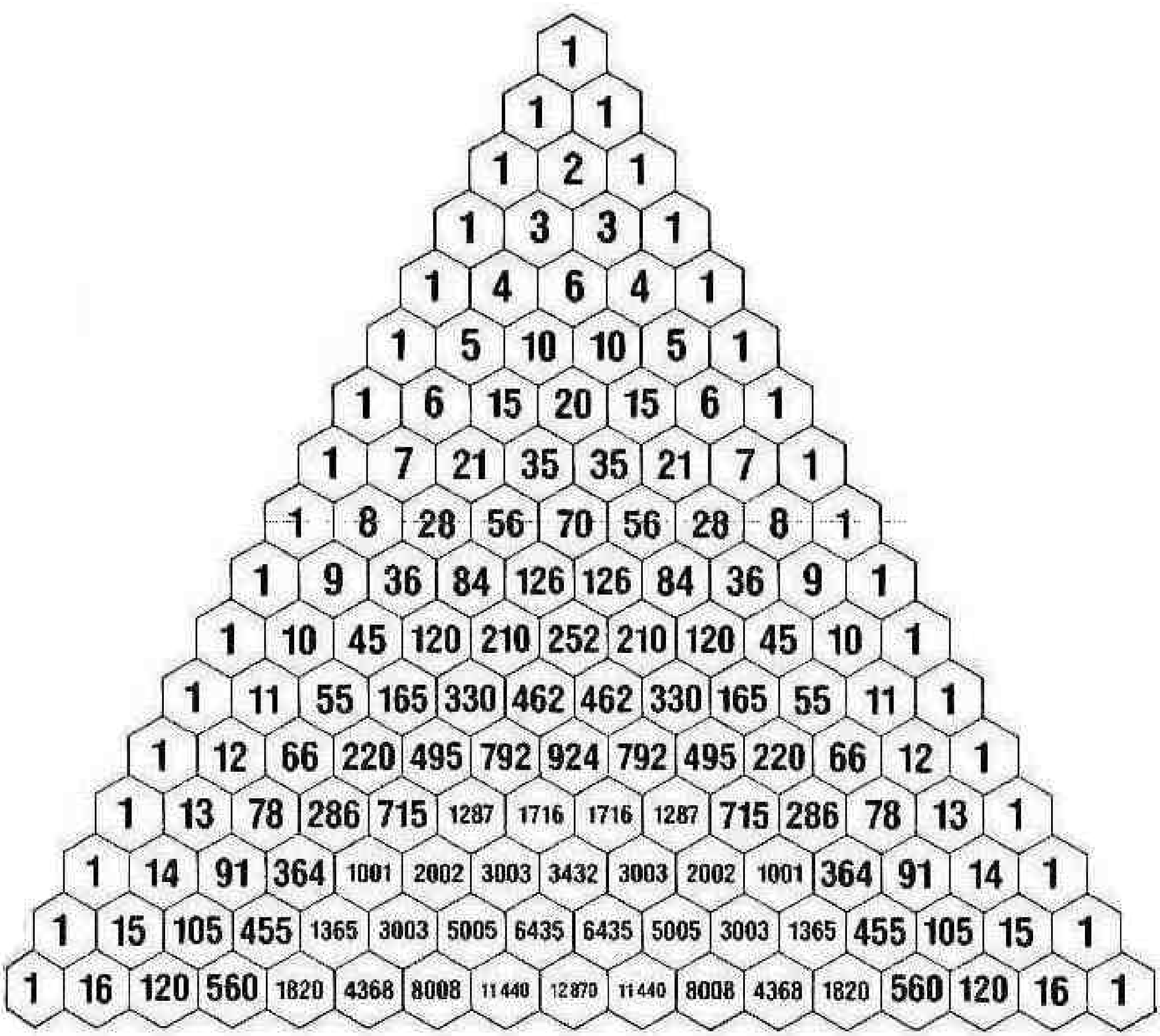
The Catalan numbers, like the Fibonacci numbers, are the solution to a surprisingly large number of problems. Some of these are shown on TSP #?. The problems all have the same answer (14) and are all special cases of whole sequences of problems which all have the same sequence of answers. The first 3 problems are fairly easily seen to be equivalent, but the last two are more difficult.

It is not easy to define what a left-right rooted tree is, so here is a figure which shows the 14 distinct left-right rooted trees with 4 edges.



The formula for the nth Catalan number is $\frac{1}{n+1}\binom{2n}{n}$. *In the previous problems, we were finding the* 4^{th} *Catalan number. The sequence begins 1, 1, 2, 5, 14, 42, ...*

Pascal’s Triangle



From “Visual Patterns in Pascal’s Triangle” by Dale Seymour Publications

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Some Patterns in Pascal's Triangle

- Each number is the sum of the two numbers above it
- The outside numbers are all 1
- The triangle is symmetric
- The first diagonal shows the counting numbers
- The sums of the rows give the powers of 2
- Each row gives the digits of the powers of 11.
- Each entry is an appropriate "choose number."
- The Fibonacci numbers are in there along diagonals.

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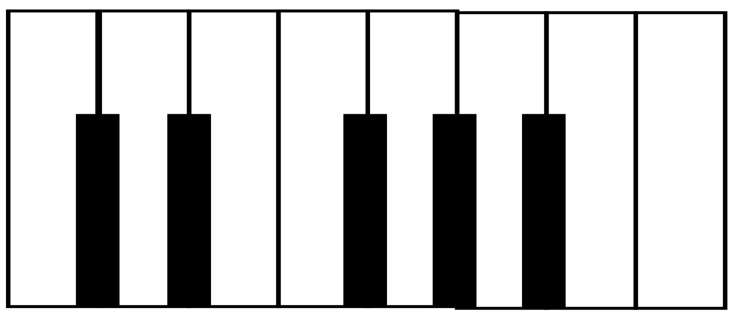
Some Applications of Choose Numbers

From a classroom with 15 boys, how many ways are there to select 4 of them for "Desk Duty?"

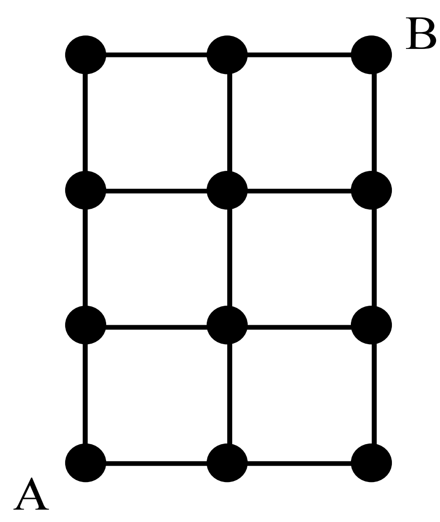
How many 3-note chords are possible within a given octave?



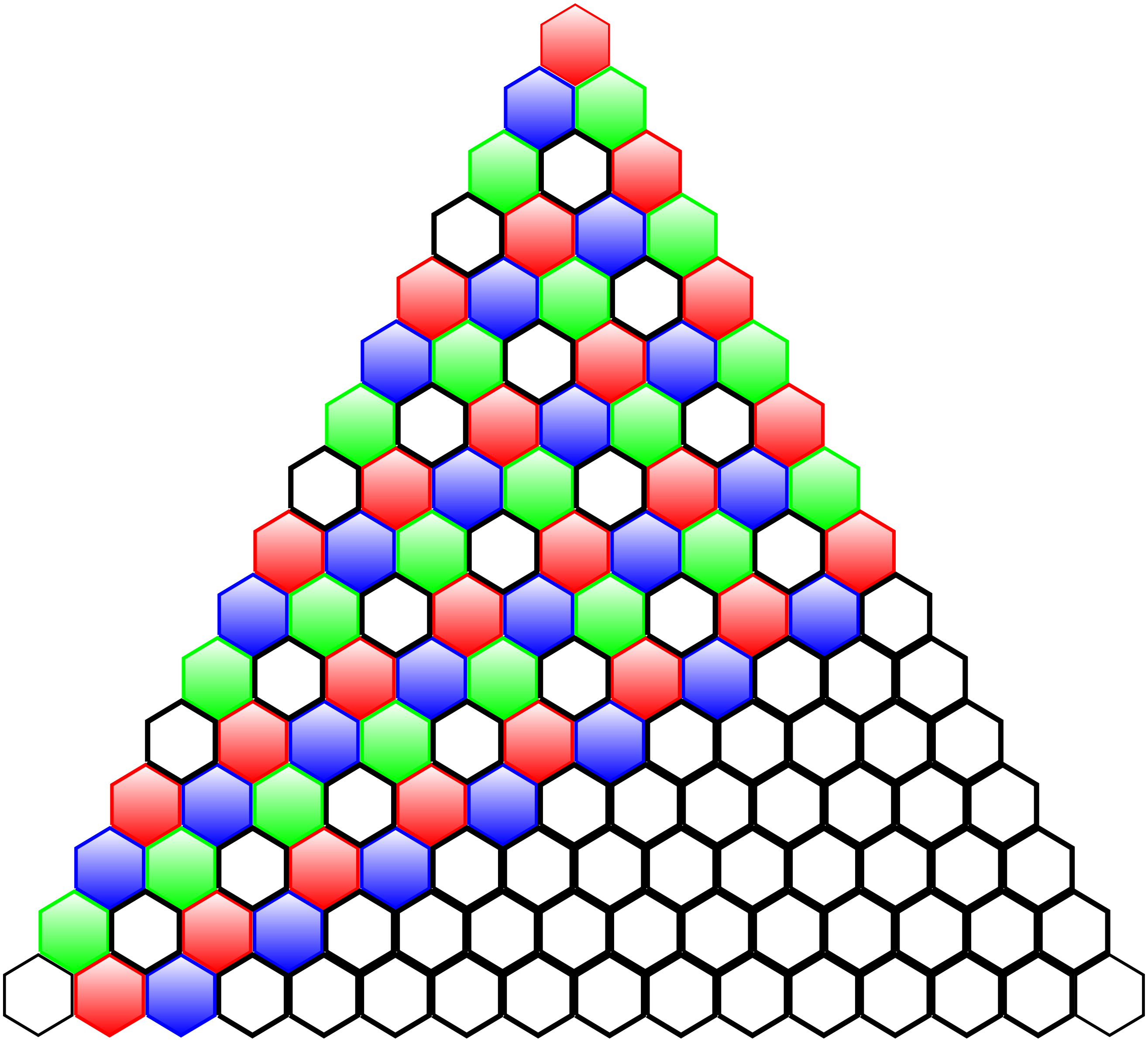
With 7 flavors available, how many ways are there to select 3 different scoops for a sundae?

If I toss 9 coins all at once, what are the chances that exactly 3 of them come up heads?

How many ways are there to walk from A to B on the grid, moving only North and East?



Fibonacci Numbers in Pascal's Triangle



Handout #2 — The Fibonacci Numbers

The Fibonacci numbers are the numbers in the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...
Each number in the sequence is the sum of the two previous numbers.

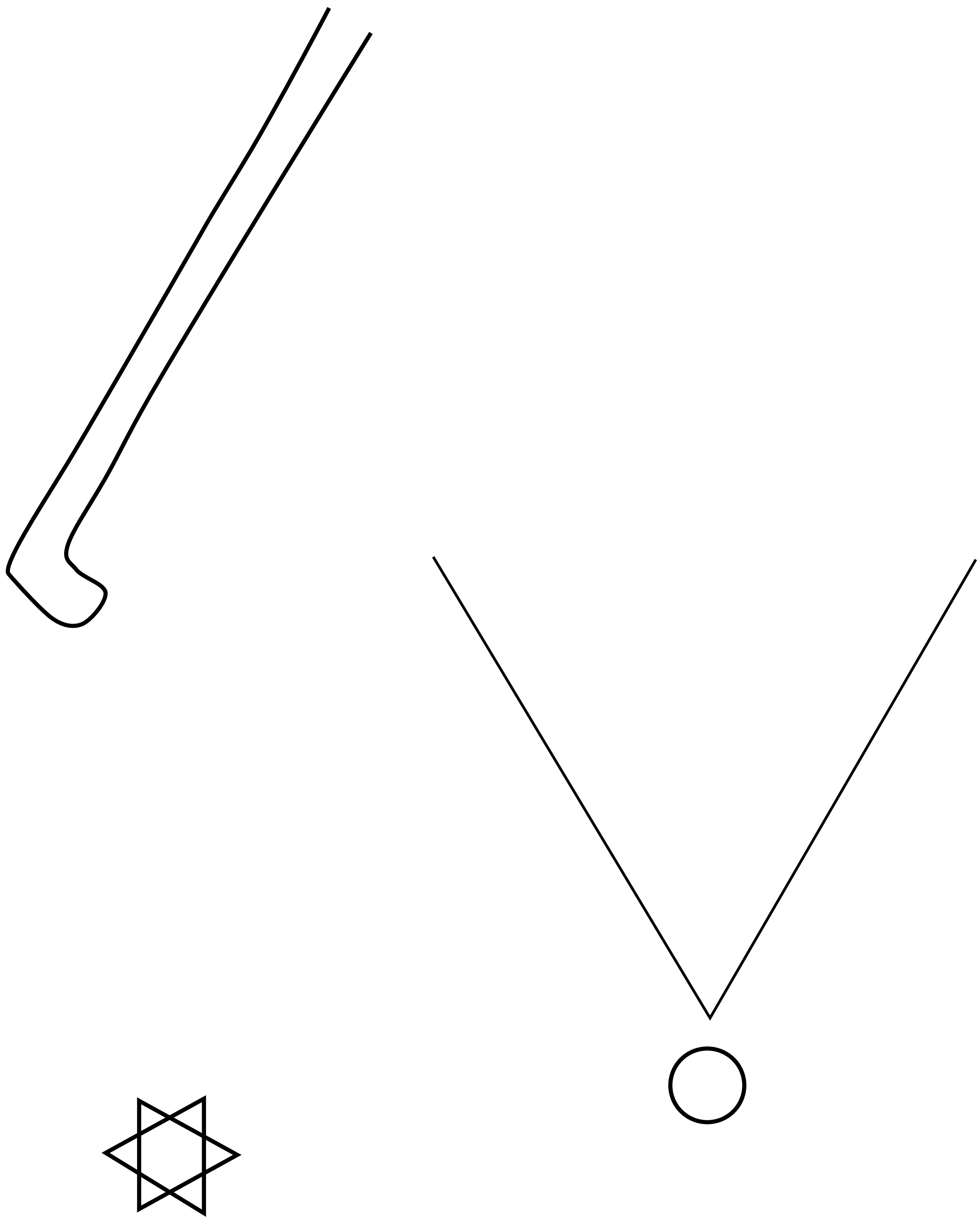
The Fibonacci numbers tell you how many ways a given number can be expressed as the sum of 1's and/or 2's. For example, the 4th Fibonacci number is 5, because there are 5 ways to represent 4 as the sum of 1's and/or 2's: 4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2.

[Note that the "zeroth" Fibonacci number is 1, as is the first Fibonacci number.]

Here we will count the number of ways to write 6 as the sum of 1's and/or 2's. Fill in the chart below:

Write 6 as the sum of 1's and 2's, using this many 2's	**These are the ways**	**This is the *number* of ways**
0		
1		
2		
3		

The Arithmetic of Pascal's Triangle



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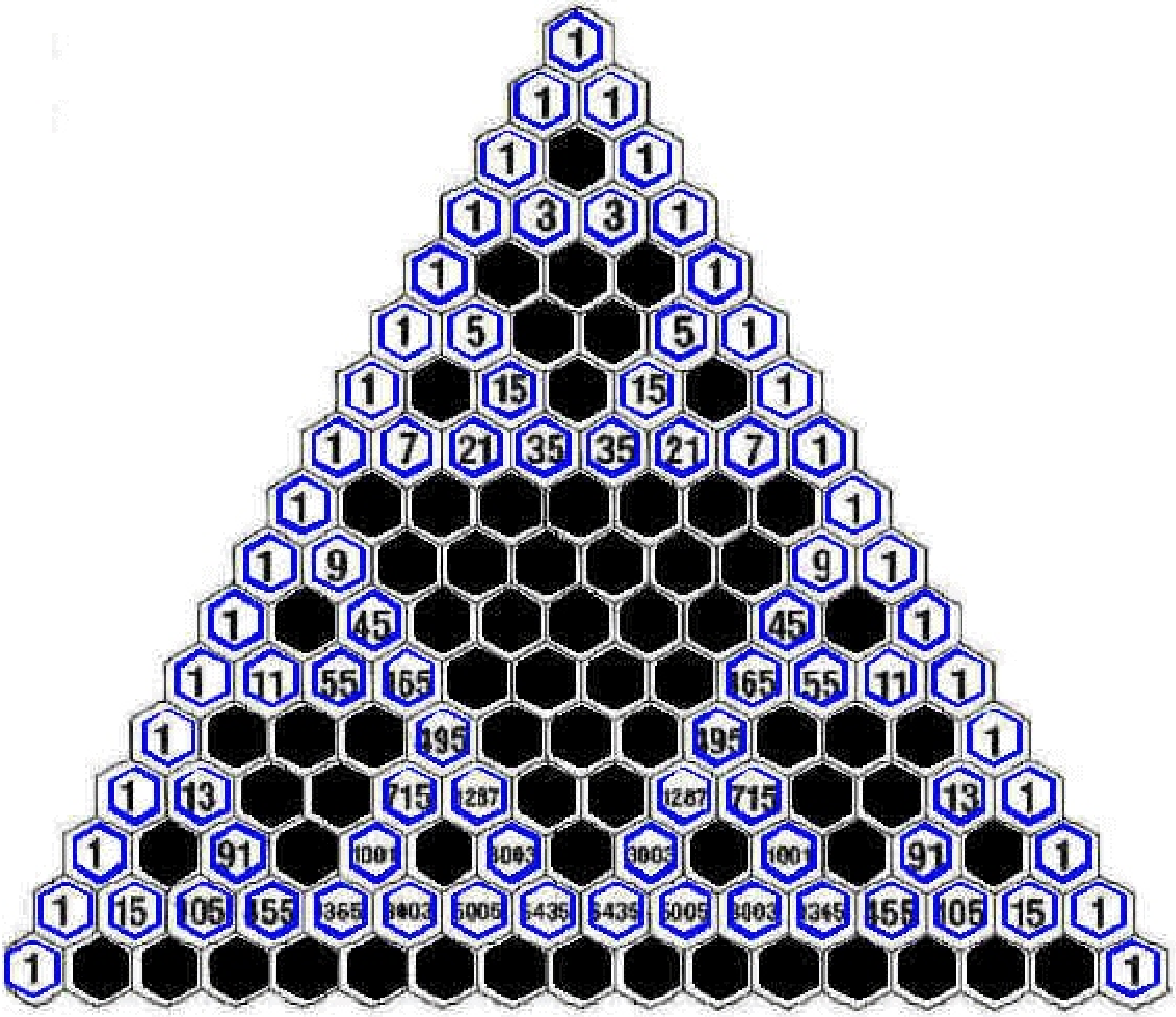
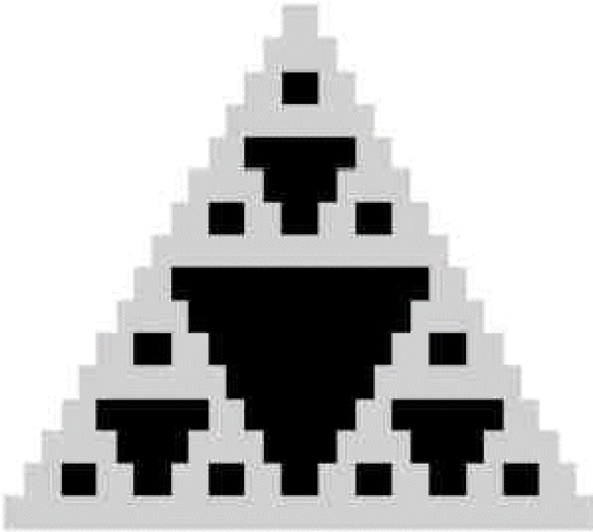
A Hidden Pattern in Pascal's Triangle



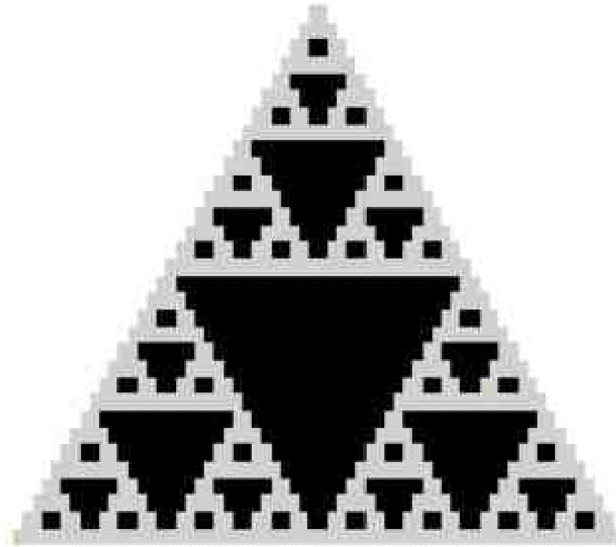
If we color all the entries which are divisible by 2, black, and all the entries which are not divisible by 2, blue, then we get this Sierpinski-like coloring.

This is called the "Mod-2" coloring

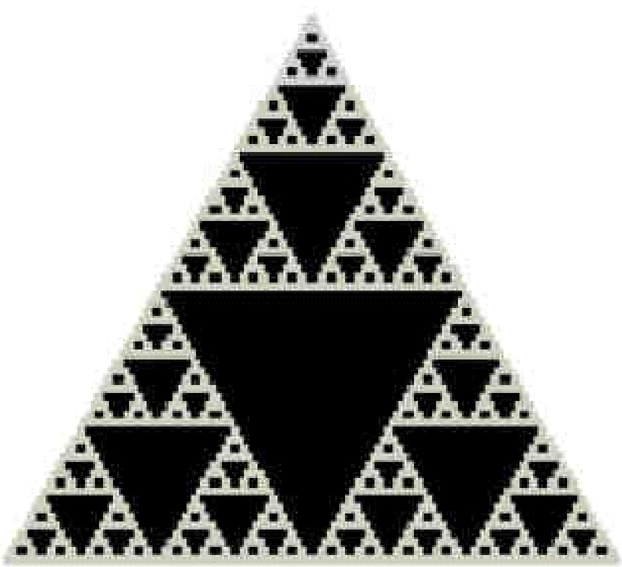
Mod-2 Coloring — First 128 Rows



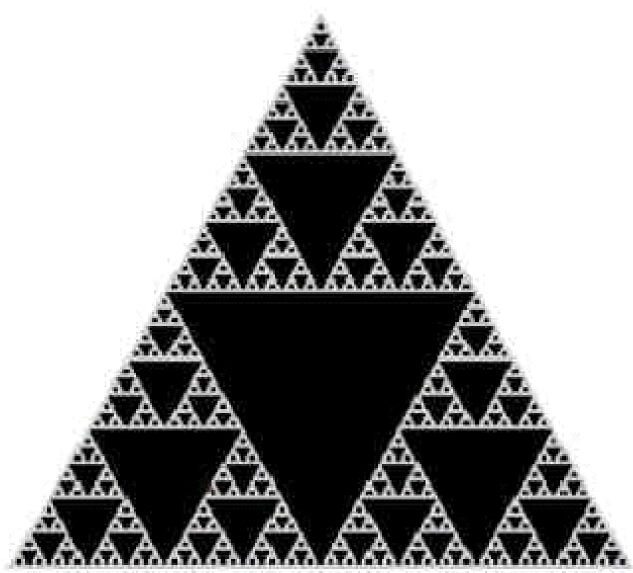
16 Rows



32 Rows



64 Rows



128 Rows

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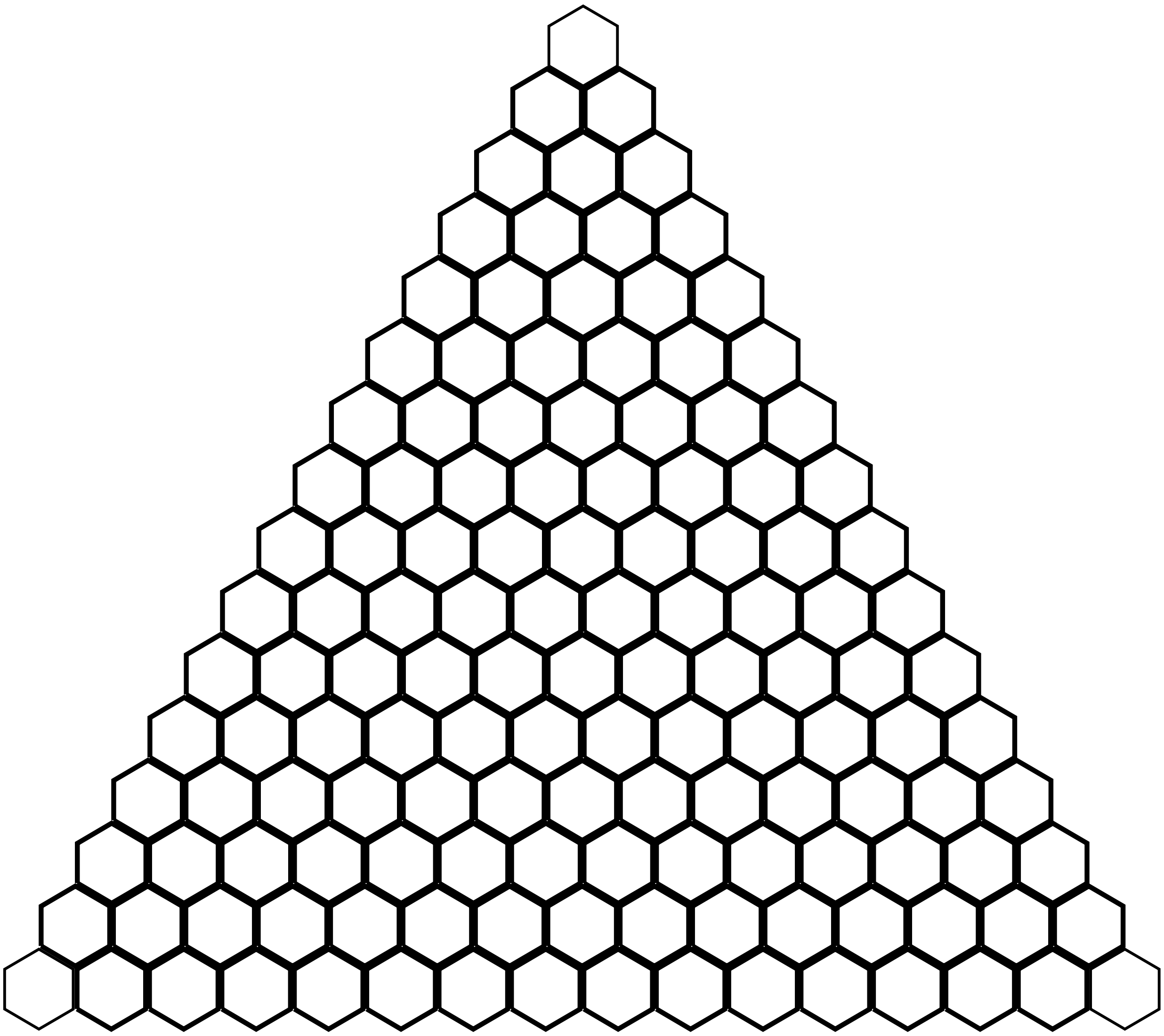
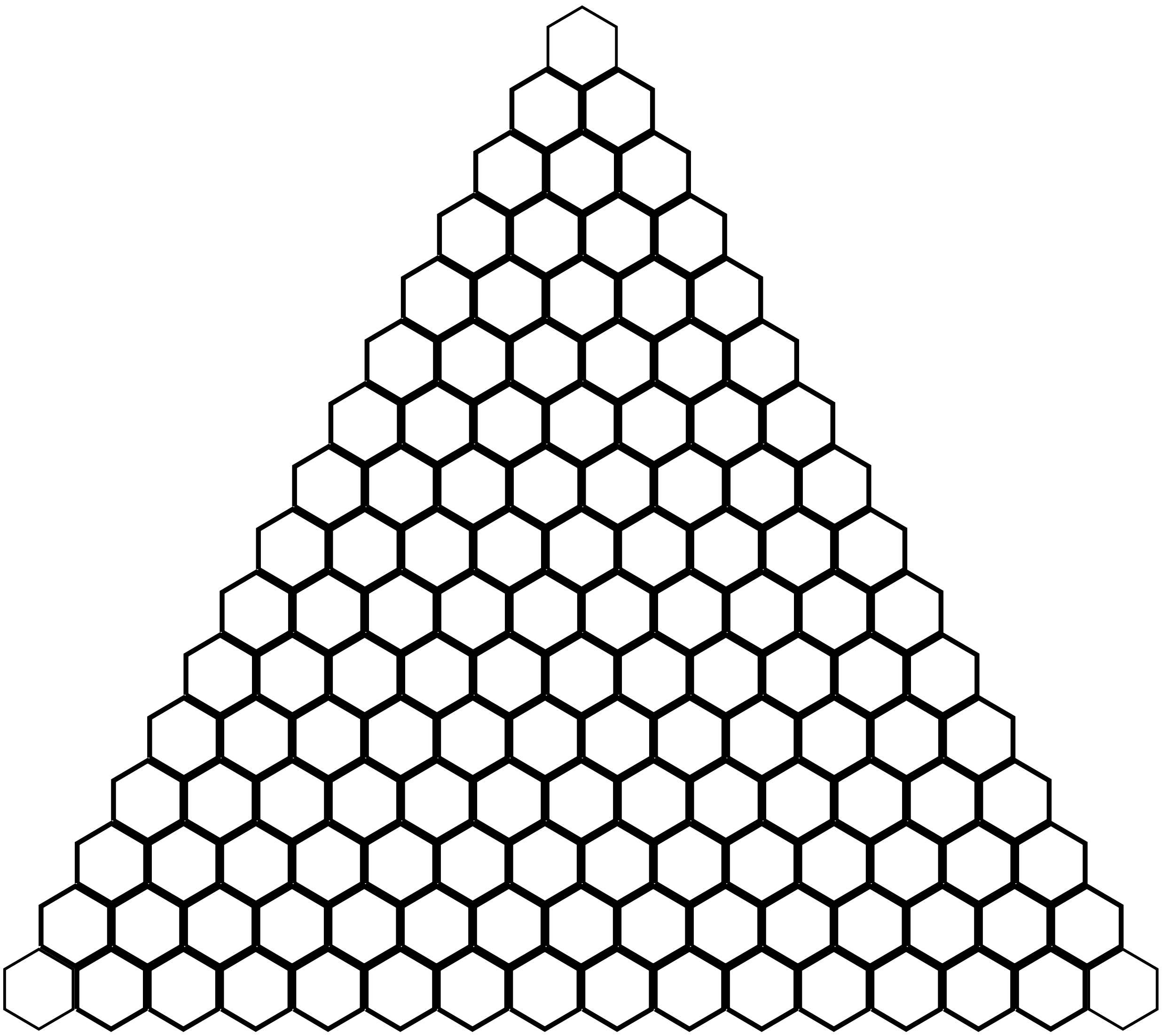
Coloring Mod 3 — Divisibility Only



Coloring Mod 3 — With Remainder



An Easier Way to Color Modularly

To find the remainder when you make a sum,

you can just find the remainders of the things you're adding...and add *them*!

+ mod 3	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

For example: If you add the entries "5" and "10," you get the sum "15." The remainder of 15, mod 3, is 0.

Alternately, you could look at the remainders of "5" and "10," namely "2" and "1" respectively, and add them. 2 + 1 = 3, and the remainder of "3" mod 3 is 0.

So we get the same answer two different ways!

Just Add Colors

Since each remainder was assigned a color

+ mod 3	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

we can forget about the numbers and remainders, and just “add” the colors themselves!

“+” mod 3			

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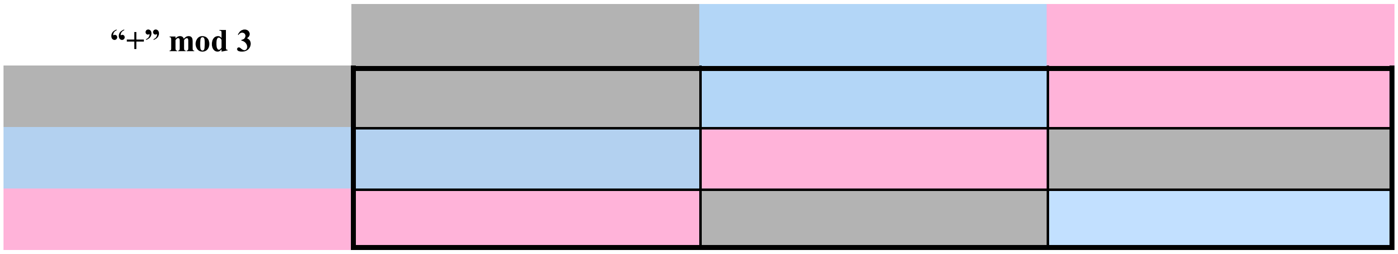
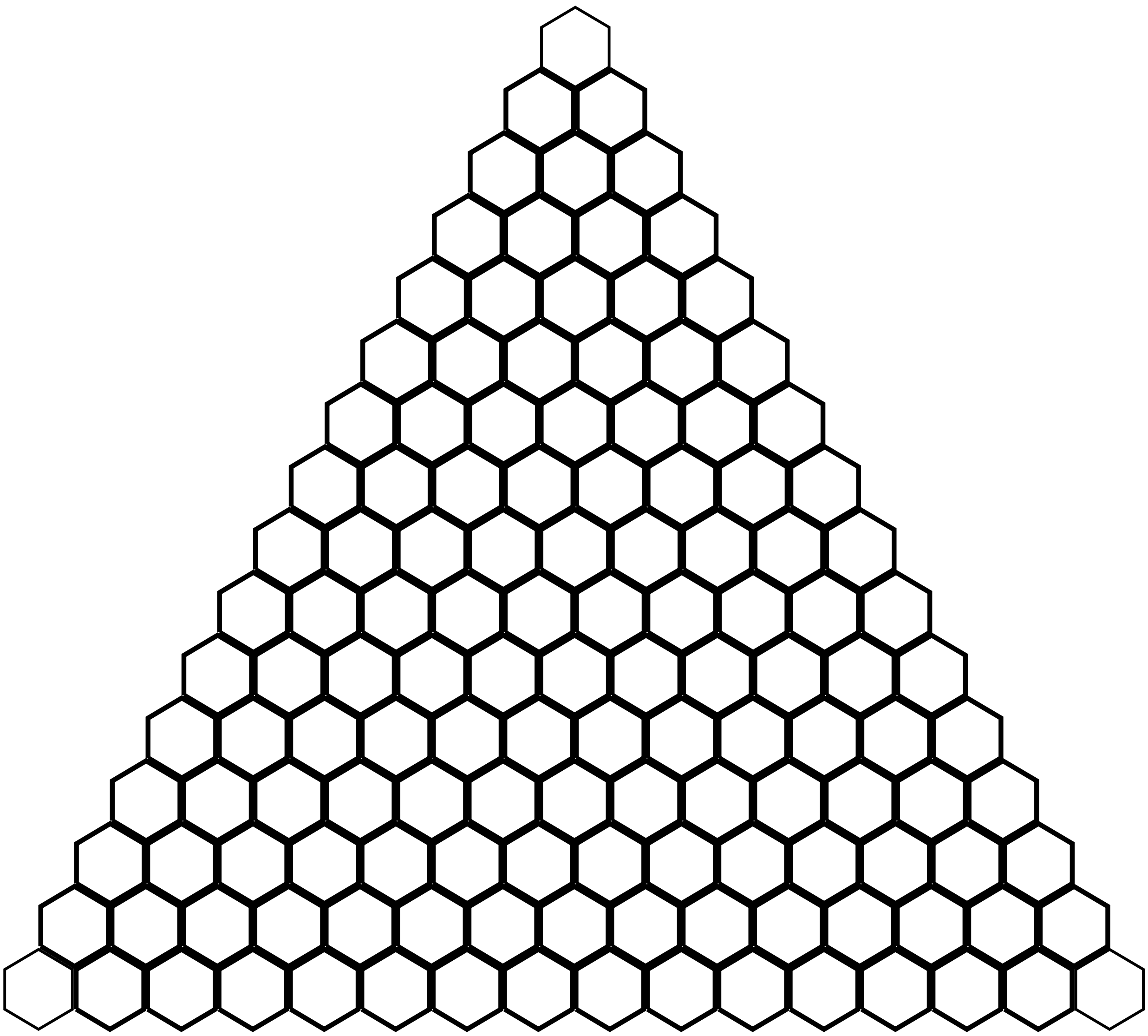
Adding Just Colors





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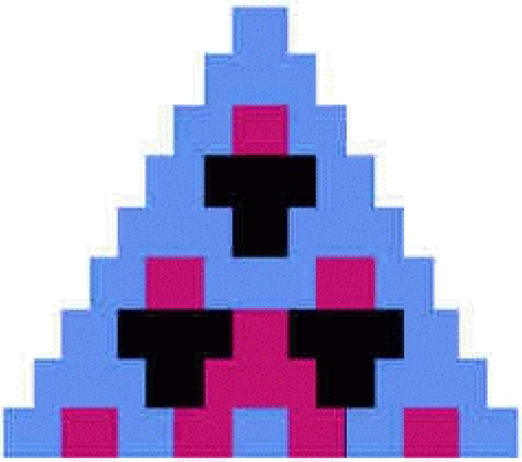
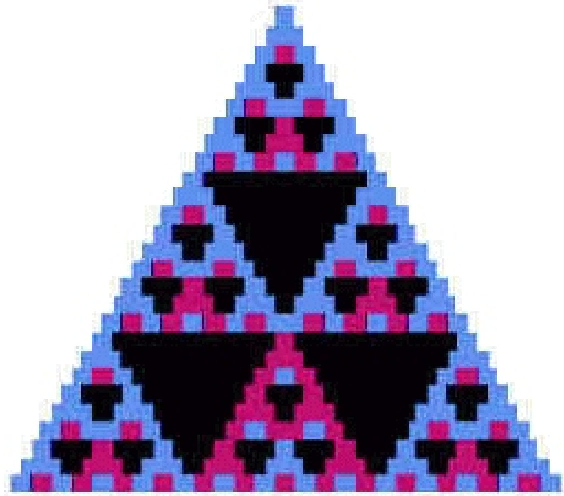
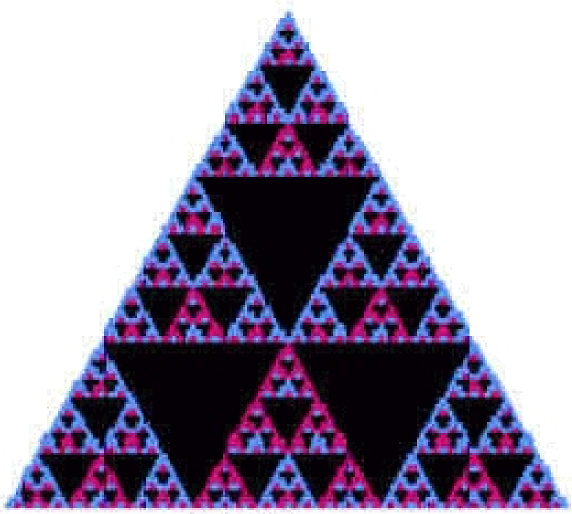
First 243 Rows with Remainder Mod 3



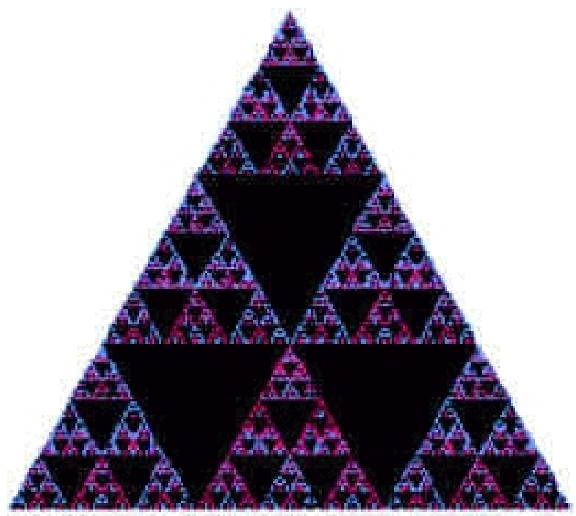
9 Rows



27 Rows

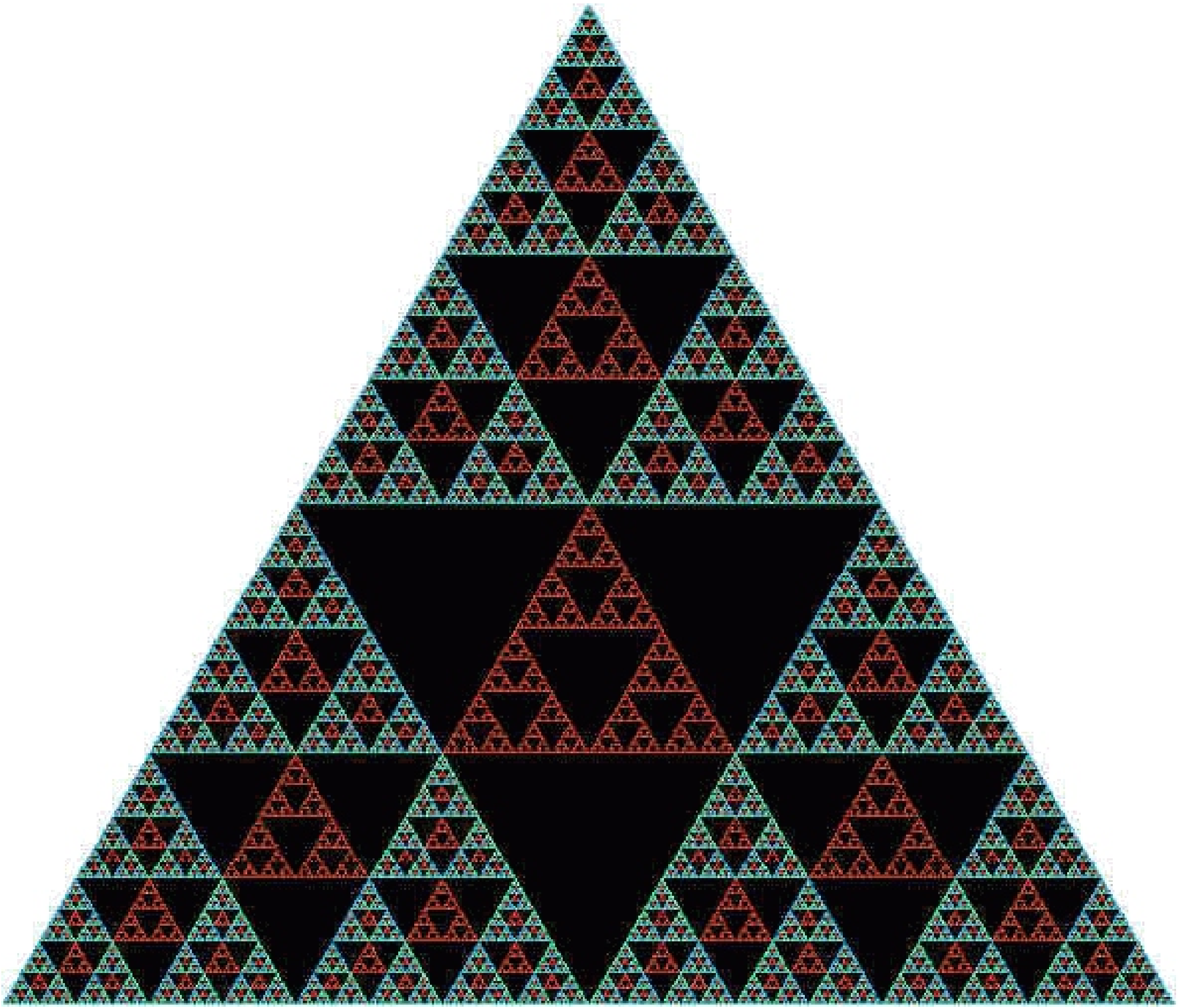


81 Rows

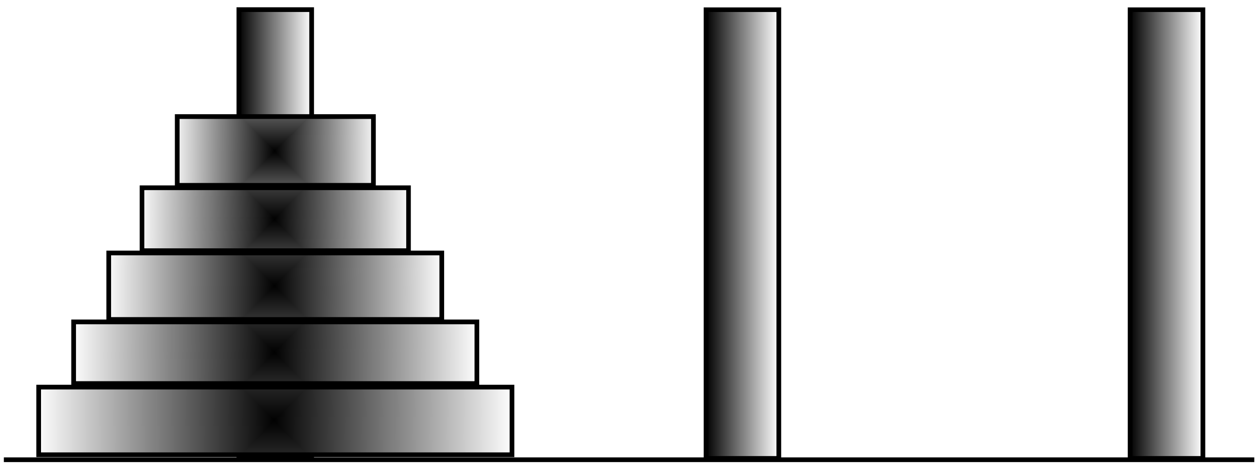


243 Rows

A Mod 4 Coloring of 125 Rows



The Tower of Hanoi Puzzle



A set of different-sized discs are stacked on one of three poles. The stack must be moved to another pole, according to the following rules:

- You may only move one disc at a time.
- During the transfer, you may never put a disc on top of a smaller disc.

How many moves will it take to move all the discs?

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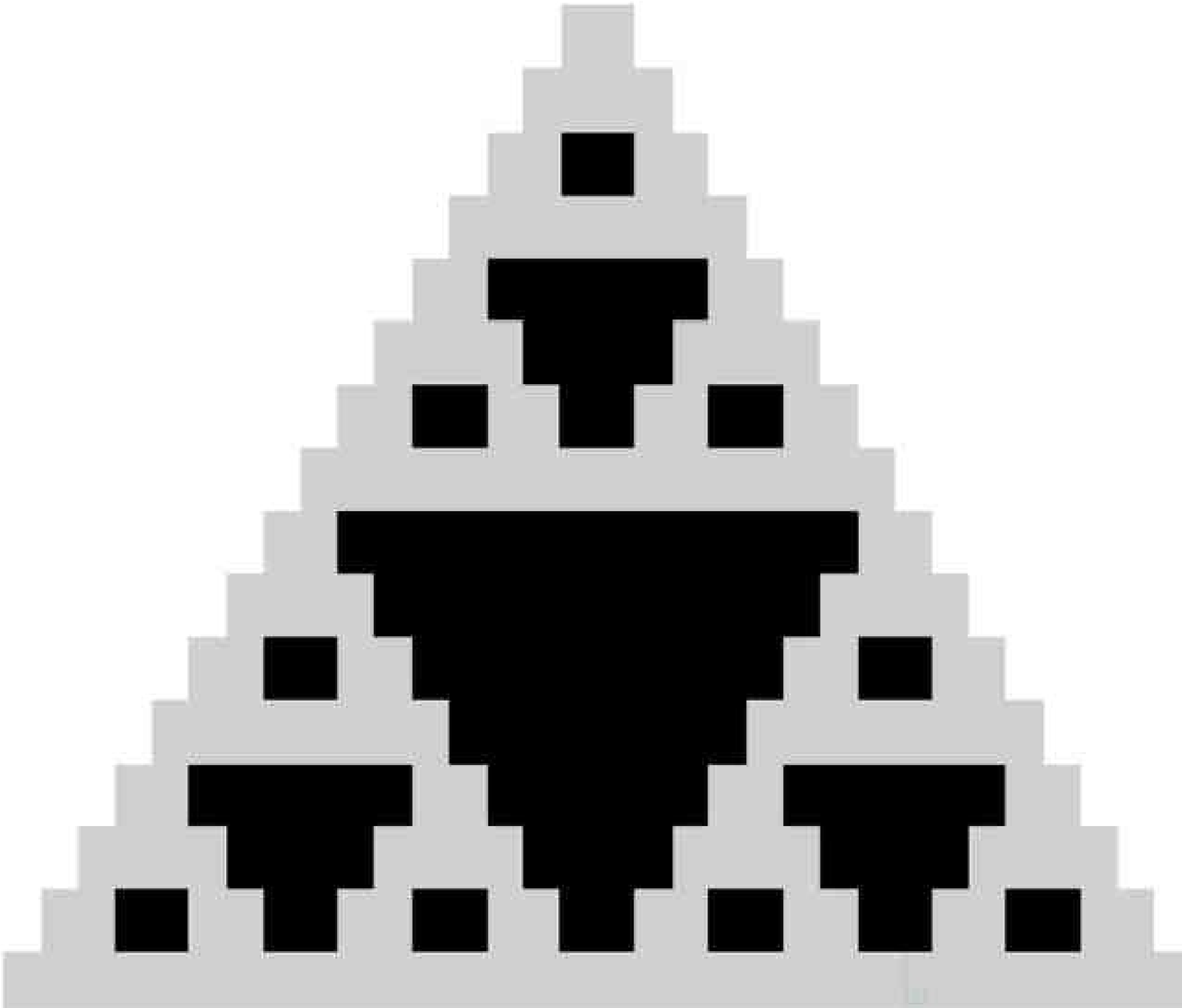
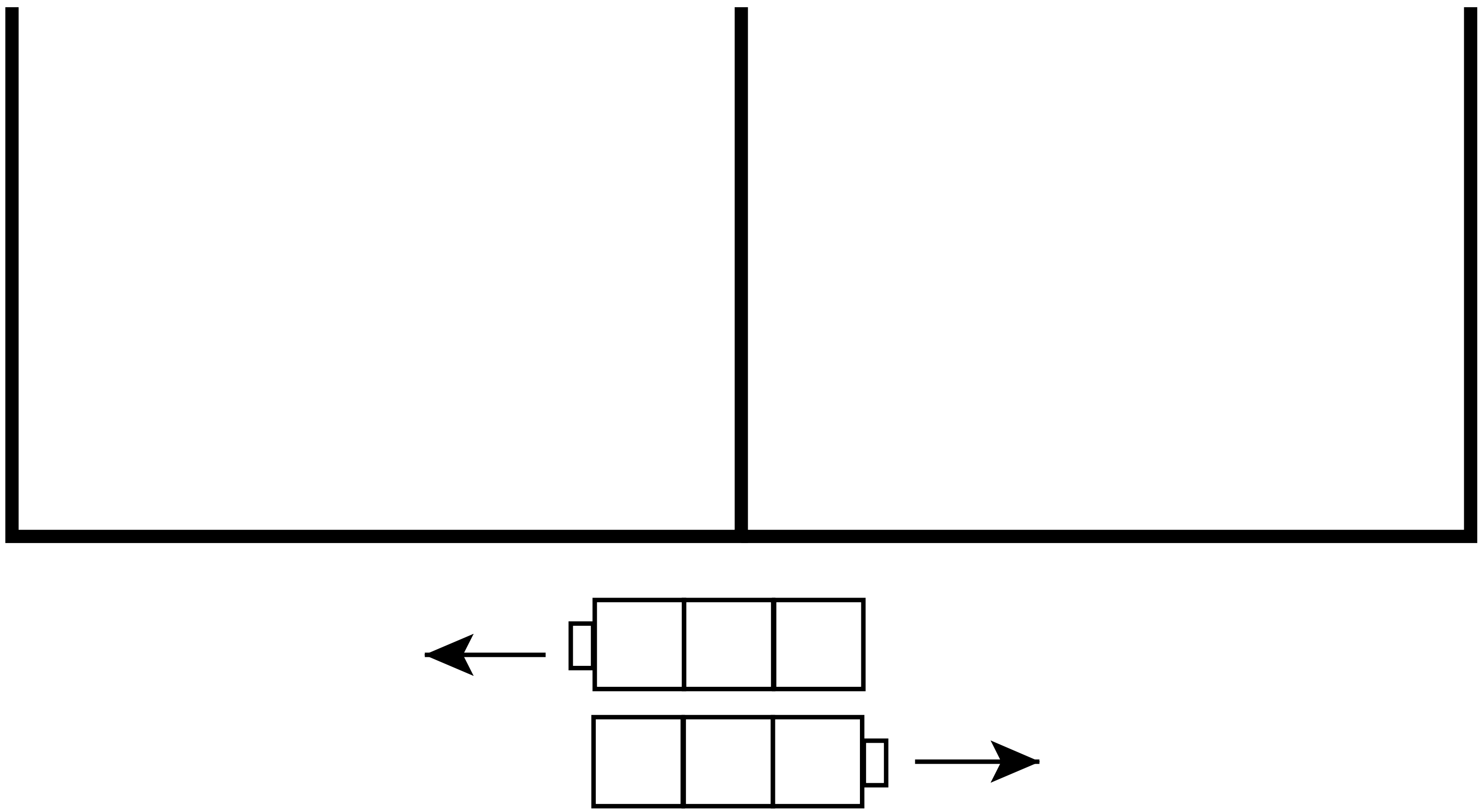
Using Pascal's Triangle to Solve the Hanoi Puzzle



Moving the Blocks

small, medium, small, large, small, medium, small, large



The blocks move in the direction in which the tab is pointing.

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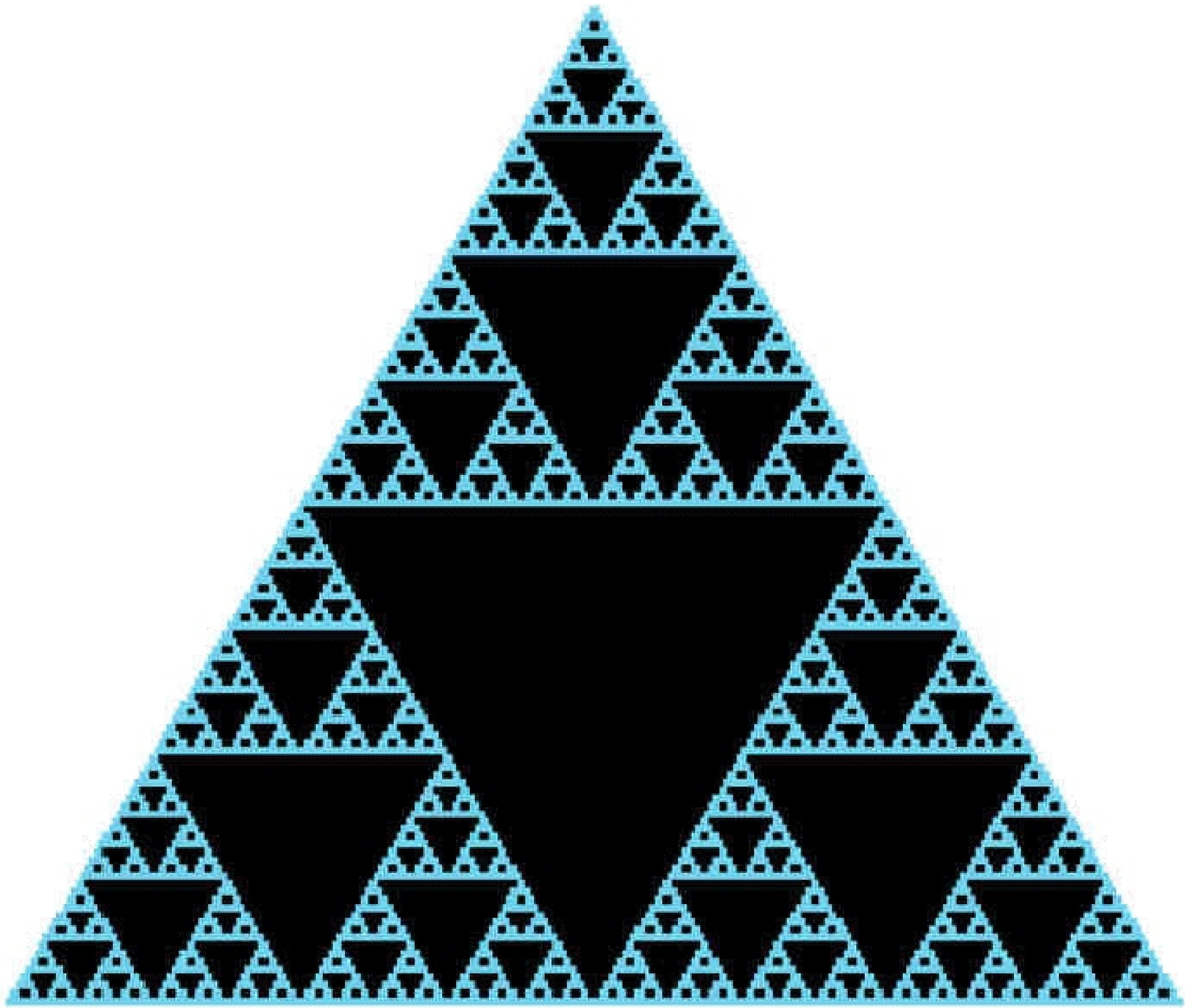
Using Pascal's Triangle to Solve the Hanoi Puzzle



Let's read together the number pattern that the bottom row gives:

Handout #5— Coin Flipping

A coin is flipped 5 times

- How many different outcomes are possible? (You can use the multiplication rule for this.)
- Make a systematic list of the possible outcomes in the chart below, listing systematically according to the number of Heads. In the bottom row, put the total number for each column.

0 Heads	1 Heads	2 Heads	3 Heads	4 Heads	5 Heads

- Why do these numbers appear in Pascal's Triangle?
- What is the probability of flipping a coin 5 times and having 2 heads come up?

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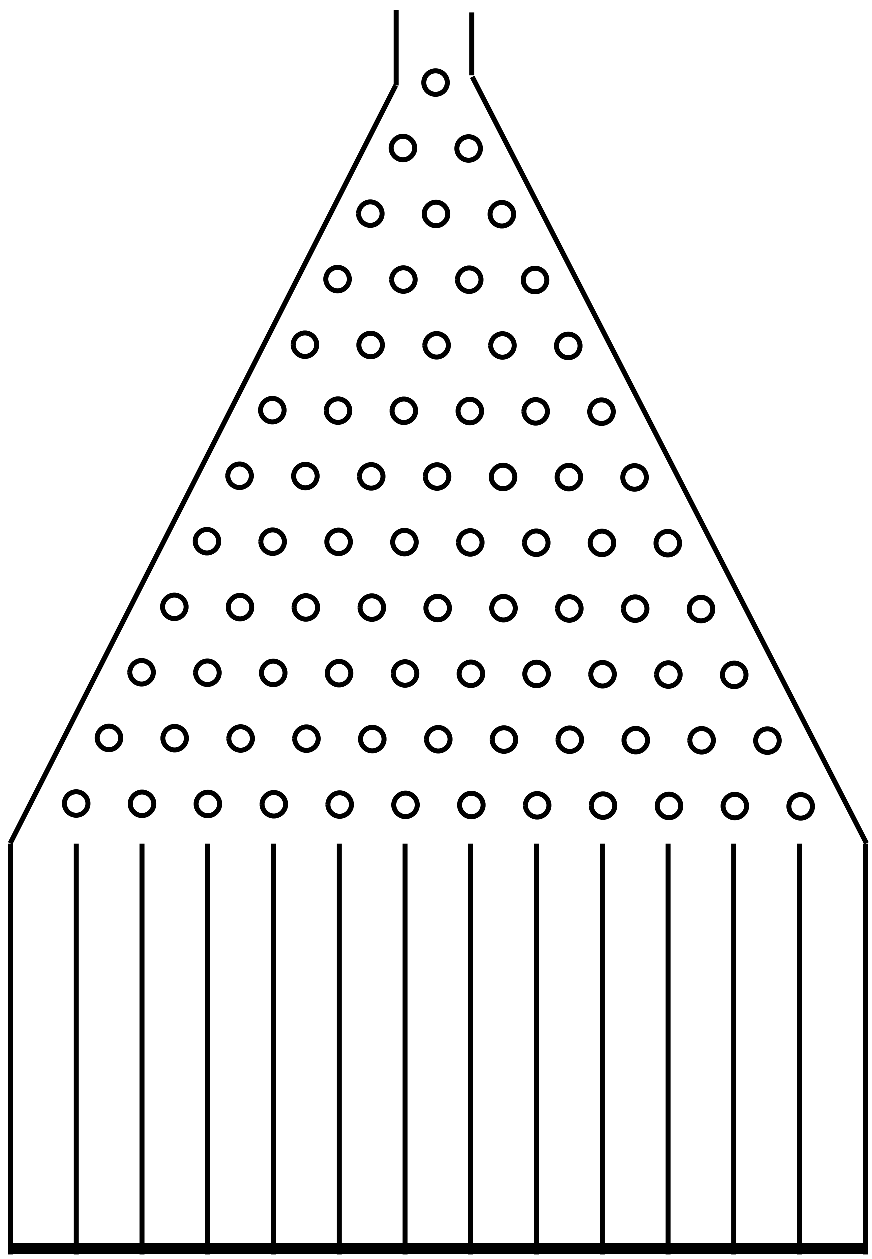
Approximating the Bell Curve
The Binomial Distribution



Suppose we drop 4096 (2^{12}) marbles into the top of this pegboard. How many balls do we expect, on average, to fall into each of the collectors at the bottom?

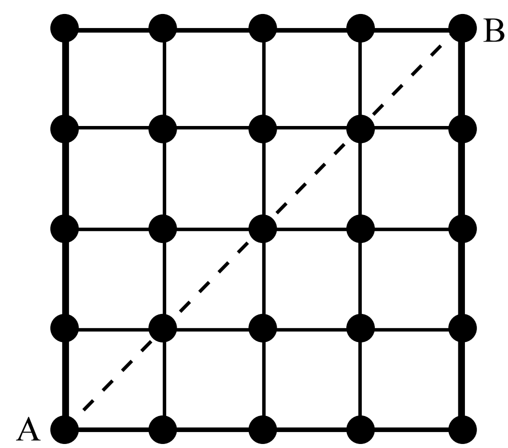
Catalan Numbers

The Catalan numbers are a sequence of numbers that turn up in a variety of surprising places.

- A stewardess selling \$5 headsets starts with no money. She receives 4 each of \$5 and \$10 bills. How many ways can she receive them so that she'll be able to give change?

- In an election, A and B each get 4 committee votes, which come in one at a time. In how many ways can A always be leading B in the election?

- How many ways are there to walk from A to B in the grid to the right without going below the diagonal?



- How many ways are there to diagonalize a hexagon?

- How many left-right rooted trees are there with 4 edges?

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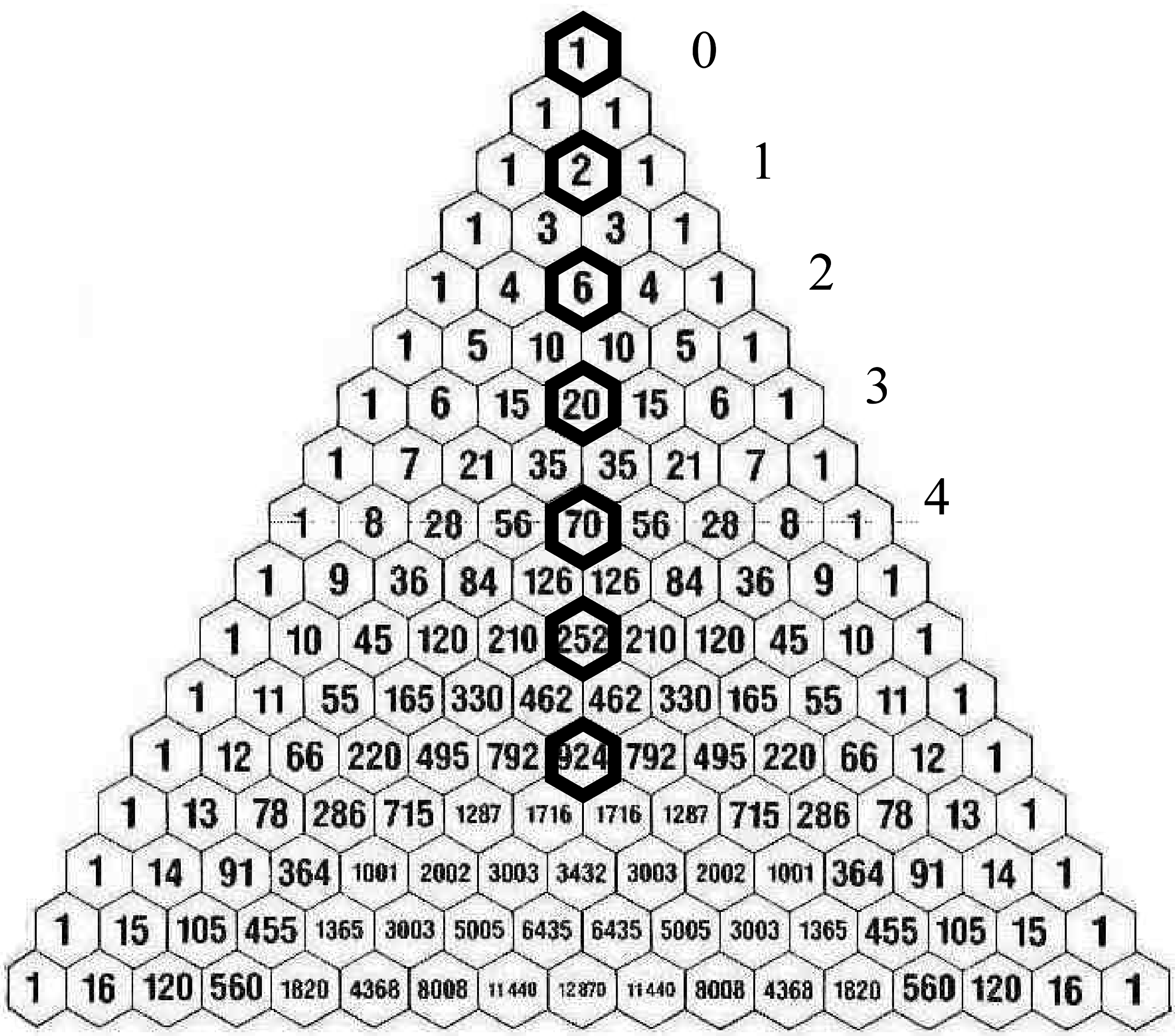
Catalan Numbers

1, 1, 2, 5, 14, 42, 132, 424, ...

The answer to all the previous questions was 14, the 4th Catalan number. And as you change the “4” in each of the problems, you get an appropriate Catalan number.

These can be found in Pascal’s Triangle:



Handout #1 — Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

From "Visual Patterns in Pascal's Triangle" by Dale Seymour Publications

Handout #2 — The Fibonacci Numbers

The Fibonacci numbers are the numbers in the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... Each number in the sequence is the sum of the two previous numbers.

The Fibonacci numbers tell you how many ways a given number can be expressed as the sum of 1's and/or 2's. For example, the 4th Fibonacci number is 5, because there are 5 ways to represent 4 as the sum of 1's and/or 2's: 4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2.

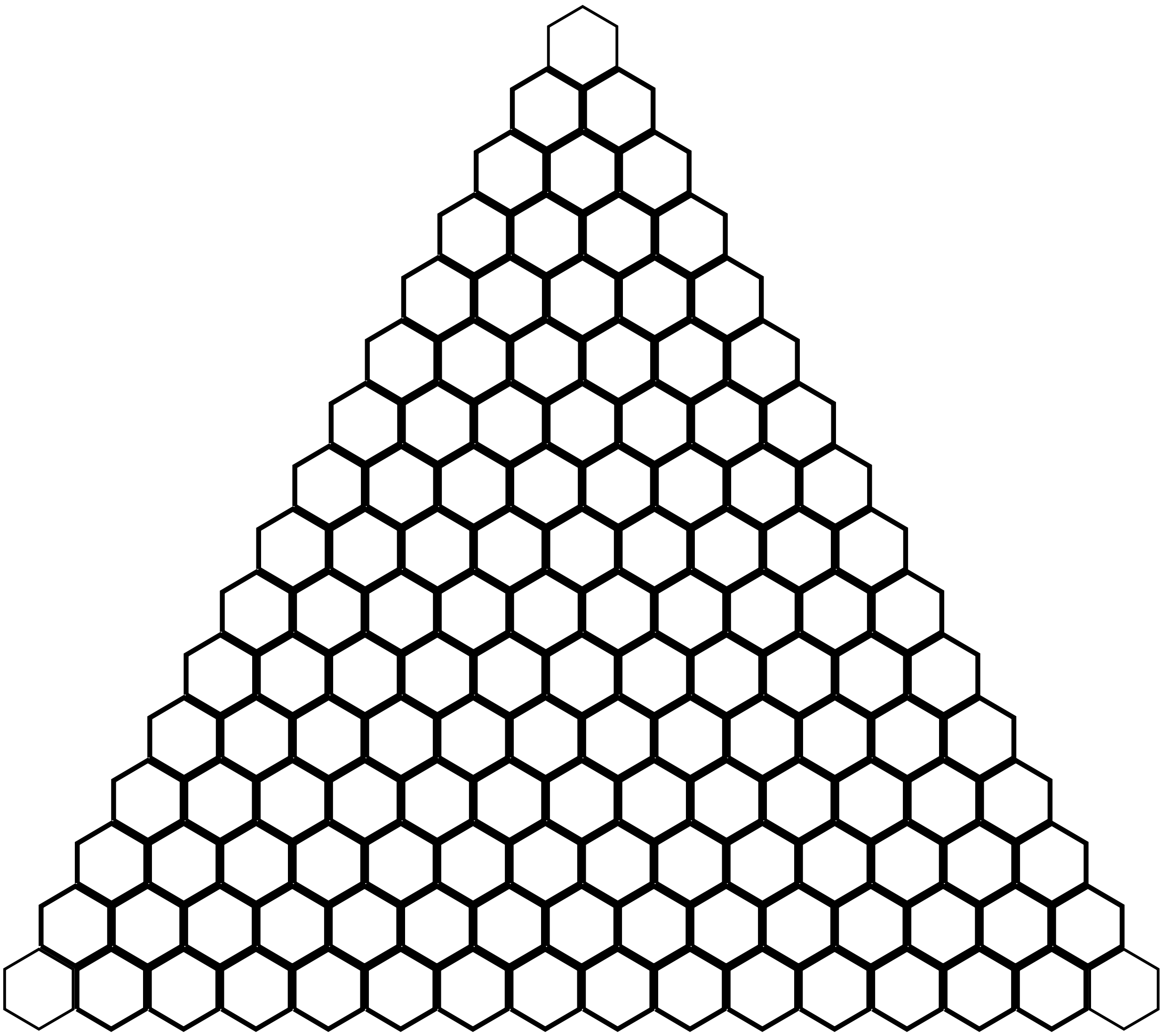
[Note that the "zeroth" Fibonacci number is 1, as is the first Fibonacci number.]

Here we will count the number of ways to write 6 as the sum of 1's and/or 2's. Fill in the chart below:

Write 6 as the sum of 1's and 2's, using this many 2's	**These are the ways**	**This is the *number* of ways**
0		
1		
2		
3		

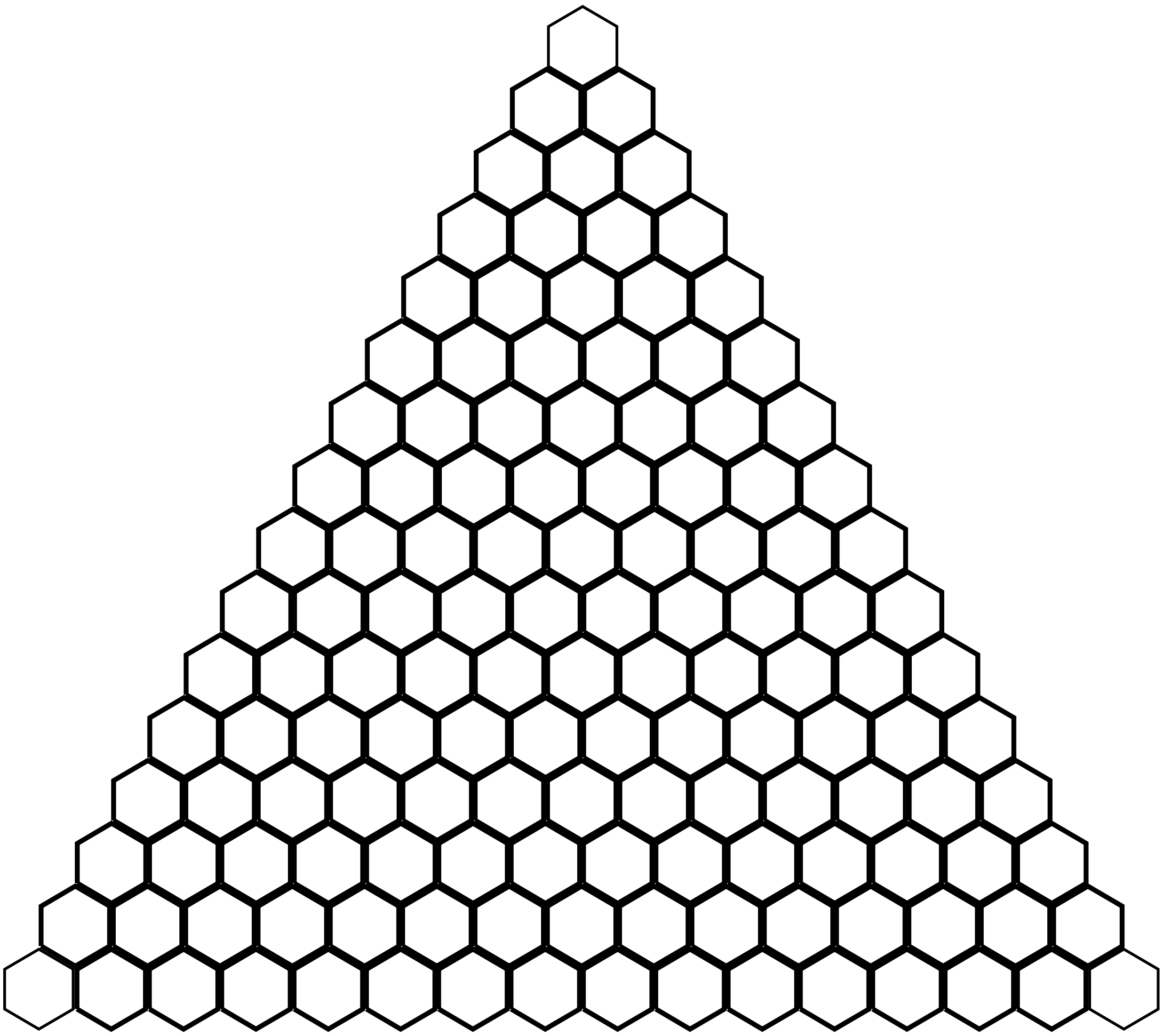
Handout #3 — Coloring the Multiples of 3 in Pascal's Triangle

Fill in the numbers in the first few rows of Pascal's Triangle. Then color all the multiples of 3 with a dark color, and the non-multiples of 3 with a lighter color.



Handout #4 — Coloring Pascal's Triangle by Remainder (mod 3).

Again, color the multiples of 3 with a darker color, but now select *two* other colors; one color for those numbers that leave a remainder of 1 when divided by 3, and the other color for those entries that leave a remainder of 2 when divided by 3.



Copyright 1999 Prepared for use in the Rutgers Leadership Program in Discrete Mathematics Blazing Pascal HO # 4

Handout #5— Coin Flipping

A coin is flipped 5 times.

- How many different outcomes are possible? (You can use the multiplication rule for this.)

- Make a systematic list of the possible outcomes in the chart below, listing systematically according to the number of Heads. In the bottom row, put the total number for each column.

0 Heads	**1 Heads**	**2 Heads**	**3 Heads**	**4 Heads**	**5 Heads**

- Why do these numbers appear in Pascal's Triangle?

- What is the probability of flipping a coin 5 times and having 2 heads come up?

Blazing Pascal

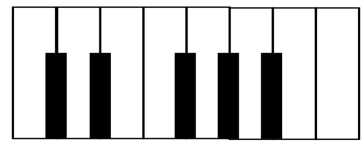
Follow-up session

December 1999

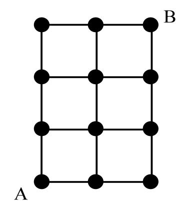
Some Patterns in Pascal's Triangle

- Each number is the sum of the two numbers above it
- The outside numbers are all 1
- The triangle is symmetric
- The first diagonal shows the counting numbers
- The sums of the rows give the powers of 2
- Each row gives the digits of the powers of 11.
- Each entry is an appropriate "choose number."
- The Fibonacci numbers are in there along diagonals.

Some Applications of Choose Numbers



- From a classroom with 15 boys, how many ways are there to select 4 of them for "Desk Duty?"
- How many 3-note chords are possible within a given octave?
- With 7 flavors available, how many ways are there to select 3 different scoops for a sundae?
- If I toss 9 coins all at once, what are the chances that exactly 3 of them come up heads?
- How many ways are there to walk from A to B on the grid, moving only North and East?



An Easier Way to Color Modularly

To find the remainder when you make a sum, you can just find the remainders of the things you're adding...and add *them*!

+ mod 3	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

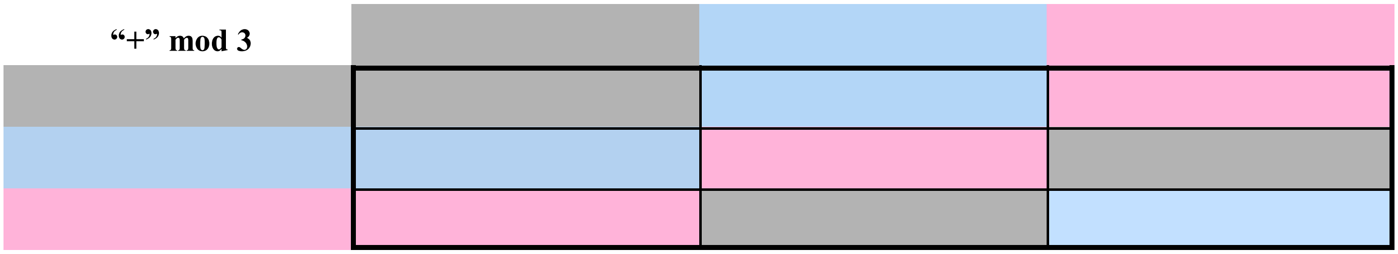
For example: If you add the entries "5" and "10," you get the sum "15." The remainder of 15, mod 3, is 0. Alternately, you could look at the remainders of "5" and "10,"namely "2" and "1" respectively, and add them. 2 + 1 = 3, and the remainder of "3" mod 3 is 0. So we get the same answer two different ways!

Just Add Colors

Since each remainder was assigned a color

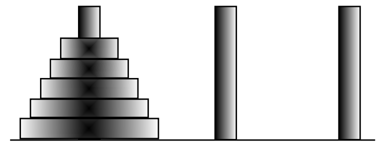
+ mod 3	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

we can forget about the numbers and remainders, and just "add" the colors themselves!



The Tower of Hanoi Puzzle

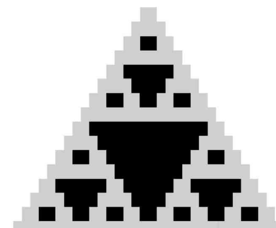
A set of different-sized discs are stacked on one of three poles. The stack must be moved to another pole, according to the following rules:



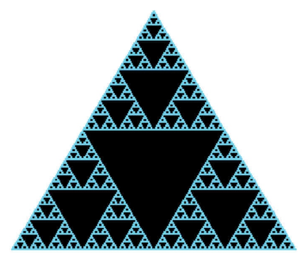
- You may only move one disc at a time.
- During the transfer, you may never put a disc on top of a smaller disc.
- How many moves will it take to move all the discs?

Using Pascal’s Triangle to Solve the Hanoi Puzzle

The first coloring, reading the black triangles along the bottom, yields the solution sequence: 1,2,1,3,1,2,1



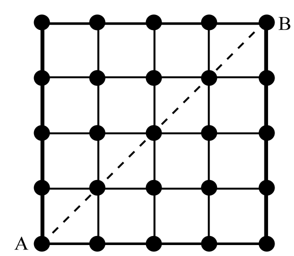
The second coloring, again reading the black triangles along the bottom, yields the solution sequence
1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,6,
1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,

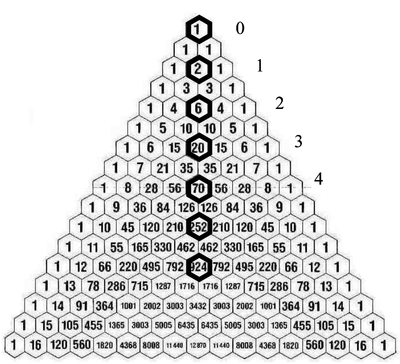


Catalan Numbers

The Catalan numbers are a sequence of numbers that turn up in a variety of surprising places.

- A stewardess selling $5 headsets starts with no money. She receives 4 each of $5 and $10 bills. How many ways can she receive them so that she’ll be able to give change?
- In an election, A and B each get 4 committee votes, which come in one at a time. In how many ways can A always be leading B in the election?
- How many ways are there to walk from A to B in the grid to the right without going below the diagonal?
- How many ways are there to diagonalize a hexagon?
- How many left-right rooted trees are there with 4 edges?

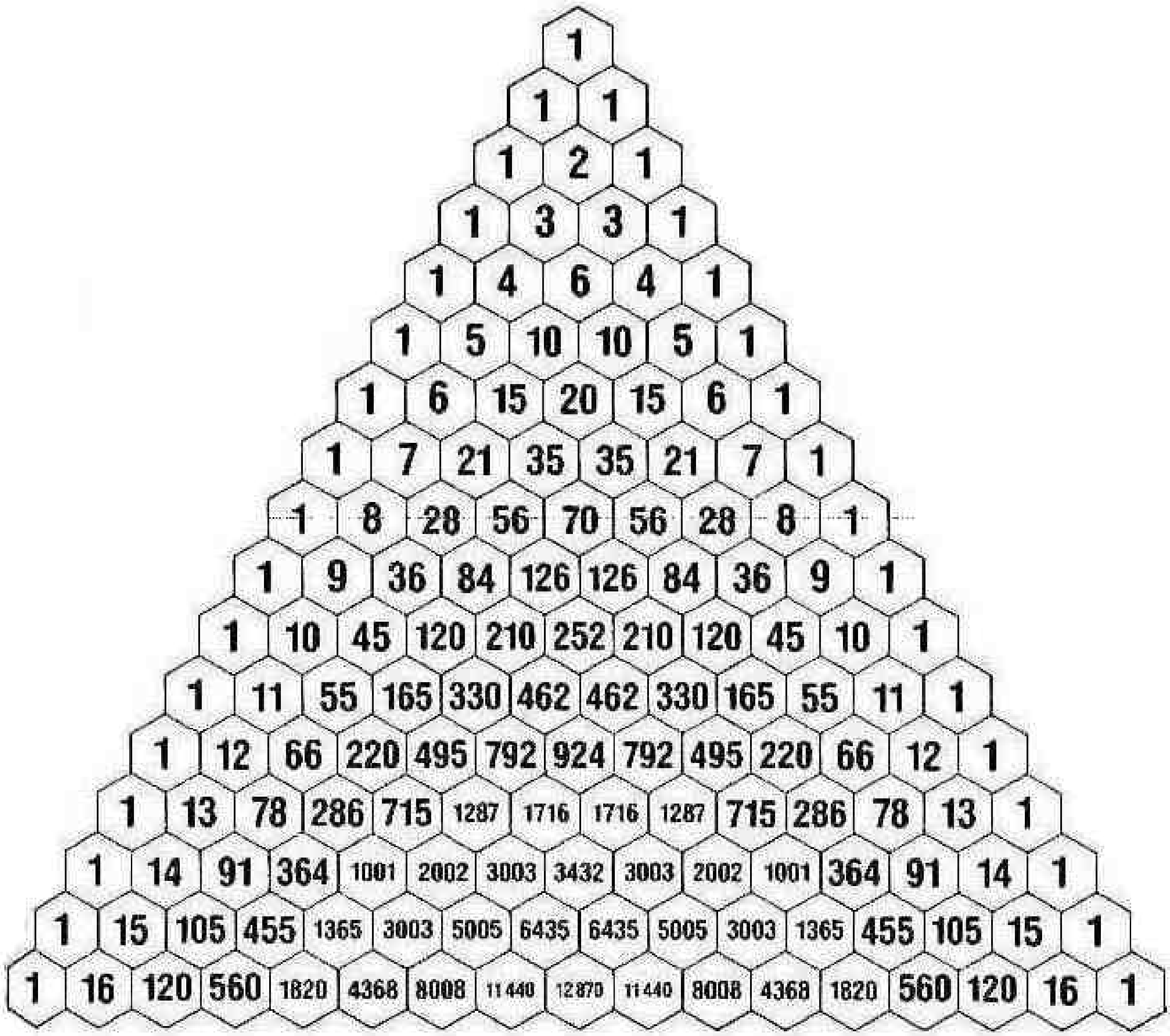




The Sequence of Catalan Numbers

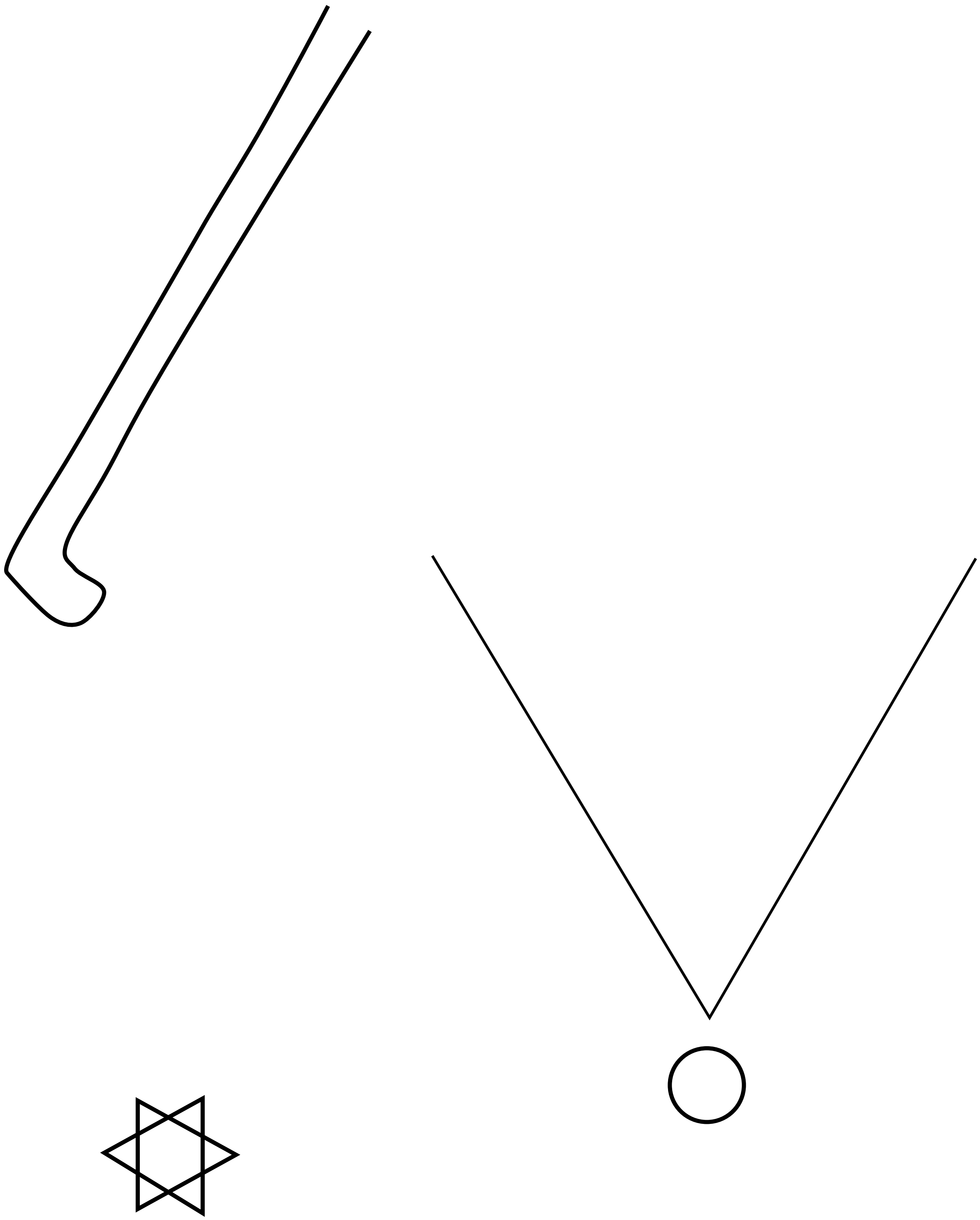
1, 1, 2, 5, 14, 42, 132, 424, ...
The answer to all the previous questions was 14, the 4th Catalan number. And as you change the “4” in each of the problems, you get an appropriate Catalan number. These can be found in Pascal’s Triangle, as shown to the left, by dividing by one more than half the row number. The formula for the nth Catalan number is
C(n) = (2*n* Choose *n*) divided by (*n*+1).

Pascal's Triangle



From "Visual Patterns in Pascal's Triangle" by Dale Seymour Publications

The Arithmetic of Pascal's Triangle



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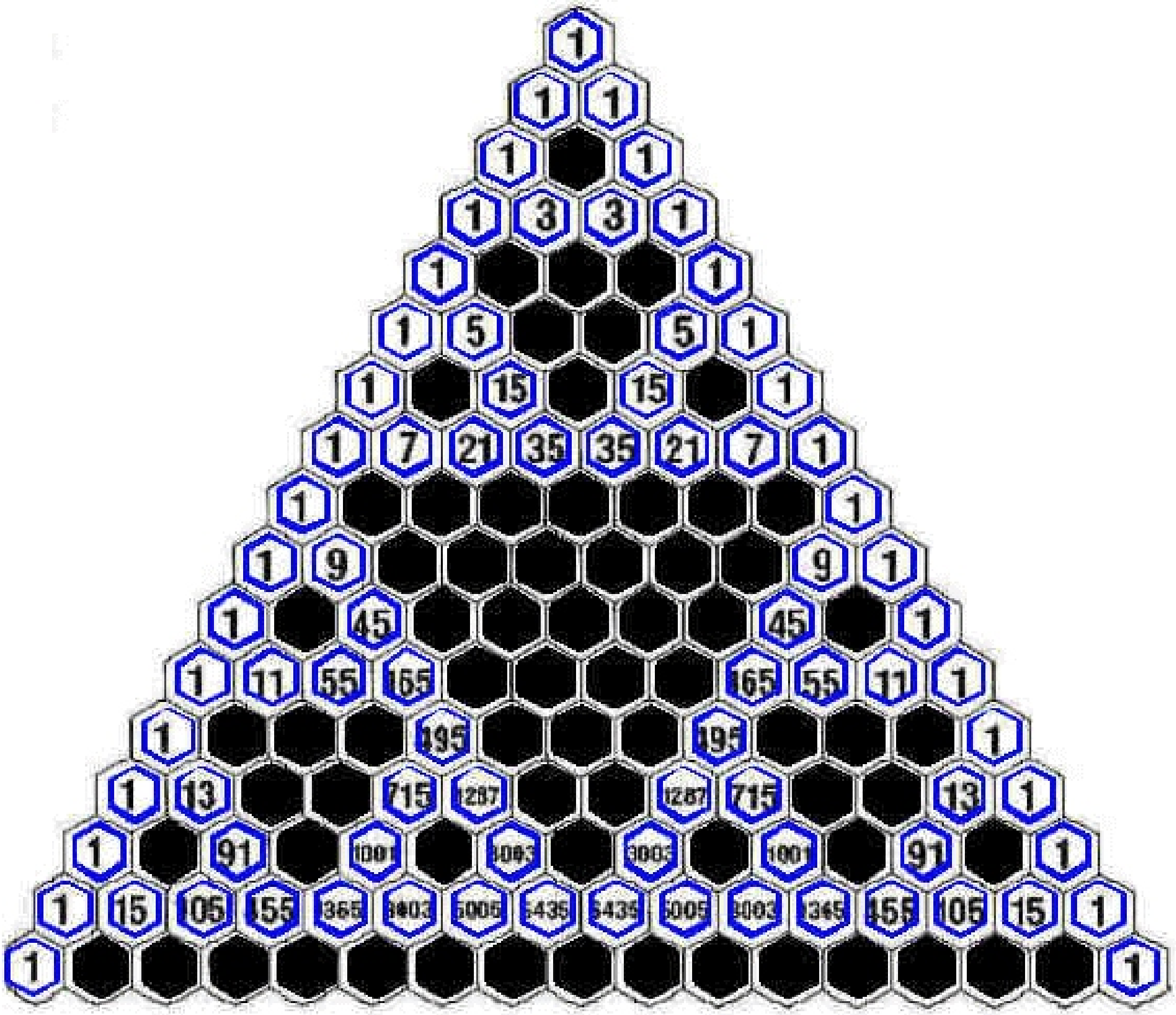
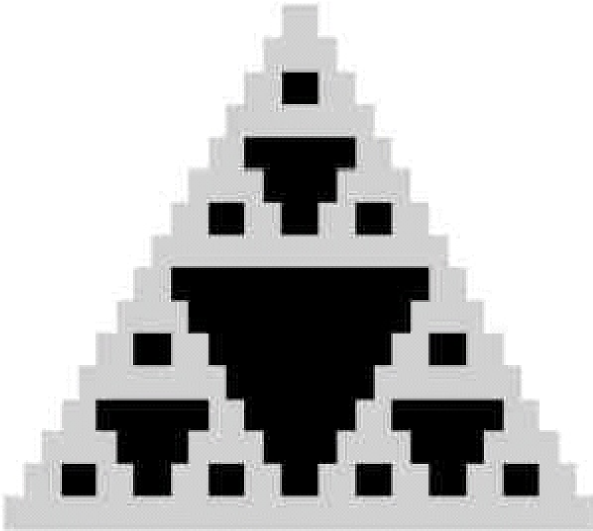
A Hidden Pattern in Pascal's Triangle



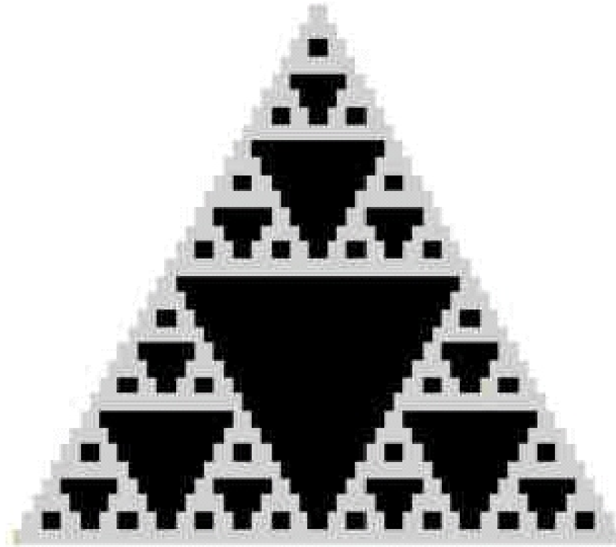
If we color all the entries which are divisible by 2, black, and all the entries which are not divisible by 2, blue, then we get this Sierpinski-like coloring.

This is called the "Mod-2" coloring

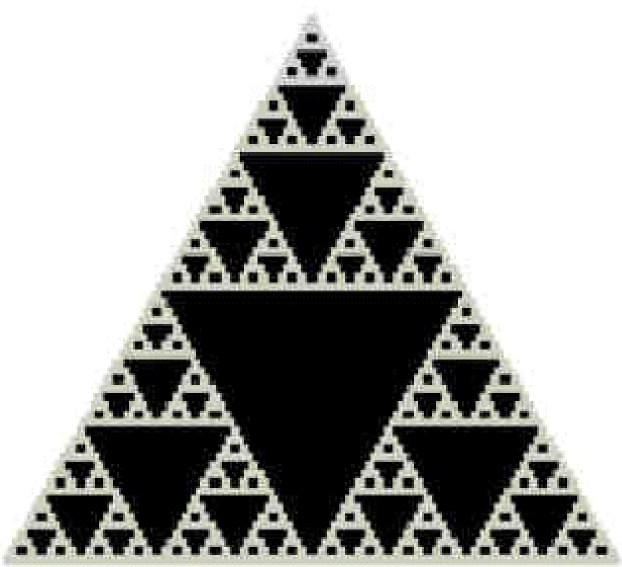
Mod-2 Coloring — First 128 Rows



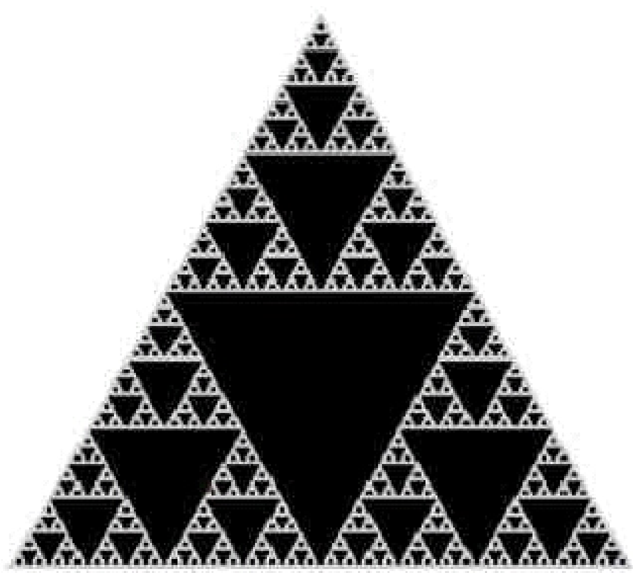
16 Rows



32 Rows

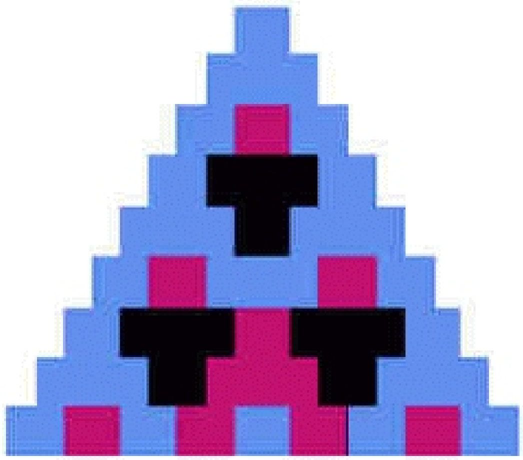


64 Rows

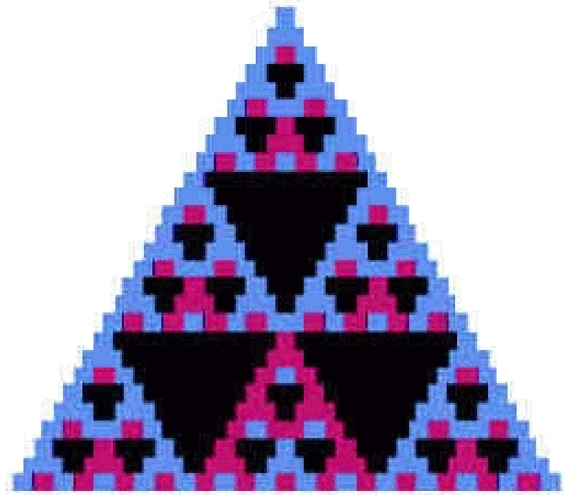


128 Rows

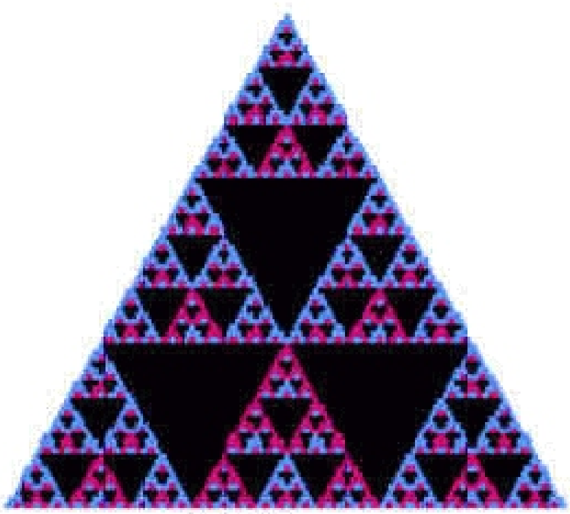
First 243 Rows with Remainder Mod 3



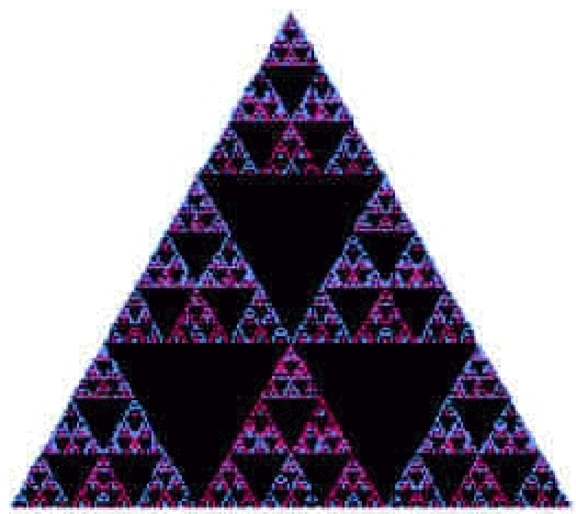
9 Rows



27 Rows

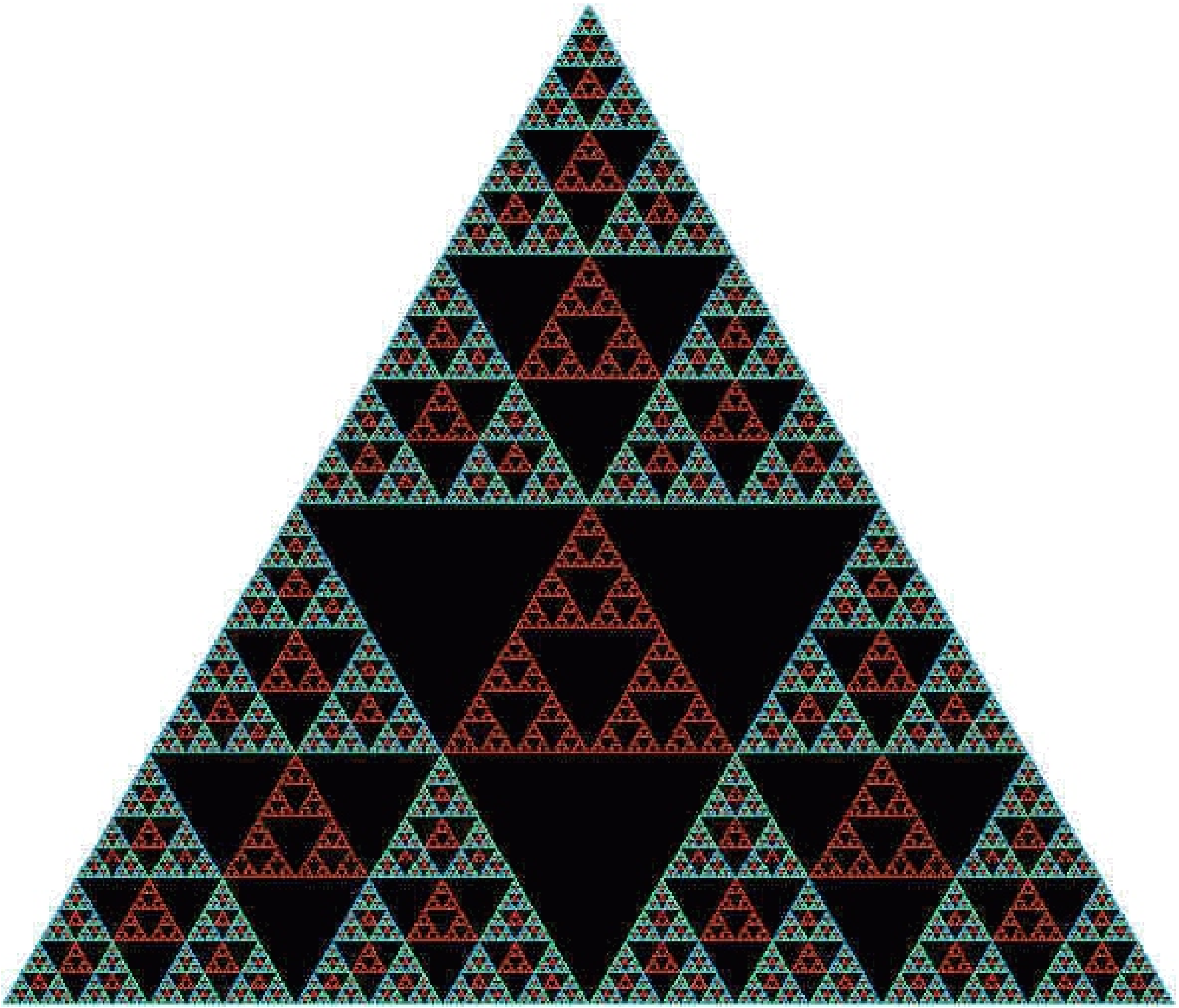


81 Rows



243 Rows

A Mod 4 Coloring of 125 Rows



Handout #5— Coin Flipping

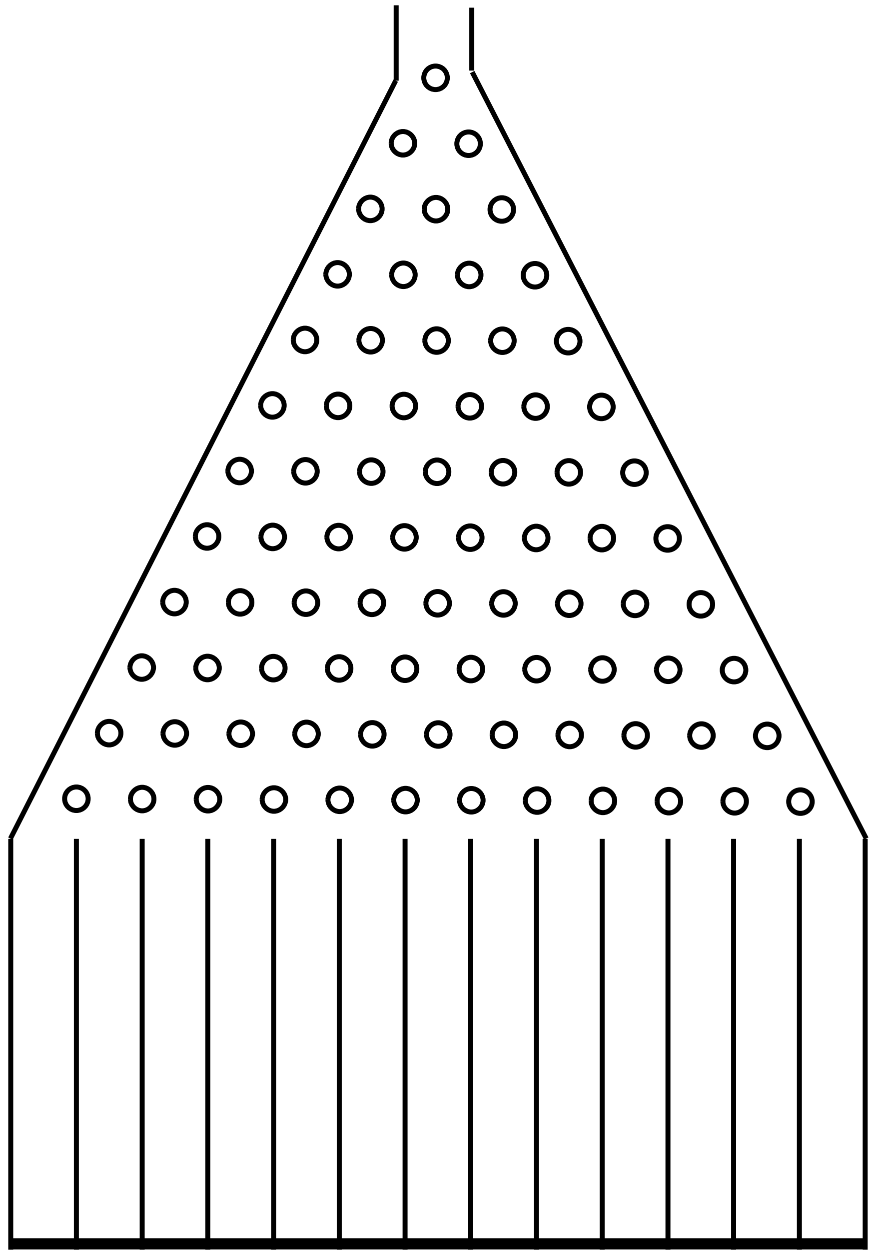
A coin is flipped 5 times

- How many different outcomes are possible? (You can use the multiplication rule for this.)
- Make a systematic list of the possible outcomes in the chart below, listing systematically according to the number of Heads. In the bottom row, put the total number for each column.

0 Heads	1 Heads	2 Heads	3 Heads	4 Heads	5 Heads

- Why do these numbers appear in Pascal's Triangle?
- What is the probability of flipping a coin 5 times and having 2 heads come up?

Approximating the Bell Curve
The Binomial Distribution



Suppose we drop 8192 (2^{13}) marbles into the top of this pegboard. How many balls do we expect, on average, to fall into each of the collectors at the bottom?

Handout #2 — The Fibonacci Numbers

The Fibonacci numbers are the numbers in the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... Each number in the sequence is the sum of the two previous numbers.

The Fibonacci numbers tell you how many ways a given number can be expressed as the sum of 1's and/or 2's. For example, the 4th Fibonacci number is 5, because there are 5 ways to represent 4 as the sum of 1's and/or 2's: 4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2.

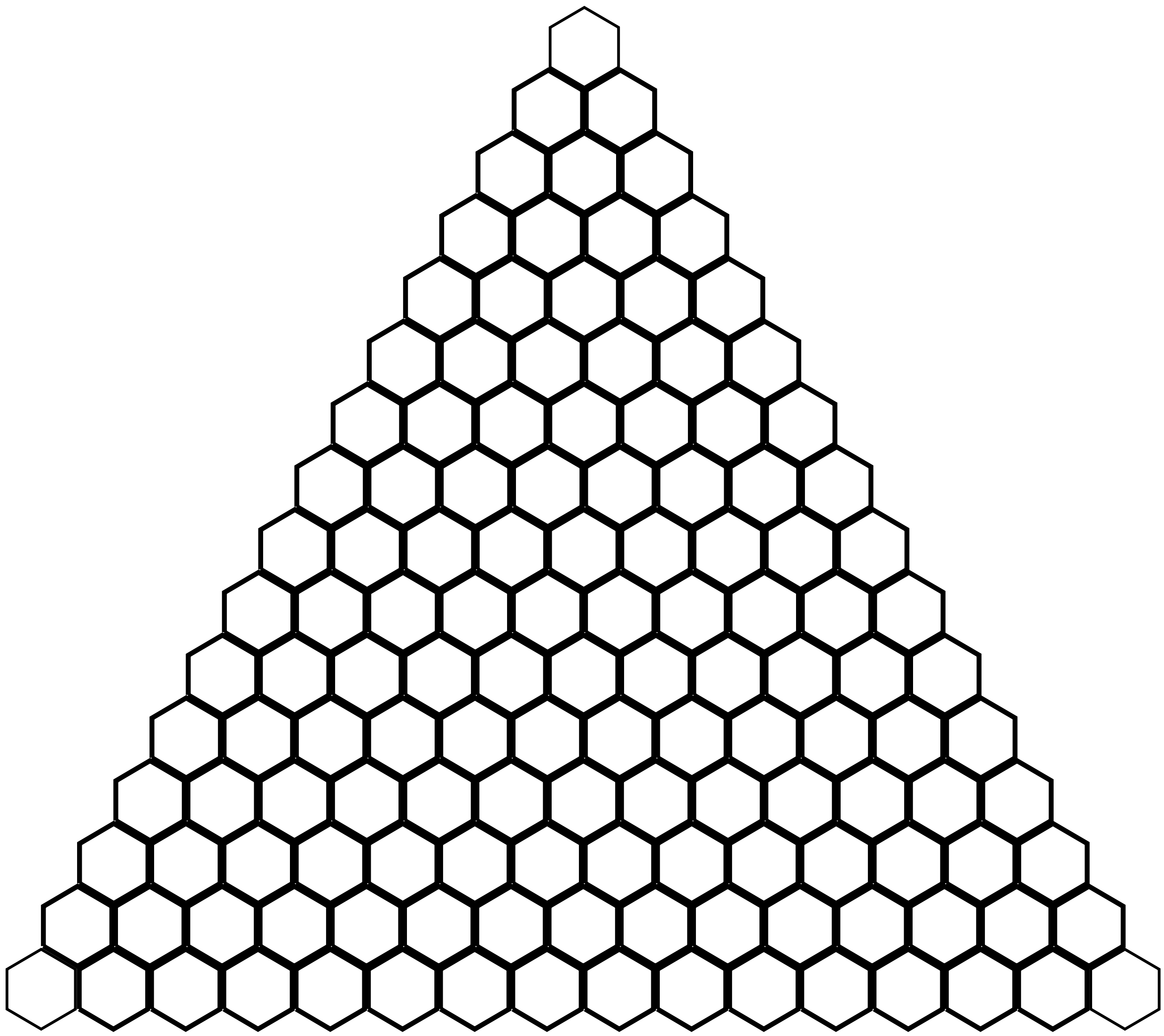
[Note that the "zeroth" Fibonacci number is 1, as is the first Fibonacci number.]

Here we will count the number of ways to write 6 as the sum of 1's and/or 2's. Fill in the chart below:

Write 6 as the sum of 1's and 2's, using this many 2's	**These are the ways**	**This is the *number* of ways**
0		
1		
2		
3		

Handout #3 — Coloring the Multiples of 3 in Pascal's Triangle

Fill in the numbers in the first few rows of Pascal's Triangle. Then color all the multiples of 3 with a dark color, and the non-multiples of 3 with a lighter color.



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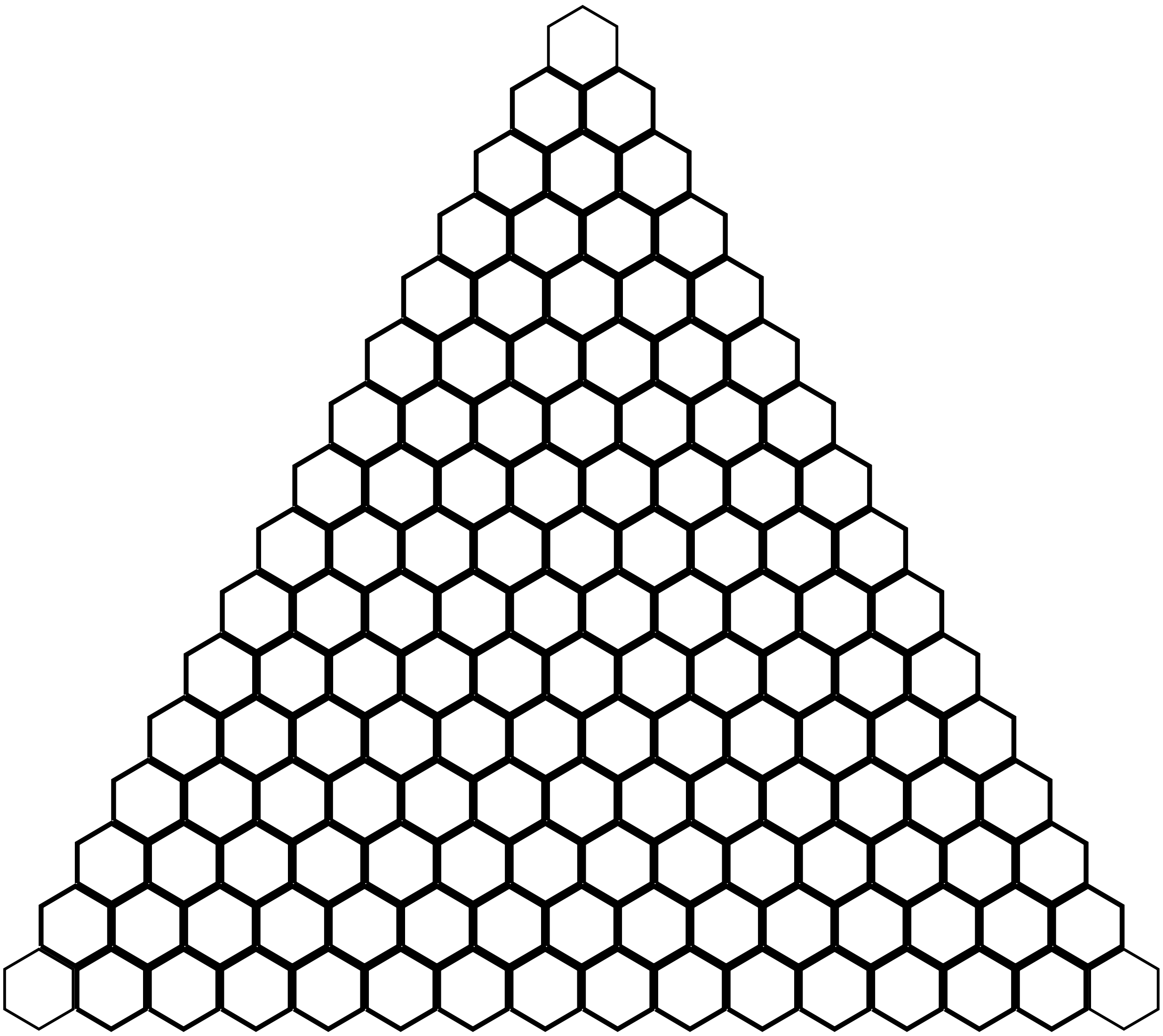
Handout #4 — Coloring Pascal's Triangle by Remainder (mod 3).

Again, color the multiples of 3 with a darker color, but now select *two* other colors; one color for those numbers that leave a remainder of 1 when divided by 3, and the other color for those entries that leave a remainder of 2 when divided by 3.



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0 Heads	**1 Heads**	**2 Heads**	**3 Heads**	**4 Heads**	**5 Heads**

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