Instructor Notes: All About Scheduling
Revised: September 14, 2001

This workshop discusses various types of scheduling situations which are modeled and solved using techniques of discrete mathematics. We begin by reviewing two types of situations which we have encountered before.

First, we remind ourselves of a scheduling situation which we discuss on the very first day of the Leadership Program summer institute. This was using graph coloring to schedule class projects (show TSP #2, which was TSP #11 from Day 1; note TSP #1 is a title page for the workshop), or to schedule garbage pick-ups. Do the example on TSP #2-3 (TSP #11-12 from Day 1), and remind people that all of the activities in today's workshop will be included in the Resource Book that they will receive at the end of the session.

Second, we remind ourselves of a scheduling situation which we discuss on the first day of the second summer program. This was using matching to schedule car pool; this is on TSP #4 (which is TSP #12 from Week 3 Day 1). Introduce the graph representation of this problem on TSP #5 (which is TSP #13 from Week 3 Day 1), briefly discuss the concept of a "matching" on the bottom of that transparency, and use that concept to solve the problem. (Be aware that some of the participants will be familiar with matchings from attended the second summer program, while others will have never seen it before.)

Point out that in both "scheduling of class projects" and "scheduling of car pools", the critical move is to represent these real-life situations using tools of discrete mathematics.

Let's introduce a complication into the "scheduling of car pools". In this situation, the five drivers — Jim, Gerrie, Bill, Ronnie, and Georgina — don't want to drive the same day every week. They would much rather, in fact, drive on each day about the same number of times. Can you make a schedule for them which will accomplish that?

Here's a more realistic version of the same problem: the office task scheduling problem — it's more realistic because the tasks are different and to ensure fairness each task should be rotated among the staff members. Show TSP #7 (The Office Staff Scheduling Problem – note that there is no TSP #6) and ask participants for their suggestions. Since many will be familiar with office procedures, some will soon suggest that is solved by creating a rotation schedule – assign each person to one

task on Monday and have them "rotate" through the tasks on consecutive days. Using TSP #8, discuss how this situation, including the rotation schedule, is modeled with a complete bipartite graph. Solicit a task assignment for the first day, and, using a colored transparency pen, color the edges of Monday's matching. Use a different colored pen for the second day, etc. Then, at the end, what do we have? An edge coloring of the graph. Introduce the notions of edge coloring and edge chromatic number with TSP #9, and find the edge-chromatic number of all cycles. Distribute HO #1 on which they are asked to find the edge chromatic number of wheels and grids, and finally complete graphs. Discuss the results for wheels and grid graphs.

Before continuing with the discussion of the edge-chromatic number of complete graphs, return to the situation with complete bipartite graphs, using TSP #10. In general, $K_{n,n}$ has edge chromatic number n — so that if there are n people with n jobs, then you can make a rotation schedule over n days. The coloring is simple — for the first color match each person with the task directly across, for the second color match each person with the task one below, for the third color match each person with the task two below, etc. This is simply the rotation schedule in its general setting.

Now return to the question of the edge-chromatic number of complete graphs. They will have verified that for K-3 and K-4 it is 3, and will be puzzled by the fact that for K-5 it appears to be 5. Why can't it be done with four colors — one way of verifying this is that each color can apply to only two edges, but the graph has 10 edges, so you need five colors. That verifies the conclusion but doesn't satisfy the curiosity.

So let's look at this problem in a different way (see TSP #11). Suppose you have six people competing in a table-tennis tournament. Everyone has to play every other person once. You can schedule three matches in each round. How many rounds do you need? You need at least five rounds, since each player has to play five other players. But can you do it in five rounds? Yes, you can.

What is you have five players? In any given round, there are only two matches, with one player getting a "bye". So the schedule for five players is the same as that for six players, with a dummy sixth player called "bye". So K-5 requires five colors, just like K-6. Similarly K-3 requires 3 colors, just like K-4. And also K-7 requires 7 colors, just like K-8. That should explain the jumps.

Participants will want to know how we found the schedule for a tournament. Tell them that you will show them if they promise not to give away the secret,

particularly to coaches of the athletic teams. Make the coaches come to them each time they need to work out a schedule.

Here's how it works. If you have an even number of teams, say 8, create a graph with eight vertices -- 7 of them forming a regular 7-gon and the other at the center of the 7-gon. Draw all of the edges to form a complete graph. Pick an edge from the center to a vertex of the 7-gon, identify the three edges of the 7-gon that are perpendicular to that edge (they are parallel to one another), and color all four of these edges with one color. Repeat this will all seven edges from the center to the outside vertices. What you end up with is an edge-coloring of the graph. Translate this to a schedule of the teams, with each color representing one round, that is, the four edges of that color are the pairings for that round. Work this strategy out for six players on the board, they show TSP #12 for eight players (this TSP is in color).

Take a break.

We now move on to another type of scheduling, presented here as a sequence of problems discussing "Scheduling music and jobs", that you should present and discuss. Show only the first two paragraphs of TSP #13, revealing one question at a time. (All items on TSP #13-14 are presented here in italics.)

> *Your daughter the folk-rock star is recording her songs onto a two-sided demonstration tape. Since you are expected to pay for the tapes, you want to be sure to minimize the total amount of tape needed, since the cost of each tape depends on its length. Which songs should be on each side?*
>
> *A. Assume that there are six songs, of length 8, 10, 11, 12, 13, and 15 minutes — and it makes no difference to her which songs go with which.*

Work on this problem at the board, helping them realize that since the total is 69 minutes, the best possible result is for one side to have 35 minutes and the other 34 minutes. Let them look at it for a few minutes and come up with two solutions — one involving 15+11+8 and the other involving 15+12+8. Let them work next on the following problem on the TSP.

> *B. Assume that there are eight songs, of length 7, 7, 8, 11, 13, 13, 20, and 21 minutes — and it makes no difference to her which songs go with which.*

Although the total of the songs is 100 minutes, there is no way of getting 50 minutes onto each side of the tape! That's not easy to see, but show them that there are two

cases — where 20 and 8 are together (and there are no two numbers which add to 22), and the other where 20 and 8 are separate (and there are no two odd numbers which add to 30 or 42). There are two solutions which involve 49 and 51 — namely 8+20+21 and 7+8+13+21. Which leads in to the next question.

> *C. Assume that she wants the lead songs on the two sides to be the 8 minute and 21 minute songs.*

In this case, you can't get 51-49 (because the only two such divisions have 8 and 21 on the same side), so the best you can do is 52-48 which can be done using 7+20+21 and 7+8+13+20 and 7+7+8+13+13.

If only tapes could have three sides . . . then we could do some other problems . . . so we change venues to "tapes have three sides". Show the first problem on TSP #14 and explain why it's the same type of problem.

> *D. Assume that the eight songs above are eight bags of spices (parsley, sage, rosemary, and thyme) and that they have to be packed in three bins of about the same size. How close to can you get them to equal size?*

It turns out that there is one solution — namely 21+13, 20+13, and 11+8+7+7 — but there are several ways of getting 34 which don't result in solutions — like 20+7+7 or 13+13+8. So you can get off on the wrong foot. One more exploration.

> *E. Assume that the 20 numbers from 1 to 20 have to be put into five boxes so that the total in each box is as close to the same as possible. What's the best that you can do?*

The first step is to realize that the total of the numbers is 210 (here's an opportunity to review the triangular numbers), so that the best possible answer is to have 42 in each of the five boxes. Now you have to have a strategy. Have them strategize a bit and maybe they will come up with the following ideas: (1) that they should use the biggest numbers first (since they are hardest to distribute), and (2) that they should try to fill one box at a time. Using those strategies, each group should arrive at the solution 20+19+3, 18+17+7, 16+15+11, 14+13+12+2+1, 10+9+8+6+5+4 — other solutions are possible. Actually, this problem is flawed because someone is likely to realize that 1+20, 2+19, 3+18, ... all add up to 21, so any two such pairs add up to 42. If someone offers that response, accept that answer gracefully, and add them to strategize for the same problem with six boxes. Are there six boxes, each containing 35? Here's a solution: 20+15, 19+16, 18+17, 14+13+9, 12+11+10+3, and

8+7+6+5+4+2+1. How about seven boxes, each with 30 — 20+10, 19+11, 18+12, 17+13, 16+14, 15+9+6, 8+7+5+4+3+2+1. How about ... ? Explain that these are examples of a topic call "bin-packing", that is discussed briefly on TSP #15.

> *F. Back to scheduling. You have five employees and 20 jobs that any one of them can do. The jobs take 1, 2, 3, ... , 20 hours to do. Give each employee a list of jobs to do so that each one has the same amount of work?*

Hopefully they will realize that this is the same problem as the previous one!

We now move on to a different type of scheduling problem, beginning with an activity followed by a discussion of "Scheduling your morning".

Give participants HO #2 labeled "Scheduling your morning". Each table should also get 50 linker cubes and two sets of labels for the 12 tasks. Using each linker cube to represent 5 minutes, have them create 10 and 15 minute blocks, and attach labels to the blocks of the appropriate sizes.

After they work on the problem for a while, it will become clear that in order to do the problem there has to be agreement on the order in which tasks must be done — in the sense that they must establish precedence rules which indicate what tasks must be completed before you can begin each individual task. Rather than having each group develop its own precedence rules, develop a precedence chart on TSP #17 so that all groups are working on the same problem. If time is short, use Joe's Precedence Rules on TSP #18 instead.

Let them get back to work, after noting that the schedule that they develop must observe these precedence rules. Ask people from several tables to draw their solutions on the blackboard, and discuss the various solutions. Note that there are variations among the solutions — e.g., some might eat before dressing (in which case they can listen to the radio while eating), or before showering — but in the end all of them require the alarm to be set at 6:50. (This assumes that Joe's Precedence Rules are used.)

Can one do better? Reviewing the diagram, we can see that no two of the tasks in the chain WU-E-S-Dr-B-WD-Br-D can be done at the same time, so that the total has to be 70 minutes — suppose you had another person to help you with your tasks, how much extra sleep could you get? (Not much! Only 5 minutes from dish-washing.)

TSP #19 (=HO #3) has another problem, a project which involves eight tasks, each of which takes a certain amount of time, and a set of precedence rules. There are two robots, or processors, which can do these tasks. What is the shortest amount of time that it will take to get the entire job done?

Do this at the board using a greedy algorithm (see TSP #20). Then distribute HO #3 and have then do this at their seats, if there is time; otherwise, distribute HO #3 but do the problem on the TSP. If there is time, put participant solutions on the board, review the solutions, and introduce the idea of a critical path — a path which has the longest sum of processing times — and then the critical path algorithm. Review the paragraph at the bottom of TSP #20. The kind of situation with which teachers are familiar in which critical path analysis could be used is creating a yearbook – yearbooks often appear late because tasks that should have been scheduled earlier are not completed until a later point, holding up the beginning of other tasks. Ironically, there was an extended article in the New York Times after the bombing of the World Trade Center in 1993 describing how critical path analysis permitted the WTC to be reopened ahead of schedule.

Close the workshop with the summary on TSP #21.

All About Scheduling — December 5, 1998

Comments

The workshop went well, except that, since we started 15 minutes late, there was not enough tie to complete the “critical path” scheduling activities. They did the “morning” activity, and I showed how the program could be described using a directed graph, and how the answer could be seen in processor format, but there was no time to look at the last activity — although I did show them the TSP and indicate that the bottom looked like a two-sided tape problem, although there would be slack times.

Judy Brown pointed out that doing the 5 person 5 job activity on a handout was not worthwhile, since they do such rotations all the time. You can elicit the simple idea of assigning jobs using two circles with one rotating and one fixed. (She suggested a variation where each person avoids one job altogether, but I’m not sure that I would add another activity, since the workshop is already long.)

It was suggested that the daughter pay the additional costs to have her two songs be lead-off on the two sides of the tape.

Interestingly, the bin packing problem involving the numbers from 1 to 20 into five bins is solved too easily — since you can get 10 21s but adding 1+20 etc.

The page with wheels, grids, and complete graphs took them more time than I expected. (It should also be on a transparency.) The wheels lead to the conjecture that the number of colors needed is the maximum degree — which doesn’t work for complete graph — I have to check the theorem — perhaps the edge-chromatic number is always the max or the max plus 1 (with the latter essentially only with complete graphs). An interesting point to make is that one vertex can have a major impact on the edge-chromatic number — though not on the vertex-chromatic number. The grids may have been unnessary. The complete graphs were interesting, because they started a pattern 3, 3, 5, ???

The Resource Book should have a discussion of scheduling the tournament — how to draw the diagram and how to select the edges for each round.

Note that adding dummy jobs allows you to conclude that the edge-chromatic number of any complete bipartite graph is the max of the number of elements in the two parts.

We don’t need to have the TSP on matchings — or do the examples there — a definition of matching should be added to the car pool problem.

Activity — The Office Staff Scheduling Problem

Bonnie, Debby, Chris, Lisa, and Jennifer are all staff members of Joe's programs. There are five general office tasks that must be done each day:

> **picking up the mail (PO),**
>
> **opening and distributing the mail (M),**
>
> **making coffee (C),**
>
> **listening to the answering machine (A), and**
>
> **answering phones during lunch-time (P).**

Since the five tasks are not equally demanding, the staff decides that they should do these tasks on a rotating basis — where each person has just one of these tasks each day. Can you develop a schedule which works?

We can model this situation with a bipartite graph — where the two parts are staff members and tasks.

Since each staff member can do any one of the tasks, we get what is called a "complete bipartite graph" — namely, a bipartite graph where every vertex is joined to all vertices on the opposite side.

Since there are five vertices in each part, the total number of edges in this graph $K_{5,5}$ is 5x5 = 25.

On each day, the five staff members are matched up with the five tasks, so each day we have a matching. On five days, we would have five matchings, so that theoretically we could use up all 25 edges on the graph with these five matchings. Can it actually be done?

An "edge coloring" of a graph is an assignment of colors to the <u>edges</u> of a graph so that two edges which meet at a vertex have a different color.

Of course, you can find an edge coloring of any graph where each edge has a different color. So, what's the least number of colors that you need for a graph? That depends — it's called the edge-chromatic number of the graph.

Example:

What's the edge-chromatic number of a cycle with three edges?

What's the edge chromatic number of a cycle with four edges?

What's the edge chromatic number of any cycle?

What is the edge chromatic number of a complete bipartite graph?

The conclusion that we came to in the previous problem — that there was a five-day rotation schedule of five people with five jobs — works for any number.

That is, if you have any n people, each of whom can do each of n jobs, then there is a rotation schedule which in n days will have each person doing each job exactly once.

In other words, the edge chromatic number of $K_{n,n}$ is n.

Suppose you have six people competing in a table-tennis tournament. Everyone has to play every other person once.

You can schedule three matches in each round. How many rounds do you need?

You need at least five rounds, since each player has to play five other players. But can you do it in five rounds?

Ann — Bob — Carl — Dave — Ethel — Fay

1. **Ann vs Bob; Carl vs Dave; Ethel vs Fay**
2. **Ann vs Carl; Bob vs Ethel; Dave vs Fay**
3. **Ann vs Dave; Bob vs Fay; Carl vs Ethel**
4. **Ann vs Ethel; Bob vs Dave; Carl vs Fay**
5. **Ann vs Fay; Bob vs Carl; Dave vs Ethel**

What if there are only 5 players — how many rounds do you need?

Activity — Scheduling music or jobs

Your daughter the folk-rock star is recording her songs onto a two-sided demonstration tape. Since you are expected to pay for the tapes, you want to be sure to minimize the total amount of tape needed, since the cost of each tape depends on its length. Which songs should be on each side?

A. Assume that there are six songs, of length 8, 10, 11, 12, 13, and 15 minutes — and it makes no difference to her which songs go with which.

B. Assume that there are eight songs, of length 7, 7, 8, 11, 13, 13, 20, and 21 minutes — and it makes no difference to her which songs go with which.

C. Assume that she wants the lead songs on the two sides to be the 8 minute and 21 minute songs.

D. Assume that the eight songs above are eight bags of spices (parsley, sage, rosemary, and thyme) and that they have to be packed in three bins of about the same size. How close to can you get them to equal size?

E. Assume that the 20 numbers from 1 to 20 have to be put into five boxes so that the total in each box is as close to the same as possible. What's the best that you can do?

F. Back to scheduling. You have five employees and 20 jobs that any one of them can do. The jobs take 1, 2, 3, ... , 20 hours to do. Give each employee a list of jobs to do so that each one has the same amount of work?

The Bin Packing Problem

You have a number of bins of different sizes and a number of objects of different sizes.

You want to place each object into one of the bins.

Question 1: Will the objects will fit into the bins?

> **That is, can you tell whether or not they will all fit into the bins?**

Question 2: Which objects should go into which bins?

> **That is, is there is an efficient way of distributing the objects among the bins?**

Activity — Scheduling your morning.

You start work at 8:00 a.m. and you like to sleep as late as possible. If you need to do all of the following tasks in the morning, and each takes the amount of time indicated, for what time should you set your alarm?

(B)	**Eat breakfast**	**10 minutes**
(Br)	**Brush your teeth**	**5 minutes**
(C)	**Prepare coffee**	**10 minutes**
(D)	**Drive to work**	**10 minutes**
(Dr)	**Get dressed (and whatever else you do)**	**10 minutes**
(E)	**Do your exercises**	**15 minutes**
(O)	**Cook oatmeal**	**10 minutes**
(R)	**Listen to the weather report on the radio**	**5 minutes**
(S)	**Take a shower**	**10 minutes**
(WC)	**Warm up the car**	**10 minutes**
(WD)	**Wash the dishes**	**5 minutes**
(WU)	**Wake up you sleepyhead, get out of bed**	**5 minutes**

Give a schedule which would allow you to complete all of these tasks in the shortest possible time. (No funny stuff!)

(Using each linker cube to represent a five-minute interval, you can represent each task with a chain of linker cubes — and label each chain appropriately.)

Precedence rules

This task . . .		must be preceded by . . .	
(B)	Eat breakfast	C, O	10
(Br)	Brush your teeth		5
(C)	Prepare coffee		10
(D)	Drive to work		10
(Dr)	Get dressed (and whatever else you do)		10
(E)	Do your exercises		15
(O)	Cook oatmeal		10
(R)	Listen to the weather report on the radio		5
(S)	Take a shower		10
(Wc)	Warm up the car		10
(Wd)	Wash the dishes		5
(Wu)	Wake up you sleepyhead, get out of bed		5

Joe's Precedence rules

This task . . .		must be preceded by . . .	
(B)	Eat breakfast	C, O	10
(Br)	Brush your teeth	B	5
(C)	Prepare coffee	WI	10
(D)	Drive to work	Br, WC	10
(Dr)	Get dressed (and whatever else you do)	R, S	10
(E)	Do your exercises	WU	15
(O)	Cook oatmeal	WU	10
(R)	Listen to the weather report on the radio	WU	5
(S)	Take a shower	E	10
(WC)	Warm up the car	Dr	10
(WD)	Wash the dishes	B	5
(WU)	Wake up you sleepyhead, get out of bed!		5

A project involves eight tasks, for each of which the times and preceding tasks are given below:

Task	Time	Preceded by ...
A	5	
B	6	
C	2	
D	3	
E	4	A
F	1	A, B
G	8	C, D
H	7	F, G

The directed graph describing this project is as follows:

Two processors (Robotka and Golembo) work on the project. Each of them can work on any task (from start to finish), but all precedent tasks must be completed before it begins that task. How long will it take the two processors to complete the project?

0

Robotka

Golembo

Algorithms for solving these scheduling problems

Alphabetically Greedy Algorithm: Order tasks alphabetically. Whenever a processor is free, assign it the first task (alphabetically) that it can do.

Better Greedy Algorithm: Order tasks by processing time. Whenever a processor is free, assign it the most time-consuming task that it can do.

Still Better Greedy Algorithm — Critical Path

Order tasks by completion time. (Of all paths from the task T to the end, the one that has the largest total sum of processing times is called the critical path from T. The completion time is the length of this longest path.) Whenever a processor is free, assign it the task that it can do with greatest completion time.

The subject of "critical path analysis" achieved prominence when the government began its large-scale projects like the Polaris Submarine Project in the 1950s and then President Kennedy's Moon Landing Project in the 1960s. To complete such projects in a timely way, it was necessary to determine at the beginning of the project those components whose timely completion were most essential to the project.

All About Scheduling

Summary

1. Scheduling class projects by vertex coloring of graphs

2. Scheduling car pools by matching

3. Scheduling rotations by edge coloring of complete bipartite graphs

4. Scheduling tournaments by edge coloring of complete graphs

5. Scheduling songs on tapes and packing objects into bins

6. Scheduling the morning activities and other complex projects using directed graphs and critical paths

Hand-out #1 — Edge chromatic numbers

Find the edge chromatic numbers of the following graphs. What patterns do you see?

wheels

grids

complete graphs

Hand-out #2 — Scheduling your morning.

You start work at 8:00 a.m. and you like to sleep as late as possible. If you need to do all of the following tasks in the morning, and each takes the amount of time indicated, for what time should you set your alarm?

(B)	**Eat breakfast**	**10 minutes**
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(C)	**Prepare coffee**	**10 minutes**
(D)	**Drive to work**	**10 minutes**
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(E)	**Do your exercises**	**15 minutes**
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Give a schedule which would allow you to complete all of these tasks in the shortest possible time. (No funny stuff!)

(Using each linker cube to represent a five-minute interval, you can represent each task with a chain of linker cubes — and label each chain appropriately.)

Hand-out #3 — Processors a-processing

A project involves eight tasks, for each of which the times and preceding tasks are given below:

Task	Time	Preceded by ...
A	5	
B	6	
C	2	
D	3	
E	4	A
F	1	A, B
G	8	C, D
H	7	F, G

The directed graph describing this project is as follows:

Two processors (Robotka and Golembo) work on the project. Each of them can work on any task (from start to finish), but all precedent tasks must be completed before it begins that task. How long will it take the two processors to complete the project?

0

Robotka

Golembo

0

Robotka

Golembo

0

Robotka

Golembo

0

Robotka

Golembo