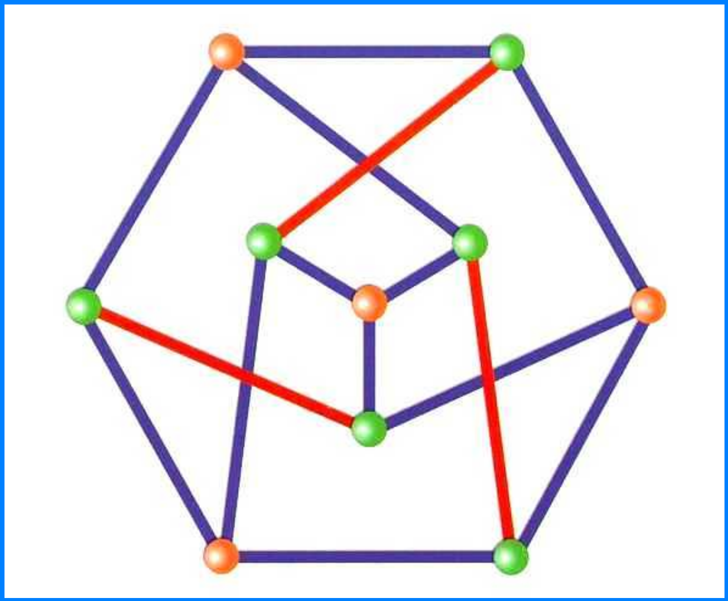
Consensus List Colorings of Graphs



Fred Roberts, Rutgers University
Joint work with N.V.R. Mahadev

The Consensus Problem

- Old problem from the social sciences
- How do we combine individual opinions or information into a decision by a group?
- Widely studied
- Large literature
- New graph-theoretical formulation
- New applications: transportation, communications, scheduling, routing, fleet maintenance, genetics, …

Table of Contents:

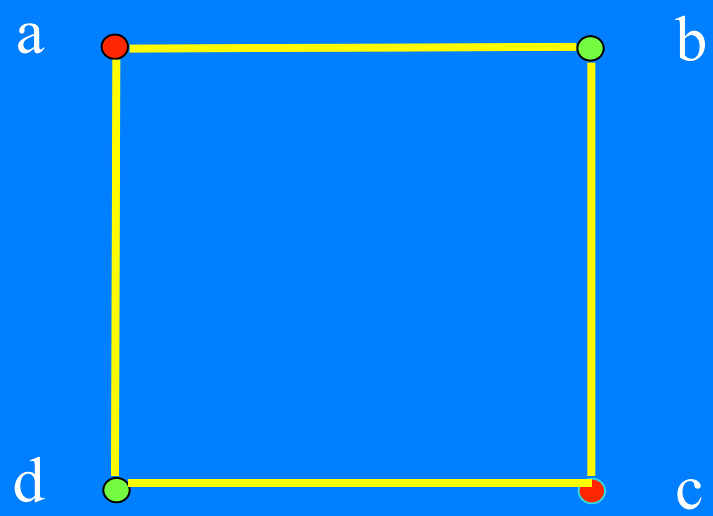
I. Graph Coloring and its Applications

II. List Colorings of Graphs and Connections to Applications

III. Consensus List Colorings: 3 Models and Implications for Sharing Information

IV. Future Research

Graph Coloring & Applications

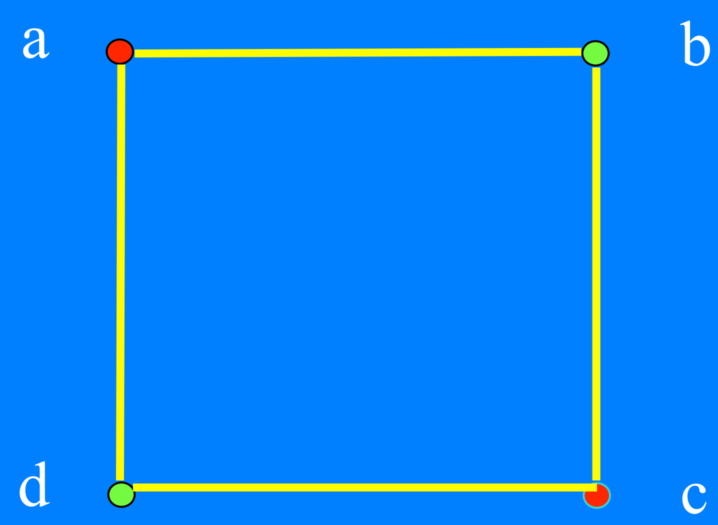


C_4

Proper coloring of graph $G = (V,E)$:

$\{x,y\} \in E \Longrightarrow f(x) \neq f(y)$

Graph Coloring & Applications



C_4

Chromatic Number

$\chi(G)$ = smallest p so that graph G has a proper coloring in p colors.

$\chi(C_4) = 2$

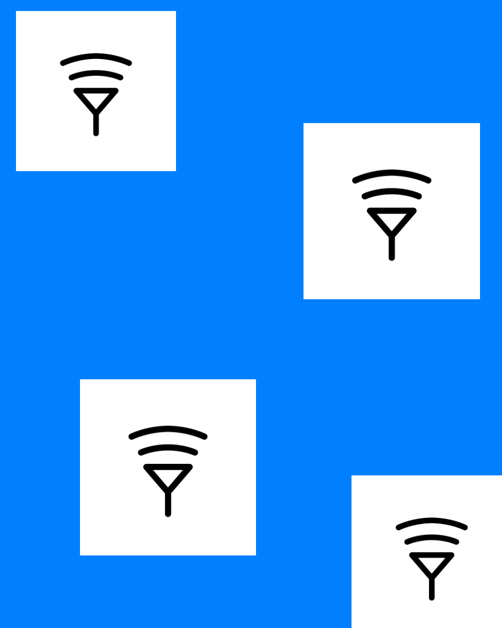
Graph Coloring & Applications

Proper coloring of graph $G = (V,E)$:

$$\{x,y\} \in E \Longrightarrow f(x) \neq f(y)$$

Channel Assignment:

V = set of transmitters
edge = interference
color = assigned channel



Graph Coloring & Applications

Traffic Phasing:

V = set of individuals or cars or …
with requests to use a facility
room, tool, traffic intersection
edge = interference
color = time assigned to the individual or car
or …

Graph Coloring & Applications

Scheduling:

V = set of committees looking for meeting times
edge = committees have a member in common
color = time assigned to committee

Similar problem in scheduling final exams for classes or meeting times for classes (in an ideal university where students first choose classes and then classes are scheduled)

Graph Coloring & Applications

Routing:

V = set of garbage or mail truck tours = order of sites visited on a given day

edge = two tours have site in common

color = day assigned to tour

Credit: wikipedia.org

Graph Coloring & Applications

Fleet Maintenance:

V = set of vehicles (trucks, cars, planes, ships) and the days they are scheduled for regular maintenance

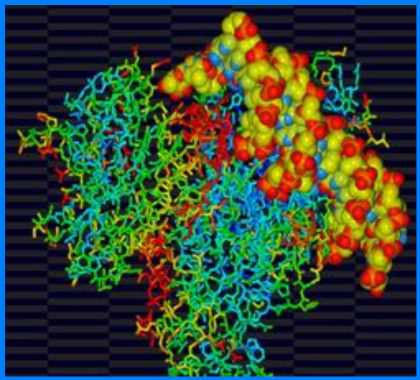
edge = two vehicles have overlapping schedule days

color = space in the maintenance facility assigned to a given vehicle

How Graph Coloring Enters into Physical Mapping of DNA

Physical Mapping 101:

- **DNA** = sequence of G, T, C, A: CGAGATGCTG
- **Physical map**: piece of DNA telling us location of certain markers along the molecule
  - ✓Markers = precisely defined subsequences
- Step 1: Make copies of the molecule we wish to map – the **target molecule**
- Step 2: Break each copy into disjoint **fragments** CG, AG, ATG, CTG
  - ✓Use restriction enzymes



How Graph Coloring Enters into Physical Mapping of DNA

- Step 3: Obtain overlap information about the fragments
- Step 4: Use overlap information to obtain the mapping

Obtaining Overlap Information

- One method used: Hybridization.
- Fragments replicated giving us thousands of **clones**
- **Fingerprinting**: check if small subsequences called probes bind to clones. Fingerprint of a clone = subset of probes that bind

How Graph Coloring Enters into Physical Mapping of DNA

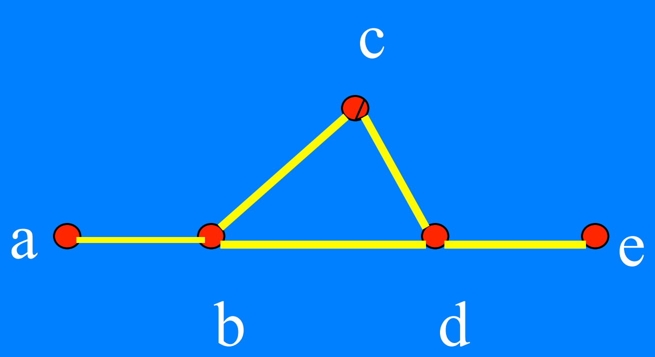
- Two clones sharing part of their fingerprints are likely to have come from overlapping regions of the target DNA.

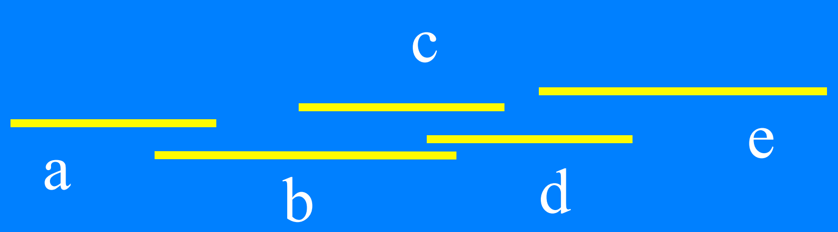
Errors in Hybridization Data

- Probe fails to bind where it should (false negative)
- Probe binds where it shouldn't (false positive)
- Human mis-reading/mis-recording
- During cloning, two pieces of target DNA may join and be replicated as if they were one clone
- Probes can bind along more than one site
- Lack of complete data

How Graph Coloring Enters into Physical Mapping of DNA: Interval Graphs

Take a family of real intervals.
Let these be vertices of a graph.
Join two by edge iff the intervals overlap.
Corresponding graph is an **interval graph**.





How Graph Coloring Enters into Physical Mapping of DNA: Interval Graphs

Good algorithms for:
- Recognizing when a graph is an interval graph.
- Constructing a “map” of intervals on the line that have the corresponding intersection pattern

How Graph Coloring Enters into Physical Mapping of DNA: Interval Graphs

- Interval graphs arose in part from the pioneering work of geneticist Seymour Benzer in early 1960s.

- He asked:

Can you tell the genetic topology without actually seeing it? Can you just use overlap information?

How Graph Coloring Enters into Physical Mapping of DNA

•From overlap information, create a **fragment overlap graph**:

V = fragments (clones)

E = fragments (clones) overlap

•***If clone overlap information is complete and correct, fragment overlap graph is an interval graph***.

•Then corresponding "map" of intervals gives relative order of fragments on the target DNA

•This gives beginning of a "physical map" of the DNA.

How Graph Coloring Enters into Physical Mapping of DNA

- But fragment overlap graph may not be an interval graph – due to errors/incomplete information
- Label each clone with the identifying number of the copy of target molecule it came from
- Think of label as a color
- Two clones coming from same copy of the target molecule cannot overlap – we broke that molecule into disjoint fragments.
- Thus: ***numbers give a graph coloring for the fragment overlap graph***.

How Graph Coloring Enters into Physical Mapping of DNA

Dealing with False Negatives:

•Here, the primary errors omit overlaps.
•Try to add edges to fragment overlap graph to obtain an interval graph.
•Require the numbering to remain a graph coloring.
•May not be doable.
•If doable, work with resulting graph.
•If several such graphs, use minimum number of added edges.

How Graph Coloring Enters into Physical Mapping of DNA

Dealing with False Positives:

- Here, the primary errors are overlaps that should not be there.
- Delete edges from fragment overlap graph to obtain an interval graph.
- Require the numbering to remain a graph coloring.
- Always doable.
- If several such graphs, use minimum number of deleted edges.

How Graph Coloring Enters into Physical Mapping of DNA

Dealing with both False Negatives and Positives:

- Here, we know some overlaps are definitely there and some are definitely not.
- Think of two edge sets E_1 and E_2 on same vertex set V, $E_1 \subset E_2$.
- Think of same coloring on each graph (V,E_i)
- Look for set E of edges such that $E_1 \subset E \subset E_2$ and (V,E) is an interval graph.
- The coloring is automatically a coloring for (V,E).
- This is called the **interval sandwich problem**.

How Graph Coloring Enters into Physical Mapping of DNA

- Determining if we can add edges to G with a coloring f to obtain an interval graph for which f is still a coloring: NP-hard
- Determining the smallest number of edges to remove to make G an interval graph: NP-hard.
- The interval sandwich problem is also NP-hard.

List Coloring

- Given graph G and list S(x) of acceptable colors at each vertex.
  - ✓ S is a **list assignment**.
- A **list coloring** for (G,S) is a proper coloring f such that $f(x) \in S(x)$ for all x.
  - ✓ **List colorable** if a list coloring exists.
- Channel assignment: list of acceptable channels
- Traffic phasing: list of acceptable times for use of facility
- Scheduling: list of acceptable meeting times
- Routing: List of acceptable days

List Coloring

- Given graph G and list S(x) of acceptable colors at each vertex.
  - ✓ S is a **list assignment**.
- A **list coloring** for (G,S) is a proper coloring f such that f(x) ϵ S(x) for all x.
  - ✓ **List colorable** if a list coloring exists.
- Fleet Maintenance: List of acceptable spaces in maintenance facility
- Physical mapping:
  - ✓ Lose or inaccurately record information about which copy of target DNA molecule a clone came from.
  - ✓ Might know set of possible copies it came from.

List Coloring

•Given graph G and list S(x) of acceptable colors at each vertex.

•Later, we will interpret S(x) as a set of pieces of information individual x has that are relevant to a decision a group of individuals has to make.

•And we will consider how sharing of information can lead to acceptable solutions to a problem when not sharing means there is no acceptable solution.

List Coloring: Complexity

- NP-complete to determine if G is colorable in at most k colors if $k \geq 3$.
- Thus, NP-complete to determine if there is a list coloring for (G,S) if $|\bigcup S(x)| \geq 3$.
- Both problems polynomial for 2.

List Coloring and Physical Mapping

- **False Negatives**: Given (G,S), can we add edges to G to obtain an interval graph H so that (H,S) is list colorable?
  - ✓Impossible if (G,S) is not list-colorable.
  - ✓Adding edges makes coloring harder.
- **False Positives**: Given (G,S), what is smallest number of edges to remove from G to obtain an interval graph H so that (H,S) is list colorable?
- **Both**: Given (V,E_1) and (V,E_2) with $E_1 \subset E_2$ and S on V. Is there a set E so that $E_1 \subset E \subset E_2$, $G = (V,E)$ is an interval graph, and (G,S) is list colorable?

What if (G,S) is not List Colorable?

Alternative approach: Don't change the fragment overlap graph, but instead modify the list assignment S.

QUESTION: ***If (G,S) is not list colorable, can we modify the lists S, getting a new set of lists S*, so that (G,S*) is list colorable?***

•This question arises in channel assignment, traffic phasing, scheduling, routing, fleet maintenance, DNA physical mapping, and other problems.

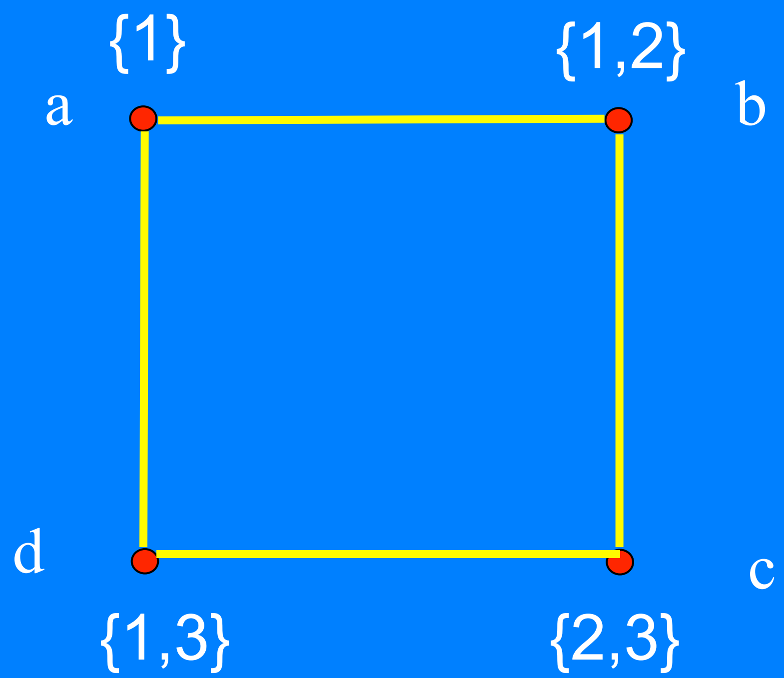
The Problem as a Consensus Problem

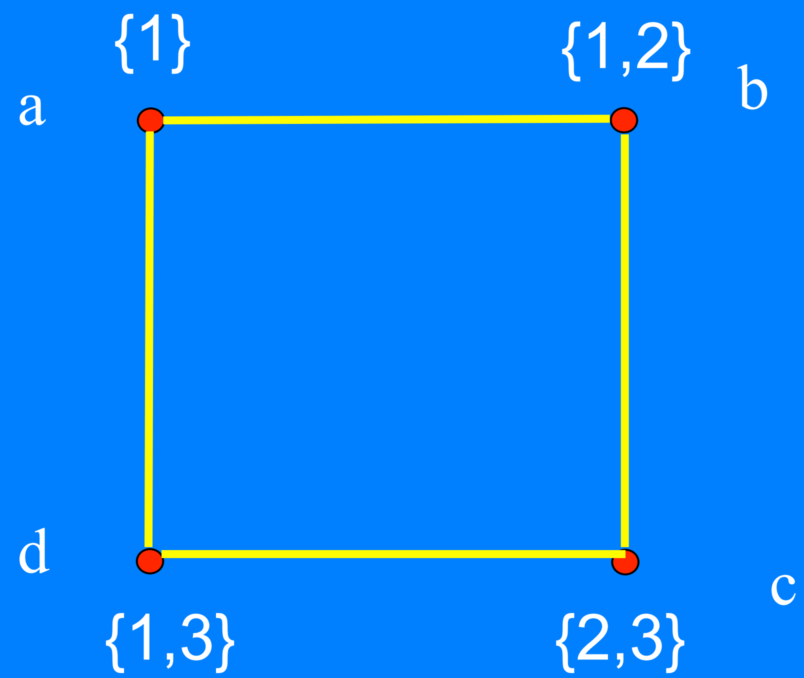
- Think of vertices as individuals.
- If (G,S) has no list coloring, some individuals will have to make sacrifices by ***expanding or changing*** their lists for a list coloring to exist.
- Three models for how individuals might change their lists.
- Think of these as providing a procedure for group to reach a consensus about a list coloring.

First Consensus Model: The Adding Model

•Each individual may add one color from set of colors already used in US(x).

✓ One acceptable channel
✓ One acceptable use time
✓ One acceptable meeting time
✓ One acceptable day for a tour
✓ One acceptable space in a maintenance facility
✓ One possible additional copy number for a clone


{1}
{1,2}
a
b
d
c
{1,3}
{2,3}

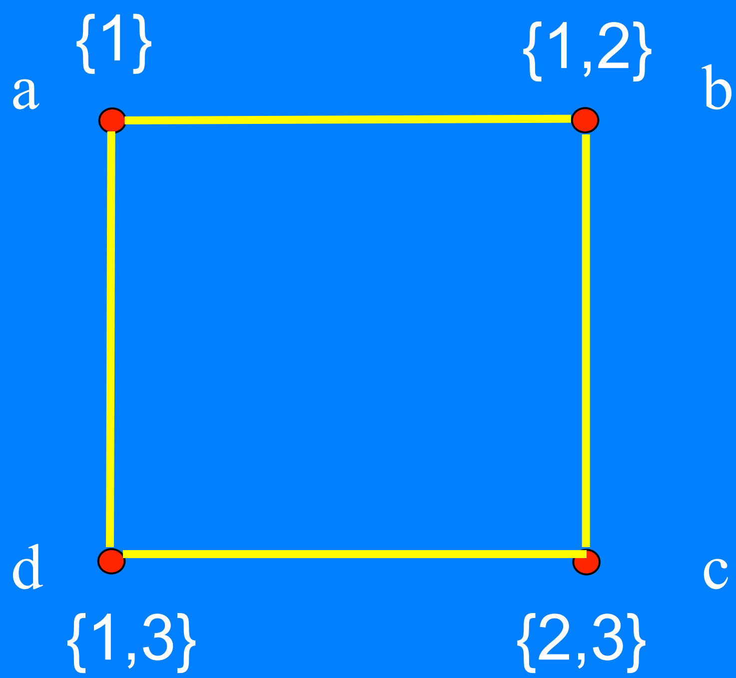


Not list colorable.

f(a) must be 1.

Thus, f(b) must be 2, f(d) must be 3.

What is f(c)?



Adding color 1 to S(c) allows us to make f(c) = 1.

Not list colorable.
f(a) must be 1.
Thus, f(b) must be 2, f(d) must be 3.
What is f(c)?

p-Addability

(G,S) is **p-addable** if there are p distinct vertices $x_1, x_2, \ldots, x_p$ in G and (not necessarily distinct) colors $c_1, c_2, \ldots, c_p$ in $\cup S(x)$ so that if

$$S^*(u) = S(u) \cup \{c_i) \text{ for } u = x_i$$
$$S^*(u) = S(u) \text{ otherwise}$$

then (G,S*) is list-colorable.

In previous example, (G,S) is 1-addable.

p-Addability

Observation: (G,S) is p-addable for some p iff

$$|\cup S(x): x \in V| \geq \chi(G). \qquad (*)$$

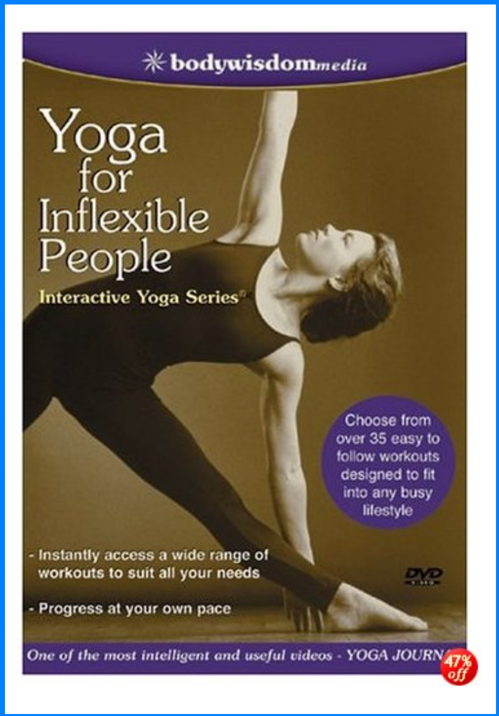
p-addable implies colorable using colors from $\cup S(x)$. So (*) holds.
If (*) holds, exists a coloring f. Let $c_i = f(x_i)$.

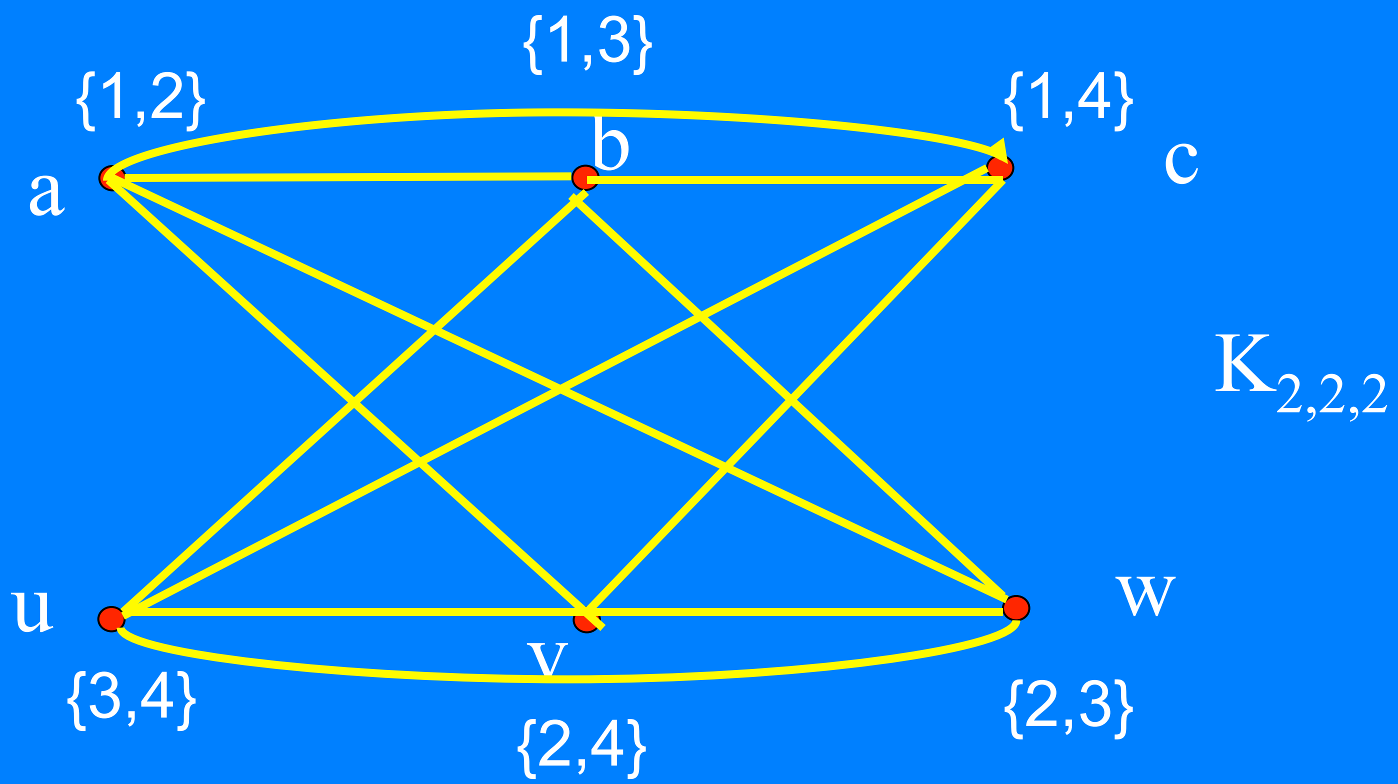
Observation: If $|\cup S(x)| \geq 3$, it is NP-complete to determine if (G,S) is p-addable for some p. (Since it is NP-complete to determine if $\chi(G) \leq k$ when $k \geq 3$.)

The Inflexibility

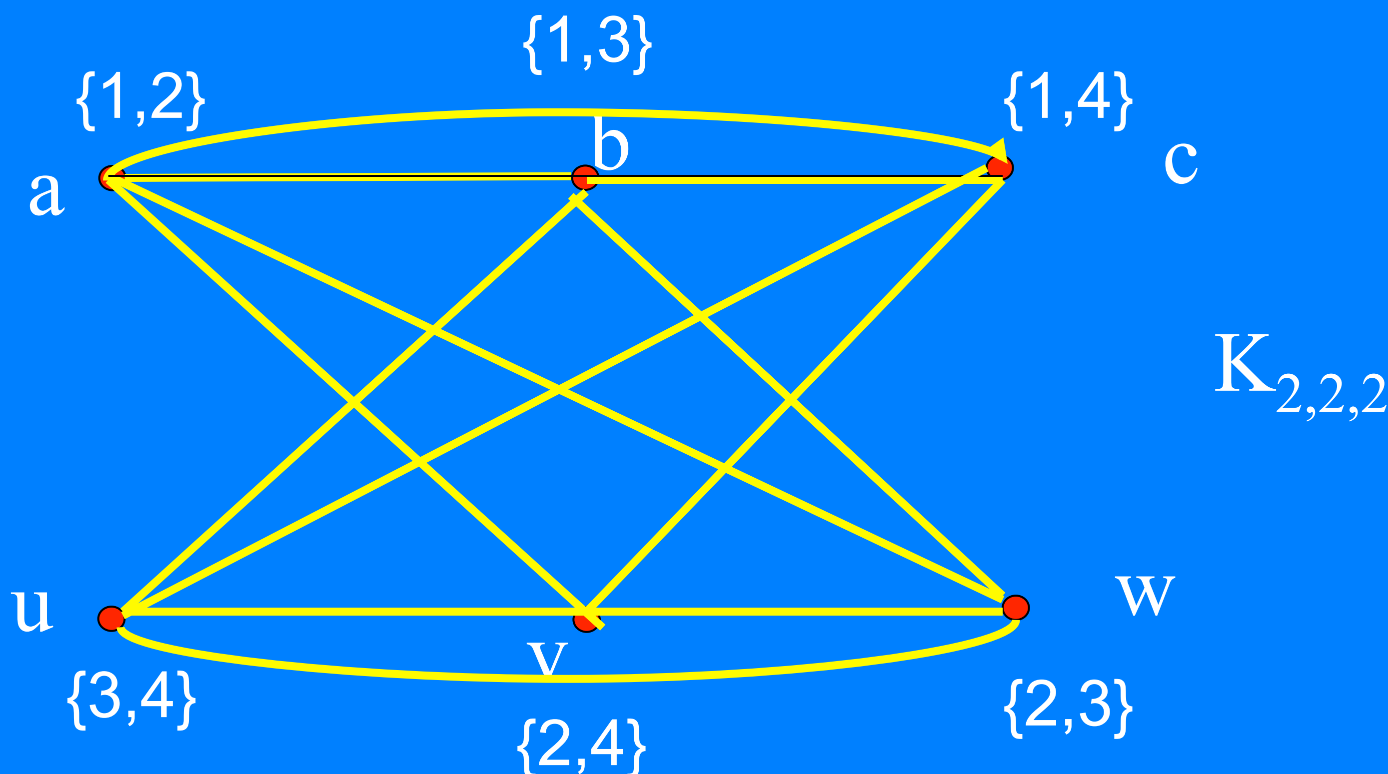
- How hard is it to reach consensus?
- What is the smallest number of "individuals" who have to add an additional acceptable choice?
- **What is the smallest p so that (G,S) is p-addable?**

- Such a p is denoted I(G,S) and called the **inflexibility** of (G,S).
- It may be undefined.



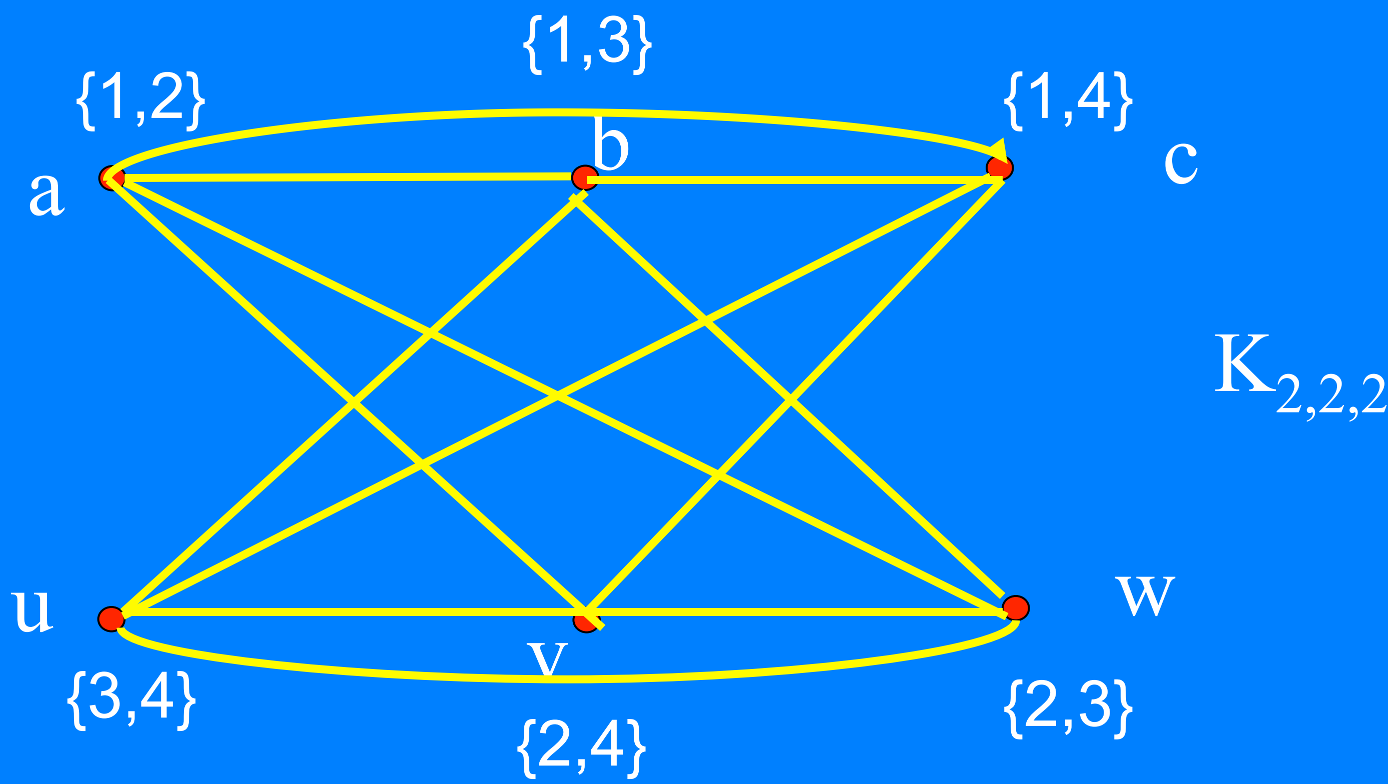


What is I(G,S)?



(G,S) is not 1-addable.

- On each partite class {x,y}, S(x) ∩ S(y) = ∅.
- List assignments need two colors for each partite class.
- If one set S(x) changes, need 4 colors on remaining two partite classes and one more color on class containing x.
- But only four colors are in ∪S(x).



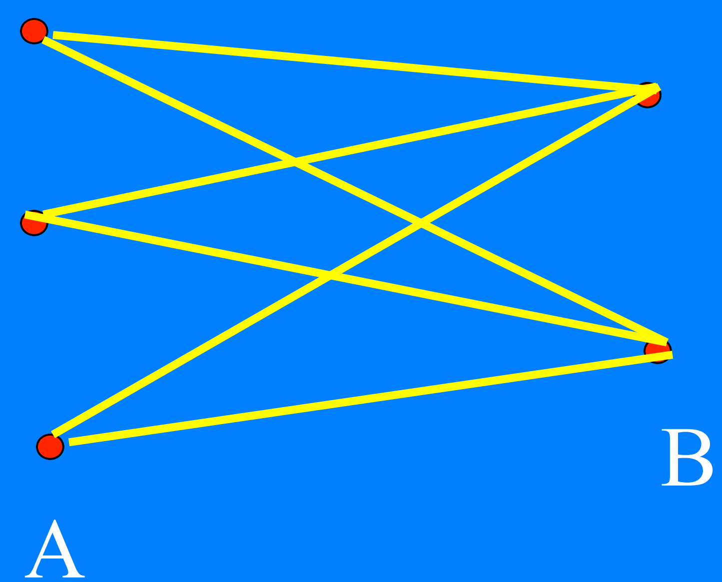
I(G,S) = 2.

Add color 1 to S(u) and color 2 to S(b).

f(u) = f(a) = 1, f(b) = f(v) = 2, f(c) = 4, f(w) = 3

Complete Bipartite Graphs

- $K_{m,n}$: There are two classes of vertices, A and B.
- A has m vertices, B has n vertices.
- Every vertex of A is joined to every vertex of B.



$K_{3,2}$

Complete Bipartite Graphs

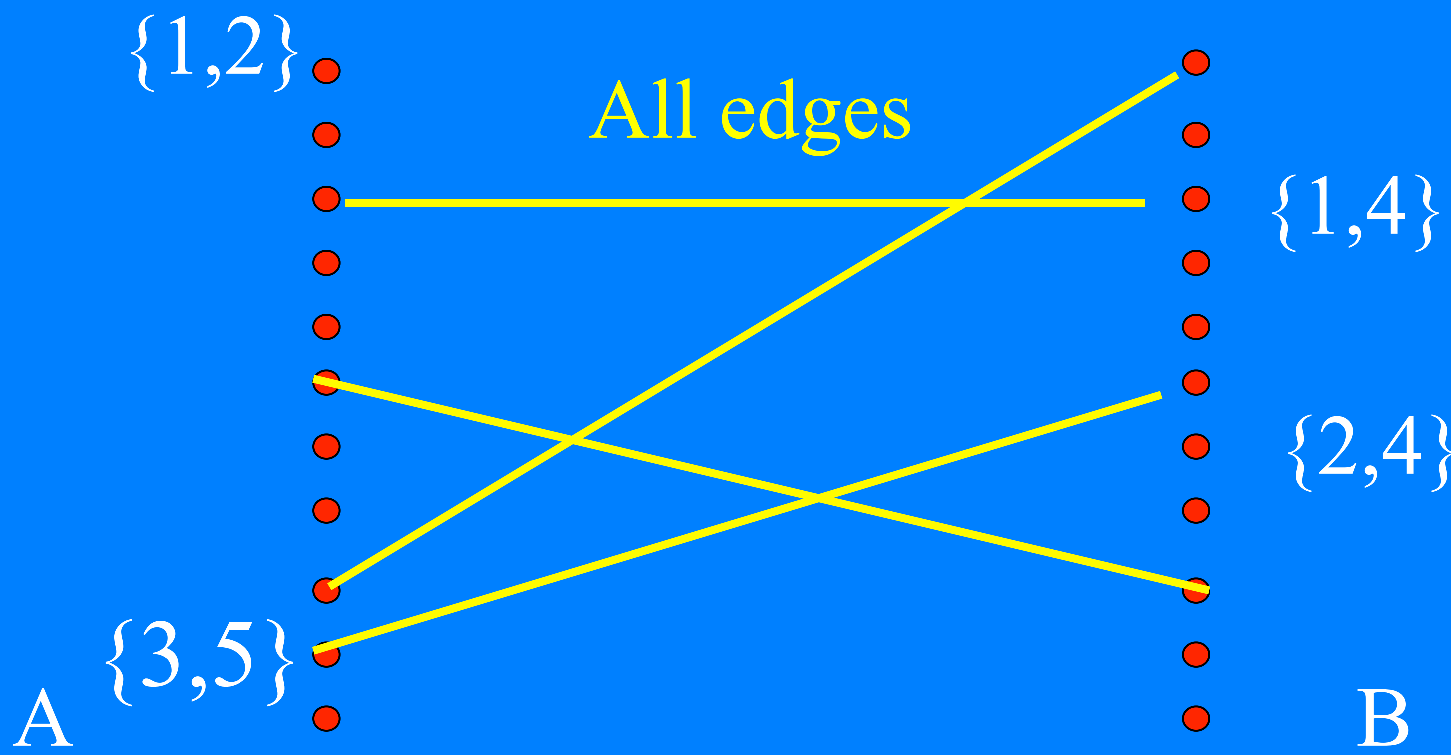
- $K_{m,n}$ has played an important role in list coloring.
- What is $I(K_{m,n}, S)$?

Sample result

- Consider $K_{10,10}$.
- Consider S: On class A, use the 10 2-element subsets of {1,2,3,4,5}. Same on B.
- What is $I(K_{10,10},S)$?

Complete Bipartite Graphs

- Consider $K_{10,10}$.
- Consider S: On class A, use the 10 2-element subsets of {1,2,3,4,5}. Same on B.
- What is $I(K_{10,10},S)$?



Complete Bipartite Graphs

- Suppose S* is obtained from S by adding colors.
- Suppose f(x) is a list coloring for $(K_{10,10}, S^*)$.
- Suppose f uses r colors on A and s on B.
- $r+s \leq 5$
- Let C(u,v) = binomial coefficient
- There are C(5-r,2) sets on A not using the r colors.
- Add one of the r colors to each of these sets.
- There are C(5-s,2) sets on B not using the s colors.
- Add one of the s colors to each of these sets.
- Get $I(K_{10,10}, S) \leq C(5\text{-}r,2) + C(5\text{-}s,2)$.
- Easy to see equality if r = 3, s = 2.
- So: $I(K_{10,10}, S) = 4$.

Complete Bipartite Graphs

• Similar construction for $K_{C(m,2),C(m,2)}$ and S defined by taking all C(m,2) subsets of {1,2,…,m} on each of A and B.

Complete Bipartite Graphs

Another Sample Result:

- Common assumption: All S(x) same size.
- Consider $K_{7,7}$ and any S with $|S(x)| = 3$, all x, and $|\cup S(x)| = 6$.
- Claim: $(K_{7,7},S)$ is 1-addable.
- Consider the seven 3-element sets S(x) on A.
- Simple combinatorial argument: There are i,j in {1,2,…,6} so at most one of these S(x) misses both
- S* obtained from S by adding i to such a set S(x).
- Take f(x) = i or j for any x in A.
- For all y in B, S*(y) = S(y) has 3 elements, so an element different from i and j can be taken as f(y).

Complete Bipartite Graphs

- Consider $K_{7,7}$ and S with $|S(x)| = 3$, all x, and $|\cup S(x)| = 7$.
- Claim: There is such an S so that $(K_{7,7},S)$ is not 0-addable.
- On A, use the seven sets $\{i,i+1,i+3\}$ and same on B, with addition modulo 7.
- Show that if f is a list coloring, $\{f(x): x \in A\}$ contains one of the sets $\{i,i+1,i+3\}$.
- This set is S(y) for some y in B, so we can't pick f(y) in S(y).

Upper Bounds on I(G,S)

- Clearly, $I(G,S) \leq |V(G)|$ if (G,S) is p-addable, some p.
- (Can add colors to at most each vertex.)

Proposition: If (G,S) is p-addable for some p, then

$$I(G,S) \leq |V(G)| - \omega(G),$$

where $\omega(G)$ = size of largest clique of G.

(Clique = set of vertices each of which is joined to each of the others.)

Upper Bounds on I(G,S)

• We know $I(G,S)/|V(G)| \leq 1$.

<u>Main Result:</u> There are (G,S) such that $I(G,S)/|V(G)|$ is arbitrarily close to 1.

• Interpretation: Situations exist where essentially everyone has to "sacrifice" by taking as acceptable an alternative not on their initial list.

• In channel assignment, there are situations where essentially every list of acceptable channels needs to be expanded. Similarly in other applications.

• Same result if all sets S(x) have same cardinality.

Upper Bounds on I(G,S)

- We know $I(G,S)/|V(G)| \leq 1$.

<u>Main Result:</u> There are (G,S) such that $I(G,S)/|V(G)|$ is arbitrarily close to 1.

- Interpretation: Situations exist where essentially everyone has to "sacrifice" by taking as acceptable an alternative not on their initial list.

- In physical mapping, there are situations where essentially every list of possible copies needs to be expanded.

- Same result if all sets S(x) have same cardinality.

Second Consensus Model: the Trading Model

- Allow side agreements among individuals.
- Allow trade (purchase) of colors from another's acceptable set.
- (The adding model paid no attention to where added colors came from.)
- In channel assignment: Allow possibility that channel was incorrectly recorded in set of possible channels of another transmitter and should be moved.

Second Consensus Model: the Trading Model

- Allow side agreements among individuals.
- Allow trade (purchase) of colors from another's acceptable set.
- (The adding model paid no attention to where added colors came from.)
- In physical mapping: Allow possibility that label was incorrectly recorded in set of possible labels of another clone and should be moved.

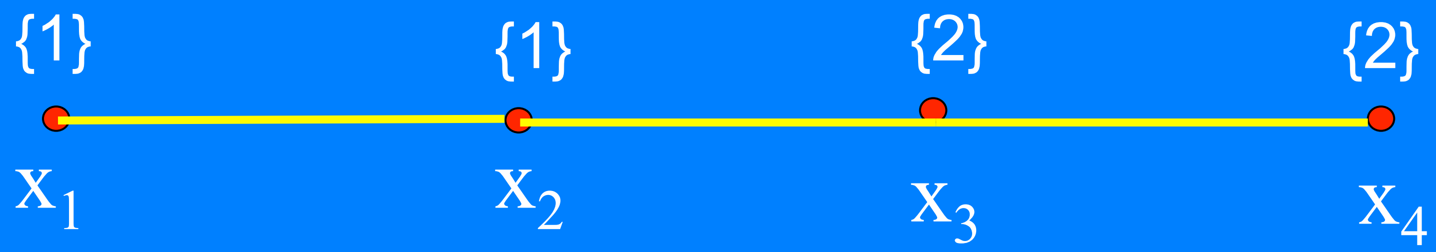
Second Consensus Model: the Trading Model

- Think of trades as taking place in sequence.
- **Trade from x to y**: Find color c in S(x) and move it to S(y).

p-Tradeability

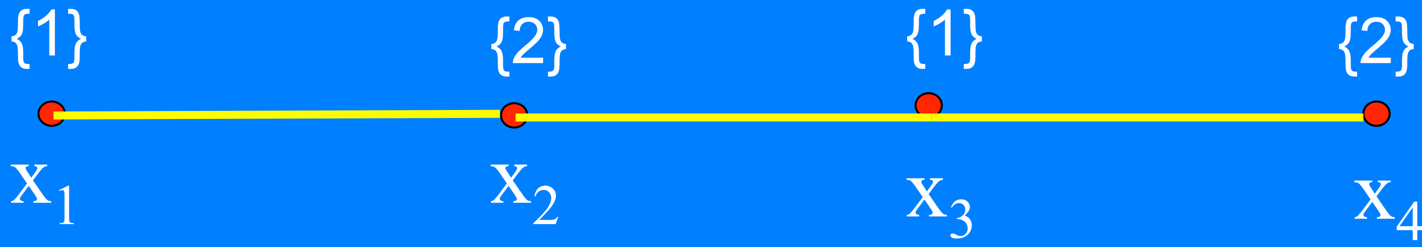
- How many trades are required to obtain a list assignment S* so that there is a list coloring?
- Say (G,S) is **p-tradeable** if this can be done in p trades.

(G,S)



If we trade color 2 from x_3 to x_2 and then color 1 from x_2 to x_3, we get (G,S*) that is list colorable.

(G,S*)



Thus, (G,S) is 2-tradeable.

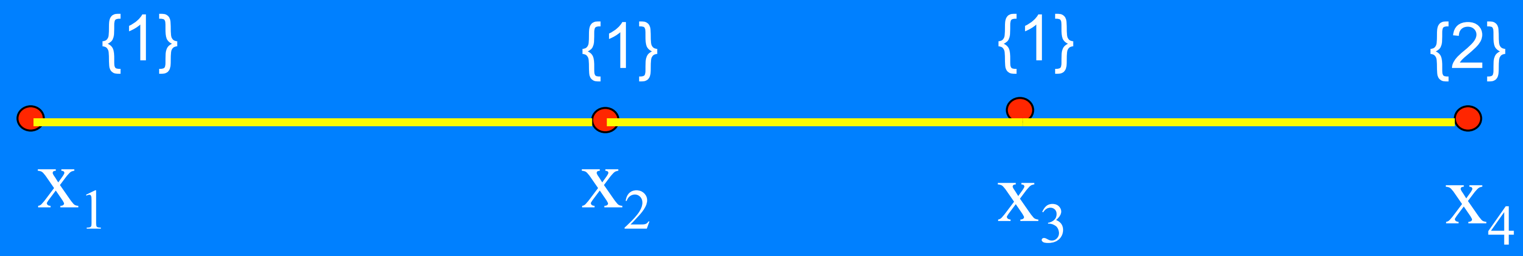
Second Consensus Model: the Trading Model

•If $|\cup S(x)| \geq 3$, then it is NP-complete to determine if (G,S) is p-tradeable for some p.

•Recall: (G,S) is p-addable for some p iff

$$|\cup S(x): x \in V| \geq \chi(G). \qquad (*)$$

•(*) not sufficient to guarantee p-tradeable for some p.



There are not enough 2's.

The Problem $(G,p_1,p_2,\ldots,p_r)$

- **Given G and positive integers p_1, p_2, …, p_r, is there a graph coloring of G so that for all i, the number of vertices receiving color i is at most p_i?**
- Let p_i = number of times i occurs in some S(x).
- Then (G,S) is p-tradeable for some p iff this Problem $(G,p_1,p_2,\ldots,p_r)$ has a positive answer.
- Problem $(G,p_1,p_2,\ldots,p_r)$ arises in "timetabling" applications (scheduling).
- DeWerra (1997): This is NP-complete even for special classes of graphs (e.g., line graphs of bipartite graphs)

The Problem $(G,p_1,p_2,\ldots,p_r)$

- Variants of this problem consider:
  - Case where all p_i are the same
  - Case where every color i must be used exactly p_i times
  - An edge coloring version
  - A list coloring version.
- See papers by
  - Hansen, Hertz, Kuplinsky
  - Dror, Finke, Gravier and Kubiak
  - Even, Itai and Shamir
  - Xu

The Trade Inflexibility

- Trade inflexibility $I_t(G,S)$ = smallest p so that (G,S) is p-tradeable. (May be undefined.)

- Observation: If (G,S) is p-tradeable for some p, then $I_t(G,S) \leq |V(G)|$.
- Proof: Suppose S* from S by sequence of trades and (G,S*) has list coloring f.
- If f(x) = i either i was in S(x) or it was added.
- So, we can arrange at most one incoming trade (namely of i) to each such set S(x) since there is no reason to add any other colors to S(x).
- So number of incoming trades can be limited to number of vertices

The Trade Inflexibility

- Main Result: There are (G,S) such that $I_t(G,S)/|V(G)|$ is arbitrarily close to 1.

- Same interpretation as for I(G,S).

The Trade Inflexibility

- Main Result: There are (G,S) such that $I_t(G,S)/|V(G)|$ is arbitrarily close to 1.

- There is also an implication for information sharing.
- The result says that if everyone has relevant information, it may be necessary for almost everyone to share some information in order for the group to reach consensus.
- Information may be your knowledge about acceptable channels for a transmitter, for example.

Trades Only Allowed to Neighbors

•Might apply in channel assignment – if interference corresponds to physical proximity.
•Not clear what this means in other applications we have talked about.

•(G,S) is **p-neighbor-tradeable** if there is a sequence of p trades, each from a vertex to a neighbor, resulting in a list-colorable list assignment.

•$I_{t,n}(G,S)$ = smallest p so that (G,S) is p-neighbor-tradeable

Trades Only Allowed to Neighbors

- In contrast to p-tradeability, $I_{t,n}(G,S)$ can be larger than $|V(G)|$
- In fact, $I_{t,n}(G,S)/\ |V(G)|$ can be arbitrarily large.
- Proof coming.

Third Consensus Model: The Exchange Model

- Instead of one-way trades, use two-way exchanges.
- A color from S(x) and a color from S(y) are interchanged at each step.
- In channel assignment: channels of two transmitters are transposed.
- In physical mapping: labels of two clones are transposed.
- Similarly in other applications
- Consider a sequence of exchanges.

Third Consensus Model: The Exchange Model

- Note that p exchanges can be viewed as 2p trades.
- However, sometimes one can accomplish the equivalent of p exchanges in less than 2p trades.

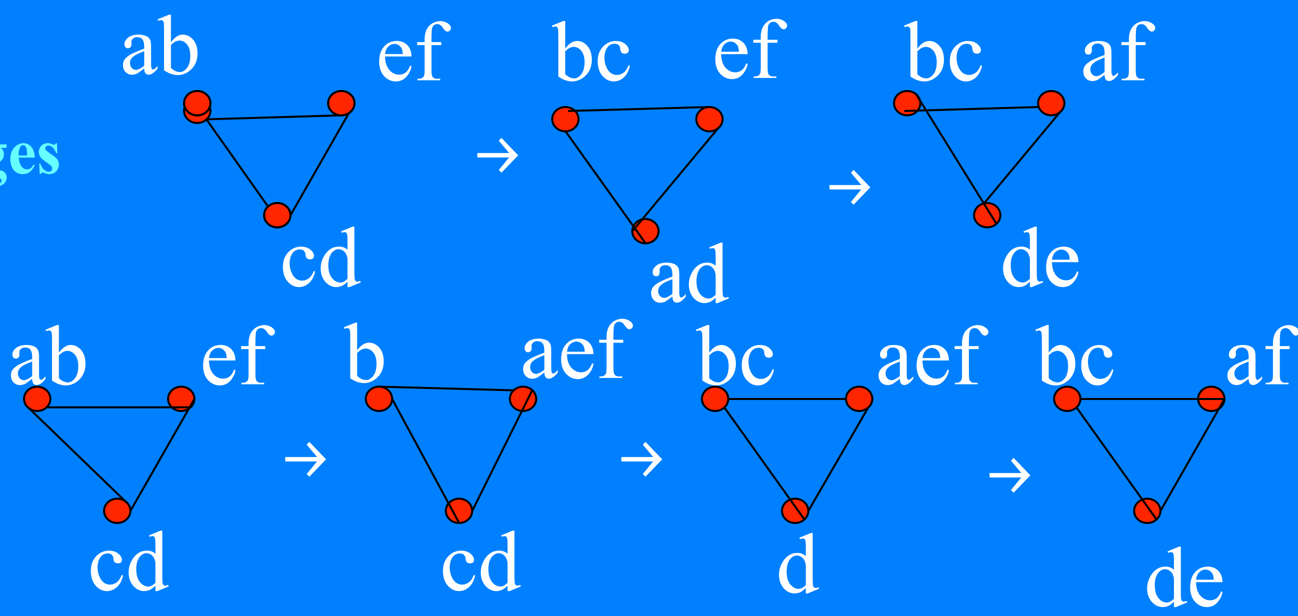
Third Consensus Model: The Exchange Model

•Note that p exchanges can be viewed as 2p trades.

•However, sometimes one can accomplish the equivalent of p exchanges in less than 2p trades.

2 exchanges

3 trades



p-Exchangeability

- How many exchanges are required to obtain a list assignment with a list coloring?

- (G,S) is **p-exchangeable** if this can be done in p exchanges.

- Clearly, (G,S) is p-exchangeable for some p iff (G,S) is q-tradeable for some q.

- Observation: If $|\cup S(x)| \geq 3$, then it is NP-complete to determine if (G,S) is p-exchangeable for some p.

The Exchange Inflexibility

- $I_e(G,S)$ = smallest p so that (G,S) is p-exchangeable. (May be undefined.)
- Observation: If (G,S) is p-exchangeable for some p, then $I_e(G,s) \leq |V(G)|$.
- Main Result: There are (G,S) such that $I_e(G,S)/|V(G)|$ is arbitrarily close to 1.
- Interpretation: As before.

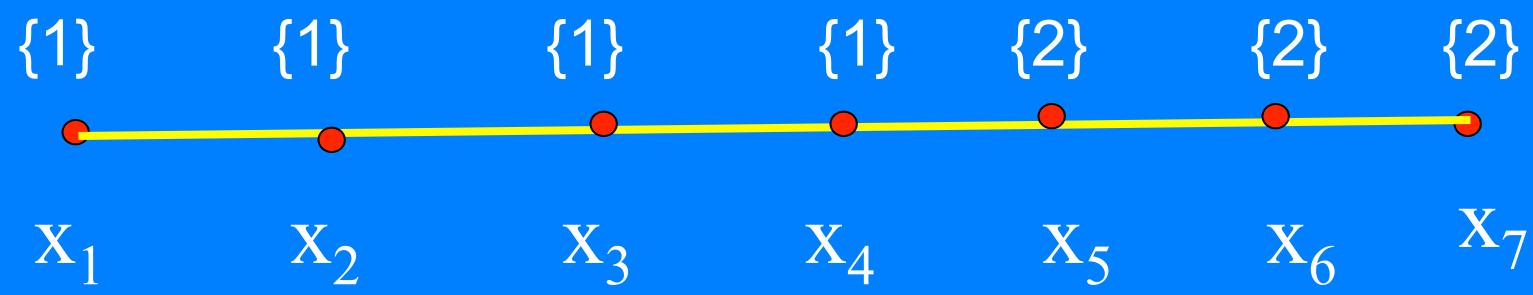
Exchanges Only Allowed between Neighbors

- (G,S) is **p-neighbor-exchangeable** if there is a sequence of p exchanges, each between neighbors, resulting in a list-colorable list assignment.

- $I_{e,n}(G,S)$ = smallest p so that (G,S) is p-neighbor-exchangeable. (Undefined if no such p.)

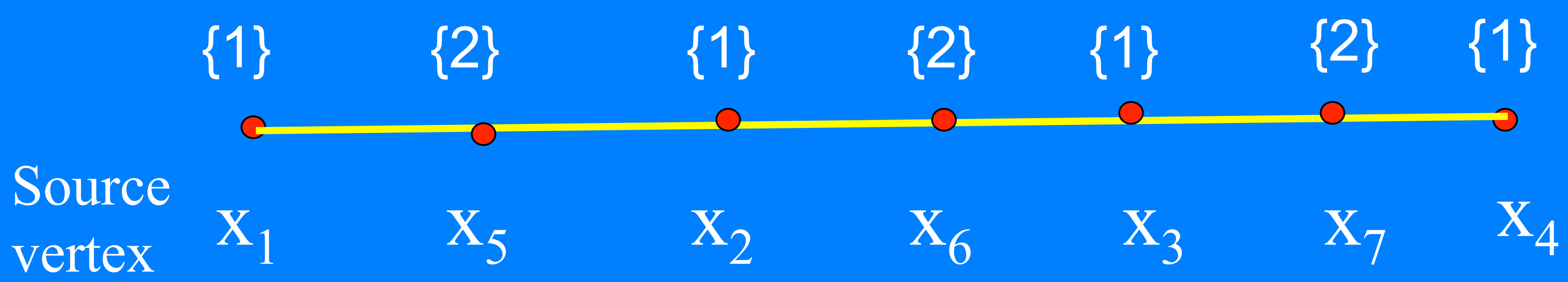
- $I_{e,n}(G,S)/|V(G)|$ can be arbitrarily large.

Consider a path of 2k+1 vertices with k+1 sets $S(x) = \{1\}$ at the beginning and k sets $S(x) = \{2\}$ at the end.

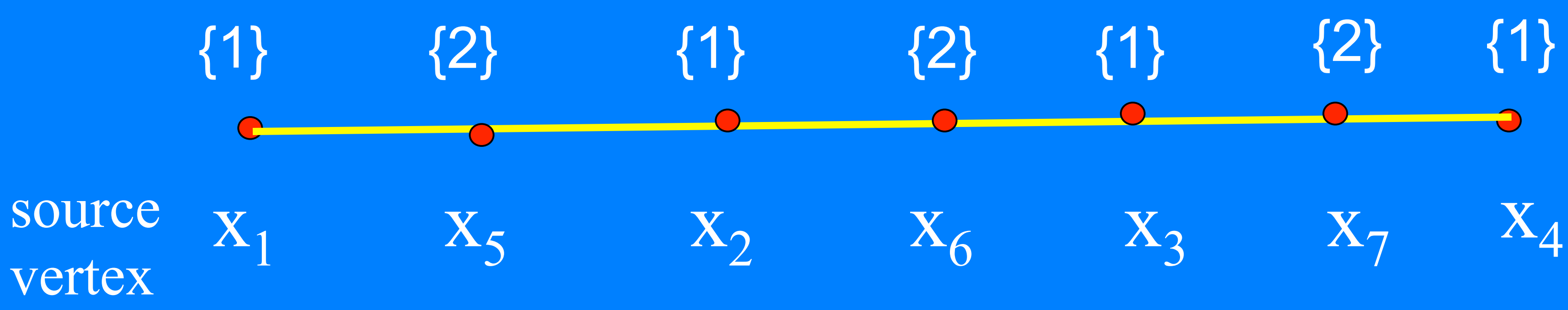


$k = 3$

The only way to color this path with colors from the set $\cup S(x) = \{1,2\}$ is to alternate colors. Thus, we must move 2's to the left in the path and 1's to the right.

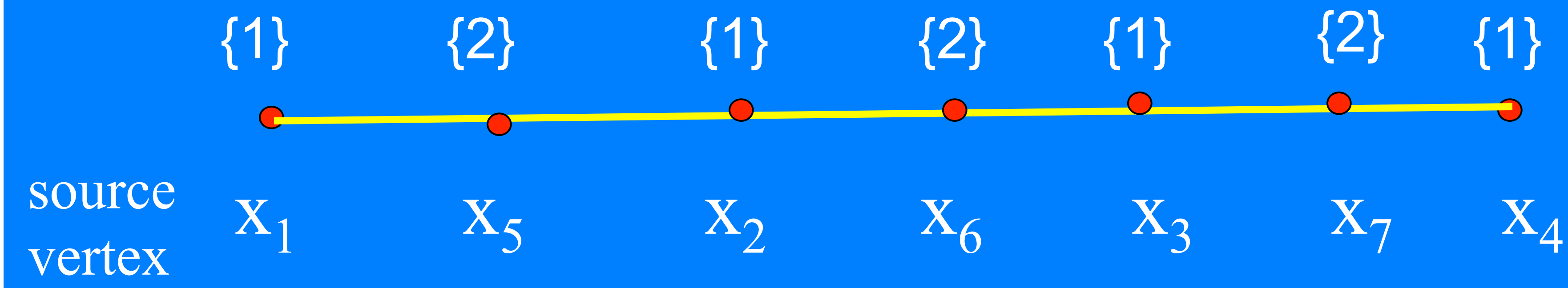


Doing this by a series of exchanges between neighbors is analogous to changing the identity permutation into another permutation by transpositions of the form (i i+1). The number of transpositions required to do this is well known (and can be computed efficiently by bubble sort).



Jerrum (1985): Number of transpositions (i i+1) required to transform identity permutation into permutation π is

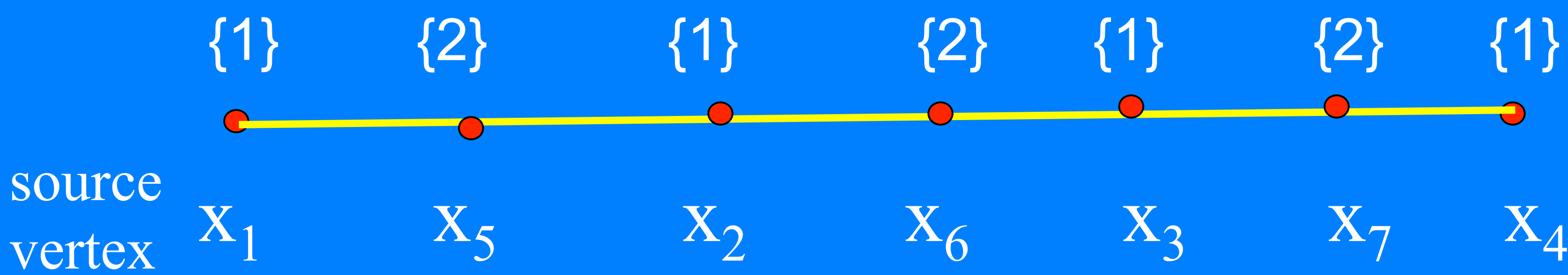
$$J(\pi) = |\{(i,j): 1 \leq i < j \leq n \;\&\; \pi(i) > \pi(j)\}|.$$



Here, $J(\pi) = k(k+1)/2$.
Thus,

$$I_{e,n}(G,S)/|V(G)| = k(k+1)/2(2k+1) \rightarrow \infty$$

as $k \rightarrow \infty$.



An analogous proof shows that $I_{t,n}(G,S)/|V(G)|$ can be arbitrarily large.

Sketch of Proof of One of Main Results

We show that $I_t(G,S)/|V(G)|$ can be arbitrarily close to 1.

Sketch of Proof of One of Main Results

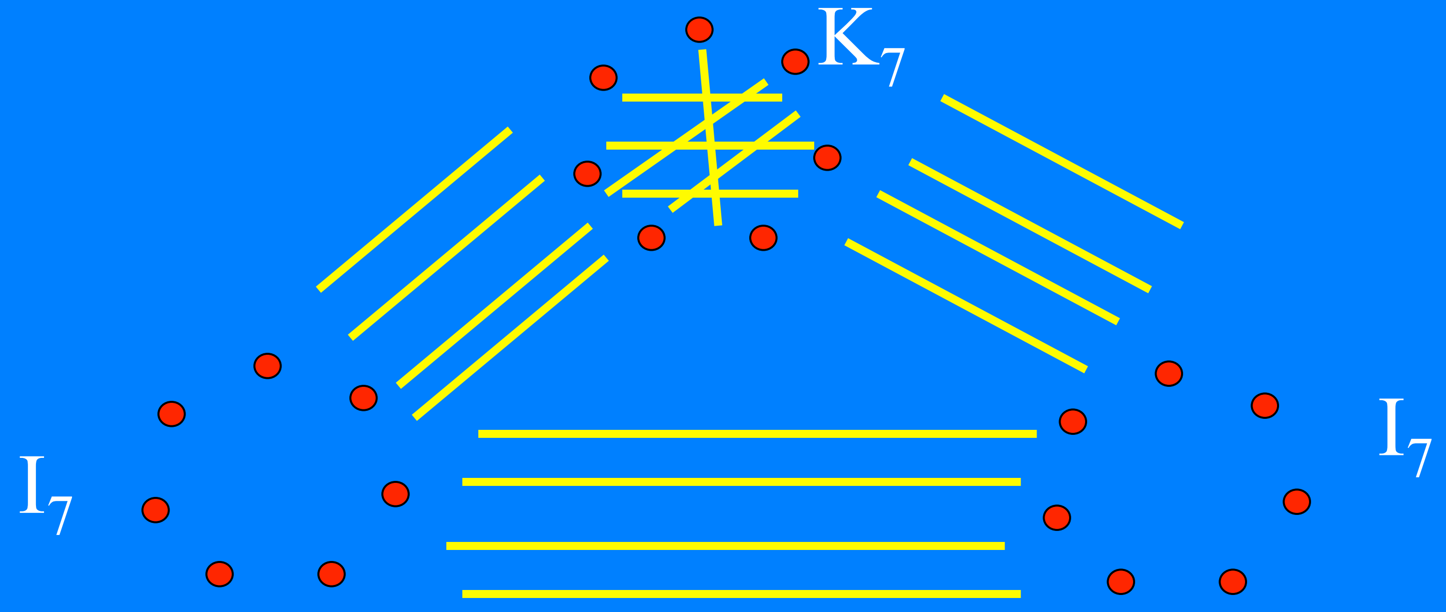
We show that $I_t(G,S)/|V(G)|$ can be arbitrarily close to 1.

K_m = complete graph on m vertices: every vertex is joined to every other vertex.

I_m = graph with m vertices and no edges.

The Graph G(m,p)

- Suppose that $m > 3p+2$.
- Take K_{m-p} and p copies of graph I_{m-p}
- Join every vertex of these p+1 graphs to every other vertex of each of these graphs.



G(9,2)

Definition of S:

On K_{m-p}: Use the sets

$\{i, i+1, m-p+1, m-p+2, \ldots, m\}$

$i = 1, 2, \ldots, m-p-1$

and the set

$\{m-p, 1, m-p+1, m-p+2, \ldots, m\}$.

On each copy of I_{m-p}, use the sets

$\{i, i+1\}$

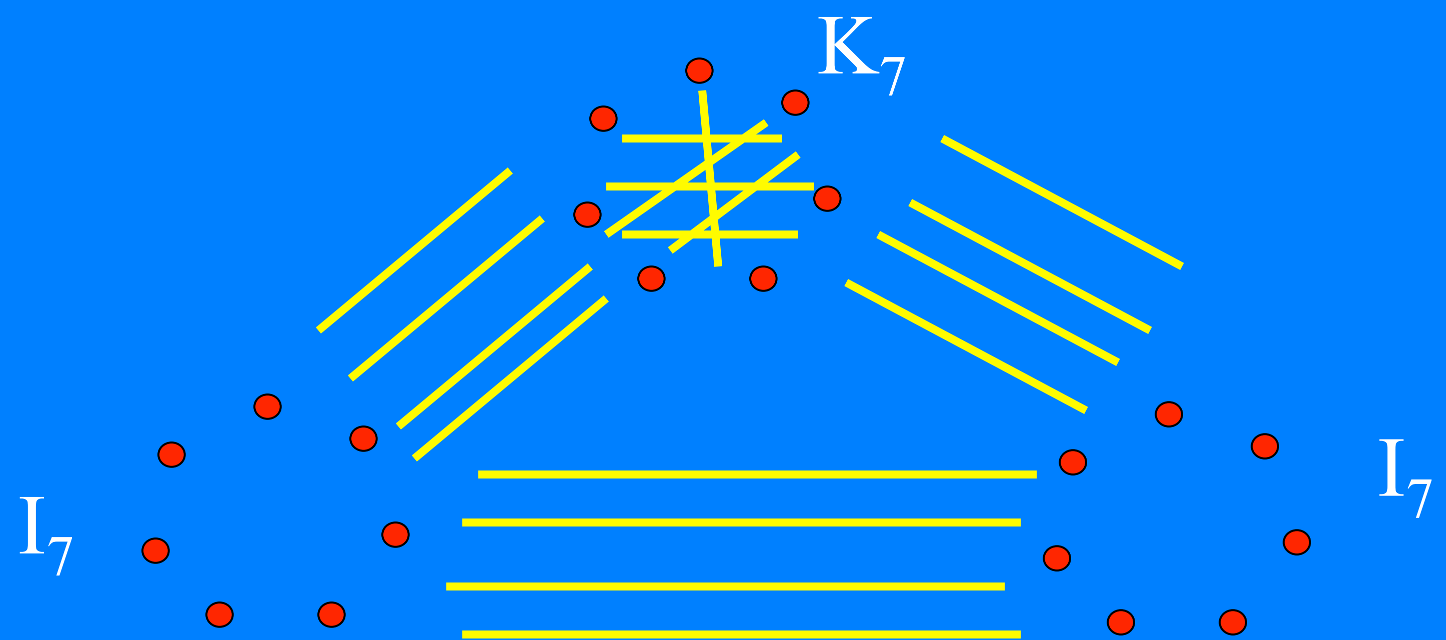
$i = 1, 2, \ldots, m-p-1$

and the set

$\{m-p, 1\}$.

The Graph G(m,p)

1289, 2389, 3489, 4589, 5689, 6789, 7189



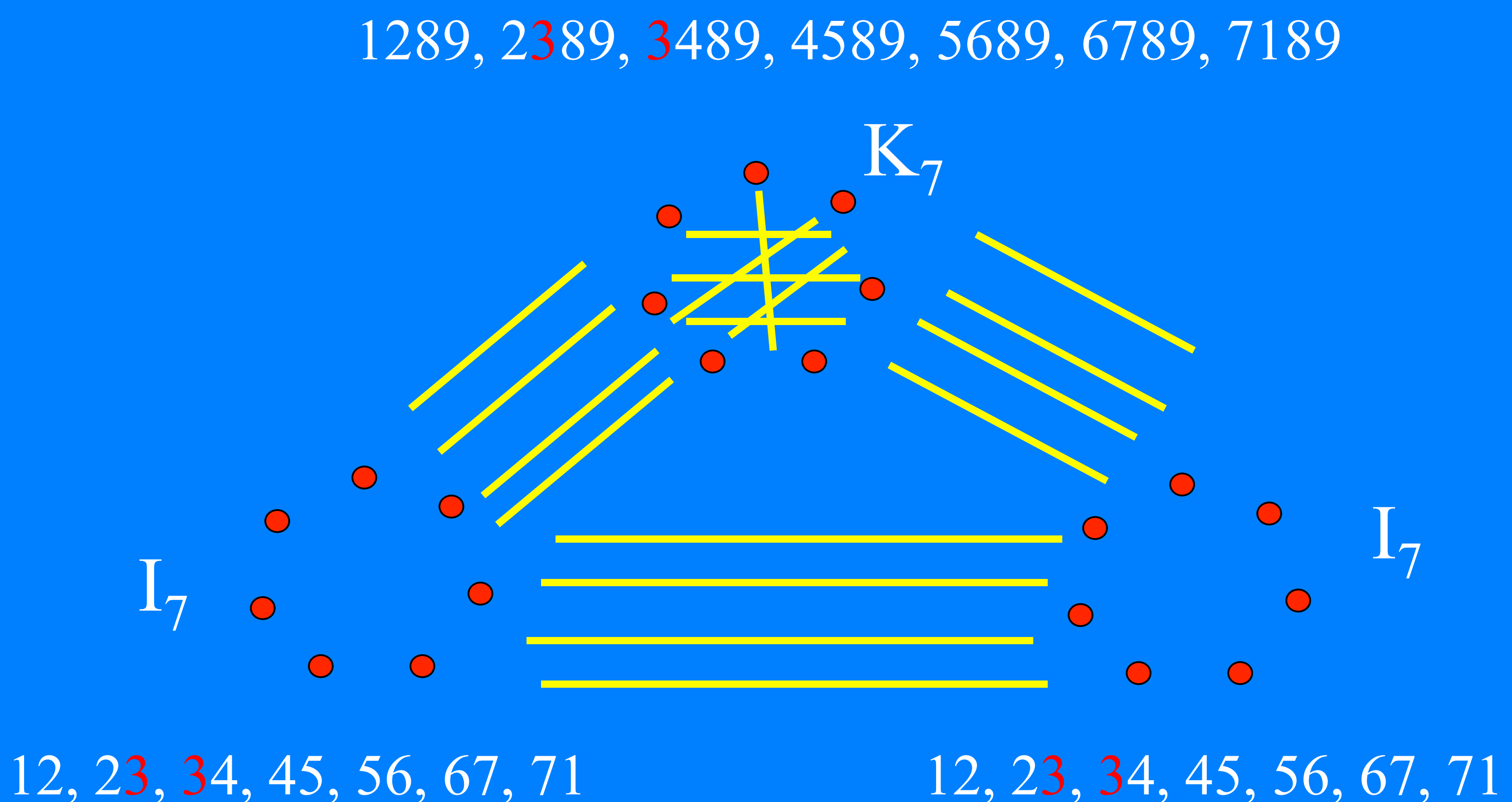
12, 23, 34, 45, 56, 67, 71   12, 23, 34, 45, 56, 67, 71

G(9,2)

Continuing the Proof:

- Let f be a list coloring obtained after trades give a new list assignment S*.
- Let $i \in \{1, 2, \ldots, m\text{-}p\}$.
- Then i appears in two sets S(x) on K_{m-p} and two sets S(x) on each I_{m-p}.
- So, i appears in 2(p+1) sets in all.
- There are m colors available.
- In f, we need m-p of them for K_{m-p}, leaving p of them for the I_{m-p}'s.
- No two I_{m-p}'s can have a color in common.
- Thus, each uses exactly one color.

The Graph G(m,p)



G(9,2)

Continuing the Proof:

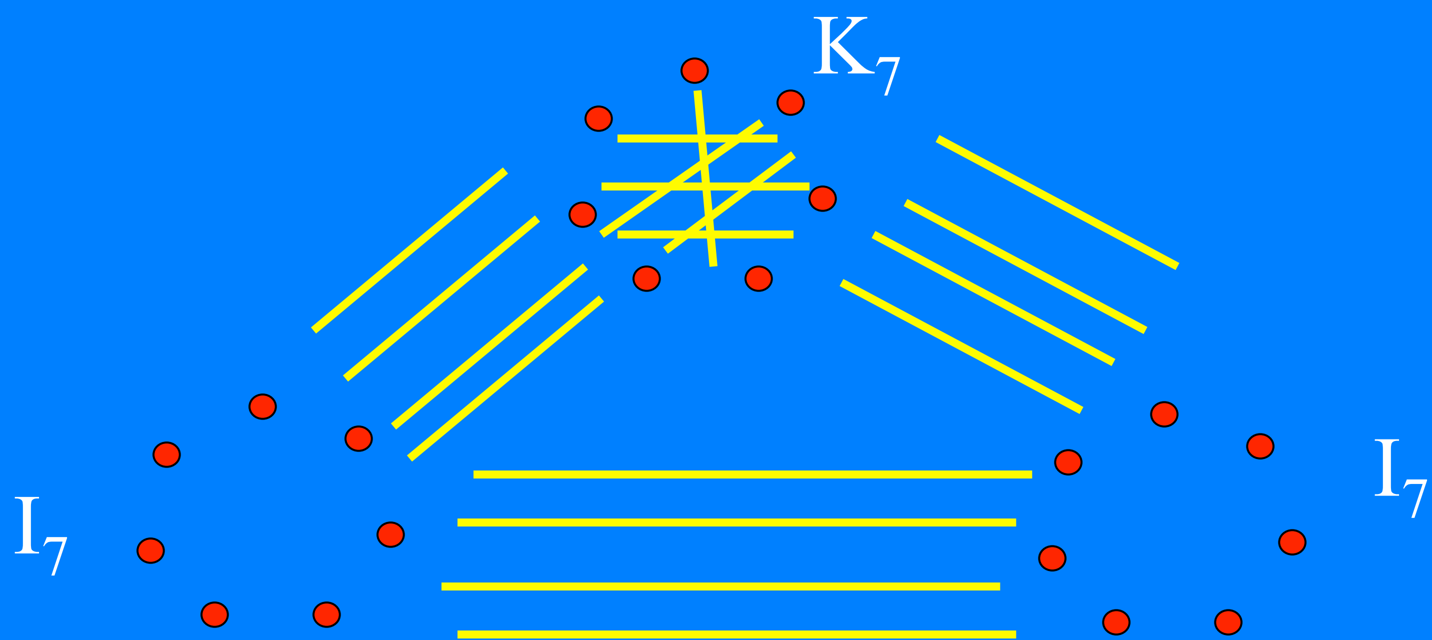
- Since $m > 3p+2$, $m-p > 2(p+1)$.
- So there are not enough copies of any color $i \leq m-p$ available to trade to the $m-p$ vertices of I_{m-p} since each appears in $2(p+1)$ sets $S(x)$.
- Hence, on each I_{m-p}, f uses a color in

  $\{m-p+1, m-p+2, \ldots, m\}$
- There are p such colors, one for each I_{m-p}.
- Thus, f on K_{m-p} must use colors 1, 2, …, m-p.
- So, f uses color m-p+1 on all vertices of one I_{m-p}, m-p+2 on all vertices of a second I_{m-p}, …, and color m on all vertices of a p^{th} I_{m-p}.

The Graph G(m,p)

Must Use Colors 1, 2, 3, …, 7 on K_7

1289, 2389, 3489, 4589, 5689, 6789, 7189



12, 23, 34, 45, 56, 67, 71 12, 23, 34, 45, 56, 67, 71

G(9,2)

Must Use Color 8 on one I_7 and 9 on other I_7

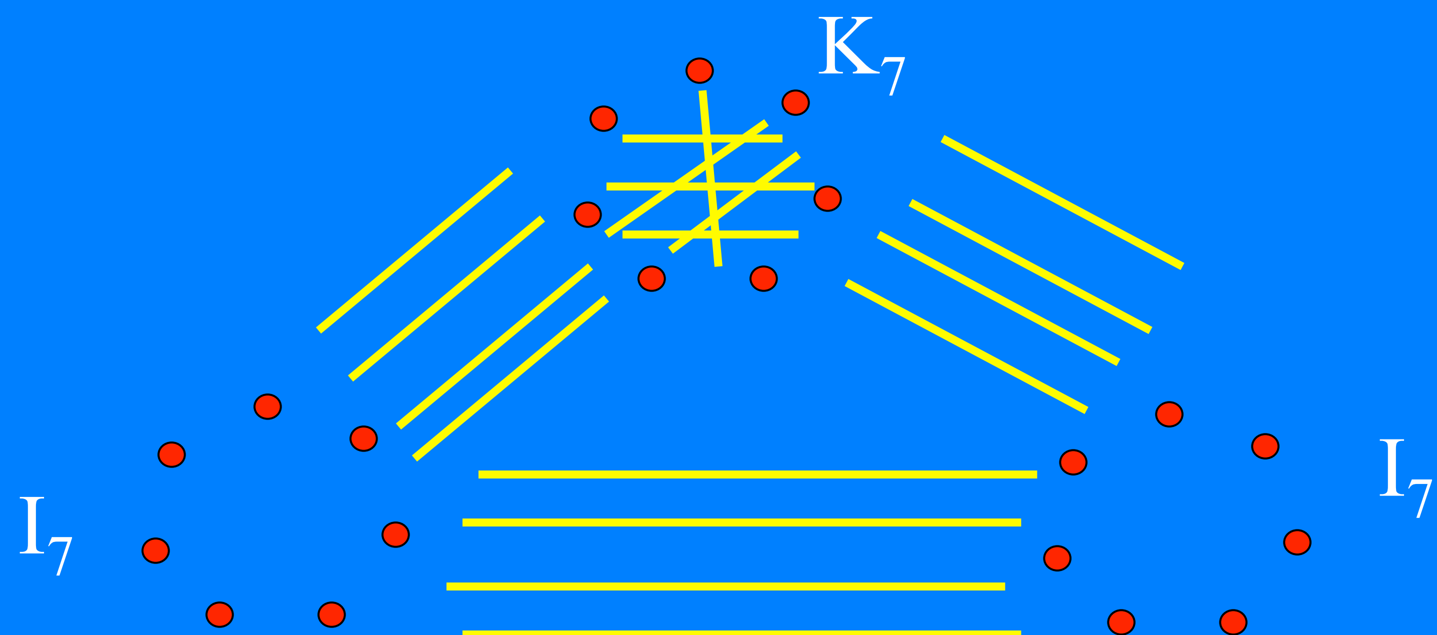
Continuing the Proof:

- To obtain S*, we must for each color among m-p+1, m-p+2, …, m find m-p copies to trade to one of the graphs I_{m-p}.
- Thus, we need a minimum of p(m-p) trades.
- This number suffices.

The Graph G(m,p)

Move all the 8's to one of the I_7's and all 9's to the other

1289, 2389, 3489, 4589, 5689, 6789, 7189



12, 23, 34, 45, 56, 67, 71 12, 23, 34, 45, 56, 67, 71

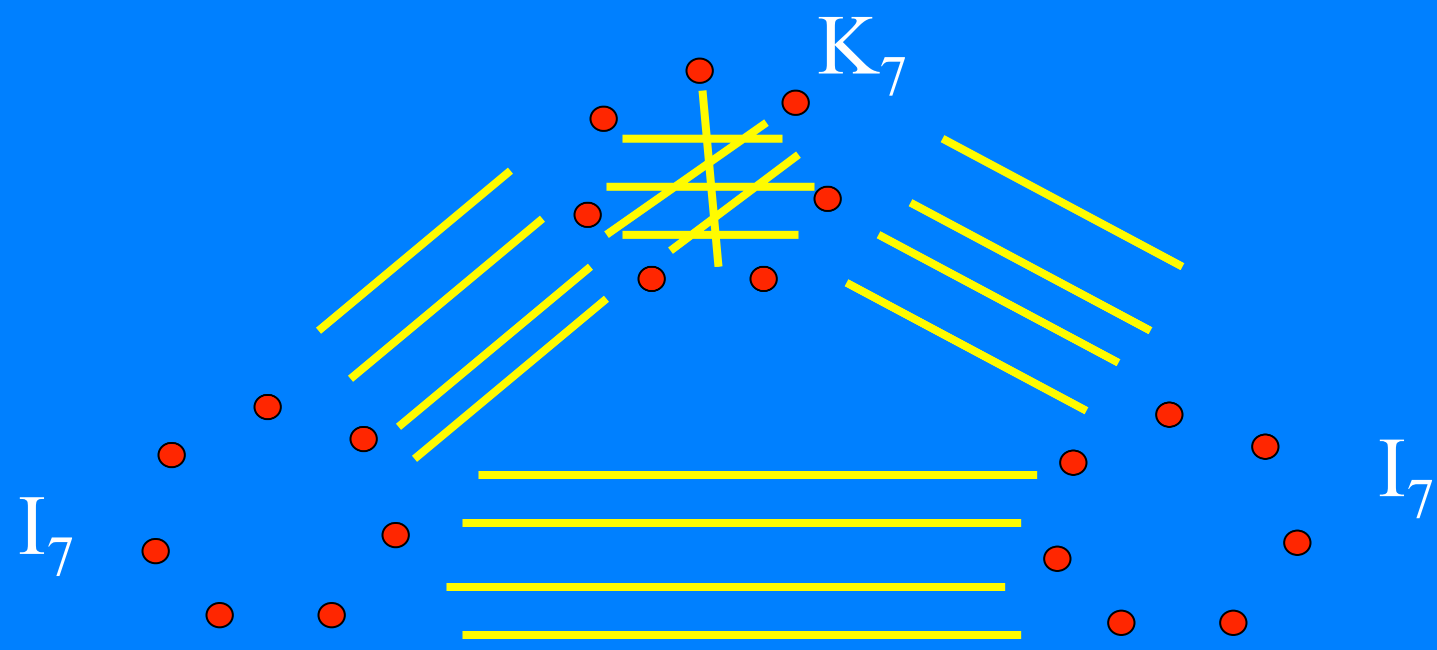
G(9,2)

Must Use Color 8 on one I_7 and 9 on other I_7

The Graph G(m,p)

Move all the 8's to one of the I_7's and all 9's to the other

12, 23, 34, 45, 56, 67, 71



128, 238, 348, 458, 568, 678, 718 129, 239, 349, 459, 569, 679, 719

G(9,2)

Must Use Color 8 on one I_7 and 9 on other I_7

Continuing the Proof:

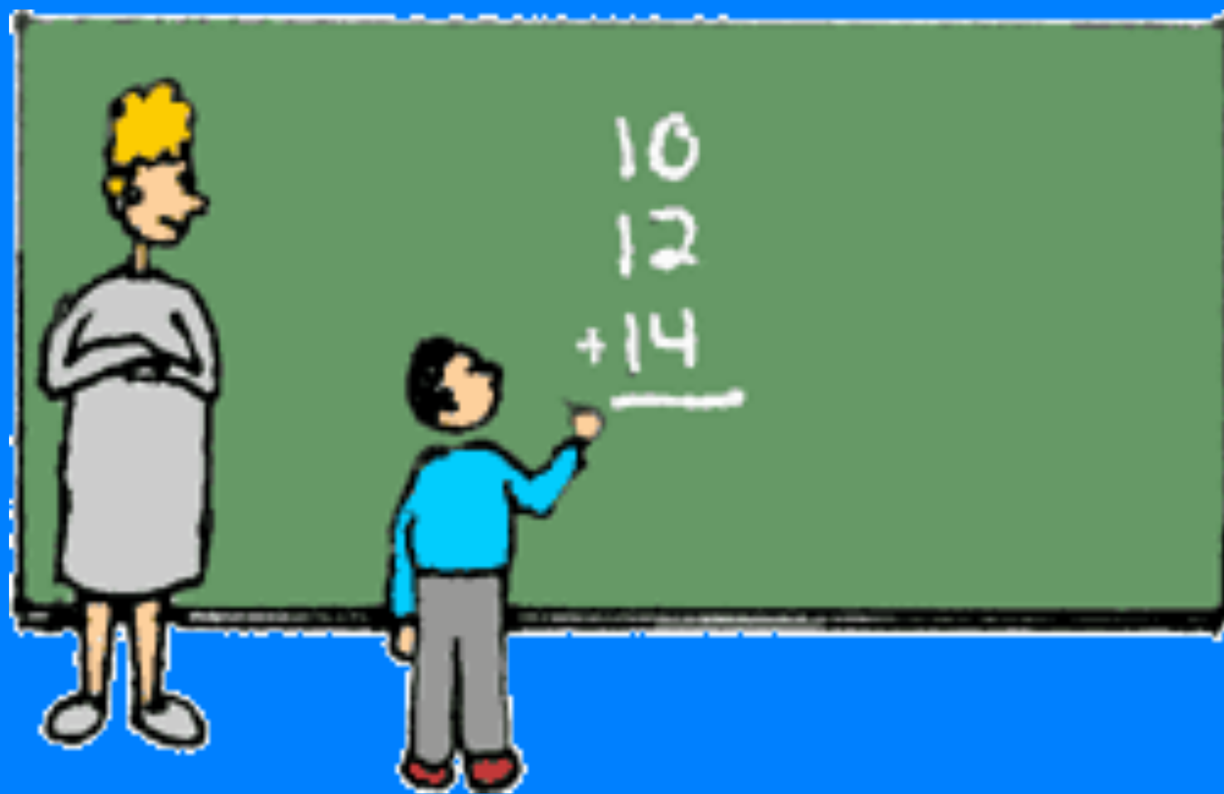
So:

$I_t(G,S) = p(m-p)$

$$I_t(G,S)/|V(G)| = p(m-p)/(p+1)(m-p)$$
$$= p/(p+1)$$
$$\rightarrow 1$$

as $p \rightarrow \infty$

Open Problems



Open Problems

We have presented three procedures for individuals to modify their acceptable sets in order for the group to achieve a list colorable situation.

So far, very little is known about these procedures.

Some Mathematical Questions:

1. Under what conditions is (G,S) p-tradeable for some p?
2. Under what conditions is (G,S) p-exchangeable for some p?

Some Mathematical Questions

3. What are the values of or bounds for the parameters $I(G,S)$, $I_t(G,S)$, $I_e(G,S)$, $I_{t,n}(G,S)$, $I_{e,n}(G,S)$ for specific graphs or classes of graphs and specific list assignments or classes of list assignments?
4. What are the values of or bounds for these parameters under the extra restriction that all sets $S(x)$ have the same fixed cardinality?
5. What are good algorithms for finding optimal ways to modify list assignments so that we obtain a list colorable assignment under the different consensus models?

Consensus Issue

Are there other examples where obtaining a consensus requires almost everyone to compromise?

Some Questions Relating to Physical Mapping

6. Given a graph G with a list assignment S, can we remove edges from G, obtaining an interval graph H, so that H with S has a list coloring? If so, what is the smallest number of edges we can remove to get such an H?

7. Given (V,E_1), (V,E_2) with $E_1 \subset E_2$, and S on V, is there a set E so that $E_1 \subset E \subset E_2$ with $G = (V,E)$ an interval graph and (G,S) list colorable?


Thank
You !!