Energy Balance Models

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Module Summary. This module introduces the student to the process of mathematical modeling. It shows how the process starts in the "real world" with a physical system and some observations or an experiment. When the laws of physics that are thought to govern the behavior of the system are translated in mathematical terms, the result is what is called a *mathematical model*. The mathematical model is subsequently analyzed for its properties and used to generate predictions about the behavior of the system in a changing environment. These predictions are tested against observations, and if there is agreement between predictions and observations, the model is accepted; otherwise, the model is refined, for example by bringing in more details of the physics, and the process is repeated. Thus, mathematical modeling is an *iterative process*.

To illustrate this iterative process, this module builds a series of *zero-dimensional energy balance models* for the Earth's climate system. In a zero-dimensional energy balance model, the Earth's climate system is described in terms of a single variable, namely the temperature of the Earth's surface averaged over the entire globe. In general, this variable varies with time; its time evolution is governed by the amount of energy coming in from the Sun (in the form of ultraviolet radiation) and the amount of energy leaving the Earth (in the form of infrared radiation). The mathematical challenge is to find expressions for the incoming and outgoing energy that are consistent with the observed current state of the climate system, and then use the resulting energy balance model to see whether the climate system admits other equilibrium states and, if so, how a transition from one equilibrium state to another could be triggered.

The module includes descriptions of several simple experiments that illustrate various concepts used in the discussion. They require little or no special equipment.

Informal Description. This module introduces the student to the mathematical modeling process by showing how to build a zero-dimensional energy balance model for the Earth's climate system. The process is an iterative one and generates various versions of the model. Successive versions include more physics to better match the observations. The emphasis in the module is on the *process*, rather than the models derived in the process, because the process is universal and independent of the complexity of the model. The process is illustrated in Figure 1.

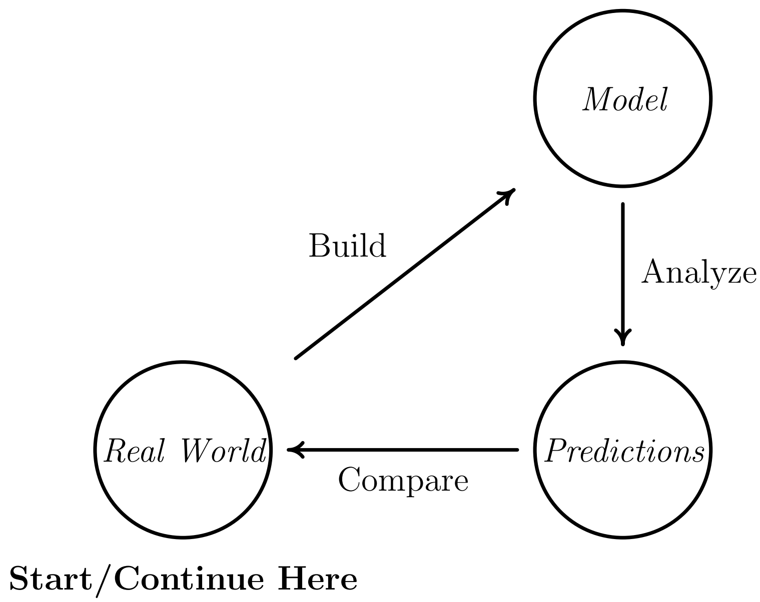

Figure 1: The modeling cycle.

The mathematical modeling process starts in the "real world" with a physical system and some observations or an experiment. We assume that the behavior of the system is governed by the laws of nature—Newton's law of motion, Fourier's law of heat conduction, etc. When these laws are formulated in mathematical terms, we obtain what we call a "mathematical model"—a set of mathematical equations that describe the state of the physical system as it evolves in time. In the next step of the modeling process, we "analyze" the model—that is, we apply our mathematical knowledge to extract information from the model, to see whether we understand and can explain what we see in the real world. In the third step we use the model to make predictions about what we will see in additional experiments and observations. We then return to the real world to test these predictions by running the experiments or collecting more observations, and either accept the model if we find that the outcome matches our predictions, or refine the model if we find that improvements are needed. Typically, we go around this modeling cycle many times, building progressively better models, thus improving our understanding of the physical system and increasing our ability to make predictions about its behavior.

In this module, the physical system of interest is the Earth's climate system—a prototypical "complex system" that has many components: the atmosphere, oceans, lakes and other bodies of water, snow and ice, land surface, all living things, and so on. The components interact and influence each other in ways that we don't always understand, so it is difficult to see how the system as a whole evolves, let alone why it evolves the way it does. For some complex system it is possible to build a physical model and observe what happens if the environment changes. This is the case, for example, for a school of fish, whose behavior we can study in an aquarium. It is also true for certain aspects of human behavior, which we can study in a social network. But in climate science this is not possible; we have only one Earth, and we cannot perform a controlled real-life experiment. The best we can do if we want to gain insight into what might have happened to the Earth's climate system in the past, or what might happen to it in the future, is to build mathematical models and "play" with them. Mathematical models are the climate scientists' only experimental tools.

The modeling process—building and testing a series of imperfect models—is the most essential brick in the foundation of climate science and an indispensable tool to evaluate the arguments for or against climate change. Models are never perfect—at best, they provide some understanding and some ability to test "what-if" scenarios. Especially in an area as complex as the Earth's climate, we cannot and should not expect perfection. Recognizing and identifying imperfection and uncertainty are key parts of all modeling and, especially, climate modeling.

Mathematical models of the Earth's climate system come in many flavors. They can be simple—simple enough that we can use them for back-of-the-envelope calculations, or they can be so complicated that we need a supercomputer to learn what we want to know. But whatever kind of models we use, we should always keep in mind that they are *simplified representations* of the real world, they are not the "real world," and they are made for a purpose, namely to better understand what is driving our climate system.

The present module looks at zero-dimensional energy balance models. They are the simplest possible description of the Earth's climate system. But as we will see, they can provide insight into possible climate states. In these models, the state of the climate system is characterized by a single variable—the temperature of the Earth's surface, averaged over the entire globe. An *energy balance equation* is a formal statement of the fact that the temperature of the Earth increases if the Earth receives more energy from the Sun than it re-emits into space, and that it decreases if the opposite is the case. The module shows how to construct energy balance models by finding mathematical expressions for the incoming and outgoing energy. The models are tested against "real-world" data and improved in successive steps of the iterative modeling process to better match the available data.

In this module, the focus is on the physics, but we emphasize that modeling the Earth's climate system is fundamentally an interdisciplinary activity. Understanding the Earth's climate requires knowledge, skills, and perspectives from multiple disciplines. For example, atmospheric chemistry explains why much of the incoming energy from the Sun (largely in the ultraviolet and visible regions of the spectrum) passes through the atmosphere and reaches the Earth's surface, but much of the blackbody radiation emitted by the Earth (largely in the infrared regions of the spectrum) is trapped by greenhouse gases like water vapor and carbon dioxide. Similarly, the life sciences help us understand the part played by the biosphere in the Earth's climate system—the effects of the biosphere on the Earth's albedo and the interactions between atmospheric chemistry and plant and animal life.

The module includes descriptions of several simple experiments that can be done to illustrate various concepts used in the module. They require little or no special equipment.

Target Audience. This module is suitable for undergraduate students in the mathematical sciences.

Prerequisites. Basic knowledge of the concept of derivatives and ordinary differential equations.

Mathematical Fields. Algebra, ordinary differential equations.

Applications Areas. Geophysics and climate science.

Goals and Objectives.

- Teach the process of "mathematical modeling."

- Show how a simple model like the zero-dimensional energy balance model can provide insight into aspects of climate dynamics.
- Show that nonlinear models can have multiple solutions.

The Module

What is Climate and Who Cares?

The following quote is from a speech given by Senator Sheldon Whitehouse (D-RI) and uploaded to YouTube on Oct. 13, 2011.[1]

> "Mr. President, I am here to speak about what is currently an unpopular topic in this town. It has become no longer politically correct in certain circles in Washington to speak about climate change or carbon pollution or how carbon pollution is causing our climate to change.
>
> This is a peculiar condition of Washington. If you go out into, say, our military and intelligence communities, they understand and are planning for the effects of carbon pollution on climate change. They see it as a national security risk. If you go out into our nonpolluting business and financial communities, they see this as a real and important problem. And, of course, it goes without saying our scientific community is all over this concern. But as I said, Washington is a peculiar place, and here it is getting very little traction."

Perhaps the silliest (although there are many contenders for this "honor") contribution (as reported by Fox Nation[2]) to this discourse is the igloo Senator James Inhofe's (R-OK) family is reported to have built for Al Gore during a Washington, DC, snowstorm (Figure 2). The Senator's family confused *weather* with *climate.*

A single snowstorm in a particular place, even a snowstorm in Washington, DC, is weather. Weather takes place over periods of hours, days, or even a few years. Climate, on the other hand, takes place over periods of tens, hundreds, and thousands of years. The difference between weather and climate is one of scale—both in time and in space. The temperature in Detroit, Michigan, today is weather, but the average temperature of a large region of the Earth and over a period of many years is climate.

Climate skeptics, those who deny the effects we humans are having on the Earth's climate, often say that, since it is difficult to forecast the weather in a particular place even a few days in advance, it is impossible to forecast climate. But, paradoxically,

[1] http://www.youtube.com/watch?v=k6VQ0vYfrAw

[2] http://mediamatters.org/blog/201002100004, accessed October 21, 2011.

Figure 2: Sen. Inhofe's family builds an igloo for Sen. Gore.

it is easier to forecast climate. This situation is not unusual—for example, it is hard to forecast who will be killed in automobile accidents on a particular New Year's Eve but easy to forecast that many will be killed.

Mathematics is a quantitative discipline. We prefer precise statements to vague, more qualitative statements. The question to ask is then: "Which measurable property gives us the most information about the state of the climate system?" There are several candidates; for example, we might track the average temperature of the entire Earth over periods of many years, or the annual average amount of precipitation over the entire Earth. In this module we will work with the average temperature of the entire Earth. One reason why this is a useful measure is that we have a considerable amount of data. For recent years we have data from large networks of sensors as well as data from satellites, and by analyzing things like tree rings, ice core samples and ocean sediments we can estimate the average temperature of the Earth for many thousands or even millions of years back into the past.

Then the next question is: "Can we explain this past record of the Earth's climate?" This question is much more difficult to answer. An even more difficult question is: "Can we use these data to make predictions about the future of the Earth's climate?" This is where mathematics and mathematicians come in. Mathematicians build *mathematical models*, which are the "instruments" that enable us to find answers to these more difficult questions. This module will show you what we mean by a mathematical model of the Earth's climate and how we can go about constructing such models.

1 Climate Model – Cycle #1

We consider the Earth with its atmosphere, oceans, and all other components of the climate system as a homogeneous solid sphere, ignoring differences in the atmosphere's composition (clouds!), differences among land and oceans, differences in topography (altitude), and many other things.

1.1 Observation

The climate system is powered by the Sun, which emits radiation in the ultraviolet (UV) regime (wavelength less than $0.4\,\mu$m). This energy reaches the Earth's surface, where it is converted by physical, chemical, and biological processes to radiation in the infrared (IR) regime (wavelength greater than $5\,\mu$m). This IR radiation is then reemitted into space. If the Earth's climate is in equilibrium (steady state), the average temperature of the Earth's surface does not change, so the amount of energy received must equal the amount of energy re-emitted.

1.2 Modeling

Units.

- Length, meter (m)
- Energy, watt (W); 1 watt = 1 joule per second.
- Temperature, kelvin (K). An object whose temperature is 0 K has no thermal energy; 0 K is *absolute zero*. Water freezes at 273.15 K and boils at 373.15 K. The Kelvin scale is closely related to the Celsius scale. The magnitude of a degree in the Celsius scale is the same as the magnitude of a kelvin in the Kelvin scale, but the zero point is different. For the Celsius scale, the zero point is the temperature at which water freezes; for the Kelvin scale, the zero point is absolute zero.

Variables.

- T, the temperature of the Earth's surface averaged over the entire globe.

Physical parameters.

- R, the radius of the Earth.
- S, the energy flux density (also referred to as the energy flux)—the amount of energy (W) flowing through a flat surface of area $1\,\text{m}^2$. From satellite observations we know that the energy flux from the Sun is $S = 1367.6\,\text{Wm}^{-2}$.
- σ (Greek, pronounced “sigma”), Stefan–Boltzmann constant; its value is $\sigma = 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}$.

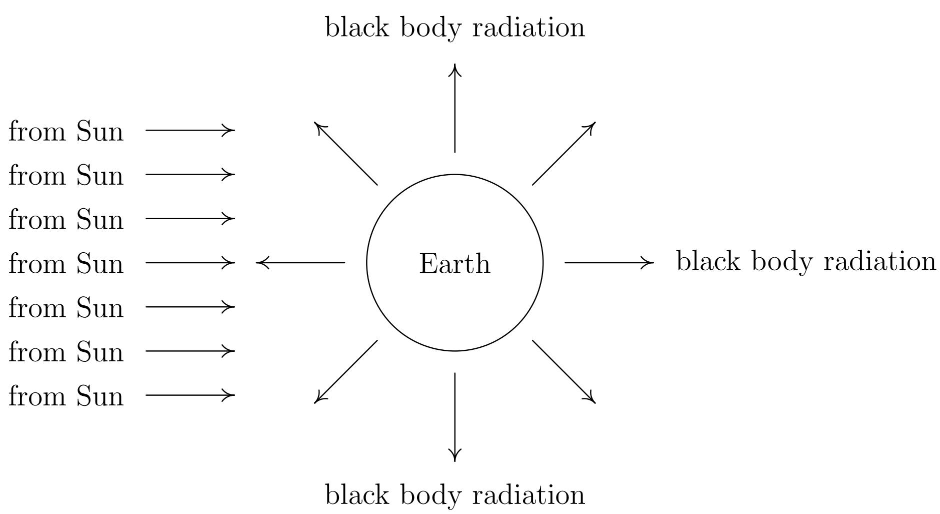

Figure 3: Simplest climate model.

Building the model.

- Viewed from the Sun, the Earth is a disk.
- The area of the disk as seen by the Sun is πR^2.
- The energy flux density is S.
- The amount of energy flowing through the disk (i.e., reaching the Earth) is

$$\text{Incoming energy (W): } E_{\text{in}} = \pi R^2 S.$$

- All bodies radiate energy in the form of electromagnetic radiation.
- The amount of energy radiated out depends on the temperature of the body.
- In physics, it is shown that for "black-body radiation" the temperature dependence is given by the Stefan–Boltzmann law (in units of Wm^{-2}),

$$F_{SB}(T) = \sigma T^4. \tag{1}$$

  (The subscript SB refers to the mathematical physicists Joseph Stefan and Ludwig Boltzmann, who first proposed this formula in the 1880s.)
- The area of the Earth's surface is $4\pi R^2$.
- The amount of energy radiated out by the Earth is

$$\text{Outgoing energy (W): } E_{\text{out}} = 4\pi R^2 \sigma T^4.$$

1.3 Analysis

If the incoming energy is greater than the outgoing energy, the Earth's temperature increases. If the incoming energy is lower than the outgoing energy, the Earth's temperature decreases. If the incoming energy balances the outgoing energy, the Earth's temperature remains constant; the planet is said to be in *thermal equilibrium*.

At thermal equilibrium, the temperature T must be such that $E_{\text{in}} = E_{\text{out}}$. Our mathematical model gives the equation

$$\pi R^2 S = 4\pi R^2 \sigma T^4 \quad \text{or} \quad \tfrac{1}{4} S = \sigma T^4.$$

It is customary to define $Q = \frac{1}{4} S$ and use Q instead of S, so the equation becomes

$$Q = \sigma T^4.$$

Solving for T, we obtain the expression

$$T = \left(\frac{Q}{\sigma} \right)^{1/4}.$$

With $\sigma = 5.67 \cdot 10^{-8}$ and $S = 1376.6$, we find

$$T = \left(\frac{\frac{1}{4} \cdot 1376.6}{5.67 \cdot 10^{-8}} \right)^{1/4} \approx 278.7\,\mathrm{K}.$$

Conclusion. Model #1 gives the average temperature at equilibrium $T \approx 278.7\,\mathrm{K}$, about 5.5 degrees Celsius.

2 Climate Model – Cycle #2

The value $T \approx 5.5$ degrees Celsius seems reasonable but is not in agreement with the known average temperature of the Earth, which is about 16 degrees Celsius. We need a better model.

2.1 Observation

Model #1 omitted a number of important factors. The first factor we want to add involves *reflection*—some of the incoming energy from the Sun is reflected back out into space. Snow, ice, and clouds, for example, reflect a great deal of the incoming light from the Sun. We use the term *albedo* to measure the Earth's reflectivity.

2.2 Modeling

Additional physical constants.

- α, albedo. The Earth's average albedo is about 0.3, which means that roughly 70% of the incoming energy is absorbed by the Earth's surface.

Building the model.

- The amount of energy reaching the Earth is

$$\text{Incoming energy (W): } E_{\text{in}} = \pi R^2 S(1-\alpha).$$

- The amount of energy radiated out by the Earth is

$$\text{Outgoing energy (W): } E_{\text{out}} = 4\pi R^2 \sigma T^4.$$

2.3 Analysis

At thermal equilibrium, the temperature must be such that $E_{\text{in}} = E_{\text{out}}$. Our mathematical model gives the equation

$$Q(1-\alpha) = \sigma T^4.$$

Solving for T, we obtain the expression

$$T = \left(\frac{Q(1-\alpha)}{\sigma}\right)^{1/4}.$$

With $\alpha = 0.3$, we find

$$T = \left(\frac{1367.6 \cdot 0.7}{4 \cdot 5.67 \cdot 10^{-8}}\right)^{1/4} \approx 254.9\,\text{K}.$$

Conclusion. Although Model #2 is better, in the sense that it includes more physics, its prediction of the temperature value at equilibrium is worse than the prediction of Model #1.

3 Climate Model – Cycle #3

It is somewhat disconcerting that we construct a better model and get a result that is not as good as that of the earlier model. But once we accept the mathematical model, we must accept the result. The only option is to look where we might have overlooked something in the model. In this cycle, we focus on the outgoing radiation.

3.1 Observation

Greenhouse gases like carbon dioxide, methane, and water, as well as dust and aerosols have a significant effect on the properties of the atmosphere. The effect on the outgoing radiation is difficult to model, but the simplest approach is to reduce the Stefan–Boltzmann law by some factor.

3.2 Modeling

Additional physical parameter.

- ε, greenhouse factor ($0 < \varepsilon < 1$). This artificial *parameter* has no immediate physical meaning. It is introduced to model the effect of greenhouse gases on the permittivity of the atmosphere; its value is unknown.

Building the model.

- The amount of energy reaching the Earth is

$$\text{Incoming energy (W): } E_{\text{in}} = \pi R^2 S(1-\alpha).$$

- The amount of energy radiated out by the Earth is

$$\text{Outgoing energy (W): } E_{\text{out}} = 4\pi R^2 \varepsilon\sigma T^4.$$

3.3 Analysis

At thermal equilibrium, the temperature must be such that $E_{\text{in}} = E_{\text{out}}$. Our mathematical model gives the equation

$$Q(1-\alpha) = \varepsilon\sigma T^4. \tag{2}$$

This equation can still be solved for T,

$$T = \left(\frac{Q(1-\alpha)}{\varepsilon\sigma}\right)^{1/4}.$$

Question. Take $\alpha = 0.3$ as before. Which value of ε gives a climate model that correctly predicts the current global average temperature $T_1^* \approx 288\,\text{K}$? [Answer: $\varepsilon = 0.66$]

Question. What happens if the combined effects of greenhouse gases, dust, and aerosols reduce the parameter ε from 0.66 to 0.5? [Answer: The equilibrium temperature T increases.]

Our climate model predicts that, if the amount of greenhouse gasses in the Earth's atmosphere increases, then the Earth will warm up. This is the well-known *greenhouse gas effect*. However, this model is certainly too simple to predict the state of our planet with any great accuracy, so we should interpret this finding with great care.

An interesting question is what actually happens when the balance of incoming and outgoing energy is perturbed. Perhaps a volcanic eruption throws dust into the atmosphere, or humans release increasing amounts of CO_2 or other greenhouse gases into the atmosphere. Greenhouse gases affect the Earth's climate by absorbing some of the outgoing radiation.

Question. What do you expect to happen to the Earth's temperature if $E_{\text{in}} > E_{\text{out}}$? What if $E_{\text{out}} > E_{\text{in}}$? [Answer: The temperature increases if $E_{\text{in}} > E_{\text{out}}$, decreases if $E_{\text{out}} > E_{\text{in}}$]

We can ask more questions. Will the temperature continue to increase or will it eventually level off at a higher value? What does the difference $E_{\text{in}} - E_{\text{out}}$ represent? How fast will the temperature change? To answer these questions, we need a fancier model.

3.4 Modeling the dynamics

The simplest model assumes that the temperature changes at a rate proportional to the energy imbalance.

Question. Rewrite the last sentence as a mathematical equation. [Answer: (most likely) $dT/dt = k\,(E_{\text{in}} - E_{\text{out}})$.]

In fact, it is traditional to formulate the equation in terms of energy densities (Wm^{-2}). Recall that E_{in} and E_{out} are energies, so they are expressed in units of watts (W). To convert to energy densities, we need to divide by the Earth's surface area (πR^2). In terms of energy densities, the *temperature evolution equation* is

$$C\frac{dT}{dt} = (1-\alpha)Q - \varepsilon\sigma T^4. \tag{3}$$

This is an *ordinary differential equation* (ODE) for the temperature T as a function of time t. The constant C is the *planetary heat capacity*, which connects the rate of change of the temperature to energy densities.

Question. What is the dimension of C? [Answer: Joule per kelvin.]

Ignoring the constant C, Eq. (5) is an ODE of the type $dT/dt = f(T)$. A visual representation helps us to understand how the Earth's temperature changes when the balance of the incoming and outgoing energy is perturbed.

Sketch the graph of $f(T) = (1 - \alpha)Q - \varepsilon\sigma T^4$ for T between 200 K and 400 K, taking $\varepsilon = 0.66$ and $\alpha = 0.3$. Then use the graph to answer the following questions.

Question.

- What does the vertical axis represent in the physical world? [Answer: Rate at which the temperature changes.]
- What is the zero of $f(T)$ in the range between 200 K and 400 K? Where have we seen this value before? What does it represent? [Answer: $f(T) = 0$ for $T = 288$ K. This equilibrium solution of the ODE is the same as the solution found in the previous section. It corresponds to the current state of the climate.]
- If the temperature is 300 K, do you expect the temperature to increase, decrease, or remain the same? Use the graph to help you.
- If the temperature is 250 K, do you expect the temperature to increase, decrease, or remain the same? Use the graph to help you.

Do the same, taking $\varepsilon = 0.5$, and compare your findings in the two cases.

3.5 Analysis

The graph of f is referred to as the *phase line*. It contains all the information about the dynamics of the system. Consider the case $\alpha = 0.3$ and $\varepsilon = 0.66$, where we found an equilibrium at $T^* = 288$ K. If the average temperature T is less than T^*, the Earth's surface will warm up; on the other hand, if T is greater than T^*, it will cool down. If T is exactly equal to T^*, it will stay the same. Thus, after any small perturbation, the average temperature tends to be restored to its equilibrium value T^*. In mathematics, we say that T^* corresponds to a *stable* equilibrium.

Question. Is the equilibrium you found for $\varepsilon = 0.5$ stable? [Answer: Yes.]

Conclusion. We can match the current climate state by taking into account the effect of greenhouse gases. Our model indicates that the current climate state is stable.

4 Climate Model – Cycle #4

By reducing the outgoing radiative energy by the factor ε, we were able to match the current climate state. But aerosols, dust, and greenhouse gases affect not only the outgoing energy, they also affect the incoming energy. By blocking the incoming solar radiation, they prevent it from reaching the Earth's surface and change the reflectivity of the Earth's atmosphere (albedo).

4.1 Observation

The effect of aerosols, dust, and greenhouse gases on the albedo is difficult to quantify. One would need extensive satellite observations, and even if the data were available, it would be difficult to account for the effect, especially in a simple energy balance model. We will model the effect indirectly by assuming that the albedo depends on the global average temperature in such a way that it decreases monotonically from a high value of 0.7 at low temperatures to a low value of 0.3 at high temperatures.

Question. How do you think the release of aerosols and dust affects the albedo, α? [Answer: α increases.]

4.2 Modeling

Additional physical parameter.

- $\alpha \equiv \alpha(T)$, temperature-dependent albedo. The albedo is small for water, large for ice, so we assume that it is a decreasing function of temperature. A possible formula is

$$\alpha(T) = 0.7 - 0.4 \frac{e^{(T-265)/5}}{1 + e^{(T-265)/5}}, \tag{4}$$

which results in the values $\alpha(T) \approx 0.7$ for $T < 250$ and $\alpha(T) \approx 0.3$ for $T > 280$.

Building the model. We use the same expressions for the incoming and outgoing energy, but include the temperature dependence of the albedo.

- The amount of energy received by the Earth is

$$\text{Incoming energy (W): } E_{\text{in}} = \pi R^2 S(1 - \alpha(T)).$$

- The amount of energy radiated out by the Earth is

$$\text{Outgoing energy (W): } E_{\text{out}} = 4\pi R^2 \varepsilon\sigma T^4.$$

The temperature evolution equation is

$$C \frac{dT}{dt} = (1 - \alpha(T))Q - \varepsilon\sigma T^4. \tag{5}$$

This equation cannot be solved analytically. Of course, it is easy to find T numerically, but that is not the point of the discussion. Something much more interesting is happening here. Take a look at Figure 4. The black curve is the graph of $T \mapsto (1 - \alpha(T))Q$, which is monotone and S-shaped. The blue curve is the graph of the function $T \mapsto \varepsilon\sigma T^4$ with $\varepsilon = 0.66$, which increases monotonically with a monotonically increasing slope. The two graphs intersect in three points. Label these points T_1^*, T_2^*, and T_3^*, where $T_1^* < T_2^* < T_3^*$. Each of these points represents an equilibrium state of the climate system. The value T_1^* is the same as before, $T_1^* \approx 288\,\text{K}$, and represents the present climate. The values of T at the other points of intersection are $T_2^* \approx 265\,\text{K}$ and $T_3^* \approx 233\,\text{K}$.

Question. Show that T_1^* and T_3^* correspond to stable equilibria.

The situation with respect to T_2^* is different. Suppose the climate is in the intermediate state at T_2^*. The dynamic equation (5) shows that $dT/dt > 0$ whenever the system receives more energy than it re-emits. In other words, any small perturbation

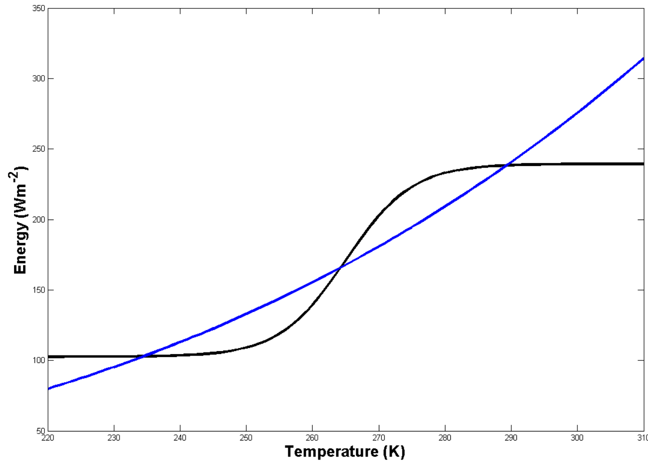

Figure 4: Greenhouse effect on the Earth's energy balance.

that causes $E_{\rm in}$ to increase above $E_{\rm out}$ leads to an increase of T. As the temperature increases, the albedo α decreases, and this will cause a further increase of T. Thus, the effect of the perturbation on the temperature is amplified. This process will continue until the system reaches the equilibrium state at the higher temperature T_1^*, where it will remain. On the other hand, if the system is perturbed and $E_{\rm in}$ decreases below $E_{\rm out}$, the opposite happens: T decreases, α increases, and the effect of the perturbation on the temperateure is amplified until the system reaches the equilibrium state at the lower temperature T_3^*, where it will remain. These arguments show that T_2^* is an *unstable* equilibrium. Any small perturbation will drive the system away from the equilibrium state at T_2^*, so this state will never be observed.

The equilibrium at T_3^* corresponds to a climate state that is more than 50 degrees colder than the current climate. Is it possible that the Earth's climate system can actually be in this equilibrium state?

Conclusion. The climate system can have *multiple equilibrium states.* The climate model (5) admits two stable and one unstable equilibrium state.

5 Climate Model – Cycle #5

So far, we have always assumed that the Earth radiates like a black body, so the outgoing energy E_{out} follows the Stefan–Boltzmann law. But satellites have been collecting data about the energy radiated out into space by the Earth since the '70s. Could we maybe use these data to come up with a better model for the outgoing energy?

5.1 Observations

The satellite data show that, in the temperature range of interest, the variation of the outgoing radiation with temperature is represented very accurately by a simple linear function,

$$F_{BS}(T) = A + BT. \tag{6}$$

(The subscript BS refers to the meteorologists Mikhail I. Budyko and William D. Sellers, who first proposed a formula of this type in the 1960s, well before satellite data became available.) If the temperature is measured in degrees C, the best fit with the observational data for the northern hemisphere is obtained with $A = 203.3\,\text{Wm}^{-2}$ and $B = 2.09\,\text{Wm}^{-2}\text{deg}^{-1}$.

Question. Although the expression (6) is based entirely on observations, the right-hand side looks like the first two terms of the Taylor expansion of F_{BS} near $T = 0$. We could do the same for the Stefan–Boltzmann law, Eq. (1), and expand F_{SB} in a Taylor series, with one difference: In the Stefan–Boltzmann law, T is measured in kelvins, so when we expand F_{SB}, we must do so near $T = 273.15$, $F_{BS}(T) = A' + B'(T - 273.15)$. What are the values of the coefficients A' and B'? How different are they from the values A and B for the Budyko–Sellers model?

5.2 Modeling

Additional physical parameters.

- A and B, constants in the Budyko–Sellers model for the outgoing energy.

Building the model.

- The amount of energy received by the Earth is

$$\text{Incoming energy (W): } E_{\text{out}} = \pi R^2 S(1 - \alpha(T)).$$

- The amount of energy radiated out by the Earth is

$$\text{Outgoing energy (W): } E_{\text{out}} = 4\pi R^2 (A + BT).$$

5.3 Analysis

At thermal equilibrium, the temperature must be such that $E_{\text{in}} = E_{\text{out}}$. Our mathematical model gives the equation

$$Q(1 - \alpha(T)) = A + BT.$$

Again, this equation is most easily analyzed graphically. The graph of $T \mapsto Q(1 - \alpha(T))$ is the same as in Figure 4, and the graph of $T \mapsto A + BT$ is a straightened-out version of the blue curve in the same figure. The graphs intersect in three points over the temperature range of interest. If we label them again as T_1^*, T_2^*, and T_3^*, then T_1^* corresponds to the current climate state and T_3^* to a climate state that is about 50 degrees colder, while T_2^* corresponds to an unstable equilibrium state, which is never observed.

Conclusion. Modern satellite data are consistent with the earlier observation that the climate system can have another, much colder equilibrium state besides the current (stable) state.

6 Snowball Earth

The conclusion of the preceding section suggests that, although the Earth's climate system is currently in a stable equilibrium state with an average global temperature of about 288 K, well above the freezing temperature of water, there is another stable equilibrium state which is perhaps 50 degrees colder. The much colder state would correspond to a complete glaciation of the Earth, with all oceans frozen to a depth of several kilometers and almost the entire planet covered with ice—a *Snowball Earth* state. The possibility of such a scenario, although hard to imagine, raises serious questions for climate science. For example, has the Earth ever been in a snowball state? If so, what caused its transition to such a state, how did it transition out of that state, and could it return to a snowball state in the future?

There is indeed fairly strong geological evidence that the Earth's climate may have been in the Snowball Earth state up to four times during the Neoproterozoic age, between 750 and 580 million years ago. The evidence comes from geological deposits that can form only during glaciations and that have been found in tropical areas around the globe at what was then sea level. In addition, there are related deposits which point to large build-ups of CO_2 in the atmosphere during these same periods, which were subsequently brought down rapidly to normal levels, and there are geological indications of very little biological activity during these times.

7 Bifurcation

To address the issue of a possible transition to Snowball Earth, we take another look at the model (2) and ask what happens to the equilibrium solutions if the energy flux emitted by the Sun and received by the Earth changes.

7.1 Observation

Recall that the solar energy flux is represented by S, and that we adopted the standard notation, using $Q = \frac{1}{4}S$ instead of S in Eq. (2), so the physical observation that the solar energy flux decreases implies that Q decreases.

Figure 4 can help us understand what happens. If Q decreases, the black curve moves down, the equilibrium states at T_1^* and T_2^* merge and then disappear, leaving only the deep-freeze state at T_3^*. On the other hand, if Q increases, the equilibrium states at T_2^* and T_3^* coalesce, and we are left with an ice-free Earth in the equilibrium state at T_1^*.

7.2 Analysis

The merging and subsequent disappearance of a stable and an unstable equilibrium state or, vice versa, the simultaneous emergence of a stable and unstable equilibrium state as the result of a gradual change in the physical environment is known as a *bifurcation*. (Literally, a "splitting into two.") In our case, the bifurcation is triggered by a gradual change (the technical term is "quasi-statically") in the quantity Q.

It is generally a good idea to use a dimensionless parameter as the quantity triggering the bifurcation. Q, which has the dimension Wm^{-2}, can be made dimensionless by dividing it by its value for the current climate, which is $Q_0 = 342\,\mathrm{Wm}^{-2}$. The dimensionless quantity Q/Q_0 is the *bifurcation parameter*.

In a bifurcation analysis, we study the behavior of the equilibrium states as a function of the bifurcation parameter. On our case, this means that we study the behavior of the equilibrium temperatures T_1^*, T_2^*, and T_3^* as functions of Q/Q_0. Figure 5 summarizes the result. The upper curve is the graph of T_1^*, the middle curve is the

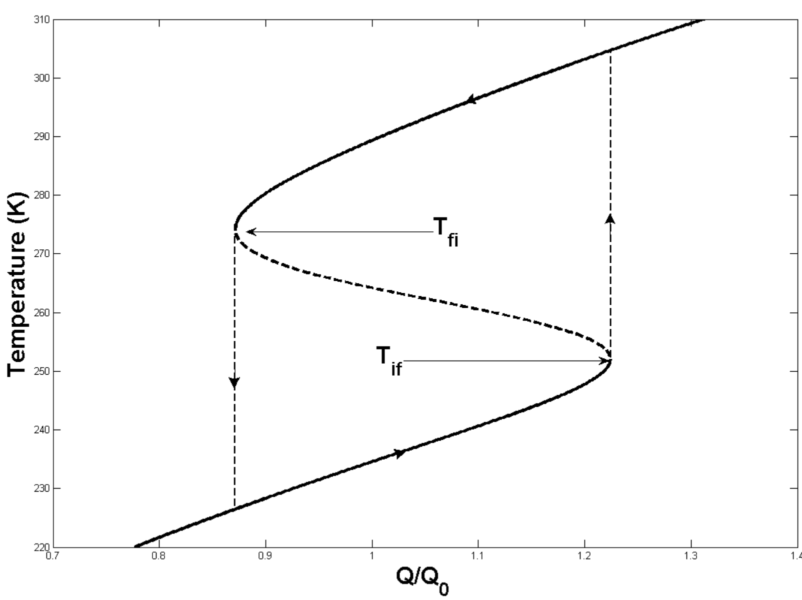

Figure 5: Mean surface temperatures at equilibrium as a function of the solar energy flux (in units of its present value).

graph of T_2^*, and the lower curve is the graph of T_3^*. It is common practice to represent stable solutions (T_1^* and T_3^*) by solid curves, unstable solutions (T_2^*) by dashed curves. A diagram of this type is called a *bifurcation diagram*. The bifurcation diagram shows that, as the solar input decreases slowly from its current value ($Q = Q_0$), the mean surface temperature T_1^* decreases until it reaches a *tipping point* at the critical value T_{fi}. The climate system transits to the lower branch, the planet turns white, and its temperature equilibrates at T_3^*. The diagram also shows the possibility of a reverse scenario. If the planet today were in the deep-freeze state and the solar input were to increase quasi-statically, the mean surface temperature T_3^* would increase until it reached another tipping point at the critical value T_{if}. All the snow and ice would melt, and the climate would settle onto an equilibrium state with temperature T_1^*. Since the paths for increasing and decreasing values of the bifurcation parameter are distinct, we see that *hysteresis* is built into the climate model.

8 Experimentation and Observation

The *scientific method* is really a variation of the modeling process depicted in Figure 1. Here, the interplay is between observation and experimentation in the real world and the mental models of the theoretician. Even though we often speak of theoretical and experimental science as if they are different categories, they are really different aspects of the same scientific enterprise. A theoretician's work is grounded in observation and experimentation, and his or her theories are validated by testing predictions against new observations and new experiments. An experimentalist does not do random experiments; rather, his or her work is guided by theory. Many important experiments, for example, are designed to test specific theories or hypotheses.

In this spirit, we have complemented this module with a series of experiments, which can be done with little or no equipment.

8.1 Black-body Radiation and Albedo

In our modeling efforts we encountered the concept of black-body radiation and the Stefan–Boltzmann law, which describes the variation of the energy emitted by a black body (like the Earth) as a function of the temperature of the body, $F_{SB}(T) = \sigma T^4$. The concept is incorporated in commercial devices like the Black and Decker TLD100 Thermal Leak Detector shown in Figure 6 (available from Amazon.com for $34.99). The idea that black or darker objects are hotter in the Sun than white or lighter objects is well grounded in everyday experience. Students can test this every (sunny)

Figure 6: Black and Decker TLD100 Thermal Leak Detector.

day by placing their hand on black pavement.[3] Students can gain some familiarity with the idea of black-body radiation and its dependence on temperature by using this device to determine the temperatures of various objects. In particular, they can learn something about the impact of albedo on the black-body radiation, by pointing the TLD100 at objects of various colors in the Sun. These experiments make a nice transition from the model without albedo (Cycle #1) to the model with albedo (Cycle #2).

8.2 Cooling Coffee, Revisited

Many students will have done experiments where they record the temperature of a hot beverage as it cools. The experiments are usually done to illustrate Newton's Law of Cooling, which says that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object (T) and the ambient temperature (i.e., the temperature of its surroundings, T_a). The mathematical equation is $dT/dt = -k(T - T_a)$, where k a positive constant. Students then fit models of the form $T(t) = T_a + Ce^{-kt}$ to the data.

Newton's Law of Cooling is important for the study of temperature change in climate science, but it is only a beginning. Figure 7 shows a simple set-up that brings in another important factor, namely evaporation. Data from one run of this experiment

[3]Caution, the phrase "hot enough to fry an egg" is not an exaggeration.

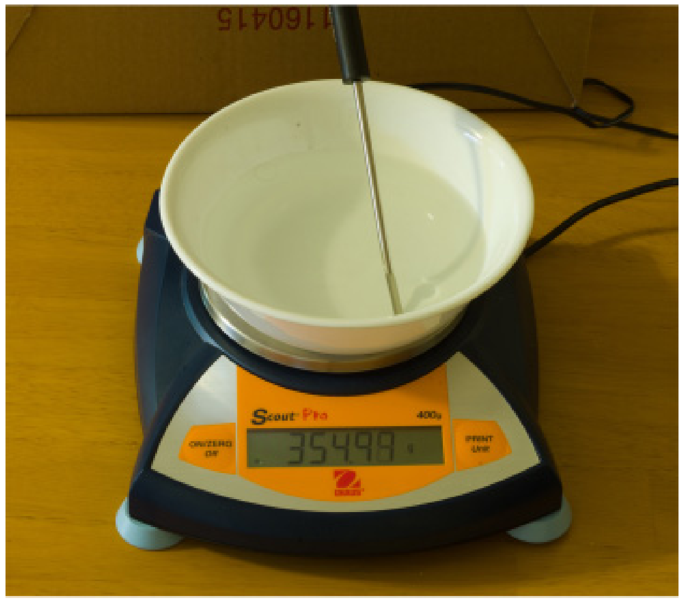

Figure 7: Evaporation and Newton's Law of Cooling – Experiment.

are shown in Figure 8. The rate at which the temperature cools cannot be explained by Newton's Law alone; evaporation is just as important. The higher the rate of evaporation, the higher the rate at which the water cools and, as we can see from Figure 8, the rate of evaporation is higher at higher temperatures.

8.3 Greenhouse Effect

The importance of the greenhouse effect can be illustrated by a simple experiment recording the temperature in the passenger compartment and trunk of a vehicle parked in the open. Figure 9 shows the data collected on the morning of August 11, 2010, from 7:30 AM until 12:40 PM. The sky was partly cloudy in the morning, and the air temperature was about 90°F at noon. On this particular morning, the temperature in the passenger compartment rose to almost 120°F and in the trunk to just under 100°F. The temperatures in the passenger compartment and the trunk of the car rise over the course of the morning for several reasons—for example, the sunlight warms the exterior of the car and some of that heat is transferred to the trunk and the passenger compartment. One big difference between the passenger compartment and the trunk is that the passenger compartment has windows. Sunlight enters the passenger compartment through the windows and heats up the interior surfaces. These

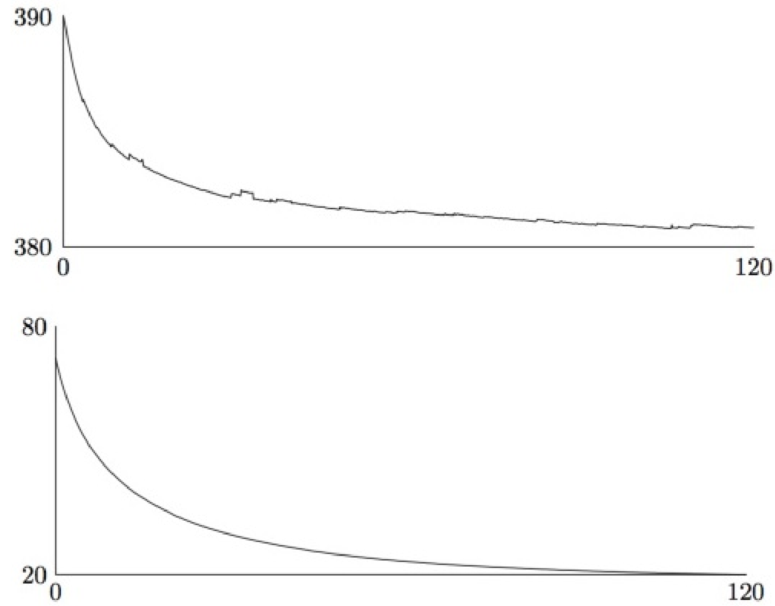

Figure 8: Evaporation and Newton's Law of Cooling – Data.

surfaces radiate heat, just like the Earth, in the form of black-body radiation. Although some of this radiation escapes through the windows, most of it is intercepted by the other interior surfaces of the car. As a result, the passenger compartment of the car became much hotter than the trunk.

This experiment is a nice transition between the climate model with full black-body radiation (Cycle #2) and the model with reduced black-body radiation (Cycle #3). Students can easily do variations on this experiment using cardboard boxes with a transparent top.

The experiment requires a warning—high temperatures are not good for batteries; in fact, the instruction manual for the LabQuest specifically warns against leaving the unit in a parked car on a sunny day.

8.4 Photosynthesis, Metabolism, and a Surprise

Figure 10 shows an experiment that was originally designed to illustrate the importance of interactions between the biosphere (living organisms) and the atmosphere. However, it produced a surprise that shed light on another important aspect of climate

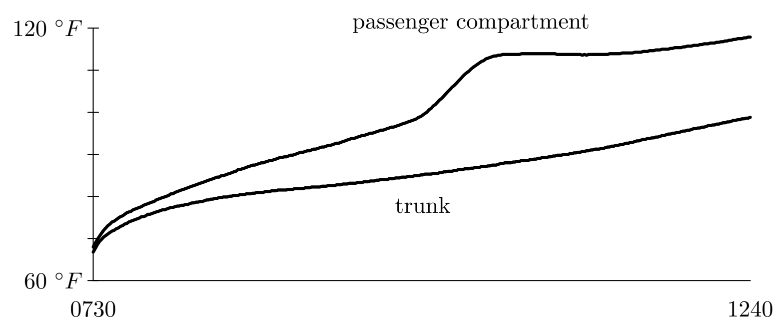

Figure 9: Temperature in the passenger compartment and trunk of a vehicle parked in the open. (Data collected with two temperature probes and recorded with LabQuest software from Vernier Software and Technology, http://www.vernier.com/.)

science, namely *feedback*.

We placed some plant material in a closed container with two Vernier probes, one recording the concentration of oxygen and the other recording the concentration of carbon dioxide, connected to a LabQuest. The apparatus was placed under an Aerogarden light that cycled on and off over a 24-hour period to simulate day and night. Because photosynthesis consumes carbon dioxide and produces oxygen, we expected that during the “day” the oxygen concentration would rise and the carbon dioxide concentration would fall. Metabolism has the reverse effect; it consumes oxygen and produces carbon dioxide, so we expected that during the “night” the oxygen concentration would fall and the carbon dioxide concentration would rise.

Figure 11 shows the results of the experiment. Notice that

- When the light went off, the oxygen level rose suddenly and unexpectedly. The carbon dioxide level also rose as expected.
- During the night, after its initial sudden jump, the oxygen level fell as expected and the carbon dioxide level rose as expected.
- When the light came on, the oxygen level suddenly and unexpectedly fell. The carbon dioxide level also fell as expected.
- During the simulated day after the initial sudden jumps the carbon dioxide and oxygen levels behaved as expected until the carbon dioxide level was quite low.

Figure 10: Photosynthesis, metabolism, and a surprise – Experiment.

After this initial surprise we came up with a reasonable hypothesis for what happened. There are three keys, and we need some fourth-grade mathematics.

- The sensors do not measure the amount of either carbon dioxide or of oxygen. They do measure concentrations—either

$$\frac{O_2}{N_2 + O_2 + H_2O + \text{other}}$$

  for oxygen or

$$\frac{CO_2}{N_2 + O_2 + H_2O + \text{other}}$$

  for carbon dioxide.

- Quotients go up when denominators go up and down when denominators go down.

- Water is more soluble in warm air than in cold air.

With all this in mind, consider what might happen when the light goes off.

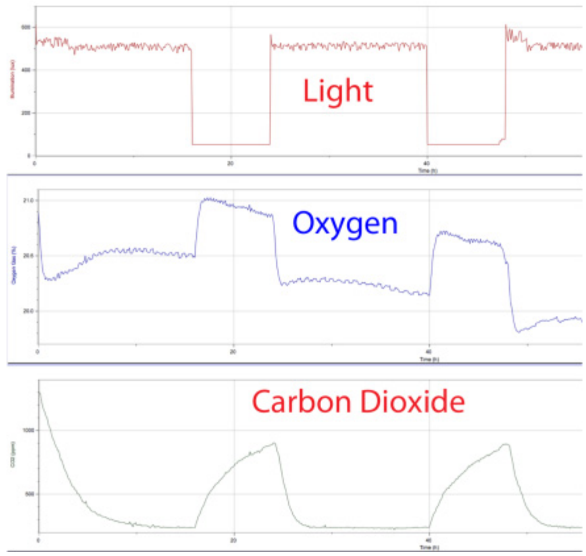

Figure 11: Photosynthesis, metabolism, and a surprise – Data for a container with plant material.

- The temperature drops and some of the water vapor in the air condenses.
- The denominator falls.
- This causes the unexpected rise in the concentration of oxygen. The concentration of carbon dioxide also rises but we expected it to rise.

Now consider what happens when the light goes on.

- The temperature rises and water is drawn into the air as water vapor.
- The denominator rises.
- This causes the unexpected drop in the concentration of oxygen. The concentration of carbon dioxide also drops but we expected it to drop.

We tested this hypothesis by running a series of experiments. Figure 12 shows the results of one such experiment. The set-up was similar to the first experiment, but the chamber was empty except for some moist towels, and we recorded relative humidity and temperature in addition to light and oxygen. We did not record carbon dioxide. Notice the results support our hypothesis. Although there is a complicated

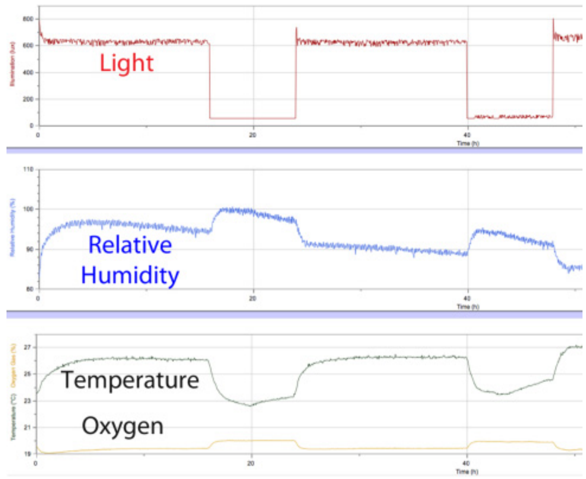

Figure 12: Photosynthesis, metabolism, and a surprise – Data for an empty container.

relationship among temperature, relative humidity, and absolute humidity, the relative humidity graph supports the conjecture that something is going on with water vapor in the air. We also took time lapse photographs of the experiment and could see water condensing on the walls of the chamber when the light went out.

Besides being a nice example of what scientists call the "scientific method" and mathematicians call the "modeling cycle" (see Figure 1)—the interplay between theory and experiment—this experiment brings up one of the most important feedback loops in the Earth's climate system. Because water is more soluble in warm air than cool air, as the Earth warms, more water is drawn into the atmosphere. Because water is a powerful greenhouse gas, more water in the atmosphere causes the Earth's temperature to rise.

An experiment motivated by the interaction between the biosphere and the atmosphere forced us to confront the interaction between the hydrosphere and the atmosphere.