Invasive Species and Probability: Percolation of the Emerald Ash Borer

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- **Module Summary:** This module explores the probabilistic spread of the Emerald Ash Borer (an invasive species) through a series of stylized landscapes.
- **Informal Description:** This module explains the ideas of percolation and their application on several different levels. Students learn through simulation or proof about percolation and percolation thresholds and how they can be applied to many areas including invasive species control.
- **Target Audience:** The module materials include a number of components (handout, simulation/lab, lecture notes on mathematical proofs) in the interest of customization (both across levels of math courses and across disciplines). The introduction, description of percolation and subsequent proofs would be an appropriate lesson for an advanced undergraduate probability class. The introduction, math background, description of percolation, and simulation component would be an appropriate lesson for an earlier course, or for a general course on quantitative methods in ecology.
- **Prerequisites:** The module can be taught at two different levels; beginner and advanced.
  - Beginner: High school through early college non-mathematics majors. Basic understanding of probability is helpful, but not necessary.
  - Advanced: Intro to proofs or beyond. Understanding of limits, probability, counting, and proofs is sufficient. Little to no background in graph theory is needed.
- **Mathematical Fields:** Probability, Simulation, Discrete Math, Proof writing, Graph Theory.
- **Application Areas:** Invasive species threaten local ecology as well as renewable economic resources systems (in this case, the timber industry). Local species may be over-consumed or out-competed for food sources: extinctions result in gaps that destabilize greater ecological processes.

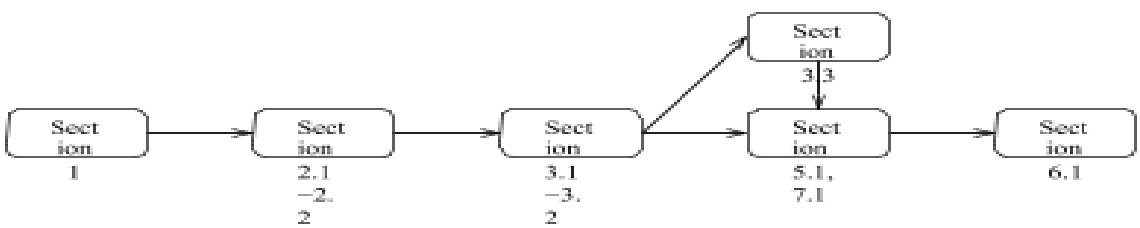
- **Goals and Objectives:** Ideas accessed: probabilistic independence

Simulation as a tool to reach an initial hypothesis idea of threshold boundary between two regimes variation due to connectivity interplay of connectivity and probability compelling pointers to unsolved problems

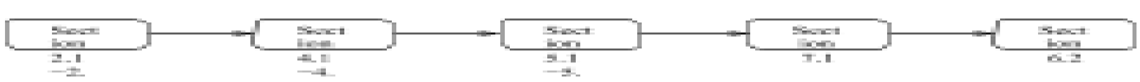
- **Technology/Software Needs:** If available, Excel or internet access allows for demonstrations, otherwise

Time: This self contained module is designed to be completed within 50-90 minutes. Suggested content based on level:

High school, beginning undergraduate, non-math major:



Advanced Undergraduate Mathematics - Proof Based



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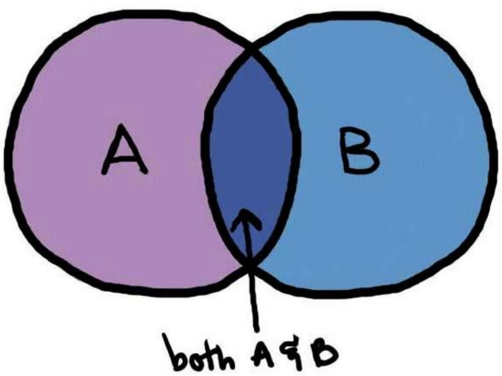
1 Math Background

This module uses the basic probability rules given below. You can skip this section if you already know basic probability.

- *Independence*: if event A has probability p_A and event B has probability p_B and events A and B are independent (that is, whether A happens does not change the probability B happens), then the probability of A and B is $p_A p_B$.

- For two independent events A and B, the probability that either A happens or B happens (or both happen) is $p_A + p_B - p_A p_B$.



2 Introduction: Invasive Species and Percolation

2.1 Invasive Species

The term *invasive species* is used to describe animal or plant species that have colonized regions outside where they are naturally found. The introduction of non-native species can cause imbalance in the environment: if the new species finds a good food source and has no natural predators, its numbers can explode. There are many examples where this explosion of population causes major problems including driving local species to extinction either through over-consumption or through direct competition for resources.

This unit focuses on the **Emerald Ash Borer**.

(Image from Tennessee Government page
http://www.tn.gov/agriculture/regulatory/eab.shtml)

- A beetle introduced to North America in the 1990s, has spread to 15 states so far. The ash borer probably hitch-hiked to Michigan on wood products imported from Asia.
- **Spread of the Ash Borer:** The ash borer is able to fly short distances from infested trees to susceptible ash trees. The ash borer infests all species of ash trees in North America, killing trees approximately three years after initial infestation.
- So far this invasive species is estimated to have killed 50-100 million ash trees. It is considered one of the most destructive non-native insects in the United States.
- The timber industry produces approximately 25 billion dollars of ash saw timber per year, and ash trees are planted extensively in urban neighborhoods throughout the Southeast. The ash borer threatens an estimated 7.5 billion ash trees in North America.

Problem: We will use ideas from probability to explore how the Emerald Ash Borer will spread through a (stylized) landscape.

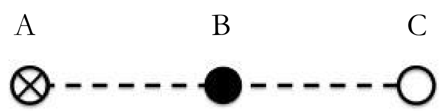
Modeling: Foresters have determined that the probability that a tree infested with ash borers will spread the infection to nearby trees depends on the distance between the two trees. For now, assume that an infested tree will have one chance to infest the trees near it before it dies, and that the ash borers remaining at a dead tree will die from lack of food.

A mathematical model for the spread of an invasive species like the emerald ash borer is *percolation*. The word *percolate* comes from the Latin word *percolare*, meaning to be filtered through. Older adults and outdoor enthusiasts may be familiar with the word percolate because they have made coffee using this process. Others may have used the word percolate to describe the way an idea gradually spreads through a social network. Scientists use the word percolate to describe the movement of a fluid through a porous medium, such as water through soil, shale, or sandstone.

Percolation theory can be used to study the movement of anything that spreads from one discrete location to another, ignoring all the space in between. For example, the emerald ash borer moves from one tree to another. Many other applications are active areas of research in applied mathematics and engineering. Your teacher can give you some examples, or you can research them for yourself.

Class Discussion - Stop and Think: I live on one end of a street with 3 ash trees lining it (the last tree is in my yard). The trees are spaced 20 feet apart. Suppose that the probability of ash borers infesting a tree within 25 feet of an infested tree is 0.6 but that if the distance is more than 25 feet then the probability of transmission is 0.

I notice that the tree at the other end of the street is infected. The following figure illustrates this situation, where the X indicates the infested tree.



We want to determine the probability of my tree becoming infected. Before reading on, discuss how to calculate the solution and explain your reasoning to each other.

For my tree to become infested, the ash borers must first move from the infested tree A to the middle tree B, then from the middle tree B to my tree C. The probability of moving from the infested tree A to the middle tree B is 0.6. The probability then of moving from the middle tree B to my tree C is another 0.6. Therefore, the probability of traveling along the path of trees from A to B to C is

$$0.6 \times 0.6 = 0.36.$$

Suppose there is a longer row of trees along the street, say a row of four trees as in the diagram below.

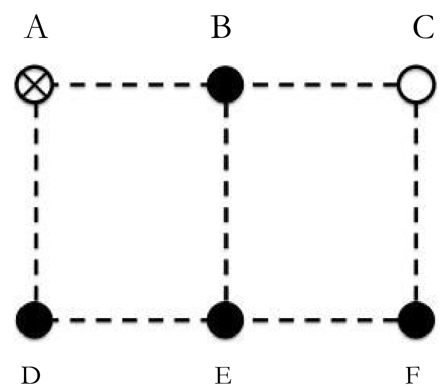
If tree A is infested and my tree is again at the end of the street, answer each of the following questions:

1. Why is there only one viable path for the infestation to travel? That is, why isn't A-B-C-B-C-D a viable path?

2. Explain why the probability that my tree will become infested is 0.216. What path did you follow to get this answer?

If there are $n+1$ trees on the street (so that there are n connections from the first tree to the last tree), what is the formula to calculate the probability of the last tree becoming infested if the first tree is infested?

Note that for the infestation to spread to a susceptible tree there must be a viable path from the infested tree to a susceptible tree. If the trees are all in one row, there is only one viable path. Suppose now that a neighboring parallel street also has 3 ash trees, as shown in the following figure.



Such a rectangular array of points (or vertices or nodes) is called a *lattice of points*, or just a *lattice*. The above lattice is a 2 by 3 lattice. An $n \times m$ lattice has n rows and m columns of vertices.

We now introduce a rule to make viable paths which will simplify our calculations. The rule is that we may only move from one vertex to the next in the directions to the right or down. The length of this path will be the number of edges traversed. Assuming the same separation distances and probability of transmission as in the linear example, answer the following questions:

1. Why is A-B-E-F a viable path for the transmission of the infestation from tree A to tree F?

2. Why is A-D-E-B-C-F NOT a viable path for transmission of the infestation from tree A to tree C?

3. List the three different viable paths of infestation transmission from tree A to tree C.

4. What do the three paths have in common?

5. One viable path is A-B-C-F. What is the probability of the infestation traveling along path A-B-C-F?

6. Another viable path is A-D-E-F. What is the probability of the infestation traveling along path A-D-E-F?

7. What is the probability of the infestation traveling along the other viable path?

8. Now that the probability has been calculated for each of the three possible paths, can we use

those probabilities to calculate the actual probability of the infestation traveling from tree A to tree F?

Because we are assuming the infestation will travel along a single path, and that these paths are distinct from each other, we can estimate the probability that tree F will eventually be infested, if tree A is infested, as

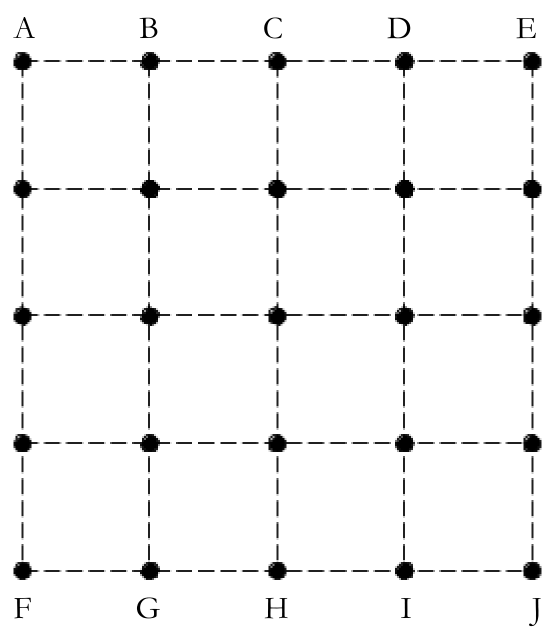
$$Pr(F \text{ is infested}) = Pr(A-B-C-F) + Pr(A-B-E-F) + Pr(A-D-E-F)$$

$$Pr(F \text{ is infested}) = 0.216 + 0.216 + 0.216 = 0.648$$

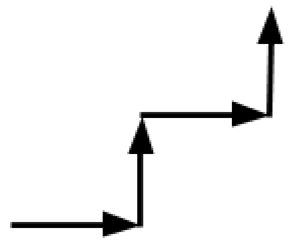
2.2 Percolation

The previous examples are known as *bond percolation* because it is the bonds between the vertices which transmit the disease from one vertex to another. We say a graph *percolates through a lattice* (or just *percolates*) if there is a connected path from a vertex at one side of the lattice to a vertex on the opposite side. This is referred to as "top-down percolation" or "left-right percolation," depending on the orientation of the lattice. This can represent many examples, such as an infestation spreading through an orchard of trees, a wildfire spreading through a forest, contaminants spreading through an ecosystem, or a disease spreading through a population.

Although it was easy to calculate the probability that a particular tree could be infested in the example in the previous section, if the lattice is large, you can imagine how tedious it can be to find and keep track of the many possible paths for the borers to follow from the infested tree. Imagine how difficult this calculation would be if you were working with the trees in the neighborhood shown below!



Class Discussion - Stop and Discuss: Spend two or three minutes counting the paths in this lattice that start at bottom and percolate to the top. Is there a way we can methodically find the number of paths so that we are sure to count them all, without recounting any? Let's start by only counting paths that only move to the right and upward (see the diagram below).



Use the following questions to help organize the process of counting the paths.

1. How many "right-up" paths are there from vertex J to vertex E?

2. Suppose the probability of an infestation moving along an edge of the lattice is p. What is the probability of a "right-up" infestation moving from J to E?

3. How many "right-up" paths are there from vertex I to vertex E? Are the same number of right-up paths from vertex H to vertex D, vertex G to vertex C, and vertex F to vertex B?

4. Suppose the probability of an infestation moving along an edge of the lattice is p. What is the probability of a "right-up" infestation moving from I to E?

5. How many "right-up" paths are there from vertex H to vertex E? Are there the same number of right-up paths from vertex G to vertex D and from vertex F to vertex C?

6. Suppose the probability of an infestation moving along an edge of the lattice is p. What is the probability of a "right-up" infestation moving from H to E?

To count the number of paths from vertex G to vertex E requires careful reasoning. There are three horizontal right steps in each path from G to E. These three horizontal steps can be taken all at once at the same "level," or two of the horizontal steps can be at the same level and one at a different level, or all three of the horizontal steps can be at different levels. If all three horizontal right steps are at the same level, there are $C(5, 1) = 5$ choices for which level the horizontal steps will take place. If the three horizontal right steps are at two different levels, then there is either one step then two steps at a higher level or there are two steps then one step at a higher level. Either way, we choose two out of five levels, $C(5,2)$ and then multiply by 2, $2C(5,2) = 20$. If the horizontal steps are all at different levels, by a similar argument, there are $C(5,3) = 10$ paths with the horizontal steps at all different levels. Therefore, there are

$$C(5,1) + 2 \times C(5,2) + C(5,3) = 5 + 2 \times 10 + 10 = 35$$

right-up paths from vertex G to vertex E. Likewise, because the paths are the same length, there are 35 right-up paths from vertex F to vertex D. Since each path from G to E is of length 7, the probability of an infestation moving from G to E (or from F to D) is $35p^7$.

By a similar argument, there are $C(5,1)+3\times C(5,2)+3\times C(5,3)+C(5,4)=70$ right-up paths from vertex F to vertex E. The probability of an infestation moving from F to E is $70p^8$.

Therefore, the total probability of an infestation percolating along a right-up path is calculated by

0 right steps:	$p^4+p^4+p^4+p^4+p^4=5p^4$
1 right step:	$4\times 5p^5=20p^5$
2 right steps:	$3\times 15p^6=45p^6$
3 right steps:	$2\times 35p^7=70p^7$
4 right steps:	$1\times 70p^8$

for a total of $f(p)=70p^8+70p^7+45p^6+20p^5+5p^4$.

The polynomial $f(p)$ is called the ***probability of percolation***. We will describe this in more detail in the next section.

Consider an $n \times n$ lattice. Let's calculate the "right-up" probability of percolation. Adapting the method from the previous example, the case for general $n>0$ is

0 right steps: $p^4+p^4+p^4+\cdots+p^4=np^{n-1}$

1 right step: $(n-1)\times np^n=n(n-1)p^n$

2 right steps: $[(n-2)\times(C(n,1)+C(n,2))]\,p^{n+1}$

⋮

k right steps: $(n-k)\left[C(n,1)+(k-1)\left(\sum_{i=2}^{k-1}C(n,i)\right)+C(n,k)\right]p^{n+k-1}$

⋮

n right steps: $\left[C(n,1)+(n-1)\left(\sum_{i=2}^{n-1}C(n,i)\right)+C(n,n-1)\right]p^{2n-1}$

and

$$f_n(p)=np^{n-1}+(n(n-1)p^n)+([(n-2)\times(C(n,1)+C(n,2))]\,p^{n+1})+\cdots+\left((n-k)\left[C(n,1)+(k-1)\left(\sum_{i=2}^{k-1}C(n,i)\right)+C(n,k)\right]p^{n+k-1}\right)+\left[C(n,1)+(n-1)\left(\sum_{i=2}^{n-1}C(n,i)\right)+C(n,n-1)\right]p^{2n-1}$$

Of course, the polynomial $f(p)$ for this example is still incomplete, since we were only considering right-up paths. Suppose we consider paths in the lattice that can move up and down, but still only move to the right. Then the calculation for $f(p)$ becomes even more complicated. For example, the first two calculations (0 right steps and 1 right step) remain the same, but 2 right steps has more calculations:

0 right steps: $p^4+p^4+p^4+p^4+p^4=5p^4$

1 right step: $4 \times 5p^5 = 20p^5$
2 right steps: $3 \times 15p^6 + 4 \times p^8 + 3 \times p^{10} + 2 \times p^{12} + p^{14} = 45p^6$
3 right steps: $2 \times 35^7 + \cdots$
4 right steps: $1 \times 70p^8 + \cdots$

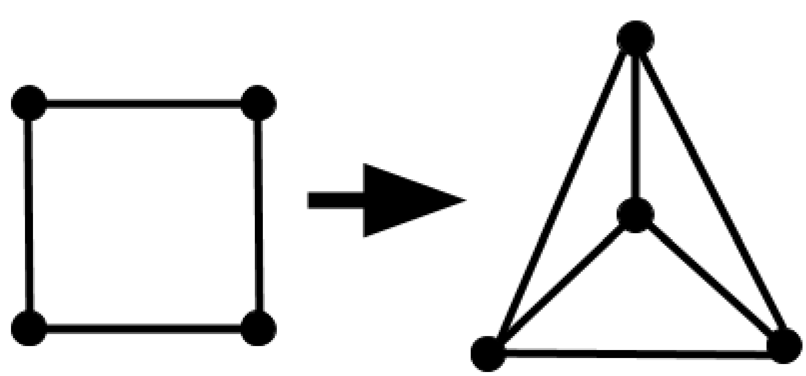
The probability of percolation will depend on the value of p, the ***probability of transmission***.

Class Discussion - Stop and Discuss: What do you suppose will happen for larger values of p? Smaller values of p? Will larger or smaller values of p more likely lead to percolation?

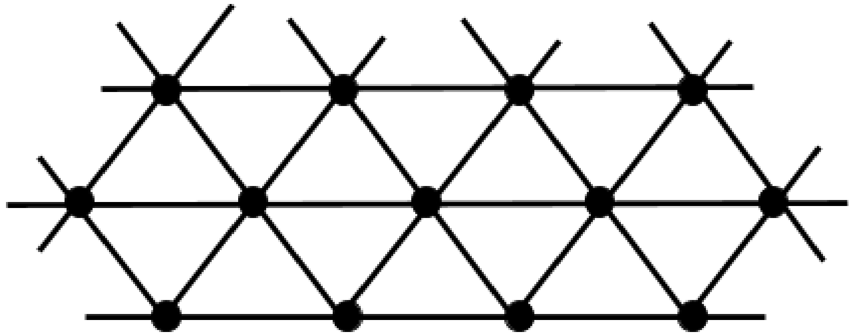
Controlling the probability of transmission can help control whether an infestation will percolate through a lattice. For example, the farther apart trees are planted, the less likely it is that an infestation will percolate through an entire orchard or ecosystem.

2.3 Percolation Threshold

In the previous section we made an attempt to calculate the probability of percolation for a 5×5 lattice, which proved to be a cumbersome process. We then briefly tried to calculate the probability of percolation for an $n \times n$ lattice. This was an even more cumbersome calculation. Lattices are not the only graphs for which we wish to calculate the probability of percolation. In general, the probability of percolation could also be calculated for a planar graph. A planar graph is a graph that can be embedded in the plane. That is, the vertices and edges can be redrawn so that the edges intersect only at their endpoints. For example, the graph on four vertices below can be redrawn so the two "diagonal" edges do not intersect.



Thus, we can generalize the concept of a lattice to include planar graphs like the one below.



The calculation for the probability of percolation for a planar graph on n vertices, represented by $f_n(p)$, can be as cumbersome or difficult as the standard lattice. Because probability of percolation is difficult to compute directly, we often look for methods to estimate this through indirect methods.

In the next section we will discuss one way in which $f_n(p)$ can be estimated through simulation. Before doing so, let's consider another example. For each value of $0 \le p \le 1$ and any positive integer n, we have a probability of percolation on a planar graph with n vertices and with a transmission probability of p. Technically speaking we define

$$f_n(p) = Pr\,(planar\ graph\ on\ n\ vertices,\ given\ the\ transmission\ probability\ is\ p).$$

Stop and Think: Suppose the probability of percolation is given by

$$f_n(p) = \frac{\left(\frac{3p}{2}\right)^n}{1+\left(\frac{3p}{2}\right)^n}$$

1. Let $p = \frac{1}{6}$. What happens to the value of $f_n(p)$ for increasing values of n? Filling in the table below will help answer this question.

Value of n	**Value of $f_n(p)$**
2	$\frac{1}{1+(\frac{1}{4})^2} = 0.058824$
3	$\frac{1}{1+(\frac{1}{4})^3} = 0.015385$
4	
5	
6	
7	

8	

This concept is referred to as the limit of $f_n(p)$ as n approaches infinity, written $\lim_{n\to\infty} f_n(p)$.

2. Let $p = \frac{5}{6}$. What happens to the value of $f_n(p)$ for increasing values of n? Filling in the table below will help answer this question.

Value of n	**Value of $f_n(p)$**
2	$\frac{1}{1+(\frac{5}{4})^2} = 0.609756$
3	$\frac{1}{1+(\frac{5}{4})^3} = 0.661376$
4	
5	
6	
7	
8	

Note that when $p = \frac{1}{6}$, $\lim_{n\to\infty} f_n(p) = 0$ and when $p = \frac{5}{6}$, $\lim_{n\to\infty} f_n(p) = 1$. This would suggest there is a ***threshold value*** for the probability p above which $f_n(p)$ will approach 1, and below which $f_n(p)$ will approach 0. In this example, what do you suppose that threshold value is? How would you determine the threshold value?

Extending this example to the general case, if $f_n(p)$ is the probability of percolation, we can ask

1. If p is close to 0, what is the value of $f_n(p)$? What about when p is close to 1?

2. What happens if n gets large?

We are now ready to more formally define the percolation threshold. The ***percolation threshold*** is a value $0 \le p_c \le 1$ (if one exists) such that

$$\lim_{n\to\infty} f_n(p) = \left\{ \frac{0 \text{ if } p<p_c}{1 \text{ if } p>p_c} \right\}$$

In non-technical terms, the percolation threshold is the critical point of a very large (infinite) graph; if the transmission probability is below the threshold, then the graph will not percolate, if it is above the threshold, then the graph will percolate.

Often times there is no particular critical value, but there is a percolation function

$$\lim_{n\to\infty} f_n(p) = f(p).$$

3 Simulation In-class Activity

3.1 Simulation Background and Notes

In an activity with your class, you will learn an alternative way to study the transmission of an invasive species through a grid graph like the one above. You will use a spinner or other random outcome to decide whether or not each line segment will allow borers to spread, and then look for paths in the graph. This method is called a *simulation*, because you are "acting out" the random process of spreading from tree to tree. A simulation requires many repetitions of the process to produce a reliable estimate of the answer.

Complete section 3.2 as a handout for all the students. Then graph the results of the simulation as a function of the transmission probability p. (Note: The probability of the graph percolating is small for $p < .5$ and large for $p > .5$. So you will most likely see a shift in the simulation near $p = .5$.)

Bond percolation is easily simulated by randomly assigning each bond, or edge, in a square lattice or other graph to be open with probability p. By simulating percolation in a finite graph, students can quickly collect data that allows them to conjecture the value of the asymptotic phase transition probability. Handouts are provided for both bond and, as an extension, site percolation.

Collectively, the class should explore a range of probabilities between 0 and 1, being sure to include the value 0.5. A rule of thumb is that each probability should be simulated at least 5 times. You might rule out very small probabilities and very large probabilities after discussing threshold behavior with the class.

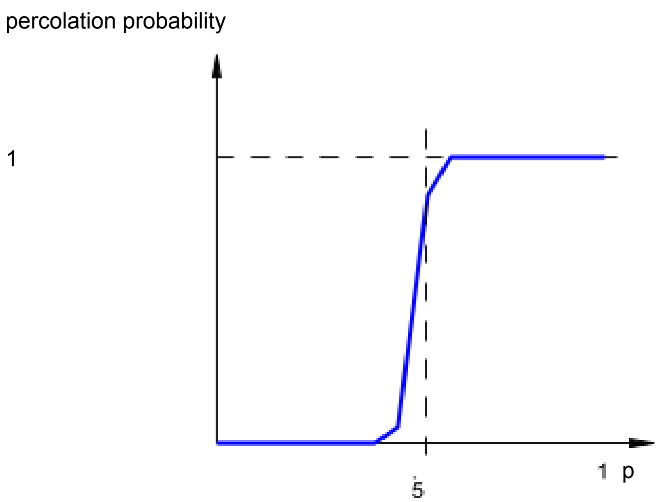
In a small class, you may wish to have every student simulate percolation for each probability value. In larger classes, you can accomplish the task more quickly by dividing into groups, and having each group do the simulation with a different probability. Electronic methods of simulation enable the class to explore larger lattices, a wider range of probabilities, and perform more replications of each. Alternatively, you could hand out strips of paper with pre-computed simulations, done with either of the methods described below, or project these strips of pre-computed simulations on the board for the whole class to evaluate together.

Choose one of the following methods to perform the simulation. For each probability value, track the proportion of simulations for which percolation occurs (there is an open path from any node at the top of the lattice to any node at the bottom). Put the results in a table like the following (with idealized results) on the board:

P(open)	Percolation Proportion

0.1	0
0.2	0
0.3	0
0.4	0.1
0.5	0.6
0.6	0.9
0.7	1.0
0.8	1.0
0.9	1.0

The module asks students to graph the results, which should look something like the graph below:



Simulation Method 1: Pencil and Paper (Handout) Students use a physical (game-board style) spinner with a fraction of the circle colored black corresponding to their assigned probability. (Note, however, the significant time investment required to build and flick physical spinners. A less time-consuming option that still lets students actively participate in the simulation is to use a random number generator. Use the rand() command in Excel, or use an online random number generator (for example, http://www.random.org/decimal-fractions/) where they can generate all their random numbers to determine whether each bond is open or closed, and fill in with a solid line each open bond. Closed bonds are left as is. Once each bond is determined to be open or not, the students must determine by eye whether or not there is percolation.

Simulation Method 2: Excel Workbook Open the Excel file BondPerc.xlsx. Each time the probability of a bond being open is changed, the entire lattice is redrawn with the simulated open and closed bonds. Students must determine by eye whether or not there is an open path, consisting of adjacent colored edges, from any node at the top of the lattice to any node at the bottom of the lattice.

With a smaller class, you might want to use BondPercExplore.xlsx, which lets each student with a computer generate 3 results for each of 4 different probabilities. Note that these probabilities can be modified as in BondPerc.xlsx.

Discussing the graph: First, look at the extremes:

If p is very close to 0, there are hardly any edges, and the graph is not likely to percolate.

If p is close to 1, there are lots of edges, and the graph is very likely to percolate.

What is happening between these two extremes?

- When $p < \frac{1}{2}$, the graph does not percolate
- When $p > \frac{1}{2}$, the graph does percolate.

This simulation is illustrating the concept of **percolation threshold**, a common flavor of results in the theory of random graphs.

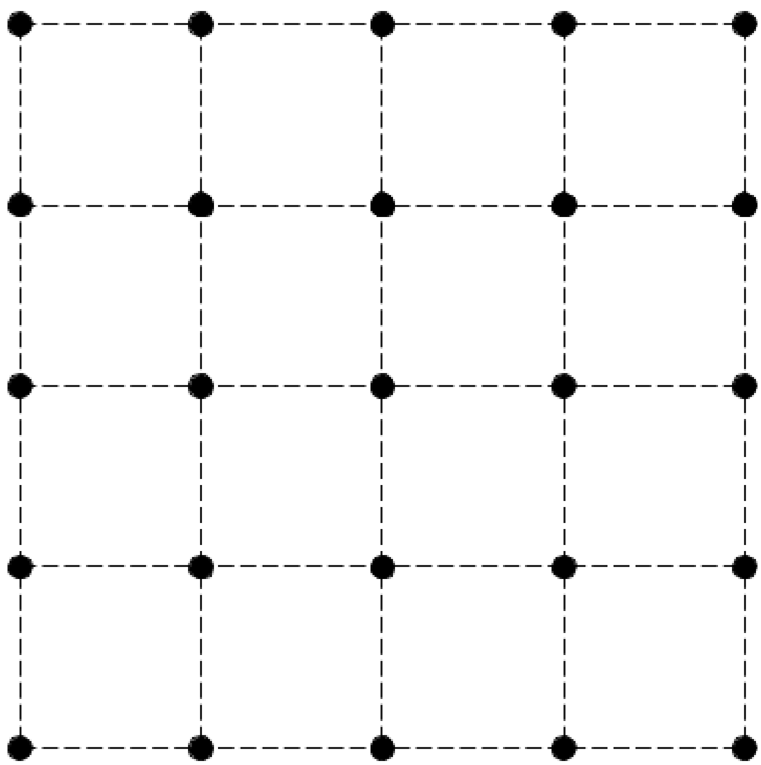
If a particular probability is above p_c then one property is exhibited, if the probability is below p_c, some other property is exhibited. We say p_c is the *threshhold* value of the property. In our case, the property is "percolation happens."

Q: What happens when the number of lattice points gets larger?

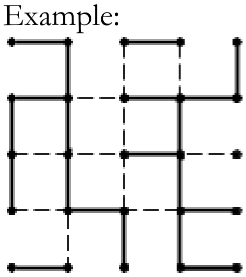
A: The graph becomes increasingly sharp.

3.2 Bond Percolation Simulation Example

Use the spinner for each of the 40 edges on the graph. If you spin and land on black, color the edge. You must go in order (top to bottom, left to right). This represents an open edge. An example is done for you.



Example:



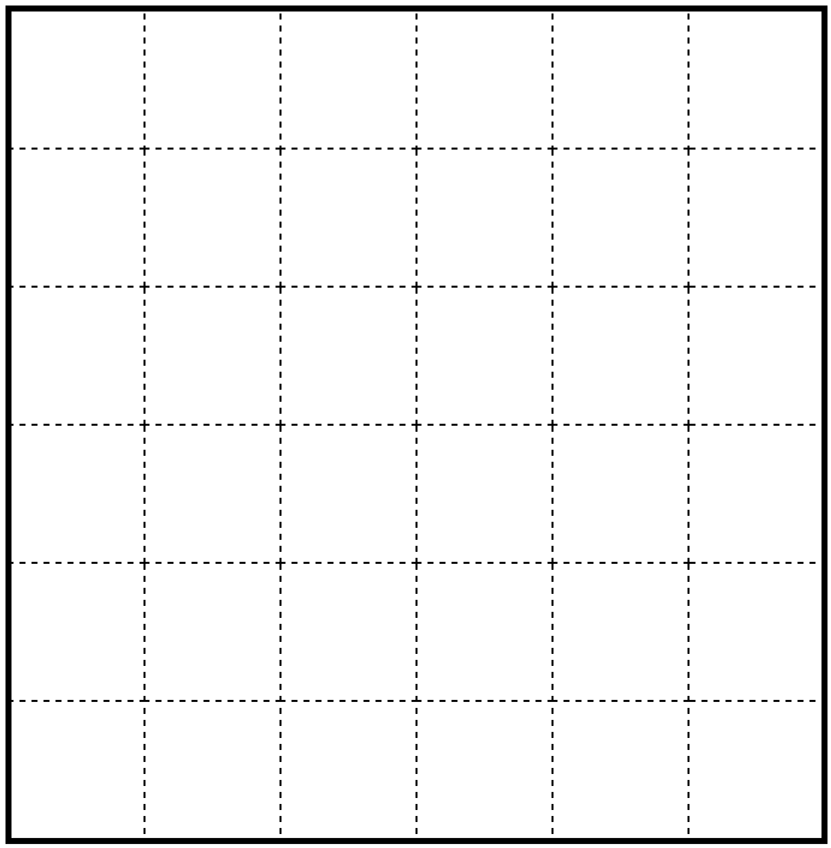
Can you find a path from the top to the bottom through the colored (open) edges? If you can, then your graph percolated.

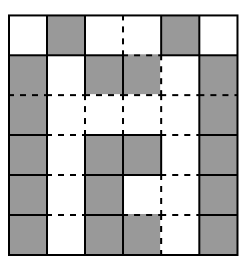
Did your random graph percolate? (Can you find a path from the top to the bottom through the colored (open) edges?)

What do you think is the bond percolation threshold?

3.3 Site Percolation Simulation Example

Use the spinner for each of the 36 blocks on the graph. If you spin and land on black, fill in the block. You must go in order (top to bottom, left to right). An example is done for you.





Can you find a path from the top to the bottom through the black boxes (open sites)? You may only move horizontally or vertically, not diagonally. If you can, then your graph percolated.

What do you think is the site percolation threshold?

4 Other Applications and Resources

Other Applications: You may use the following as discussion, handouts, or an assignment.

4.1 Other Applications

- Spread of disease in trees:

Consider a farmer who would like to plant the trees in an orchard in order to minimize the spread of disease between trees, yet maximize the yield from the orchard. The increased distance between trees represents a smaller probability of the spread of a disease. What is the optimal distance and lattice structure desired?

- Forest Fires:

Often forest rangers would like to be able to predict how far and how quickly a fi would be able to spread. Giving a time component to a simple percolation model allows forest rangers to make these predictions based on wind speed and density of the forest.

- Oil Field:

Often gas or oil is found insides porous rocks. The pores in rock form a network in which the oil or gas flows. Percolation models are used to predict how much oil or gas can be found in rocks of diff t porosity.

- Electrical network grid:

Electricity is passed from one component to another through connections (edges) which could represent power lines. For example, there is a nice structure for the power grid in a neighborhood of houses, or they may represent larger power lines connecting neighborhoods in a particular city. In order for power to pass from one city to the next, the power must percolate through the connections. What confi
should we use for the network in order to make it reliable yet cost effective?

- Communication network and social media:

Recently information has been able to spread through the use of social media and communication networks. Often the spread of text messages, tweets, etc can be analyzed through the use of a percolation network on graphs which model social network structures. The model percolates as information is passed from one person to another.

- Epidemiology:

The transmission of disease through a particular species is of great importance. If a particular individual or group becomes infected, how long will the disease propagate? Transmitability and virulence are two key factors in this model. The big question: Will there be an epidemic?

- Child immunization:

Children that get immunizations for particular diseases act as a buff for the spread of disease. If the percentage of children who receive a particular vaccine drops below a particular point, then the probability of an outbreak in children who are not immune increases dramatically.

- Gelatinous substance:

As a substance forms, bonds are made between neighboring chemicals. This formation allows the liquid to become a gelatinous substance. For example: the process of boiling an egg. These small clusters eventually bond together to form larger and larger molecules.

- Structural integrity of material:

Most materials have imperfections in them. These imperfections or impurities often make a substance weaker. Under a large amount of stress a crack often forms between these impurities. To percolate, the substance would have a crack from one end to the other, thus breaking the substance. How 'pure' must we make our material in order to have a particular strength?

- Groundwater flow:

As water flows through the soil, it percolates through the soil layers by moving through cracks and capillaries. This flow of water can be studied by a percolation model for different types of soils.

- Rumors:

Given a social network structure, if a rumor is started with an individual how far will it percolate? The social network structure would be highly dependent on the number of friends each person has and the strength of the friendship.

- Others: Lightning, Brine Ice Formation, underground lava flow.

4.2 Other Resources

References:

For more information on percolation we suggest the following texts. This list is not meant to be an extensive list, but serve as a beginners reference list.

1. 'Introduction to Percolation Theory' - Dietrich Stauff and Ammon Aharony
2. 'Percolation' - Bela Bollobas and Oliver Riordan
3. 'Percolation' - Geoffrey Grimmett
4. 'Applications of Percolation Theory' - M Sahimi

In addition, you can find several apps online which demonstrate different percolation models.

5 Teacher's Notes

5.1 Generalizations and Brief History

Bond Percolation:

The mathematical concepts of percolation were first introduced by Broadbent and Hammersley (1957: "Percolation processes I. Crystals and mazes.") Given a $n \times n$ lattice, what is the probability a path forms from the top of the lattice to the bottom of the lattice if each edge is present with probability p independent of the other edges? As is often the case, it is easier to compute the probability of percolation (forming an infi cluster) assuming an infinite lattice structure, i.e. n tends to infinity. Based on this model, for any given value of p, Kolmogorov's zero-one law tells us that the lattice either percolates with probability 1 or 0. Therefore there is some critical value p_c so that if $p < p_c$, then the model will percolate with probability 0, and if $p > p_c$, then the model will percolate with probability 1. In a very celebrated result, Harry Kesten (1982 - Percolation theory for mathematicians) proved that the critical value for the square lattice Z^2 was $p_c = 1/2$.

Site Percolation:

A similar questions can be asked for site percolation; each site is open with probability p and closed with probability $1 - p$. Is there a path from the top of the lattice to the bottom through open sites? For the square lattice Z^2, bounds have been able to show that $p_c \approx .59$, although an exact answer is still unknown! (This is a great way to show students that mathematics is alive and people are still working on very interesting and useful problems.)

Other Lattice Structures:

Other generalizations include different lattice structures. Many of these lattice structures have exact results while others do not. In most simple cases, if the number of neighbors increases, this implies there are a larger number of possible percolation paths, and therefore the value of p_c tends to decrease. However this may not be the case on lattice structures which have some vertices have very few neighbors. The wikipedia page on percolation threshold has a very comprehensive list of 2D lattice structures and bond and site percolation thresholds and bounds. (http://en.wikipedia.org/wiki/Percolation_threshold#Thresholds_on_other_2d_lattices)

Higher Dimensional Variants:

Percolation models have also been studied in higher dimensional variants. For example, consider the 3D square lattice where each vertex on the interior has 6 neighbors. Although exact values and bounds are more diffi to prove, there have been some results for simple structures. In almost all cases, the higher dimensional percolation thresholds p_c decrease significantly because of the increased number of possible paths because of the lattice structure.