When is Price Discrimination Profitable? †

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Abstract

The canonical model of a firm selling to heterogeneous, but indistinguishable, consumers implies that the firm should offer multiple products and distort its product quality relative to the efficient level, yet in practice many firms adopt a single product strategy. This tension can be resolved by recognizing that in many instances the firm's choice of product quality is constrained. We analyze a model of a quality-constrained monopolist's product line decision that encompasses a variety of important examples of second-degree price discrimination, including intertemporal price discrimination, coupons, advance purchase discounts, versioning of information goods, and damaged goods. We derive necessary and sufficient conditions for price discrimination to be profitable that generalize existing results in the literature. Specifically, we show that when a continuum of product qualities are feasible, price discrimination is profitable if and only if the ratio of the marginal social value from an increase in quality to the total social value of the good is increasing in consumers' willingness to pay. We also find that allowing price discrimination can result in a Pareto improvement, though in general the welfare effects are ambiguous.

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1. Introduction

Sellers often price discriminate by offering multiple product qualities at different prices. When consumers have heterogeneous valuations for quality (or quantity), Mussa and Rosen (1978) (and Maskin and Riley, 1984) showed that such price discrimination is always profitable for a monopolist. In this paper, we show that in the presence of technological quality constraints, the necessary and sufficient conditions for price discrimination are more restrictive. In particular, we show that price discrimination is profitable only if the ratio of the marginal social value from increased quality to the total social value of the good is increasing in consumers' willingness to pay. This condition is simple, intuitive, and easily testable. While it is implicit in many existing papers on price discrimination, and is even explicitly stated by Johnson and Myatt (2003), our paper unifies many existing results in the literature with this one simple condition.

Constraints on quality play a major role in our analysis. Absent constraints, or some equivalent limitation on product quality choice, our sufficient condition for price discrimination is always satisfied. With the exception of Salant (1989) and Acharyya (1998), research on the optimality of second-degree price discrimination has not explicitly emphasized the role of quality constraints.[1] Most research that asks when is price discriminate profitable, including Anderson and Song (2004), Deneckere and McAfee (1996), Hahn (forthcoming), Bhargava and Choudhary, (2001a, 2001b, 2004), Jones and Mendelson (1997), Johnson and Myatt (2003), and Gabszewicz et. al. (1986)

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[1] Salant (1989) showed that a simplified version of Stokey's (1979) model of intertemporal price discrimination is equivalent to Mussa and Rosen's model with a quality constraint. Acharyya (1998) also shows that adding a quality constraint to the Mussa and Rosen model can lead to pooling of monopoly quality levels.

has introduced quality constraints by restricting the firm to produce from a finite set of exogenous quality levels.

Despite not always making it explicit, the literature has implicitly accepted the assumption of quality constraints because it is quite natural. Firms are endowed with a given product technology, which bounds the maximum level of quality. And perhaps more importantly, the technologies available for lowering product quality (e.g., coupons, travel restrictions, disabling product features, and delaying delivery times) are often much richer and more diverse than the technologies available for raising quality.

In addition to emphasizing the role of quality constraints, our paper tries to offer a single framework for understanding the extensive, and somewhat disparate, literatures on many important types of price discrimination: intertemporal price discrimination, damaged goods, advance purchase discounts, coupons (rebates), and information goods. Price discrimination is particularly appealing to sellers of information goods because they incur a large fixed cost to produce their highest quality product, have extremely low marginal costs of production and bear little fixed costs of introducing lower quality product variants[2]. This type of second-degree price discrimination is so prevalent that industry jargon refers to it as versioning. Shapiro and Varian (1999) advise sellers to design a high-end product and then "start turning features off" to serve consumers with lower willingness to pay. Bhargava and Choudhary (2001a) derive a necessary condition for versioning for the case where the marginal cost is zero (pure information goods). We generalize their results by allowing for positive costs, allowing greater flexibility in

[2] Information goods include software, newspapers, books, movies, music, Internet service, telephone service, etc. Versioning information goods can occur in a variety of ways such as delay, user interface, convenience, image resolution, speed of operation, flexibility of use, capability, features/functions, comprehensiveness, annoyance, and support (Shapiro and Varian, 1999). A common example is software products that are offered in varying degrees of functionality such as student and professional versions.

product quality decision, and most importantly by deriving sufficient conditions for versioning.

We also use our result to generated more general sufficient conditions for the optimality of intertemporal price discrimination. Stokey (1979) showed that for reasonable assumptions on costs and preferences, intertemporal price discrimination is not optimal even when it is feasible. Stokey showed that when consumers' valuations and discount rates are correlated, the monopolist will engage in intertemporal price discrimination. Stokey, and later Salant (1989), also showed that when costs decline sufficiently rapidly, a monopolist would engage in intertemporal price discrimination. Using our framework we can easily generalize both of these results.

A third application is the provision of damaged goods. Deneckere and McAfee (1996) first analyzed whether a seller should intentionally offer a damaged version of a product to price discriminate. They demonstrate that it can be a Pareto improvement to offer a damaged version of a product even when a firm faces greater marginal costs for lower quality versions. The motivating examples are ones in which firms standardize on a product to exploit economies of scale in manufacturing and research and development, but then offer multiple varieties of their product by disabling features. They emphasized that this might be profitable even when the cost of disabling makes the low quality good more expensive than the high quality good. Hahn (forthcoming) considers a dynamic pricing extension of Deneckere and McAfee's static model and shows that the sufficient conditions for the profitability of introducing a damaged good are much weaker for a monopolist who lacks commitment power, but that the introduction of damaged goods is less likely to be efficient.

We also generalize existing results on the use of manufacturer coupons. It is well known that offering coupons can be profitable (Hess and Gerstners, 1991) and it has also been shown that coupons can result in a Pareto improvement (Anderson and Song, 2005). Our model offers general conditions on when price discrimination will be optimal.

Finally, we apply our result to advance purchase discounts. Many papers have argued that advance purchase discounts can increase profitability when consumers have private information about their demand. Some of these papers emphasize that firms can extract more surplus from ex post heterogeneous consumers by selling to them before their demand is known.[3] Others emphasized that advance purchase pricing can improve capacity utilization.[4] We develop a simple model in which advance purchase discounts are used to segment ex ante heterogeneous consumers who must choose between purchasing before their preferences are known or waiting and paying a higher price. In this simple model, e show that advance purchase discounts are profitable only when ratio of the marginal social value from relaxing the advance purchase requirement to the total social value of the relaxing the advance purchase requirement is increasing in consumers' willingness to pay.

We view our paper as reconciling differing results and approaches in the literature on price discrimination. In this sense we are continuing the agenda in Salant (1989) who reconciled Mussa and Rosen with the pooling result in Stokey's (1979) model of intertemporal price discrimination by showing that a special case of Stokey's model is

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[3] Shugan and Xie (2001), Courty (2003), and Gale and Holmes (1993) emphasize that the firm may be able to extract more surplus from consumers if they sell to them before their preferences are fully known. Dana (1998) identifies a related result holds in a competitive market when the spot market is imperfect.

[4] Shugan and Xie (2001) show that advance purchase discounts help to improve capacity utilization for exogenous, but limited, capacity. Gale and Holmes (1992) emphasize that selling to consumers before their preferences are fully known may allow the firm to economize on capacity. Tang et. al. (2004) emphasizes that advance purchase discounts allow firms to adjust their capacity, or inventory, for sale in the spot market using information about demand obtained from advance purchase sales.

equivalent to Mussa and Rosen's model with a quality constraint. We find much more general necessary and sufficient conditions for price discrimination.

The remainder of the paper is organized as follows. Section 2 presents a simple example with two types of consumers and two exogenously given product qualities. Section 3 solves the monopolist's problem when it faces two types of consumers and faces a quality constraint. Section 4 considers the more general problem in which a monopolist sells to continuum of consumers and faces a product quality constraint. Section 5 shows how our result ties together, and in some cases generalizes, existing results for information goods, intertemporal price discrimination, coupons, and damaged goods.

Before we proceed, note that is there a large literature on what constitutes price discrimination. We believe that selling different products to different consumers *when it would have been more efficient to sell them the same product* constitutes price discrimination. This definition of price discrimination is appealing because it corresponds to asking whether the solution to the monopolist's problem is a separating or a pooling solution and this is our primary definition throughout the paper. However in some of our analysis it may be efficient to sell different products to different consumers. In this case we say that the firm is price discriminating when the monopolist is distorting the quality of some of its products away from the efficient level in order to increase its profits.

2. An Example with Two Consumer Types

Consider a monopolist who can sell either or both of two products, one with quality $\underline{q}$ and another with quality $\bar{\bar{q}}$, to two distinct groups of consumers, high types (H) and low types (L). The monopolist cannot directly distinguish between consumer types, but can sell a different product to each type as long as the purchase decision is individually rational and incentive compatible. Consumers have unit demands and maximize their consumer surplus, $V_L(q) - p(q)$ and $V_H(q) - p(q)$ respectively. The firm has unit costs of production, $c(q)$, that vary with product quality.

Figure 1 gives a graphical depiction of the problem. Note that in the Figure, the willingness to pay is depicted as decreasing in *q* and the marginal cost is depicted as increasing in *q*, however our example is more general. We assume $V_H(q) > V_L(q), \forall q$, and $V_H(\bar{\bar{q}}) - V_L(\bar{\bar{q}}) > V_H(\underline{q}) - V_L(\underline{q})$. These last two assumptions say that the consumers that are willing to pay the most for a low quality product are also the consumers that are willing to pay the most to increase the quality from low to high, which is the well-known sorting condition. These three assumptions are equivalent to assuming that the areas of the regions A, B, C, and D depicted in the figure are all positive.

If the firm served all consumers with a single quality $\bar{\bar{q}}$ at a single price, its profit on each sale would be A + C. The high type would capture surplus D + B while the low type captures 0 surplus. If, instead, the firm chose to offer both a high and a low quality product, $\underline{q}$, it would lose A (capture only C) on each sale to a low type but gain an additional amount B (capture C+A+B) on each sale to a high type. So the firm's profits would increase as long as $B\overline{n} > A\underline{n}$, or $\frac{A}{A+B} < \frac{\overline{n}}{\underline{n}+\overline{n}}$, where $\underline{n}$ is the number of low-type consumers and $\overline{n}$ is the number of high-type consumers.

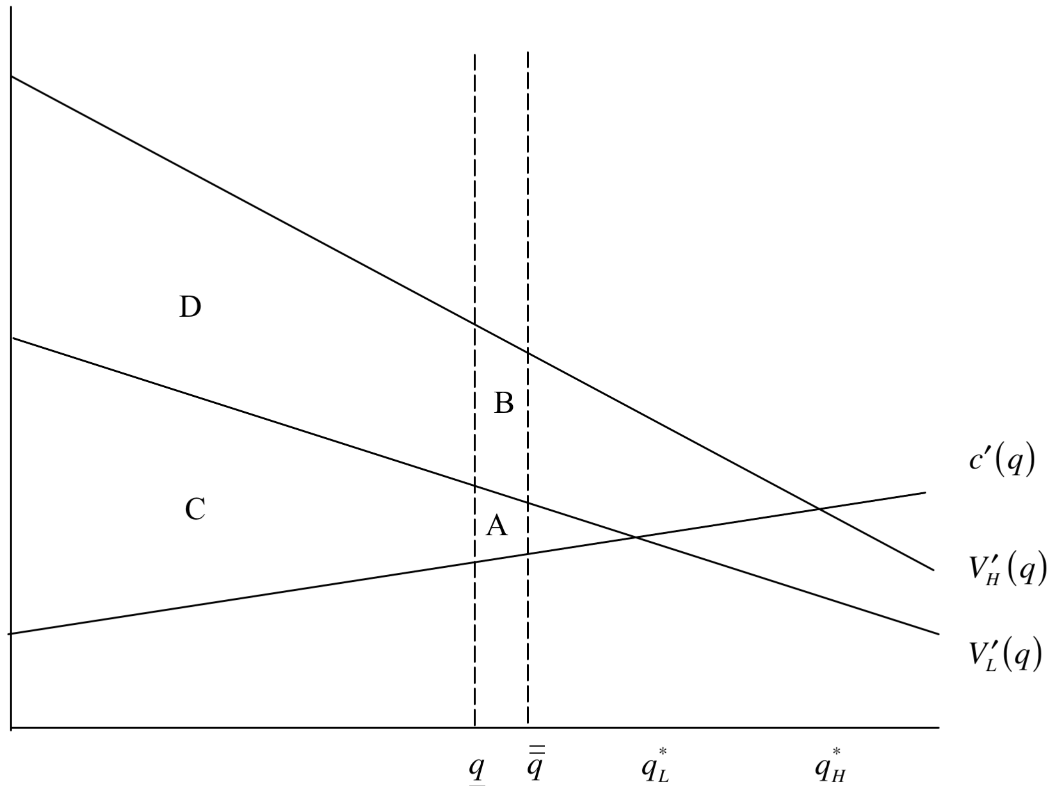

Figure 1

If the firm served only the high type consumers it would be able to capture producer surplus A + B + C + D. If instead the firm chose to offer both high and low quality, its profits would increase only if the profit earned on the new low-type consumers covered the lower margin on high-type customers, that is, if $C\underline{n} > D\overline{n}$, or

$$\frac{C}{C+D} > \frac{\overline{n}}{\underline{n}+\overline{n}}.$$

Hence the firm is willing to offer both product qualities if only if

$$\frac{C}{C+D} > \frac{\overline{n}}{\underline{n}+\overline{n}} > \frac{A}{A+B} \text{ or}$$

$$\frac{V_L(\overline{q})-c(\overline{q})}{V_L(\underline{q})-c(\underline{q})} < \frac{\overline{n}}{\overline{n}+\underline{n}} < \frac{V_H(\overline{q})-c(\overline{q})}{V_L(\underline{q})-c(\underline{q})}. \tag{1}$$

This condition says that both products are offered only if the ratio of the high types total surplus to the low types total surplus must be increasing in quality. Equivalently both products are offered only if the marginal surplus from an increase in quality to the total surplus from quality is increasing in the consumer type. Only if this condition is met will there exist values of $\underline{n}$ and $\overline{n}$ such that offering both products is optimal.

This example is easily generalized to allow the firm to choose its product quality optimally. Suppose there are $\underline{n}$ buyers of type $\underline{\theta}$ and $\overline{n}$ buyers of type $\overline{\theta}$. Buyer θ's utility from purchasing a product of quality q at price t is $V(q,\theta)-t$, and buyers purchase the product that gives them the greater consumer surplus. Assume the firm's cost for selling n units of quality q is $nc(q)$. Quality is constrained to be less than or equal to one. We assume that V and c are continuously differentiable with respect to q, that $V_q > 0$ and $V_q(q,\theta)-c'(q) > 0$, and that $V(q,\theta)$ and $V_q(q,\theta)$ are increasing in θ. Finally we assume $q^*(\theta) \equiv \arg\max_{q\leq 1} V(q,\theta)-c(q) = 1, \forall\theta$, that is, that the quality constraint binds on both types.

Proposition 1

Let N^* *denote the open interval* $\left(\frac{V_q(1,\underline{\theta})-c'(1)}{V_q(1,\overline{\theta})-c'(1)},\max_{\hat{q}}\frac{V(\hat{q},\underline{\theta})-c(\hat{q})}{V(\hat{q},\overline{\theta})-c(\hat{q})}\right)$.

a) *The firm will offer multiple qualities only if* $V(q,\theta)-c(q)$ *is log supermodular at* $q=1$.

b) *If* $V(q,\theta)-c(q)$ *is log supermodular at* $q=1$, *then* N^* *is non-empty and the firm will offer multiple qualities if and only if* $\frac{\overline{n}}{\overline{n}+\underline{n}}\in N^*$.

Proof: All proofs are in the appendix.

Proposition 1 shows that when quality is constrained, a monopolist will price discriminate only if the ratio of the marginal social value of quality to the total social value of quality is increasing in the consumer's type, θ. Proposition 1 also characterizes the distributional conditions that, along with log supermodularity of the surplus function, are sufficient for price discrimination to be profitable.

An implication of Proposition 1 is the following.

Corollary A

If $V(\theta,q)=\theta q$ *and if* $c'(q)>c(q)/q$ *for all* $q\in[0,1]$, *then offering multiple products is optimal for all* n_L *and* n_H *satisfying*

$$\max_q \frac{\theta_L q-c(q)}{\theta_H q-c(q)}>\frac{n_H}{n_H+n_L}>\frac{\theta_L-c'(1)}{\theta_H-c'(1)}. \tag{2}$$

3. The General Model

In this section we analyze a general model in which there are a continuum of buyers of type $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, with probability distribution $f(\theta)$ and cumulative distribution $F(\theta)$, and the firm can produce any number of products of any quality, q, subject to the constraint that $q \in [0,1]$. The unit cost of production is $c(q)$. Consumers maximize their consumer surplus, equal to their strictly positive utility, $V(q,\theta)$, less the price, $p(q)$. We assume that V and c are twice continuously differentiable and the V satisfies $V_q > 0$, $V_\theta > 0$, and $V_{q\theta} > 0$. Letting $S(q,\theta)$ denote the surplus function $V(q,\theta) - c(q)$, we also assume that $S_{qq} \leq 0$ and $S_{qq\theta} \geq 0$.

Finally, we assume that $V\left(1,\overline{\theta}\right) - c(1) > 0$ and

$$\left(V\left(1,\underline{\theta}\right) - c(1)\right) f\left(\underline{\theta}\right) - V_\theta\left(1,\underline{\theta}\right) < 0. \tag{3}$$

These assumptions guarantee that a monopolist selling a single product of quality 1 will serve some, but not all, consumers.

As above, we let $S(q,\theta)$ denote the surplus function $V(q,\theta) - c(q)$. Also as above, the monopolist's product quality decision depends on whether or not the surplus function is log supermodular, that is on whether or not $S(q_1,\theta)/S(q_2,\theta)$ is increasing in θ for all $q_1 > q_2$. When $S(q,\theta)$ is twice continuously differentiable, log supermodularity of $S(q,\theta)$ is also equivalent to $\partial^2 \ln S/\partial q \partial\theta = \left[S_{\theta q} S - S_\theta S_q\right]/S^2 > 0$. It is worth noting that any function that is multiplicatively separable in θ and q is not log supermodular because $S_{\theta q} S - S_\theta S_q = 0$. However it is easy to see that the functions $a + f(q) g(\theta)$ and

$a+h(q)+f(q)g(\theta)$ are log supermodular as long as $a>0$ and *f, g,* and *h* are positive, increasing functions.

The seller's problem is to choose the menu of prices and qualities $(p(\theta),q(\theta))$ on $[\theta_L,\bar{\theta}]$ which maximizes

$$\max_{\theta_L,p(\theta),q(\theta)} \int_{\theta_L}^{\bar{\theta}} \left[p(\theta)-c(q(\theta))\right]f(\theta)d\theta, \tag{4}$$

subject to $q(\theta)\leq 1$ as well as incentive compatibility and individual rationality constraints. Without loss of generality we assume the seller chooses to serve a range of consumers that includes the highest type, $\bar{\theta}$.

This is a standard mechanism design problem.[5] The solution to (4) satisfies

$$H(\theta,q(\theta))\equiv \frac{\partial V(q(\theta),\theta)}{\partial q}-c'(q(\theta))-\frac{(1-F(\theta))}{f(\theta)}\frac{\partial^2 V(q(\theta),\theta)}{\partial\theta\partial q}=0 \tag{5}$$

where $0<q(\theta)<1$ and satisfies $H(\theta,q(\theta))\geq 0$ where $q(\theta)=1$ and $H(\theta,q(\theta))\leq 0$ where $q(\theta)=0$.

We assume $H(\theta,q)$ is increasing in θ and decreasing in *q*. Given our assumptions on *V* and *c*, this holds if $F(\theta)$ satisfies a monotone likelihood ratio property.

Finally, the lowest type buyer that the firm chooses to serve, θ_L, must satisfy either

$$J(\theta_L,q(\theta_L))\equiv -V(q(\theta_L),\theta_L)+c(q(\theta_L))+\left[\frac{1-F(\theta_L)}{f(\theta_L)}\right]\frac{\partial V(q(\theta_L),\theta_L)}{\partial\theta}=0 \tag{6}$$

[5] Note that by appropriately rescaling the quality measure this problem can be written with linear costs, but for ease of application we chose not to make this simplification.

or $\theta_L = \underline{\theta}$ and $J\left(\underline{\theta}, q\left(\underline{\theta}\right)\right) \leq 0$.

Let $q^*(\theta) \equiv \arg\max_{q \leq 1} V(q,\theta) - c(q)$ denote the optimal quality for each consumer type, subject to the constraint that quality be less than or equal to one. Without loss of generality we assume $q^*\left(\overline{\theta}\right) = 1$.

Under our assumptions, $q^*(\theta)$ is weakly increasing in θ. We analyze three separate cases. First, we consider the case in which $q^*(\theta)$ is strictly increasing. Second we consider the case in which $q^*(\theta) = 1$ for all θ. And finally, we consider the case in which $q^*(\theta) = 1$ for only some θ.

If the quality constraint does not bind, that is if $q^*(\theta)$ is strictly increasing, then the firm offers multiple qualities and the surplus function is log supermodular.

Proposition 2

If $q^*(\theta)$ is strictly increasing, then $V(q,\theta) - c(q)$ is log supermodular for all θ and $q = 1$, and the firm strictly prefers to sell multiple product qualities.

If the quality constraint does bind, then it may not be optimal for the firm to offer multiple qualities; it no longer follows that $V(q,\theta) - c(q)$ is necessarily log supermodular, and it no longer follows that the firm will necessarily offer multiple product qualities. And these two properties are related.

The following is our main result. We show that log supermodularity of the surplus function is both necessary and sufficient for offering multiple products to be optimal.

Proposition 3

If $q^*(\theta) = 1, \forall\theta$ *, then*

a) if $V(q,\theta) - c(q)$ *is log submodular then the firm sells a single quality, and*

b) if $V(q,\theta) - c(q)$ *is log supermodular then the firm sells multiple qualities.*

As we show in the proof of Proposition 3, it isn't necessary that $V(q,\theta) - c(q)$ be log supermodular everywhere for multiple products to be optimal, but only that $V(1,\theta) - c(1)$ be locally log supermodular . Also, if $S_{qq} < 0$ and $S_{qq\theta} > 0$, then if $V(1,\theta) - c(1)$ is locally log submodular, the firm will sell a single product.

An immediate implication of Proposition 3 is that when consumers have utility $V(q,\theta) = \theta q$, the firm will offer multiple products as long as the marginal cost of quality is higher than the average cost for all quality.

Corollary B

If $V(q,\theta) = \theta q$ *, then multiple products are optimal if* $c'(q) > c(q)/q$ *for all* $q \in [\hat{q},1]$*,* $0 \le \hat{q} < 1$*, and multiple products is not optimal if* $c'(q) \le c(q)/q$ *for all* $q \in [\hat{q},1]$*,* $0 \le \hat{q} < 1$.

Another implication is that for multiplicatively separable utility and strictly positive costs, multiple products are more likely when the marginal value of quality is lower than the average value of quality.

Corollary ??

If $V(q,\theta) = g(q)h(\theta)$ *and* $c(q) > 0$, *then multiple products are optimal if*

$\frac{c'(q)}{c(q)/q} > \frac{g'(q)}{g(q)/q}$ *for all* $q \in [\hat{q},1]$, $0 \le \hat{q} < 1$, *and multiple products is not optimal if*

$\frac{c'(q)}{c(q)/q} \le \frac{g'(q)}{g(q)/q}$ *for all* $q \in [\hat{q},1]$, $0 \le \hat{q} < 1$.

This clearly implies that if cost is strictly positive and independent of quality, $c(q) = c > 0$, then multiple products are optimal if $g'(q) < g(q)/q$ not optimal if $g'(q) > g(q)/q$. It also implies that for any multiplicatively separable utility function $g'(q) < g(q)/q$ and $c'(q) > c(q)/q$ are sufficient for multiple products to be optimal. Some of the intuition for these results can be seen in Figure 1 which depicts utility functions and cost functions that satisfy both these conditions.

Finally, a third implication of Proposition 3 is that if the firm is only able to produce a finite number of products, log supermodularity of $V(q,\theta) - c(q)$ is still a necessary condition for the firm to offer multiple products. However for a finite set of products log supermodularity is no longer sufficient.[6]

When the efficient quality is only partially constrained, it follows that a social planner would always offer multiple products. It is still useful to be able to compare the monopolist to the social planner in this case as well. Proposition 4 affirms that the standard Mussa and Rosen results still hold when the monopolist faces a quality constraint.

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[6] To see this, let $V(\theta, q_l) = 1 + \theta$ and $V(\theta, q_h) = 1000 + 1000\theta$ and $\theta \sim U[0,1]$. It is easy to see that $V(\theta, q_h)/V(\theta, q_l)$ is increasing in θ, yet it is optimal for the firm to sell only product q_h.

Proposition 4

If $q^*(\theta)$ is weakly increasing, the monopolist's quality satisfies

$q^m(\overline{\theta}) = q^*(\overline{\theta})$, $q^m(\theta) \leq q^*(\theta), \forall \theta$, *and* $q^m(\theta) < q^*(\theta)$ *whenever*

$0 < q^*(\theta) < 1$.

4. Welfare

When there are just two types of buyers, it is easy to show that allowing price discrimination can lead to a Pareto improvement. Specifically, a Pareto improvement occurs when both types are served when price discrimination is allowed, but only the high type is served when price discrimination is banned.

Proposition 5

If there are two types of consumers, $V(q,\theta) - c(q)$ *is log supermodular in a neighborhood of* $q = 1$, $\frac{\overline{n}}{\overline{n} + \underline{n}} \in N^*$, *and* $\frac{\overline{n}}{\overline{n} + \underline{n}} > \frac{V(1,\underline{\theta}) - c(1)}{V(1,\overline{\theta}) - c(1)}$, *then offering multiple qualities results in a Pareto improvement.*

The regions described in Proposition 5 are illustrated graphically in Figure 2. The figure depicts a surplus function, $V(q,\theta) - c(q)$, which is log supermodular. It also shows both the region in which both products are offered and the sub-region in which offering both products leads to a Pareto improvement. A Pareto improvement occurs if the seller chooses to serve more buyers when he or she is allowed to price discriminate than when he or she is not allowed to price discriminate. Note that outside the depicted regions, the firm offers only one product.

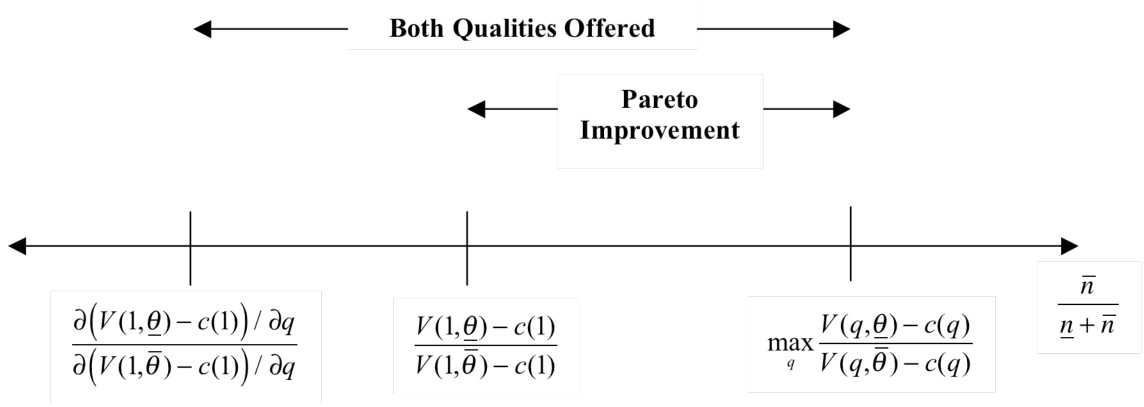

Figure 2: Regions of Price Discrimination and Pareto Improvement

When there is a continuum of buyer types, a similar intuition holds. Holding the quality offered to consumers who would have bought otherwise fixed, the consumers who would not have bought otherwise are clearly better off, and those who would have bought otherwise must also be better off because they must be receive a weakly lower price to induce them not to switch to the lower quality product.

But with a continuum of buyer types there is a second effect. In this case the quality offered to consumers who would have bought otherwise will also change. Holding the number of consumers served fixed, the monopolist can lower the quality to some of these buyers in order to raise the price to others. Despite receiving a lower quality, the marginal consumer when price discrimination is not allowed is strictly better off when price discrimination is allowed since his or her surplus is zero when price discrimination is not allowed. But the highest type consumer may be worse off as a consequence of price discrimination since the price he or she pays may increase.

The examples we have considered so far suggest that the second effect undermines the first, so that with a continuum of types, lifting a ban on price discrimination cannot result in a Pareto improvement. However it still clear that lifting a ban on price discrimination can increase consumer plus producer surplus.

5. Applications

A. Intertemporal Price Discrimination

Stokey (1979) established that intertemporal price discrimination is never optimal when consumers' utility functions are $U(\theta,t)=\theta\delta^t$ and the unit cost of production is $k(t)=\kappa\delta^t$, that is, when the cost is independent of time except for the time value of money. This well-known and important result follows immediately from Proposition 3. The monopolist maximizes profits by choosing a menu of prices (paid at time 0) and delivery times, subject to the constraint that $t\geq 0$, to maximize profits. With a change of variables, $q=\delta^t$, the firm's problem is to choose the profit-maximizing menu of prices and qualities, given utility $V(\theta,q)=\theta q$ and costs $c(q)=cq$, subject to the constraint that $q\leq 1$. Clearly $V(\theta,q)-c(q)=\theta q-cq$ is not log supermodular and $q=1$ is the efficient quality for all θ. So by Proposition 3, intertemporal price discrimination is never optimal, even though it is clearly feasible.

Stokey (1979), and later Salant (1989), both demonstrate that intertemporal price discrimination is optimal with more general cost functions. They replace the cost function with a more general one, $k(t)=\kappa(t)\delta^t$, which implies $c(q)=\kappa\left(\frac{\log q}{\log\delta}\right)q$. The surplus function, $V(\theta,q)-c(q)=\theta q-c(q)$, is log supermodular if

$$\frac{\left(qc'(q)-c(q)\right)}{\left(\theta q-c(q)\right)^2}>0,$$

or $c'(q)>c(q)/q$. So if the marginal cost of quality is positive and greater than the average cost of quality, intertemporal price discrimination is profitable.[7]

Salant showed that $c'(q)>c(q)/q$ was necessary and that

$$\frac{\theta_L-c'(0)}{\theta_H-c'(0)}>\frac{n_H}{n_H+n_L}>\frac{\theta_L-c'(1)}{\theta_H-c'(1)}$$

was sufficient for intertemporal price discrimination. Using Proposition 1, we generalize this result in the following Corollary.

Corollary A

With two consumer types, if $c'(q)>c(q)/q$ for any $q\in[0,1]$, then for a all n_L and n_H satisfying

$$\max_q \frac{\theta_L q-c(q)}{\theta_H q-c(q)}>\frac{n_H}{n_H+n_L}>\frac{\theta_L-c'(1)}{\theta_H-c'(1)} \tag{7}$$

intertemporal price discrimination is optimal.

Similarly, using Proposition 3 we generalize Salant's result to a continuum of types:

Corollary B

With a continuum of types, if $c'(q)>c(q)/q$ for all $q\in[\hat{q},1]$, $0\le\hat{q}<1$, then intertemporal price discrimination is optimal.

[7] Johnson and Myatt (2003) have a related result about the product range of a multiproduct monopolist.

Of course, $c'(q) > c(q)/q$ is easily interpreted in terms of the cost of production.

Since $c(q) = \kappa\left(\frac{\log q}{\log \delta}\right)q$, $c'(q) > c(q)/q$ holds if and only if $\kappa'(t) < 0$, because it can be written

$$\kappa\left(\frac{\log q}{\log \delta}\right) + \kappa'\left(\frac{\log q}{\log \delta}\right)\frac{1}{\log \delta} > \kappa\left(\frac{\log q}{\log \delta}\right),$$

where $\delta < 1$ and $\log \delta < 0$. So, if the firm's production costs are declining over time, then the firm will offer declining prices and inducing some consumers to delay their purchases.[8]

Finally, note that intertemporal price discrimination is also profitable when consumers have heterogeneous valuations and a common discount rate, *r*, that is higher than the firm's rate, r_f (see Landsberger and Meilijson, 1985). We can write consumers' utility, $U(t,\theta) = \theta e^{-rt}$, as $V(\theta, q) = \theta q$, where $q = e^{-rt}$, and the firm's cost function as $c(q) = \kappa q^{r_f/r}$, which implies $c'(q) > c(q)/q$, so by Corollary B intertemporal price discrimination is clearly profitable.

B. Information Goods

Information goods is a term used to describe goods like software, books, music, newspapers and magazines, which have high fixed costs of production and small or negligible variable costs. The practice of selling multiple versions of information goods, know popularly called "versioning," has been described by informally by Shapiro and

[8] As Stokey points out, when $\kappa'(t) > r = -\log \delta$ for some *t* competitive markets will also exhibit this pattern of prices and purchases. But by Proposition 4, when competitive market exhibit such delay, the monopoly market exhibit weakly greater delay for all consumers and strictly greater delay for any consumers who don't purchase immediately.

Varian (1998), and more formally by Varian (1995 & 2001) and Bhargava and Choudhary (2001b, 2004). Bhargava and Choudhary (2001a, 2001b, 2004) consider exogenous quality levels and describe when versioning is profitable in an environment where there the costs of production are zero. Bhargava and Choudhary (2001a) show that versioning is never profitable if consumers' quasi-linear utilities are $V(\theta,q)=\theta q$ and Bhargava and Choudhary (2001b) consider a two good model with a uniform distribution of consumer types and show that the firm will produce both the high and low quality good only if $V(q_H,\theta)/V(q_L,\theta)$ is increasing in θ.

We extend this literature by considering more general distributions of consumer preferences, by allowing for a general product choice, and most importantly by showing that when there are a continuum of consumer types and a continuum of product quality choices, log supermodularity is also a sufficient condition for versioning to be profitable.

In the case of information goods the natural source of the constraint is the technology available to the firm. In many instances this constraint is likely to arise endogenously as the result of fixed costs of technology development. We can think of an information good producer as solving the following problem:

$$\max_{\theta_L, p(\theta), q(\theta), \bar{\bar{q}}} \int_{\theta_L}^{\bar{\theta}} p(\theta) f(\theta) d\theta - C(\bar{\bar{q}}), \tag{8}$$

subject to $q(\theta) \le \bar{\bar{q}}$ and the incentive compatibility and individual rationality constraints. Here C denotes the fixed costs of developing the technology, but the variable costs of production are zero. In other words, total cost depends only on the quality of the highest quality product sold, not on the volume produced.

C. Coupons

Consider a simple model of coupon-based price discrimination based on Anderson and Song (2004). Assume that consumers are uniformly distributed on $\left[\underline{\theta},\bar{\theta}\right]$ the unit interval and that their utility is $V\left(\theta,N\right)=\alpha+\theta\varphi$ if they do not use a coupon and $V(\theta,C)=\alpha+\theta\varphi-H(\theta)$ if they do use a coupon. The function $H(\theta)$ represents the cost of using a coupon and is assumed to be increasing in the consumer's type. The parameters α and φ are positive scalars. The firm chooses, *d,* the face value of the coupon, and *p,* the shelf price. The constant marginal cost of the good is *c* and the cost of printing the coupons is λ per coupon user.

From Proposition 3 coupons are profitable only if $V(\theta,q)-c(q)$, $q\in\{C,N\}$, is log supermodular, and $V(\theta,q)-c(q)$, $q\in\{C,N\}$, is log supermodular if

$$\frac{\varphi}{\alpha+\theta\varphi-c}>\frac{\varphi-H'(\theta)}{\alpha+\theta\varphi-H(\theta)-c-\lambda}$$

or equivalently

$$\frac{\varphi}{\alpha+\theta\varphi-c}<\frac{H'(\theta)}{H(\theta)+\lambda}.$$

This inequality is satisfied for $H(\theta)=\theta H$ as long as $\alpha-c$ is positive and λ is negligible. More generally, the use coupons is more likely to be profitable the larger is $\alpha-c$, the smaller is λ, and the larger is $H'(\theta)-H(\theta)/\theta$.

D. Damaged Goods

A clear implication of Proposition 3 and Corollary B is that damaged goods strategies are not profitable when consumer's utility is $V(q,\theta)=\theta q$. A damaged good is one for which $c'(q) \le 0$, that is, it is weakly more expensive to produce lower quality goods. But clearly $\theta q - c(q)$ is log submodular when $c(q) \ge c'(q)/q$ (see Corollary B).

Deneckere and McAfee (1996) demonstrate that when producing a low quality "damaged" good is more expensive than producing an undamaged good, it can nevertheless be profitable, and even Pareto improving, to sell both the damaged and undamaged good[9]. They assume a continuum of types with unit demands, and restrict attention to two product qualities, q_L and q_H. They assume consumers have quasi-linear utilities $V(q_H, \theta) = \theta$ and $V(q_L, \theta) = \lambda(\theta)$.

It is easy to see that the necessary condition derived by Deneckere and McAfee is a special case of our more general condition. Specifically, in Deneckere and McAfee's model, $V(q, \theta) - c(q)$ is log supermodular if and only if

$$\frac{1}{\theta - c_H} > \frac{\lambda'(\theta)}{\lambda(\theta) - c_L},$$

or $\lambda(\theta) - c_L - (\theta - c_H)\lambda'(\theta) > 0$. The price a single product firm would charge is $p = V(q_H, \underline{\theta}) = \underline{\theta}$ where $\underline{\theta}$ is defined by

$$\underline{\theta} - c_H - \frac{1 - F(\underline{\theta})}{f(\underline{\theta})} = 0$$

So $V(q, \underline{\theta}) - c(q)$ is log supermodular if and only if

$$\lambda(\hat{\theta}) - c_L - \left(\frac{1 - F(\hat{\theta})}{f(\hat{\theta})} \right) \lambda'(\theta) > 0$$

[9] Here we discuss the second of the two models that Deneckere and McAfee (1996) analyze.

which is the necessary and sufficient condition for the provision of damaged goods derived by Deneckere and McAfee.

E. Advance Purchase Discounts

Advance purchase requirements are another potential instrument for price discrimination (see Shugan and Xie 2000, Courty and Li 2000, and Gale and Holmes 1992, 1993).[10,11] Purchasing in advance requires consumers to give up flexibility in their purchase decision, departure time, or destination. Suppose the firm can set one price, p_0, for travel if the ticket is purchased at time 0 (e.g., 14-days in advance) and another price, p_1, for travel if the ticket as purchased at time 1 (e.g., one day in advance). Suppose there are two types of consumers, business travelers and leisure travelers, who differ in their valuations for the product and in their cost of planning. Specifically, their value for travel is v_B and v_L if they buy in the spot market and is $v_B - x_B$ and $v_L - x_L$ if they buy in advance.[12] And suppose the cost, *c*, is independent of the purchase time.

The firm has three pricing options. It can sell to all the business travelers at price v_B ($p_0 = p_1 = v_B$), or sell to all buyers at price v_L ($p_0 = p_1 = v_L$), or sell to leisure travelers at price $p_0 = (v_L - x_L)$ at time 0 and to sell to business travelers at price

[10] Price discrimination can help the firm extract greater surplus from heterogeneous consumers (see Shugan and Xie 2001, Courty and Li 2000, and Dana, in progress) and also enable the firm to increase capacity utilization (see Gale and Holmes, 1992, 1993, and Dana, 1998, 1999).
[11] Advance purchase discounts can also benefit the firm in other ways. First, advance purchase discounts can be used to improve production efficiency of production by giving the firm better forecast of spot market demand (Tang et. al. 2004, and McCardle et. al. 2004). Also, firms may find it more profitable to sell in advance when consumers have an imperfect forecast of their spot market preferences (Shugan and Xie 2001, and Courty 2003).
[12] The literature on advance purchase discounts derives the value of flexibility explicitly from consumers' demands – consumers who buy in advance are either uncertain about their spot market valuations (Courty and Li, 2000, Dana 1998, and Shugan and Xie 2000) or about their departure time preferences (Gale and Holmes 1992, 1993 and Dana 1999).

$p_1 = p_0 + x_B$ at time 1. By Proposition 1, option three, which is the price discrimination option, is the most profitable if and only if

$$\frac{x_B}{v_B - x_B - c} > \frac{x_L}{v_L - x_L - c}.$$

Price discrimination is profitable only if the expected disutility for a business traveler from buying early is greater *as a percentage of the surplus generated by a business traveler* than the expected disutility for a tourist traveler from buying early *as a percentage of the surplus generated by a tourist traveler*. The intuition generated by Mussa and Rosen is that if consumers' valuations are correlated with the value they place on flexibility, then a monopolist can benefit from using advance purchase discounts. However it is clear that $x_B > x_L$, or positive correlation if x and v, is not sufficient.

6. Conclusion

This paper offers a general theory for the optimality of price discrimination that is useful in analyzing many types of price discrimination, including intertemporal price discrimination, the use of coupons, the versioning of information goods, the practice of crimping or selling intentionally damaged goods, and the use of advance purchase discounts. We derive new necessary and sufficient conditions for price discrimination to be profitable, as well as linking the common elements of many existing, but disparate, literatures into a general theory.

Our paper asks when price discrimination will be profitable, but we found it was sometimes easier to discuss when it would not be profitable. We began with a generalized Mussa and Rosen environment in which price discrimination is always profitable and looked at some modifications of that environment in which price discrimination does not occur. We found that with a continuum of consumer types a cap, or constraint, on quality is sufficient to guarantee the firm to offers a single product only if the surplus function is log submodular. With just two consumer types, price discrimination may fail to be profitable either because quality is constrained and the surplus function is log submodular, or because there are too few of one type of consumer.

7. Proofs

Proof of Proposition 1

The seller selects quality levels, $\underline{q}$ and $\overline{q}$, and transfers, $\underline{t}$ and $\overline{t}$, subject to incentive compatibility and participation constraints:

$$\max_{\underline{q},\overline{q},\underline{t},\overline{t}} I\left(V(\underline{q},\underline{\theta})-\underline{t}\right)\underline{n}\left(\underline{t}-c\left(\underline{q}\right)\right)+I\left(V(\overline{q},\overline{\theta})-\overline{t}\right)\overline{n}\left(\underline{t}-c\left(\underline{q}\right)\right) \tag{9}$$

subject to

$$V(\overline{q},\overline{\theta})-\overline{t} \geq V(\underline{q},\overline{\theta})-\underline{t}, \tag{IC-1}$$

$$V(\underline{q},\underline{\theta})-\underline{t} \geq V\left(\overline{q},\underline{\theta}\right)-\overline{t}, \tag{IC-2}$$

and

$$\underline{q} \leq \overline{q} \leq 1$$

where I is the indicator function (consumers purchase only if their surplus is non-negative).

Clearly any solution to (9) satisfies $\overline{q}=1$. Hence (9) has three possible solutions, which we label S1, S2, and S3. The first strategy, S1, is to sell a single quality $\overline{q}$ to only the high type buyers at $\overline{t}=V(1,\overline{\theta})$ and profit $\overline{n}\left(V(1,\overline{\theta})-c\right)$. The second strategy, S2, is to sell a single quality, $\underline{q}=\overline{q}=1$, to all buyer types at price $\underline{t}=V\left(1,\underline{\theta}\right)$ and profit $\left(\underline{n}+\overline{n}\right)\left(V(1,\underline{\theta})-c\right)$. The third strategy, S3, is to offer multiple qualities and sell to both buyer types. The low-type buyer pays $\underline{t}=V\left(\underline{q},\underline{\theta}\right)$ for quality $\underline{q}$ and the high type buyer pays $\bar{t} = V(1,\bar{\theta}) - \left(V(\underline{q},\bar{\theta}) - V(\underline{q},\underline{\theta})\right)$ for quality $\bar{q} = 1$ and the firm earns a profit

$\underline{n}\left(V\left(\underline{q},\underline{\theta}\right) - c\underline{q}\right) + \bar{n}\left(V(1,\bar{\theta}) - c - \left(V(\underline{q},\bar{\theta}) - V(\underline{q},\underline{\theta})\right)\right)$.

When the firm offers multiple qualities (strategy S3) the low quality level solves

$$\max_{\hat{q}} \quad \underline{n}\left(V\left(\hat{q},\underline{\theta}\right) - c\left(\hat{q}\right)\right) + \bar{n}\left(V(1,\bar{\theta}) - c\left(1\right) - \left(V(\hat{q},\bar{\theta}) - V(\hat{q},\underline{\theta})\right)\right) . \tag{10}$$

The first order condition,

$$G\left(q\right) = \underline{n}\left(\frac{\partial V\left(q,\underline{\theta}\right)}{\partial q} - c'\left(q\right)\right) + \bar{n}\left(\frac{\partial V\left(q,\underline{\theta}\right)}{\partial q} - \frac{\partial V(q,\bar{\theta})}{\partial q}\right) = 0 , \tag{11}$$

has a strictly interior solution, $\hat{q} \in (0,1)$, only if $G\left(0\right) > 0$ and $G\left(1\right) < 0$. Under our assumptions, $G\left(0\right) > 0$ and the second order condition is satisfied. So $\hat{q} < 1$, or equivalently strategy S3 strictly dominates strategy S2 if and only if $G\left(1\right) < 0$, or

$$\underline{n}\left(\partial V\left(1,\underline{\theta}\right) / \partial q - c\left(1\right)\right) + \bar{n}\left(\partial V(1,\underline{\theta}) / \partial q - \partial V(1,\bar{\theta}) / \partial q\right) < 0 .$$

Strategy S3 strictly dominates strategy S1 if and only if

$$\underline{n}\left(V\left(\hat{q},\underline{\theta}\right) - c\left(\hat{q}\right)\right) + \bar{n}\left(V(1,\bar{\theta}) - c\left(1\right) - \left(V(\hat{q},\bar{\theta}) - V(\hat{q},\underline{\theta})\right)\right) > \bar{n}\left(V(1,\bar{\theta}) - c\left(1\right)\right),$$

or equivalently $\underline{n}\left(V(\hat{q},\underline{\theta}) - c\left(\hat{q}\right)\right) - \bar{n}\left(V(\hat{q},\bar{\theta}) - V(\hat{q},\underline{\theta})\right) > 0$, for some $\hat{q}$.

These two conditions can be written as

$$\frac{\partial V(1,\underline{\theta}) / \partial q - c'\left(1\right)}{\partial V(1,\bar{\theta}) / \partial q - c'\left(1\right)} < \frac{\bar{n}}{\bar{n} + \underline{n}} , \tag{12}$$

and

$$\frac{V(\hat{q},\underline{\theta}) - c\left(\hat{q}\right)}{V(\hat{q},\bar{\theta}) - c\left(\hat{q}\right)} > \frac{\bar{n}}{\bar{n} + \underline{n}} \text{ for some } \hat{q} . \tag{13}$$

So a necessary condition for these two conditions to hold simultaneously is

$$\frac{\partial\left(V\left(1,\underline{\theta}\right)-c(1)\right)/\partial q}{V\left(\hat{q},\underline{\theta}\right)-c\left(\hat{q}\right)}<\frac{\partial\left(V\left(1,\overline{\theta}\right)-c(1)\right)/\partial q}{V\left(\hat{q},\overline{\theta}\right)-c\left(\hat{q}\right)}, \tag{14}$$

for some $\hat{q}$.

If $V(q,\theta)-c(q)$ is everywhere log submodular (14) never holds, (12) and (13) cannot both be satisfied, and either strategy S1 or S2 dominates and the firm produces only high quality.

If $V(q,\theta)-c(q)$ is log supermodular on $\{\underline{\theta},\overline{\theta}\}\times\{\hat{q},1\}$ for some $\hat{q}<1$ then (14) holds and there exists $\underline{n}$ and $\overline{n}$ such that (12) and (13) are both satisfied, that is, such that S3 dominates both S1 and S2, and the firm offers both a high and low quality product.

Proof of Proposition 2:

When $q^*(\theta)$ is increasing, $V(\theta,q)-c(q)$ it follows that

$S_q(\theta,1)=\partial V(\theta,1)/\partial q-\partial c(1)/\partial q<0$, $S_{\theta q}>0$, and $S_\theta>0$, so

$S_{\theta q}(\theta,1)S(\theta,1)-S_\theta(\theta,1)S_q(\theta,1)>0$ and $V(\theta,q)-c(q)$ is log supermodular in a neighborhood on 1.

Lemma: *If $V(\theta,q)-c(q)$ is log supermodular in a neighborhood on 1, then the firm offers multiple products.*

Suppose the firm offers a single product quality, so $q(\underline{\theta})=1$. Recall that either (6) holds, that is $J(\theta_L,q(\theta_L))=0$, or $\theta_L=\underline{\theta}$.

First, consider the case in which $\theta_L = \underline{\theta}$. Equation (3) and $q(\underline{\theta}) = 1$ imply $J(\underline{\theta}, q(\underline{\theta})) > 0$, which implies $\theta_L > \underline{\theta}$, so if $\theta_L = \underline{\theta}$ then $q(\underline{\theta}) < 1$ which is a contradiction.

Second, consider the case in which $J(\theta_L, q(\theta_L)) = 0$. Then

$$\left[V(1,\underline{\theta}) - c(1) - \left[\frac{1 - F(\underline{\theta})}{f(\underline{\theta})} \right] \frac{\partial V(1,\underline{\theta})}{\partial \theta} \right] = 0 \tag{15}$$

uniquely defines $\underline{\theta}$. Because $H(\theta, q(\theta)) \geq 0$ where $q(\theta) = 1$ we also have

$$\frac{\partial V(1,\underline{\theta})}{\partial q} - c'(1) - \left[\frac{1 - F(\underline{\theta})}{f(\underline{\theta})} \right] \frac{\partial^2 V(1,\underline{\theta})}{\partial q \partial \theta} \geq 0. \tag{16}$$

Equations (15) and (16) imply

$$\frac{\partial\left(V(1,\underline{\theta}) - c(1)\right)}{\partial \theta} \frac{\partial\left(V(1,\underline{\theta}) - c(1)\right)}{\partial q} \geq \left(V(1,\underline{\theta}) - c(1)\right) \frac{\partial^2 \left(V(1,\underline{\theta}) - c(1)\right)}{\partial q \partial \theta}, \tag{17}$$

which implies $\dfrac{\partial^2 \ln\left(V(1,\underline{\theta}) - c(1)\right)}{\partial \theta \partial q} \leq 0$. So if $V(1,\underline{\theta}) - c(1)$ is log supermodular, offering a single product cannot be optimal.

Proof of Proposition 3:

a) From (5), and our assumptions on *H,* the quality sold to the lowest-type buyer served is lower than the quality sold to the highest type buyer, i.e., $q(\theta_L) < q(\overline{\theta}) = 1$, if and only if $H(\theta_L, 1) < 0$, or

$$\frac{\partial V(1,\theta_L)}{\partial q} - c'(1) - \left[\frac{1 - F(\theta_L)}{f(\theta_L)} \right] \frac{\partial^2 V(1,\theta_L)}{\partial q \partial \theta} < 0. \tag{18}$$

The lowest-type buyer served is θ_L only if $J\left(\theta_L, q\left(\theta_L\right)\right) \leq 0$. Together $J\left(\theta_L, q\left(\theta_L\right)\right) \leq 0$ and (18) imply that $q\left(\theta_L\right) < 1$ only if

$$\frac{\partial V\left(q\left(\theta_L\right), \theta_L\right)}{\partial \theta}\left[\frac{\partial V\left(1, \theta_L\right)}{\partial q} - c'(1)\right] < \left[V\left(q\left(\theta_L\right), \theta_L\right) - c\left(q\left(\theta_L\right)\right)\right]\frac{\partial^2 V\left(1, \theta_L\right)}{\partial q \partial \theta}. \tag{19}$$

Inequality (19) can be re-written as

$$\begin{aligned}&\frac{\dfrac{\partial V\left(q\left(\theta_L\right), \theta_L\right)}{\partial \theta}}{\dfrac{\partial V\left(1, \theta_L\right)}{\partial \theta}}\left[\left(\frac{\partial V\left(1, \theta_L\right)}{\partial q} - c'(1)\right)\frac{\partial V\left(1, \theta_L\right)}{\partial \theta}\right] \\ &\qquad < \frac{V\left(q\left(\theta_L\right), \theta_L\right) - c\left(q\left(\theta_L\right)\right)}{V\left(1, \theta_L\right) - c(1)}\left[\left(V\left(1, \theta_L\right) - c(1)\right)\frac{\partial^2 V\left(1, \theta_L\right)}{\partial q \partial \theta}\right],\end{aligned} \tag{20}$$

which implies either

$$\frac{\partial V\left(1, \theta_L\right)}{\partial \theta}\left(\frac{\partial V\left(1, \theta_L\right)}{\partial q} - c'(1)\right) < \left(V\left(1, \theta_L\right) - c(1)\right)\frac{\partial^2 V\left(1, \theta_L\right)}{\partial q \partial \theta} \tag{21}$$

or

$$\frac{\partial V\left(q\left(\theta_L\right), \theta_L\right)}{\partial \theta} \Big/ \frac{\partial V\left(1, \theta_L\right)}{\partial \theta} < \frac{V\left(q\left(\theta_L\right), \theta_L\right) - c\left(q\left(\theta_L\right)\right)}{V\left(1, \theta_L\right) - c(1)}. \tag{22}$$

Equations (21) and (22) can be rewritten as

$$\begin{aligned}&\frac{\partial\left(V\left(1, \theta_L\right) - c(1)\right)}{\partial \theta}\left(\frac{\partial\left(V\left(1, \theta_L\right) - c(1)\right)}{\partial q}\right) \\ &\qquad < \left(V\left(1, \theta_L\right) - c(1)\right)\frac{\partial^2\left(V\left(1, \theta_L\right) - c(1)\right)}{\partial q \partial \theta}\end{aligned} \tag{23}$$

and

$$\frac{\partial\left(V\left(q\left(\theta_L\right),\theta_L\right)-c\left(q\left(\theta_L\right)\right)\right)}{\partial\theta}\Bigg/\frac{\partial\left(V\left(1,\theta_L\right)-c\left(1\right)\right)}{\partial\theta}<\frac{V\left(q\left(\theta_L\right),\theta_L\right)-c\left(q\left(\theta_L\right)\right)}{V\left(1,\theta_L\right)-c\left(1\right)}. \tag{24}$$

But we shall now see that neither equation (23) nor (24) holds unless $V\left(\hat{q},\theta_L\right)-c\left(\hat{q}\right)$ is log supermodular for some $\hat{q}$. Clearly (23) holds if and only if $V\left(\hat{q},\theta_L\right)-c\left(\hat{q}\right)$ is log supermodular. Similarly (24) holds if only if $\dfrac{V\left(1,\theta_L\right)-c\left(1\right)}{V\left(q\left(\theta_L\right),\theta_L\right)-c\left(q\left(\theta_L\right)\right)}$ is increasing in θ at $\theta=\theta_L$. But if $\dfrac{\partial^2\ln\left(V(\hat{q},\theta_L)-c\left(\hat{q}\right)\right)}{\partial\theta\partial\hat{q}}\leq 0$ for all $\hat{q}$ then

$$\begin{aligned}\int_{\hat{q}}^{1}\frac{\partial^2\ln\left(V(q,\theta_L)-c\left(q\right)\right)}{\partial\theta\partial q}dq &= \frac{\partial\left(\ln\left(V\left(1,\theta_L\right)-c\left(1\right)\right)-\ln\left(V\left(\hat{q},\theta_L\right)\right)-c\left(\hat{q}\right)\right)}{\partial\theta}\\ &= \frac{\partial\ln\left(\dfrac{V\left(1,\theta_L\right)-c\left(1\right)}{V\left(\hat{q},\theta_L\right)-c\left(\hat{q}\right)}\right)}{\partial\theta}\leq 0\end{aligned} \tag{25}$$

so (24) cannot hold. Therefore neither (23) nor (24) holds hold unless $V\left(\hat{q},\theta_L\right)-c\left(\hat{q}\right)$ is log supermodularity for some $\hat{q}$. That is, log supermodularity of $V\left(\hat{q},\theta_L\right)-c\left(\hat{q}\right)$ for some $\hat{q}$ is a necessary condition for the firm to sell multiple products.

Finally, since $S_{qq\theta}\left(q,\theta\right)>0$ and $S_{qq}\left(q,\theta\right)<0$, $S\left(1,\theta\right)S_{q\theta}\left(1,\theta\right)-S_q\left(1,\theta\right)S_\theta\left(1,\theta\right)<0$ for all θ implies that $S\left(q,\theta\right)S_{q\theta}\left(q,\theta\right)-S_q\left(q,\theta\right)S_\theta\left(q,\theta\right)$ is increasing in q, and so $S\left(q,\theta\right)S_{q\theta}\left(q,\theta\right)-S_q\left(q,\theta\right)S_\theta\left(q,\theta\right)<0$ for all $q\leq 1$. In other words, if $V\left(1,\theta\right)-c\left(1\right)$

is log submodular then $V(q,\theta)-c(q)$ is log submodular for all θ and all $q \le 1$. So log supermodularity of $V(1,\theta_L)-c(1)$ implies that the firm will not sell multiple products.

b) See the lemma in the proof of Proposition 2.

Proof of Proposition 4:

The firm's optimal product line is described by $H(\theta,q(\theta))=0$ where $0<q(\theta)<1$, $H(\theta,q(\theta))\ge 0$ where $q(\theta)=1$, and $H(\theta,q(\theta))\le 0$ where $q(\theta)=0$. First, it is clear these imply $q^m(\bar{\theta})=q^*(\bar{\theta})=1$. Second $q^m(\theta)<q^*(\theta)$ whenever $0<q^*(\theta)<1$ follows immediately from $H(\theta,q(\theta))=0$. Finally $q^m(\theta)\le q^*(\theta)$ whenever $q^*(\theta)=1$ follows from the constraint, so $q^m(\theta)\le q^*(\theta),\forall\theta$.

Proof of Proposition 5:

If a seller is restricted to offering a single quality, it will sell only high quality. Also, it will sell exclusively to the high types if and only if

$$\bar{n}\left(V(1,\bar{\theta})-c(1)\right)>\left(\bar{n}+\underline{n}\right)\left(V(1,\underline{\theta})-c(1)\right)$$

or

$$\frac{V(1,\underline{\theta})-c(1)}{V(1,\bar{\theta})-c(1)}<\frac{\bar{n}}{\bar{n}+\underline{n}}. \tag{26}$$

Log supermodularity implies

$$\frac{\partial V(1,\underline{\theta})/\partial q-c'(1)}{\partial V(1,\bar{\theta})/\partial q-c'(1)}<\frac{V(1,\underline{\theta})-c(1)}{V(1,\bar{\theta})-c(1)}<\frac{V(\hat{q},\underline{\theta})-c(\hat{q})}{V(\hat{q},\bar{\theta})-c(\hat{q})}, \tag{27}$$

so in the subinterval $\left(\frac{V(1,\underline{\theta}) - c\left(1\right)}{V(1,\bar{\theta}) - c\left(1\right)}, \frac{V(\hat{q},\underline{\theta}) - c\left(\hat{q}\right)}{V(\hat{q},\bar{\theta}) - c\left(\hat{q}\right)} \right)$ of N^* allowing price discrimination results in a Pareto improvement. That is, it weakly increases seller profits by revealed preference, weakly increases type $\underline{\theta}$ buyers' consumer surplus because they were not previously served, and strictly increases type $\bar{\theta}$ buyers' consumer surplus from zero to something positive because their incentive compatibility constraint strictly binds. QED.

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