Capacity of Multiantenna Gaussian Broadcast Channel

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Joint Work with David Tse, UC Berkeley

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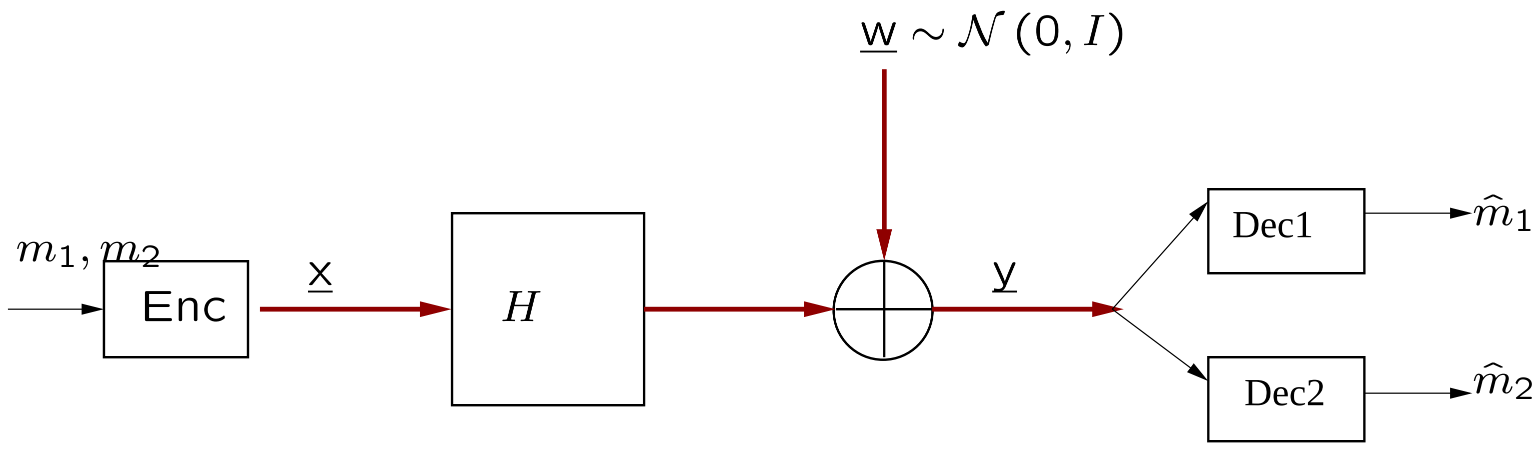
Multiple Antenna Broadcast Channel

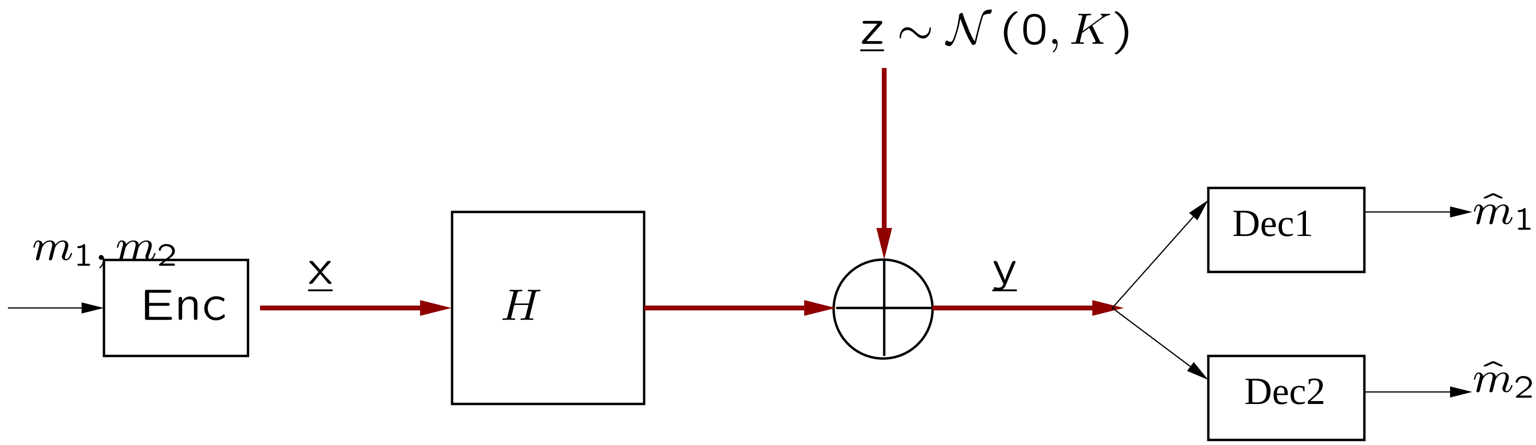


$$\underline{y} = H\underline{x} + \underline{w}$$

Overview

- Nondegraded Broadcast channel
  - Capacity region unknown
- Known Result:
  - Sum Capacity for 2 users (Caire,Shamai '00)
- Our Result:
  - Sum Capacity for general number of users and antennas
  - Simple proof and interpretation
  - The Capacity region

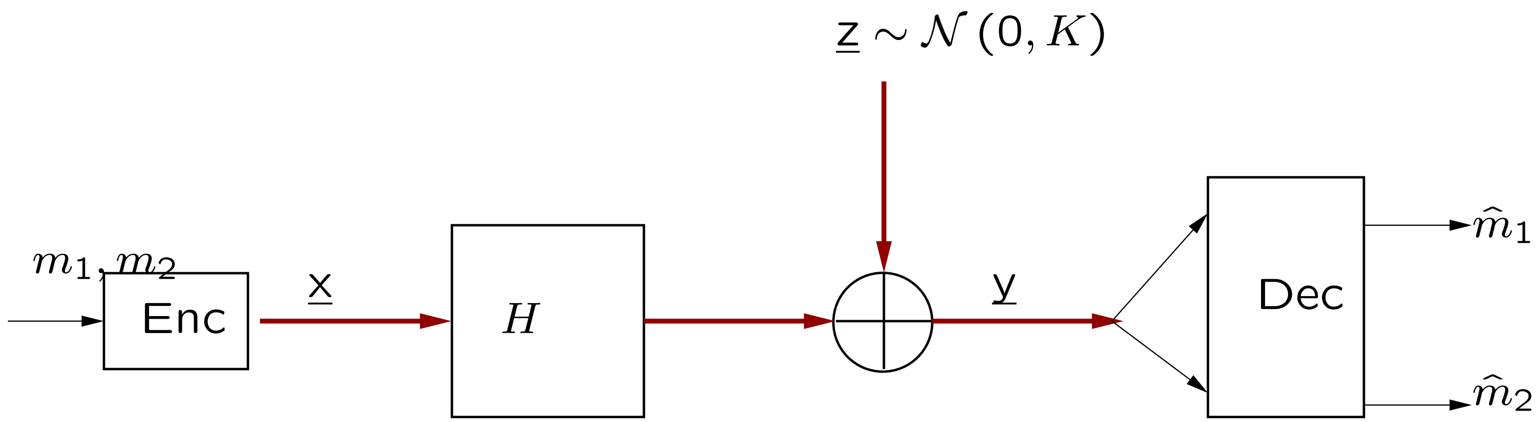
Multiple Antenna Broadcast Channel



$$\underline{y} = H\underline{x} + \underline{z}$$
$$K_{ii} = 1$$

- Only marginal channels matter
- Noise correlation arbitrary

Sato Upper Bound



$$K_{ii} = 1$$

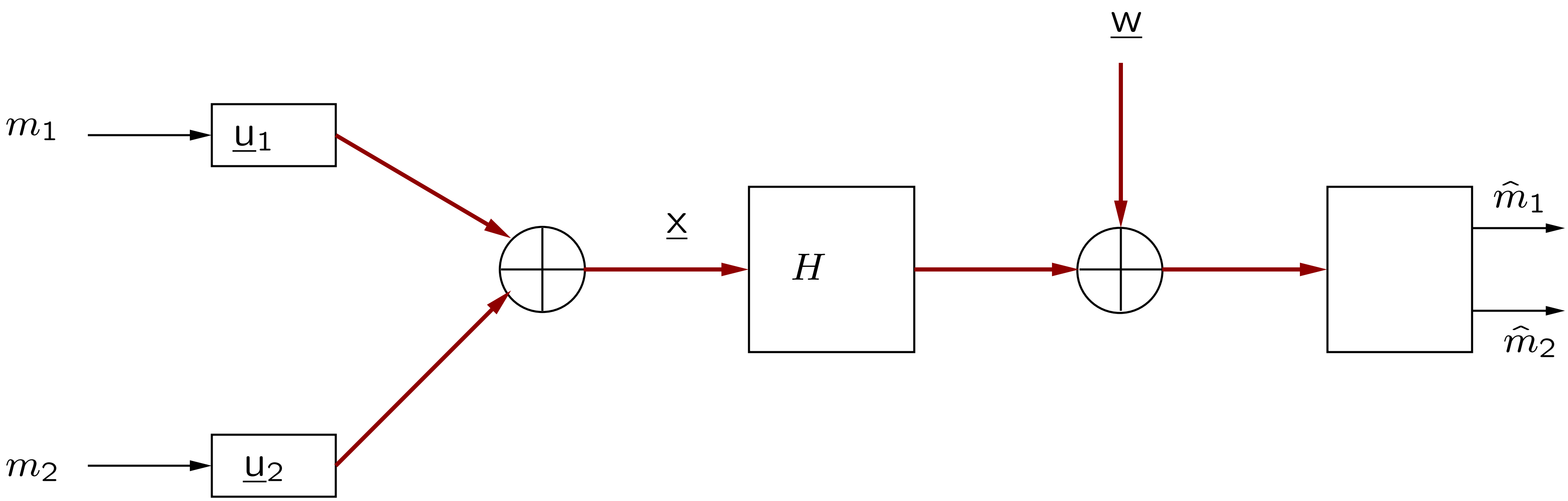
- Receivers cooperate

Upper Bound

Broadcast channel

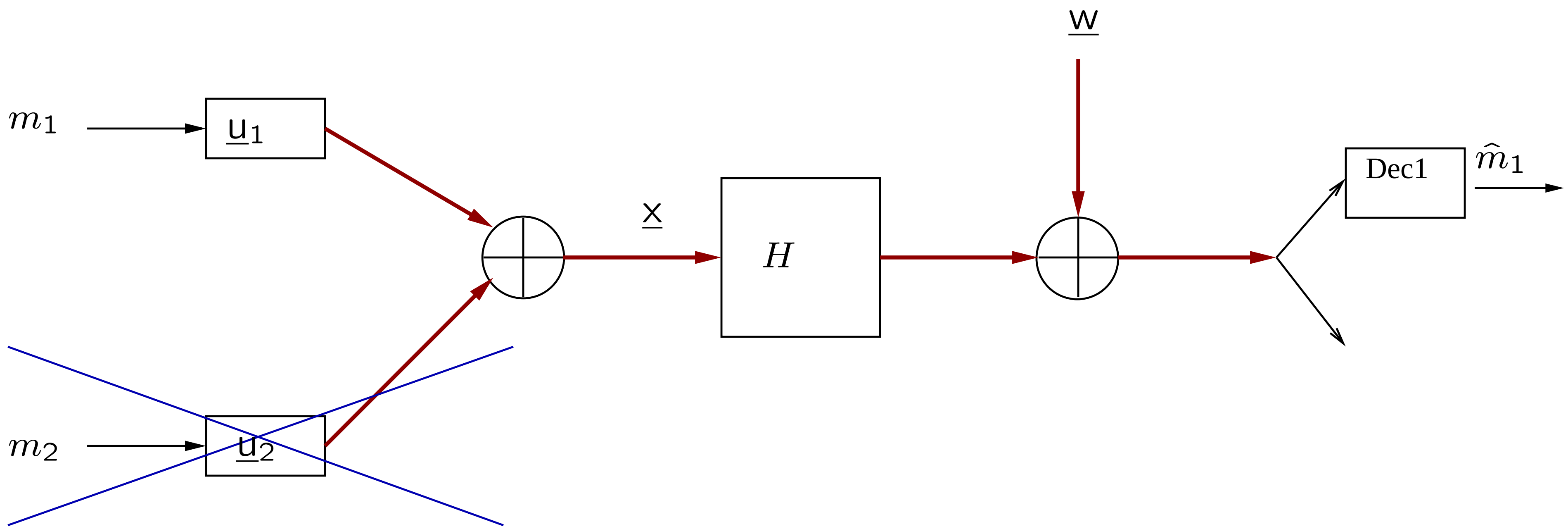
Cooperating Receivers

Achievable Rates: Costa Precoding



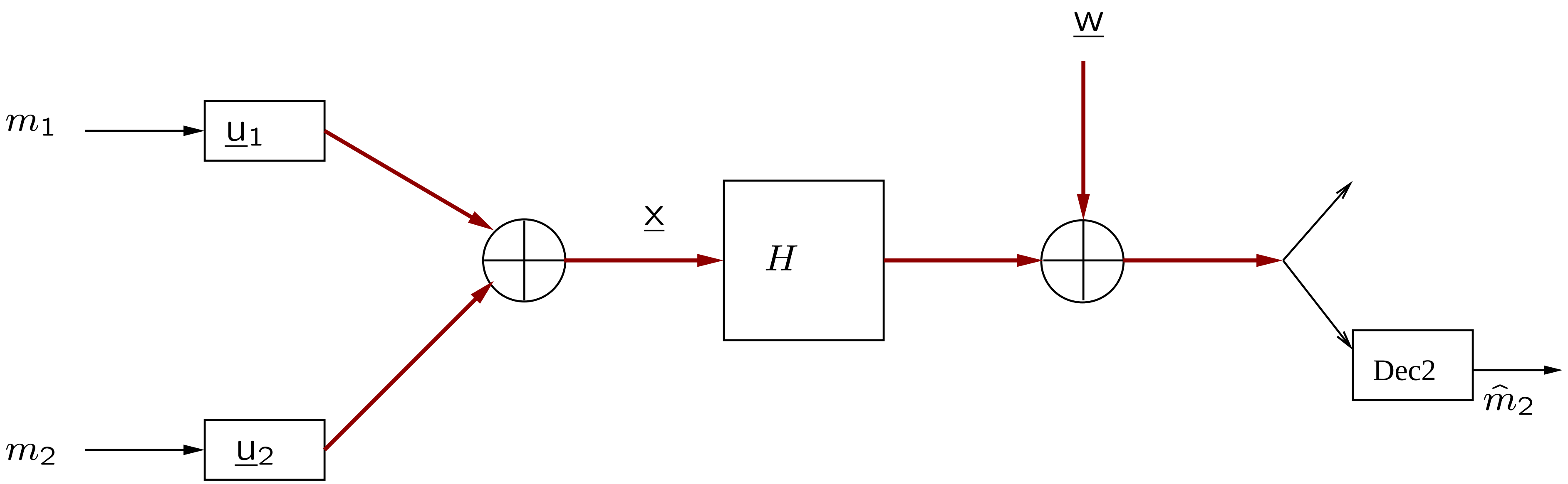
- Users' data modulated onto spatial signatures $\underline{u}_1, \underline{u}_2$

Stage 1: Costa Precoding



- Encoding for user 1 treating signal from user 2 as known interference at transmitter

Stage 2



- Encode user 2 treating signal for user 1 as noise

Costa Strategy and Upper Bound

- Want to find K such that sum rate with Costa precoding is same as capacity of cooperating receivers channel

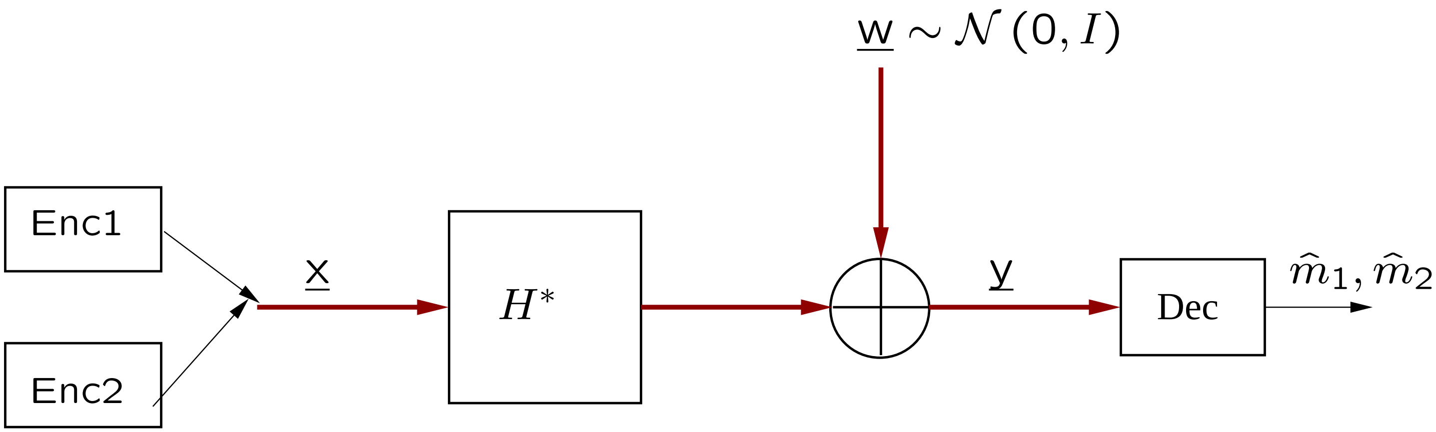
DL-UL Duality

- A representation of sum rate achievable by Costa strategy
- Present a form of duality in multiantenna channels
  - a change of variable and a conservation law
- Applications of this observation

Applications

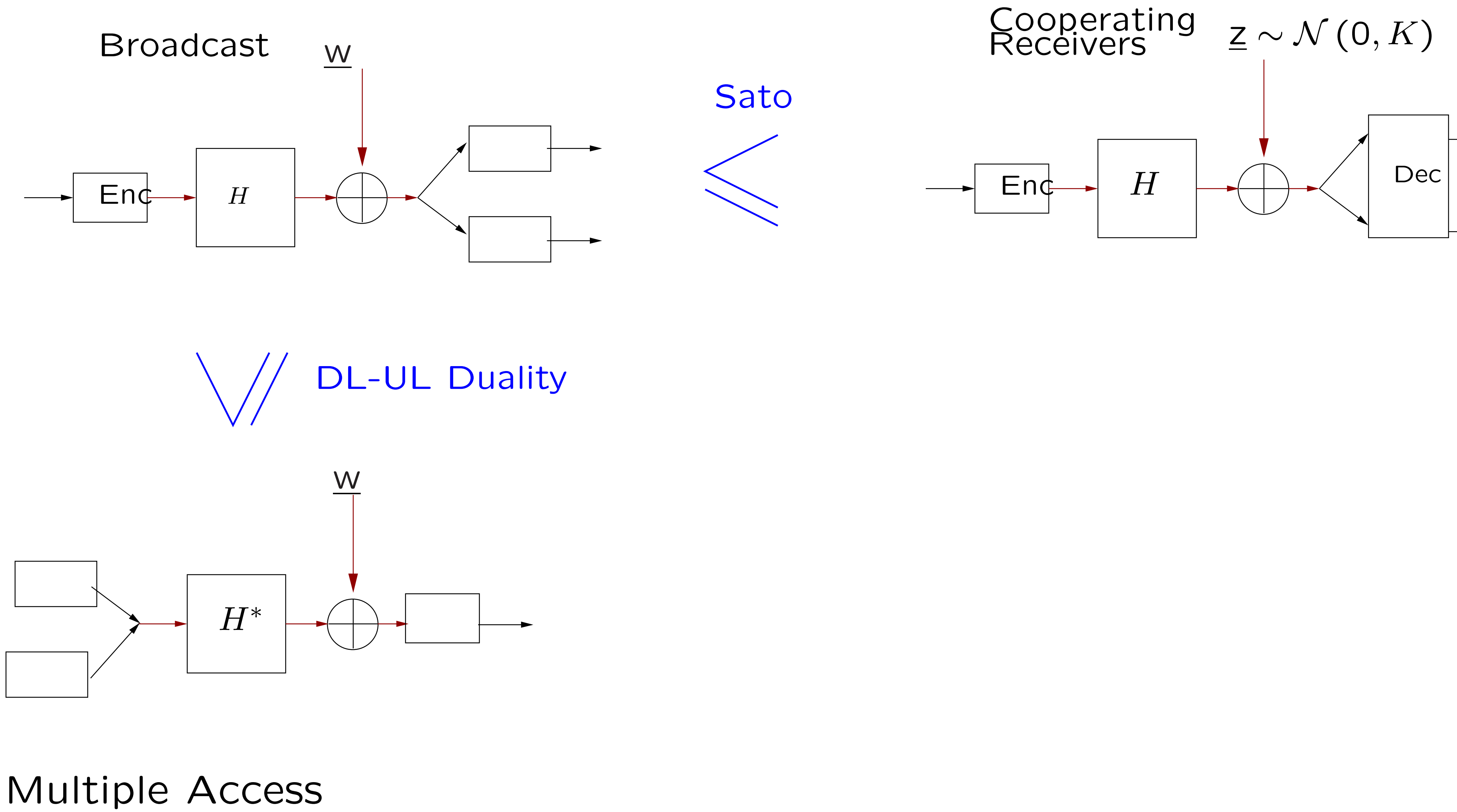
- Unifies duality observations under various guises
  - Reciprocity - Telatar (99)
  - Virtual Uplink channel - R-Farrokhi (97), Visotsky (99)
  - Duality between MAC and BC - Jindal et al (01)
- Extension of results on uplink to downlink
  - Performance of linear receivers - Tse and Hanly (99)
- Achievable rate region for multiantenna broadcast channel
  - Marton Region for Gaussian inputs

DL-UL Duality



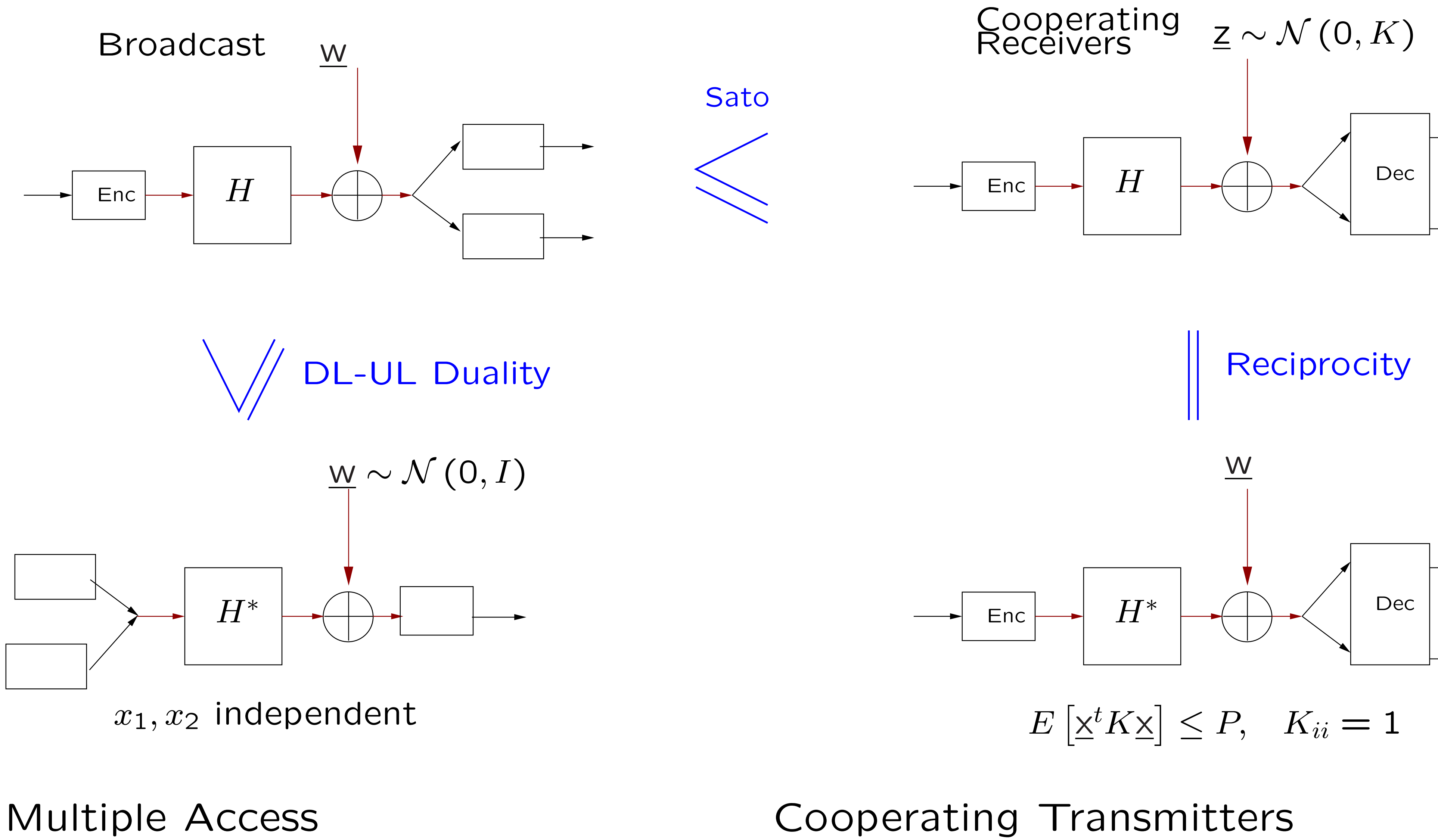
- Costa precoding over all $\underline{u}_1, \underline{u}_2$ achieves same region as uplink
  - Jindal and Goldsmith ('00)

Summary

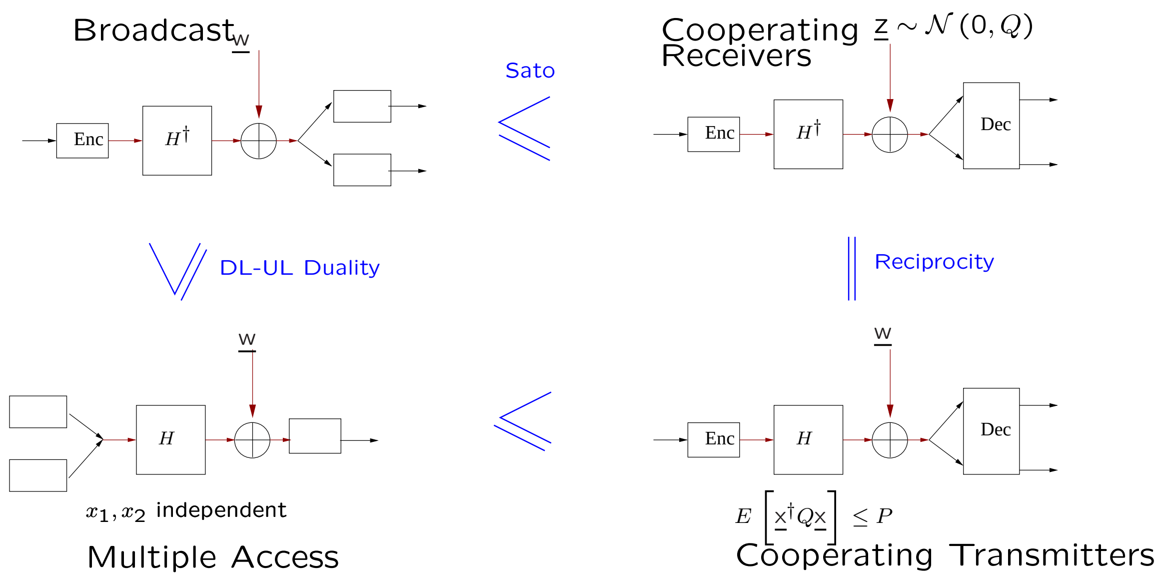


Broadcast
w
Enc
H
Sato
Cooperating Receivers
$\underline{z} \sim \mathcal{N}(0, K)$
Enc
H
Dec
DL-UL Duality
w
H^*
Multiple Access

Reciprocity



Summary



Multiple Access

Cooperating Transmitters

MAC and Cooperating Transmitters Channel

Multiple Access	Cooperating Transmitters Channel
$\max_{\Sigma_x} I\left(\underline{x}\,;\,\underline{y}\right)$	$\max_{\Sigma_x} I\left(\underline{x}\,;\,\underline{y}\right)$
$E\left[\underline{x}^t\underline{x}\right] \leq P$	$E\left[\underline{x}^t K \underline{x}\right] \leq P$
x_1, x_2 independent	$K_{ii} \leq 1$

MAC and Cooperating Transmitters Channel

Multiple Access	Cooperating Transmitters Channel
$\max_{\Sigma_x} I(\underline{x}\,;\,\underline{y})$	$\max_{\Sigma_x} I(\underline{x}\,;\,\underline{y})$
$\mathrm{tr}\,[\Sigma_x] \leq P$	$\mathrm{tr}\,[\Sigma_x K] \leq P$
$(\Sigma_x)_{ii} \geq 0, \;\; (\Sigma_x)_{ij} = 0, \; i \neq j$	$K_{ii} \leq 1$

Convex Duality

MAC Problem:

$$\max_{\Sigma_x} [I(\underline{x} ; \underline{y})] \quad \text{such that} \quad \text{tr}[\Sigma_x] \leq P, \quad (\Sigma_x)_{ij} = 0, \ (\Sigma_x)_{ii} \geq 0$$

Convex Duality

MAC Problem:

$$\max_{\Sigma_x} \left[I\left(\underline{x}\,;\,\underline{y}\right) \right] \quad \text{such that} \quad \text{tr}\left[\Sigma_x\right] \leq P, \quad \left(\Sigma_x\right)_{ij} = 0, \; \left(\Sigma_x\right)_{ii} \geq 0$$

Convex Dual:

$$\min_{\lambda \geq 0, \lambda_{ii} \geq 0, \lambda_{ij}} \max_{\Sigma_x} \left[I\left(\underline{x}\,;\,\underline{y}\right) - \lambda\left(\text{tr}\left[\Sigma_x\right] - P\right) - \sum_{i,j} \lambda_{ij} \left(\Sigma_x\right)_{ij} \right]$$

Convex Duality

MAC Problem:

$$\max_{\Sigma_x} [I(\underline{x}\,;\,\underline{y})] \quad \text{such that} \quad \text{tr}[\Sigma_x] \leq P, \quad (\Sigma_x)_{ij} = 0, \; (\Sigma_x)_{ii} \geq 0$$

Convex Dual:

$$\min_{\lambda \geq 0, \lambda_{ii} \geq, \lambda_{ij}} \max_{\Sigma_x} \left[I(\underline{x}\,;\,\underline{y}) - \lambda (\text{tr}[\Sigma_x] - P) - \sum_{i,j} \lambda_{ij} (\Sigma_x)_{ij} \right]$$

In Matrix Form: $K_{ii} = 1 - \lambda_{ii}/\lambda, K_{ij} = \lambda_{ij}/\lambda$

$$\min_{K, K_{ii} \leq 1, \lambda \geq 0} \max_{\Sigma_x} [I(\underline{x}\,;\,\underline{y}) - \lambda (\text{tr}[K\Sigma_x] - P)]$$

Convex Duality

MAC Problem:

$$\max_{\Sigma_x} [I(\underline{x}\,;\,\underline{y})] \quad \text{such that} \quad \mathrm{tr}[\Sigma_x] \le P, \quad (\Sigma_x)_{ij} = 0$$

Convex Dual:

$$\min_{\lambda\ge 0,\lambda_{ii}\ge 0,\lambda_{ij}} \max_{\Sigma_x} \left[I(\underline{x}\,;\,\underline{y}) - \lambda\left(\mathrm{tr}[\Sigma_x] - P\right) - \sum_{i,j} \lambda_{ij} (\Sigma_x)_{ij} \right]$$

In Matrix Form: $K_{ii} = 1 - \lambda_{ii}/\lambda, K_{ij} = \lambda_{ij}/\lambda$

$$\min_{K,K_{ii}\le 1,\lambda\ge 0,} \max_{\Sigma_x} [I(\underline{x}\,;\,\underline{y}) - \lambda\left(\mathrm{tr}[K\Sigma_x] - P\right)]$$

Finally: $K_{ii} \le 1, \quad \mathrm{tr}[K\Sigma_x] \le P$

$$\min_{K} \max_{\Sigma_x} [I(\underline{x}\,;\,\underline{y})]$$

Convex Duality: Positive Semidefinite Constraints

Convex Dual of MAC Problem: $K_{ii} \leq 1, \quad \text{tr}\left[K\Sigma_x\right] \leq P$

$$\min_{K} \max_{\Sigma_x} \left[I\left(\underline{x}\,;\,\underline{y}\right)\right]$$

Cooperating Transmitters Channel: $K_{ii} \leq 1, \quad \text{tr}\left[K\Sigma_x\right] \leq P$

$$\min_{K \succeq 0} \max_{\Sigma_x \succeq 0} \left[I\left(\underline{x}\,;\,\underline{y}\right)\right]$$

Convex Duality: Positive Semidefinite Constraints

Convex Dual of MAC Problem: $K_{ii} \leq 1, \quad \text{tr}\left[K\Sigma_x\right] \leq P$

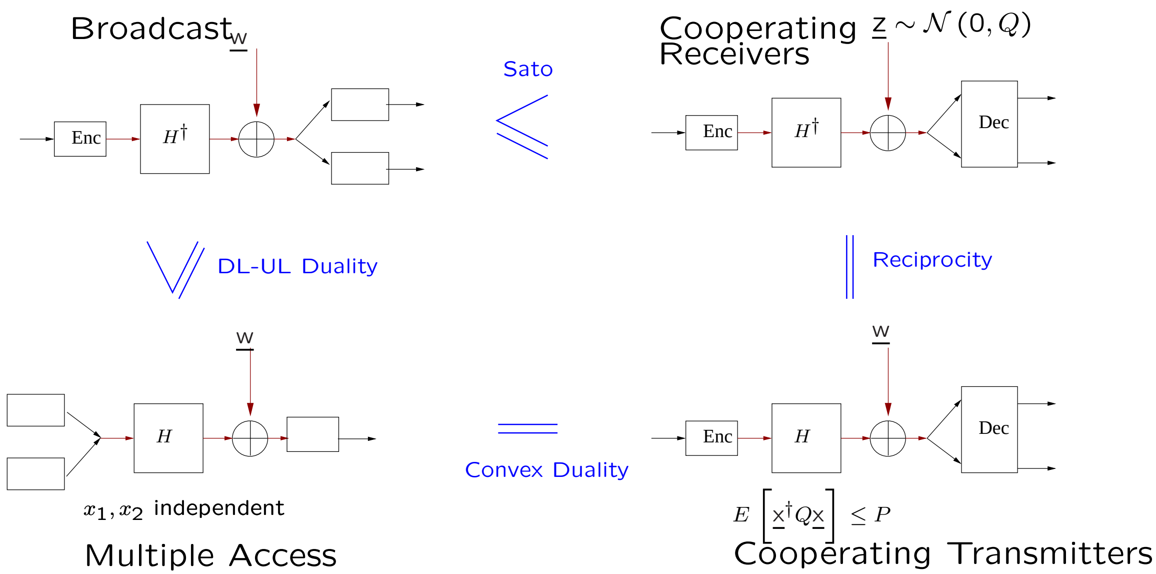
$$\min_{K} \max_{\Sigma_x} \left[I\left(\underline{x}\,;\,\underline{y}\right)\right]$$

$$\Big\| \quad \text{p.s.d. constraints}$$

Cooperating Transmitters Channel: $K_{ii} \leq 1, \quad \text{tr}\left[K\Sigma_x\right] \leq P$

$$\min_{K \succeq 0} \max_{\Sigma_x \succeq 0} \left[I\left(\underline{x}\,;\,\underline{y}\right)\right]$$

Main Result



Multiple Access

Cooperating Transmitters

Capacity Region

- Focus on

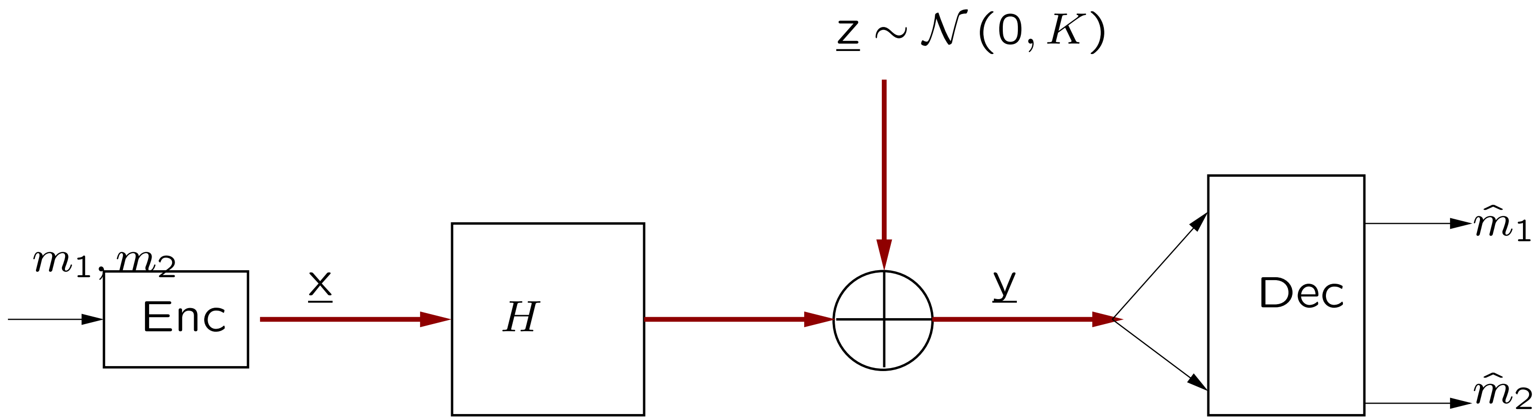
$$aR_1 + R_2$$

- R_1 = rate of user 1

- R_2 = rate of user 2

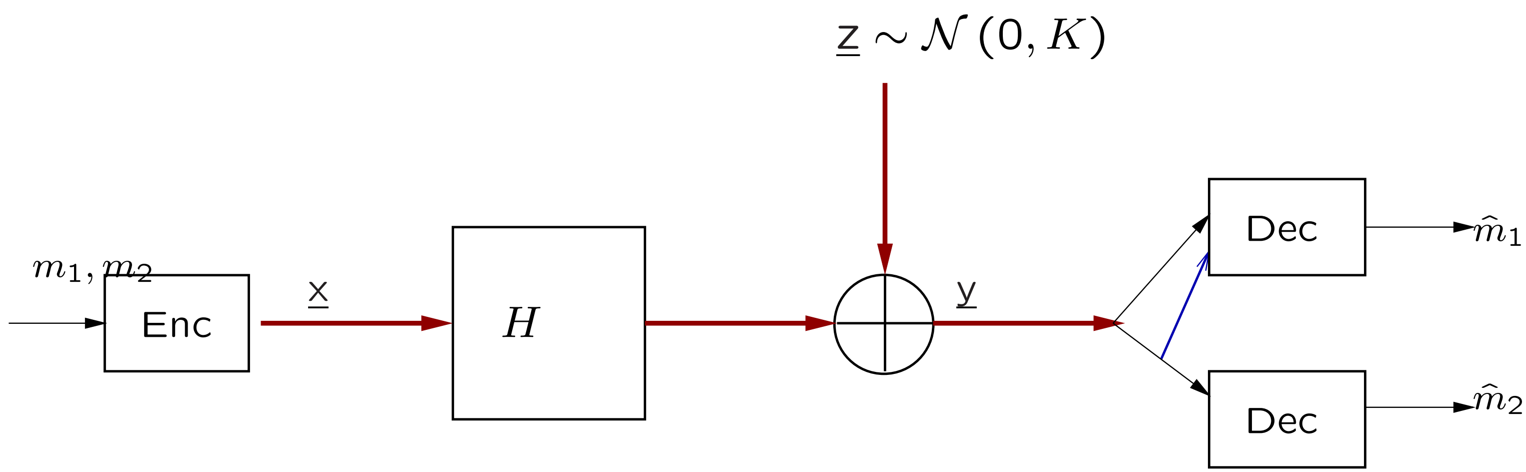
- $a < 1$: user 1 has less weight

Cooperating Upper Bound Doesnt Work



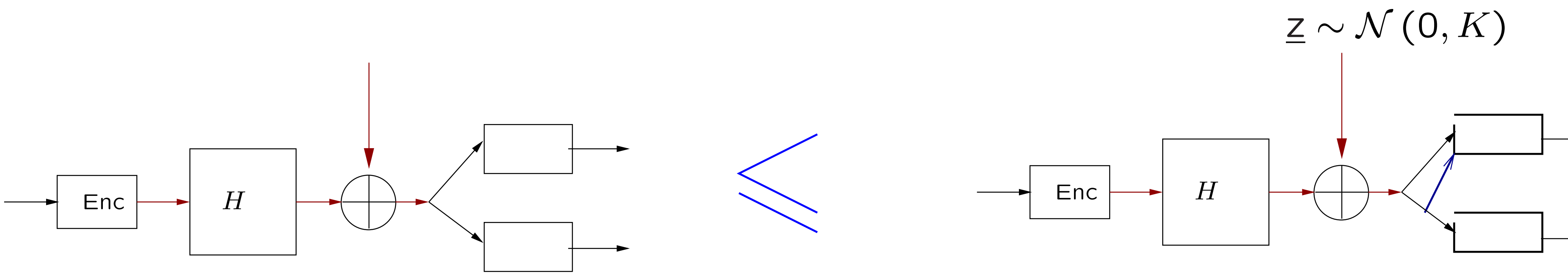
- No separation of rates R_1, R_2
- Hence no control over $aR_1 + R_2$

Degraded Receivers



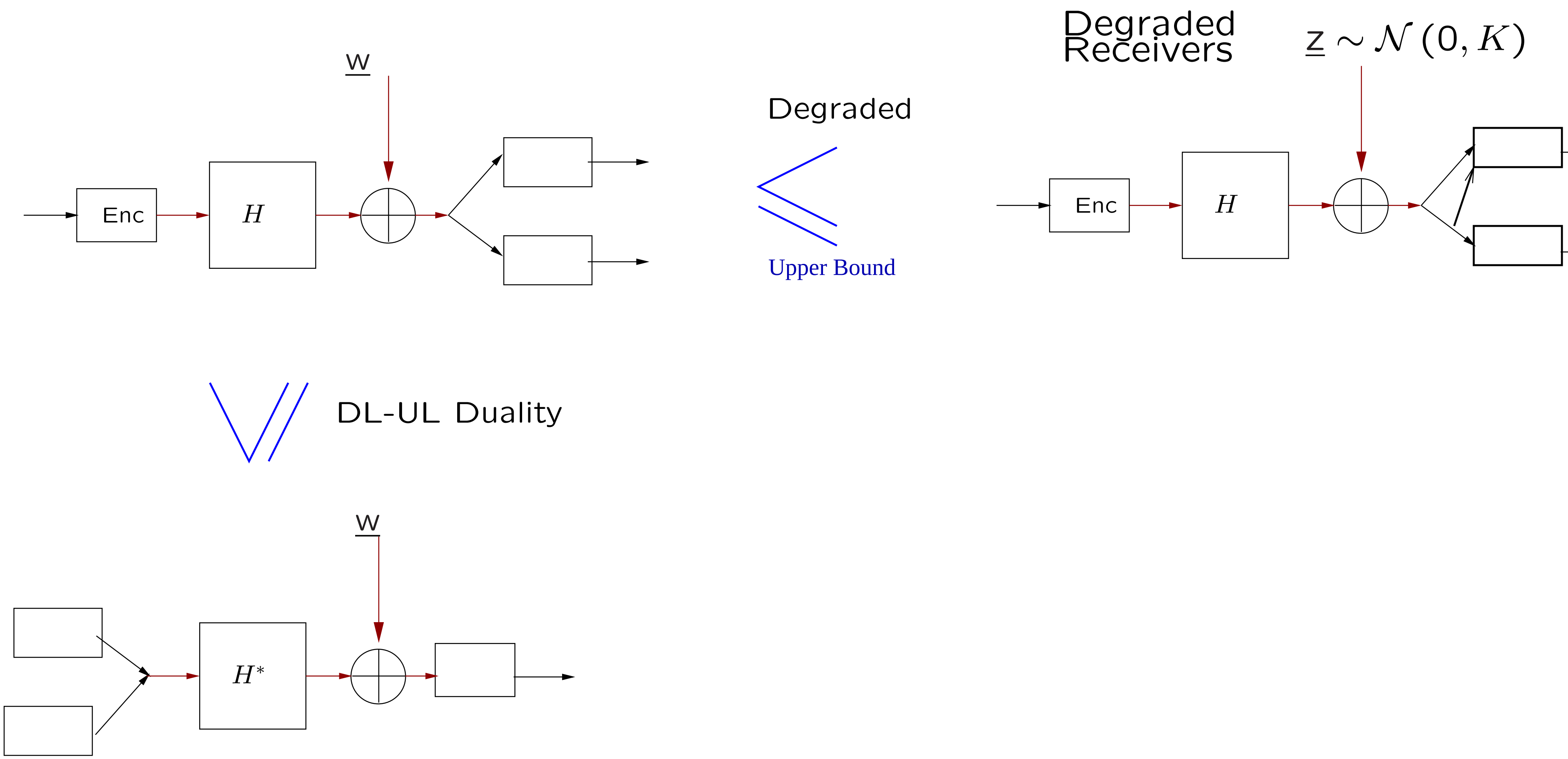
- User 1 is privy to signal of user 2
- Now a degraded Gaussian broadcast channel

Degraded Upper Bound



$$K_{ii} = 1$$

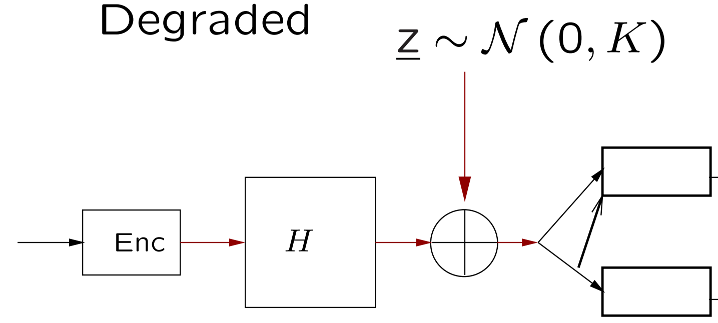
Costa Coding Achievability

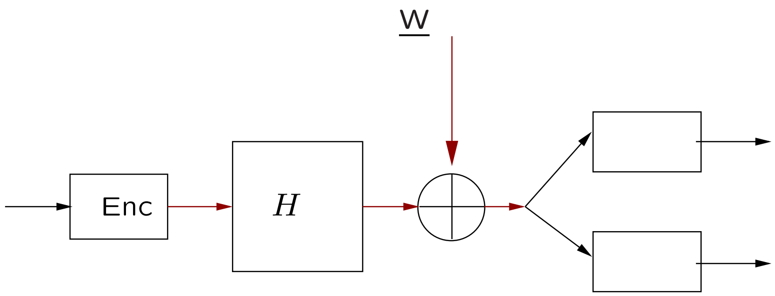


Multiple Access

What is the 4th System?

Degraded

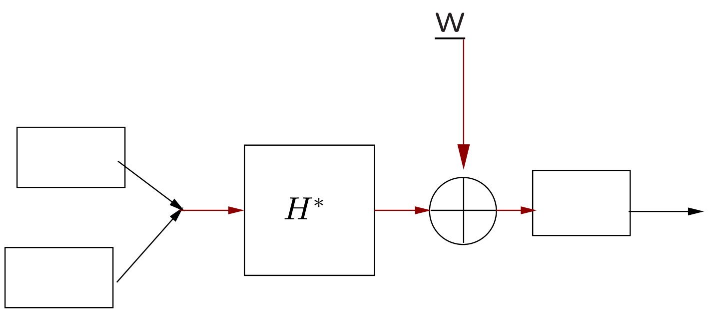




Degraded

Upper Bound

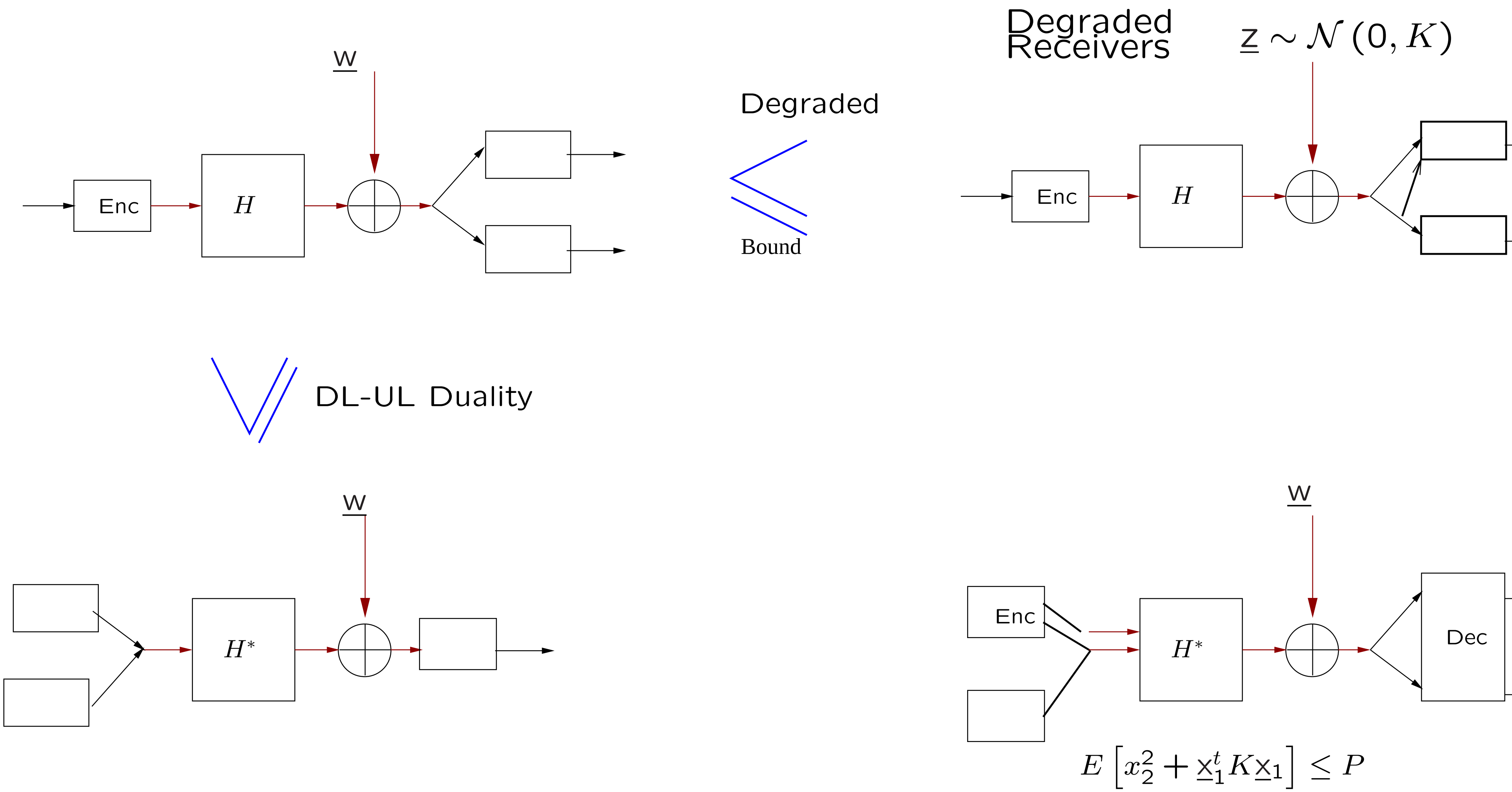
DL-UL Duality



?

Multiple Access

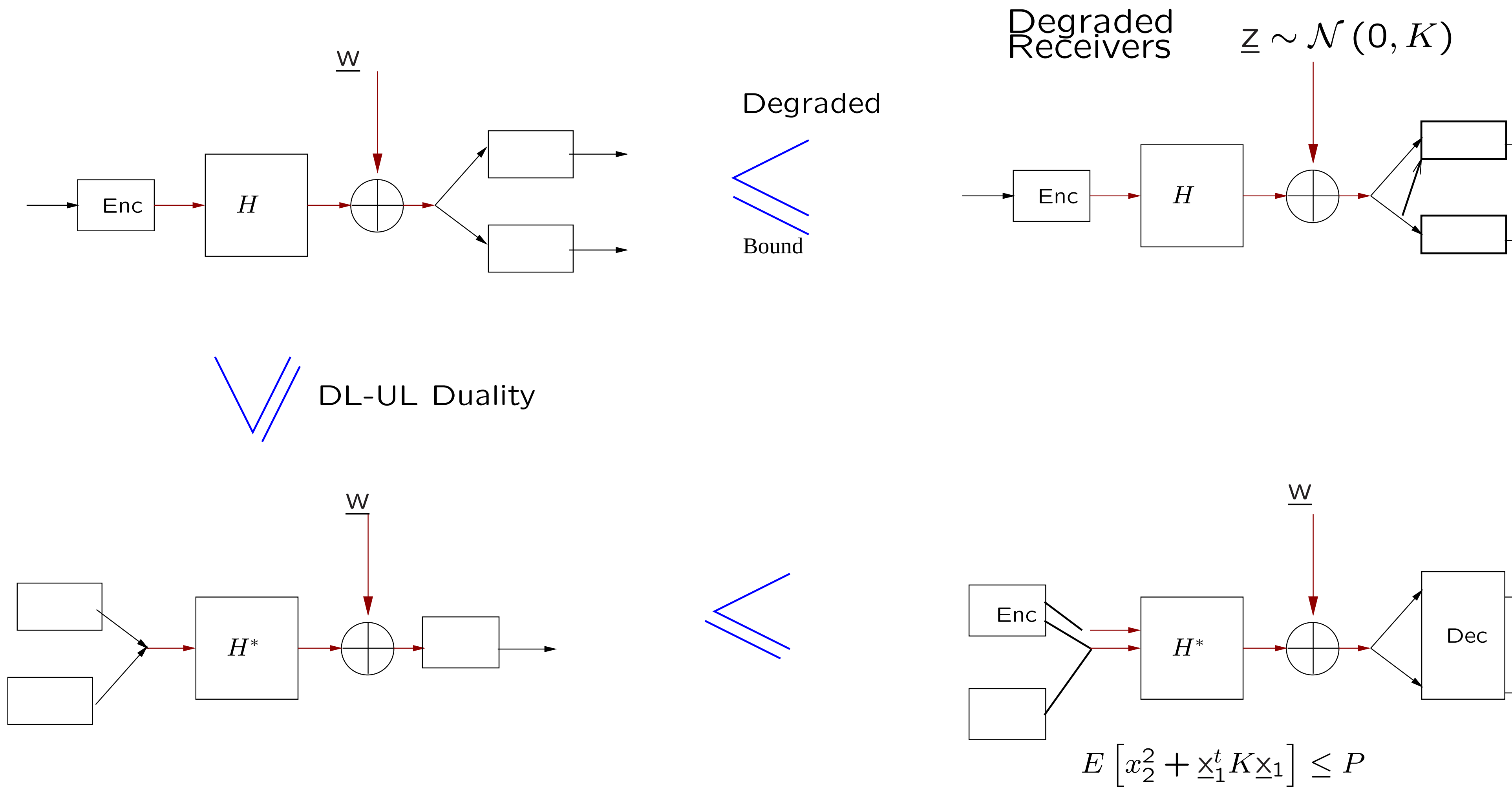
Degraded Transmitters MAC



Multiple Access

Degraded Transmitters MAC

Degraded Transmitters Bound



Multiple Access

Degraded Transmitters MAC

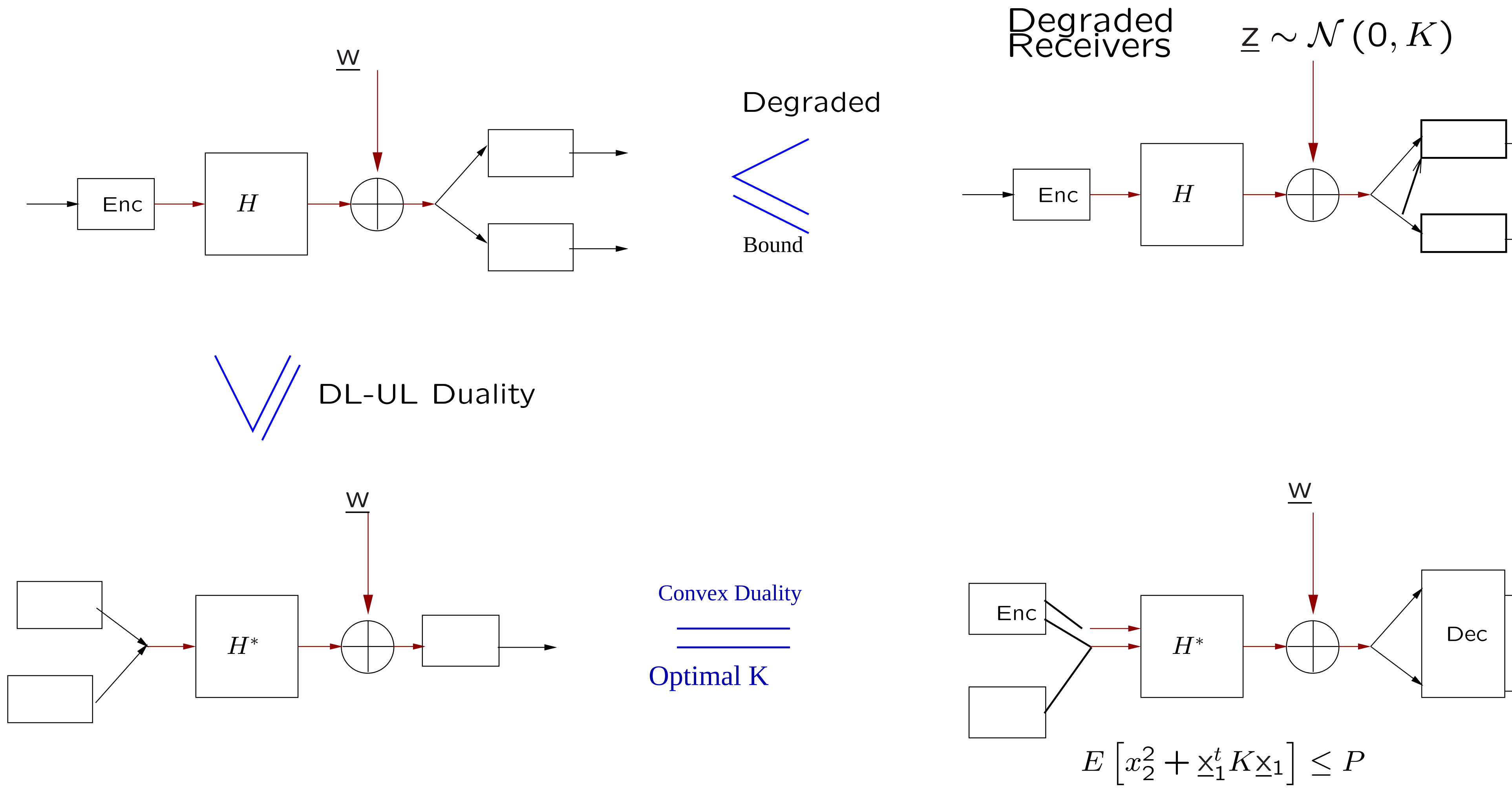
Convex Duality

- Choose cost function K such that

  User 1 (stronger one) does not use user 2's input.

- for input that maximizes $aR_1 + R_2$

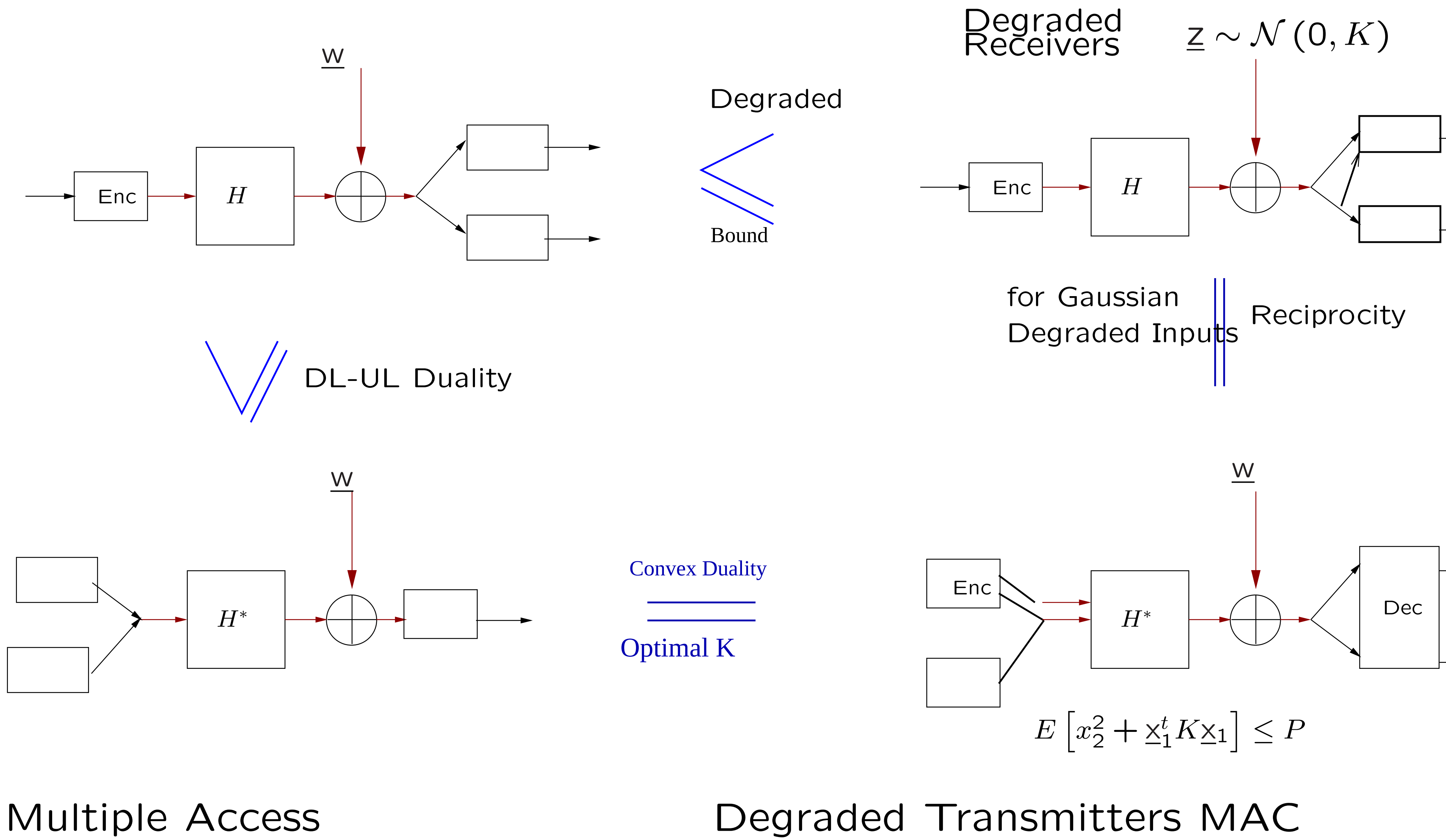
Convex Duality



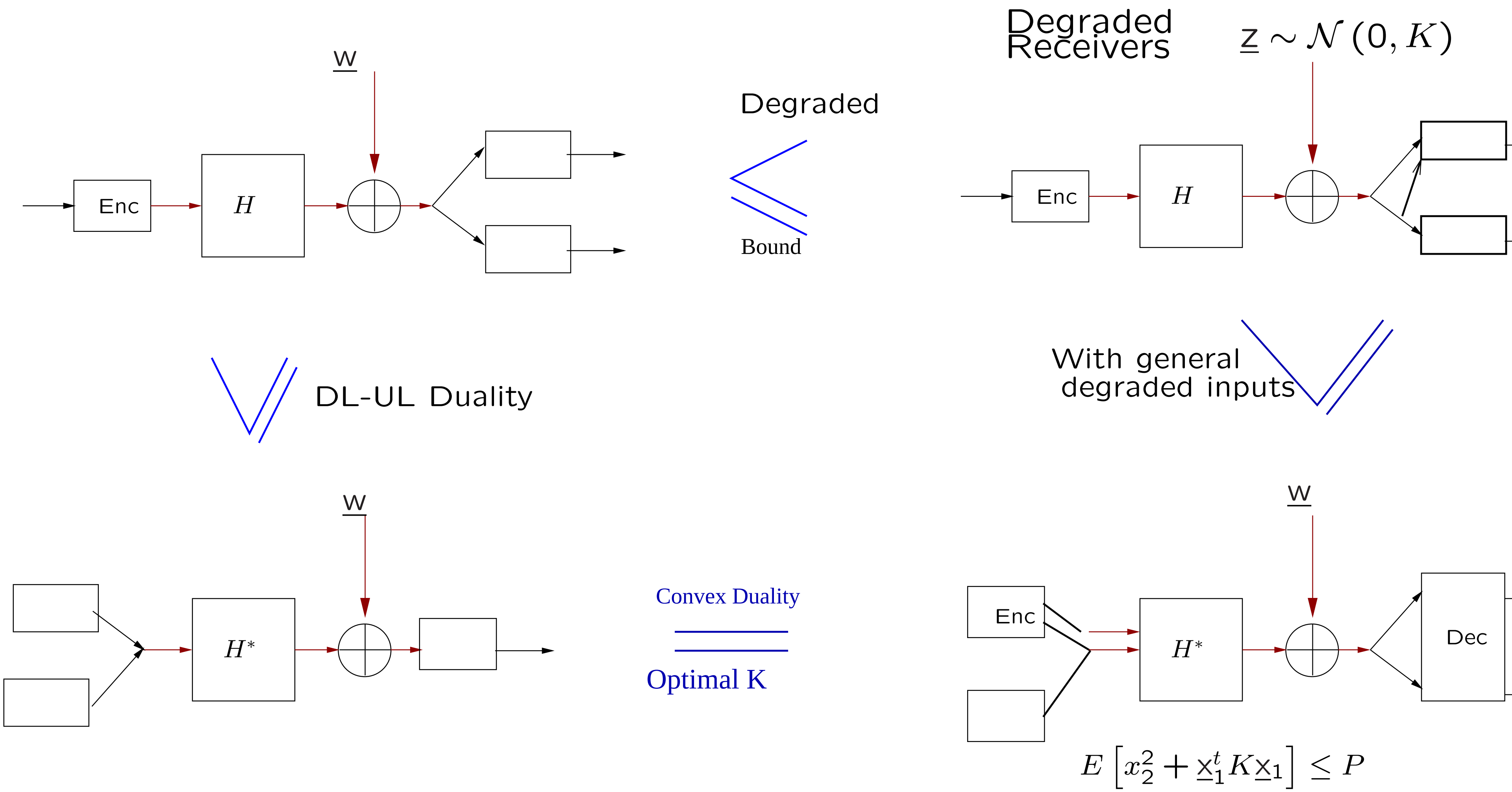
Multiple Access

Degraded Transmitters MAC

Reciprocity: Almost There



Inequalities Not in the Correct Direction



Multiple Access

Degraded Transmitters MAC

The Final Step