Non-linearity in Davenport-Schinzel Sequences

Seth Pettie

University of Michigan

Isomorphism and Subsequences

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  - **BUSH** is isomorphic to **GORE**

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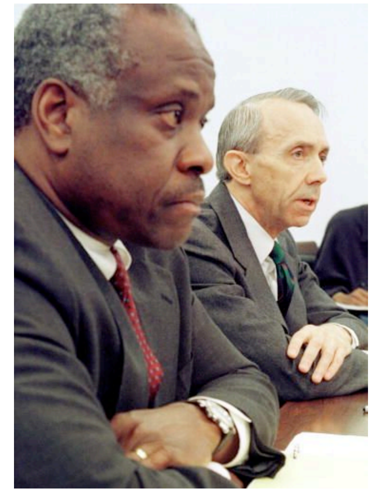

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  - **WITH_WHOM_WOULD_I_RATHER_HAVE_A_BEER?**

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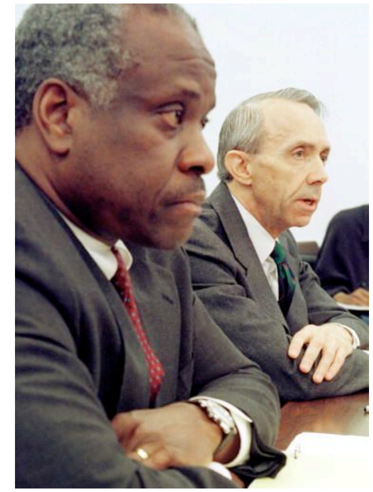

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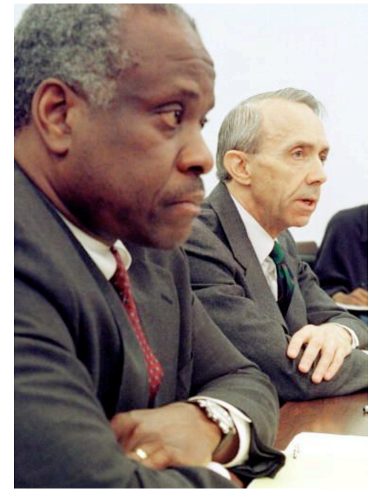

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Definitions

- $x \subset y$: x is ***isomorphic*** to a ***subsequence*** of y

- $Ex(\sigma,n) = \max |S|$:

  $S \in \{1,\ldots,n\}^*$

  $\sigma \not\subset S$

  S is $|\sigma|$-regular (technical condition)

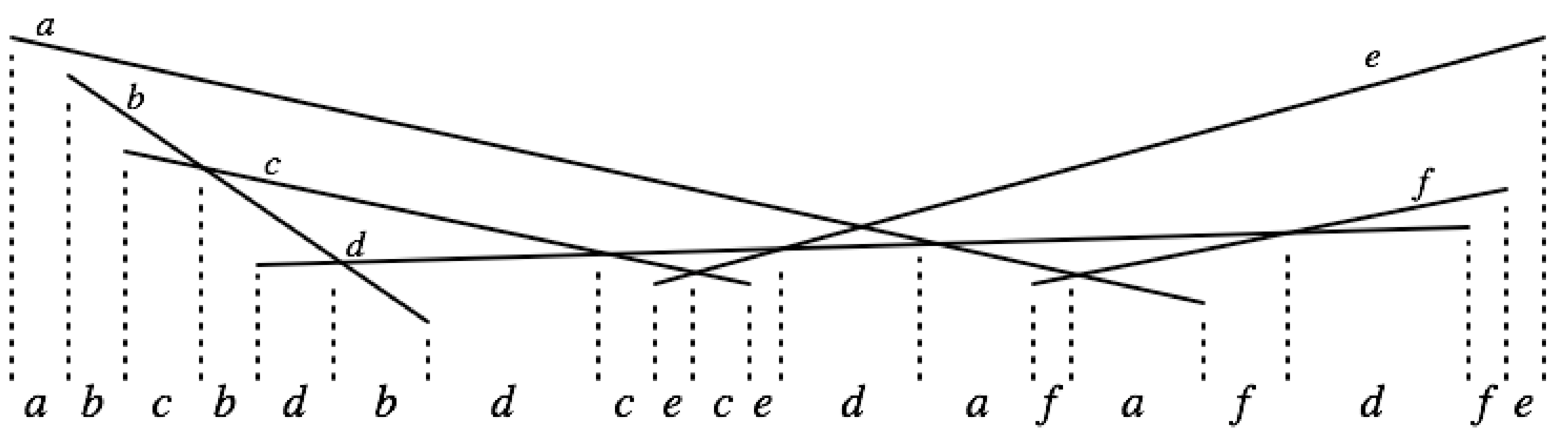
- How fast does $Ex(\sigma,n)$ grow as a function of n?

Original application: lower envelopes

(1) Give each object (line segment, quadratic, etc.) a symbol

(2) Map the lower envelope to a sequence |S|

(3) Show |S| ≤ Ex(σ,n) for some ***forbidden subseq.*** σ



this sequence does not contain **ababa**

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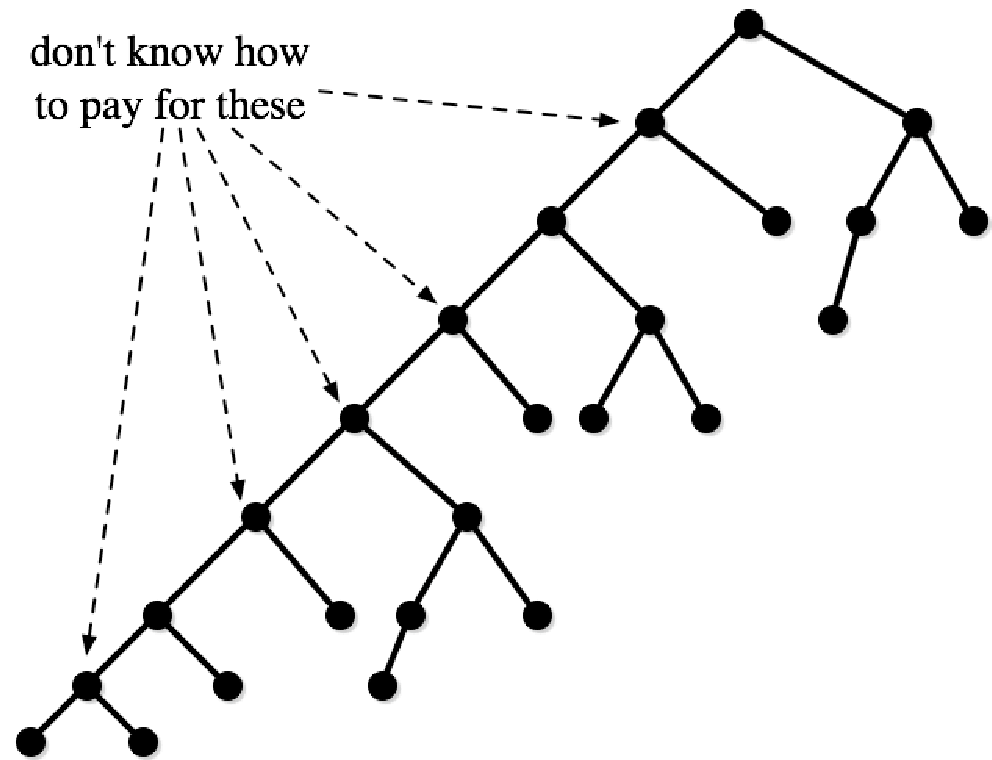
standard case: σ = ababab…a ← length k+2

"order k Davenport-Schinzel sequence"

Splay trees and Davenport-Schinzel sequences

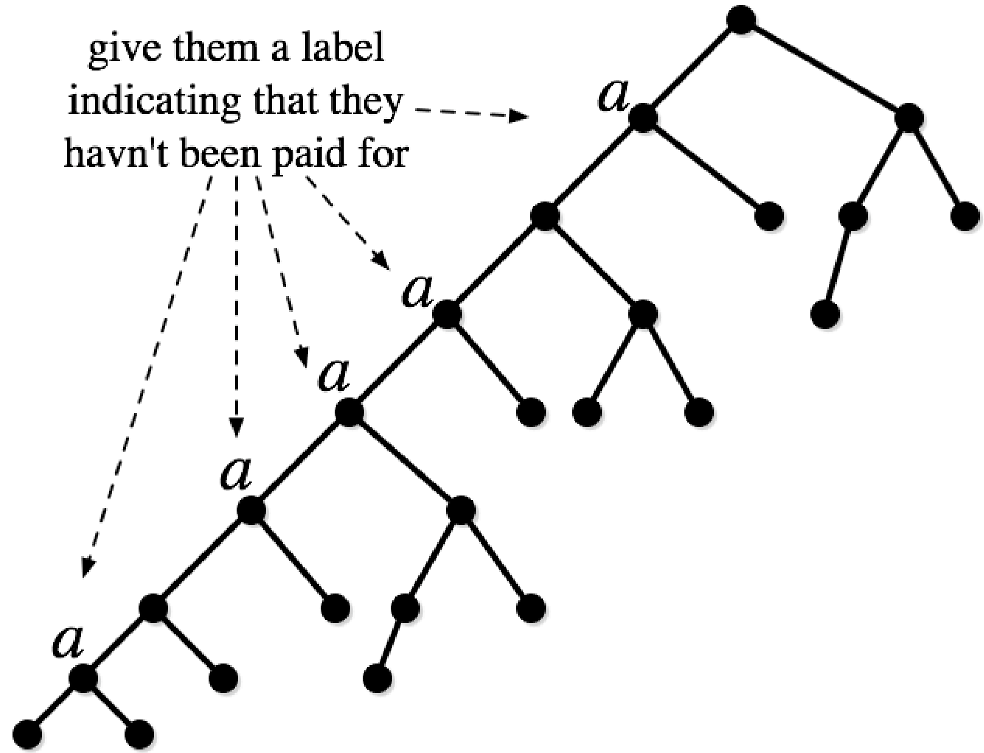
- Amortized analysis: Normally pay for time consuming ops with a reduction in potential



Splay trees and Davenport-Schinzel sequences

- New kind of amortized analysis:
- ***Label nodes*** that cannot be paid for by other means
- ***Transcribe the labels*** as a sequence S:

  $|S| \leq Ex(\sigma,n)$



- In [SODA'08] σ = abaabba or ababab a

  Thm. n deque operations take $O(n\alpha^*(n))$ time

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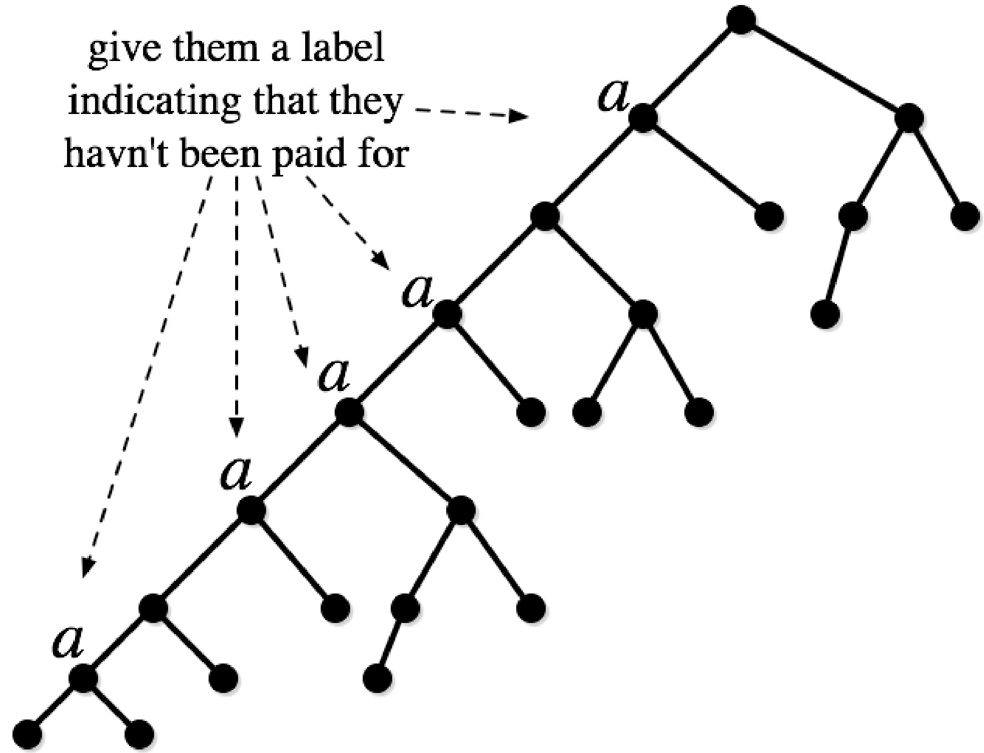

A much better way to end the proof:

… where $Ex(\sigma,n) = O(n)$

Standard Davenport-Schinzel seqs.

- $\alpha = \alpha(n)$ *α is the inverse-Ackermann function*

trivial →

Ex(**aba**, n)	n
Ex(**abab**, n)	2n-1

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	Ex(**abababab**, n)	$n \exp(\Theta(\alpha^2))$
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Klazar	Ex($\boldsymbol{\sigma}$, n)	n $\exp(O(\alpha^{\lvert\sigma\rvert}))$

Two-Letter Forbidden Subsequences

[Adamec-Klazar-Valtr]

$$\mathrm{Ex}(abbaab,n) = O(n)$$

The Two-Letter Theorem:

For any $\sigma \in \{a,b\}^*$

$\mathrm{Ex}(\sigma,n) = \omega(n)$ *if and only if* $ababa \subset \sigma$

(i.e., there is only one “cause” of superlinearity over two symbols)

The *Three-Letter Theorem*

- [Klazar-Valtr]

For $\sigma \in \{a,b,c\}^*$

$Ex(\sigma,n) = O(n)$

unless…

$ababa \subset \sigma$ or ← *non-linear*

$abcacbc \subset \sigma$ or ← *status still open*

$abcbcac \subset \sigma$ ← *status still open*

or their reversals

Recipe for linear forbidden sequences

[Klazar-Valtr]

(1) $Ex(a^i, n) = O(n)$

Recipe for linear forbidden sequences

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(1) $Ex(a^i,n) = O(n)$

(2) If $Ex(\mathbf{uw},n) = O(n)$ and $Ex(\mathbf{v},n) = O(n)$

$$Ex(\mathbf{uvw},n) = O(n)$$

uw and **v** have disjoint alphabets

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aaaa

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aa**bb**aa**bbb**

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(3) If $Ex(uawa,n) = O(n)$

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ee

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More than one cause of non-linearity

- [Klazar]
  - σ is a sequence without repetitions
  - (x,y) is in $G(\sigma)$ iff $xyyx \subset \sigma$ or $yxyx \subset \sigma$

  - If $G(\sigma)$ is ***strongly connected*** then

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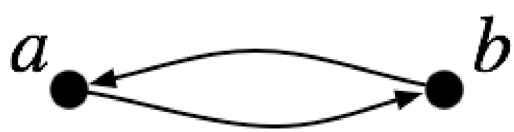
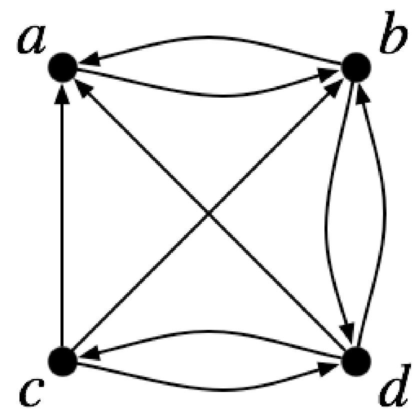


G(ababa)



G(abcbadadbcd)

only two examples known

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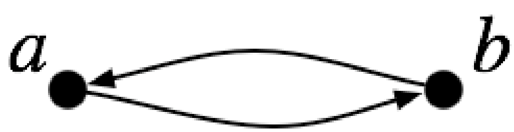
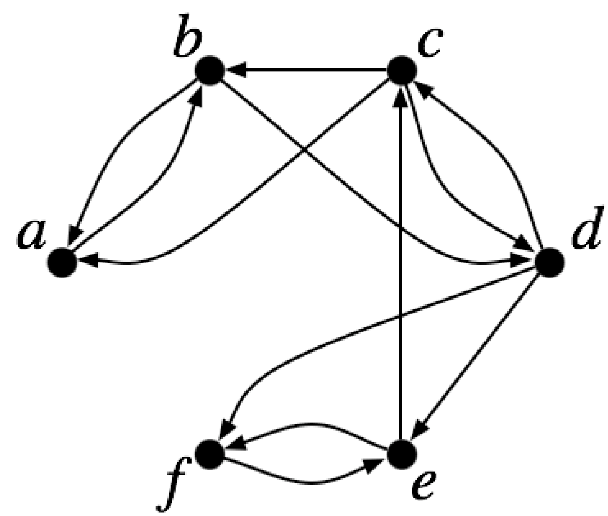
Another cause of non-linearity

- [Klazar]
  - σ is a sequence without repetitions
  - (x,y) is in G′(σ) iff xyyx ⊂ σ or ~~yxyx ⊂ σ~~
  - If G′(σ) is ***strongly connected*** then

$$\mathrm{Ex}(\sigma,n) = \sout{\Omega(n\alpha(n))} \quad \Omega(n2^{\alpha(n)})$$



G′(ababab)



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- Defn. Φ = minimal non-linear forbidden seqs.
- What we know about Φ:
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  - $|\Phi| \geq 2$ (the other a subseq of abcbadadbcd)

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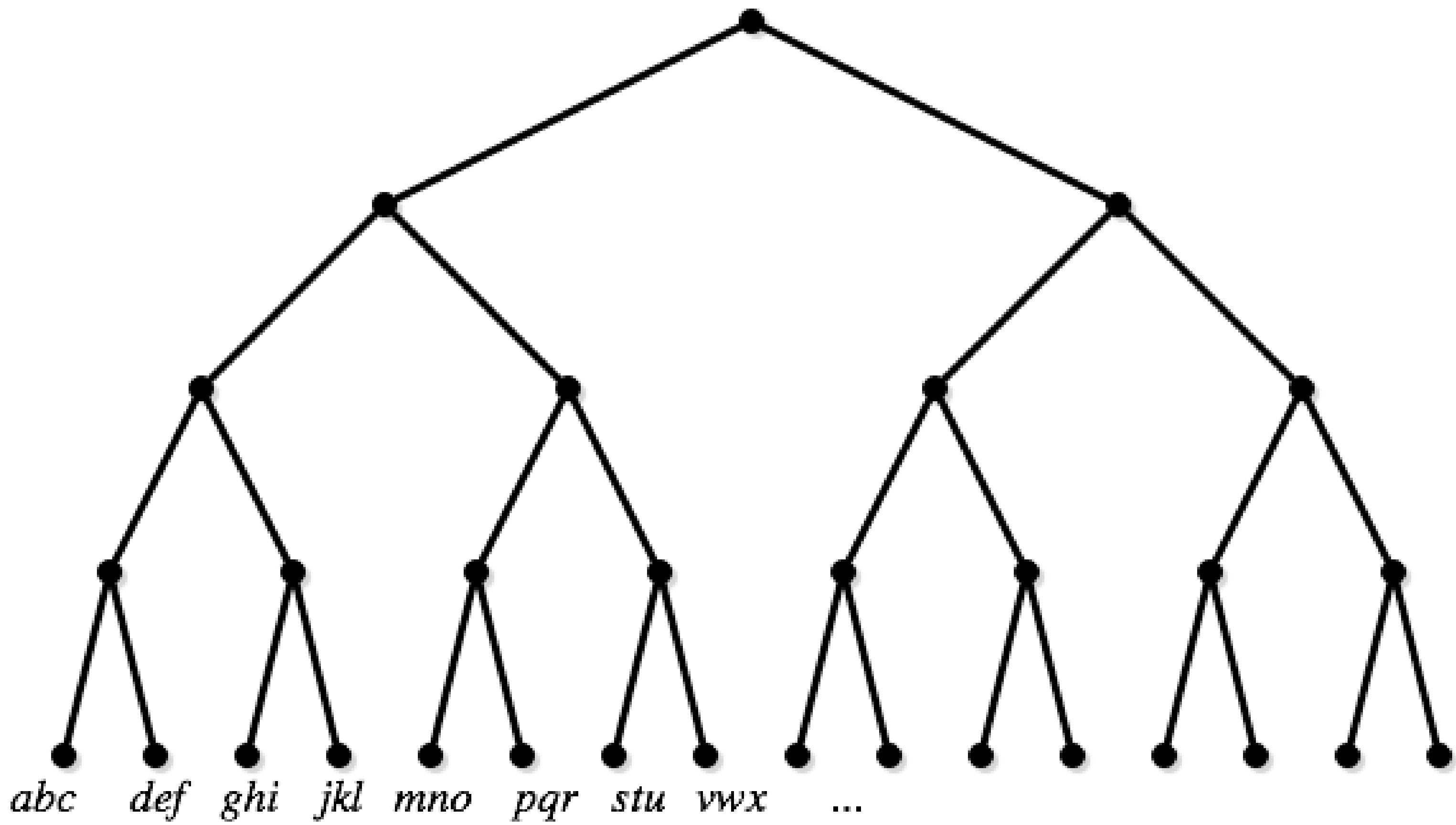
- ***A***: Still Open. But we have a candidate!
- ***Q***: How big is it Φ?
- ***A***: New result: $|\Phi| \geq 5$

Constructing Sequences

- T(1,j) : a binary tree with height j+1
  j distinct letters at each leaf



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Constructing Sequences

- T(1,j) : a bin. tree w/height j+1,   j letters at each leaf
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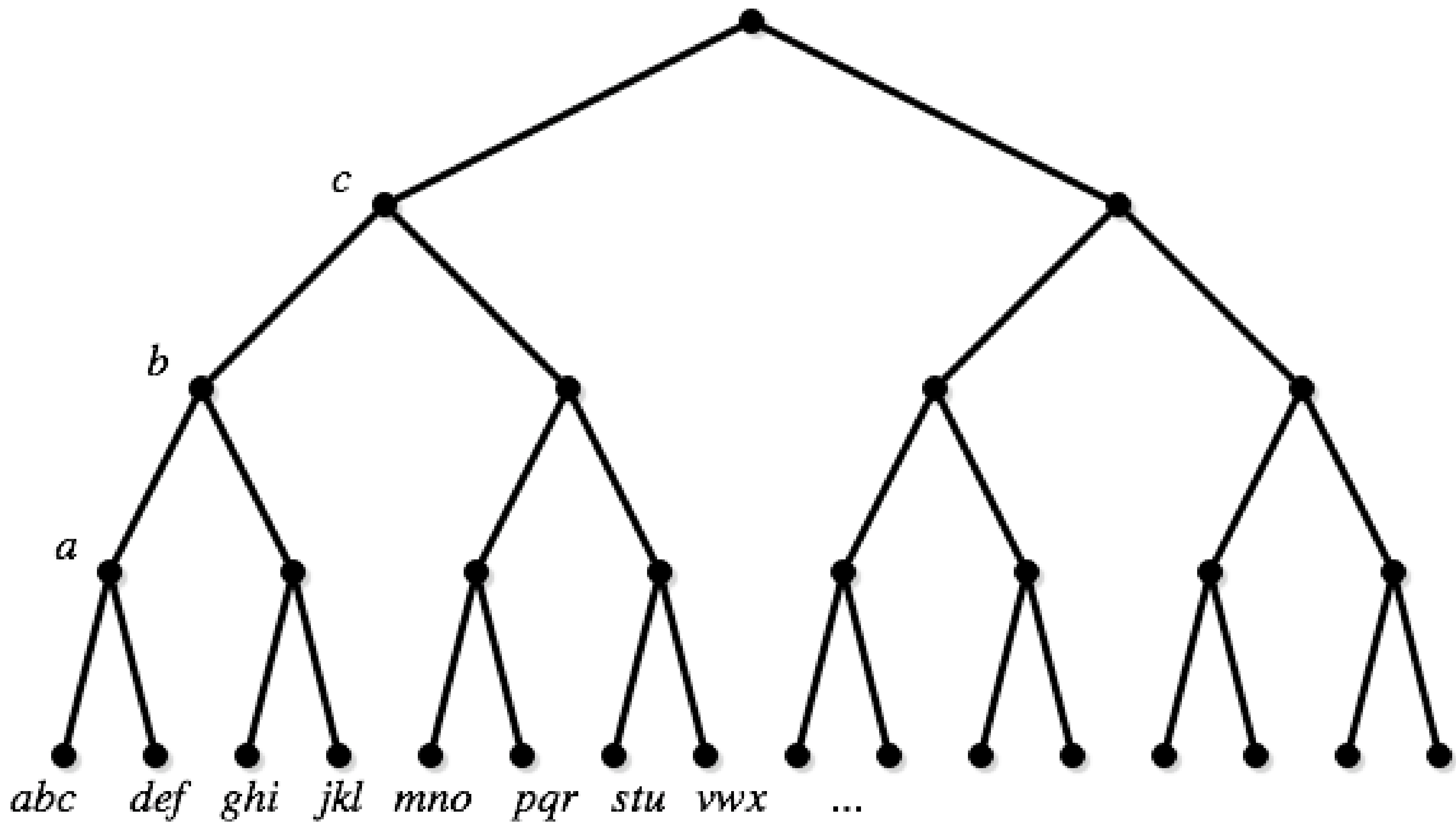

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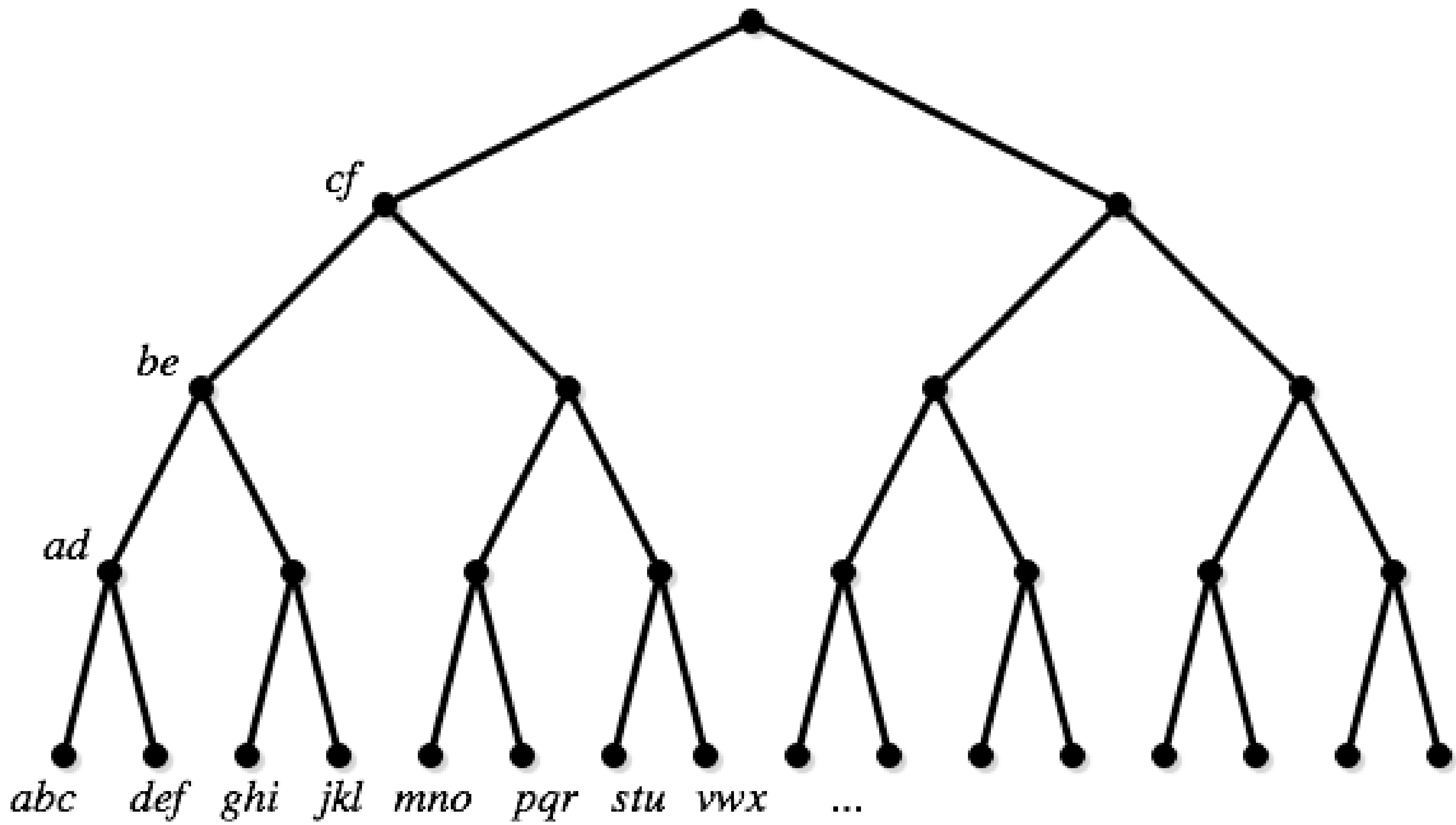

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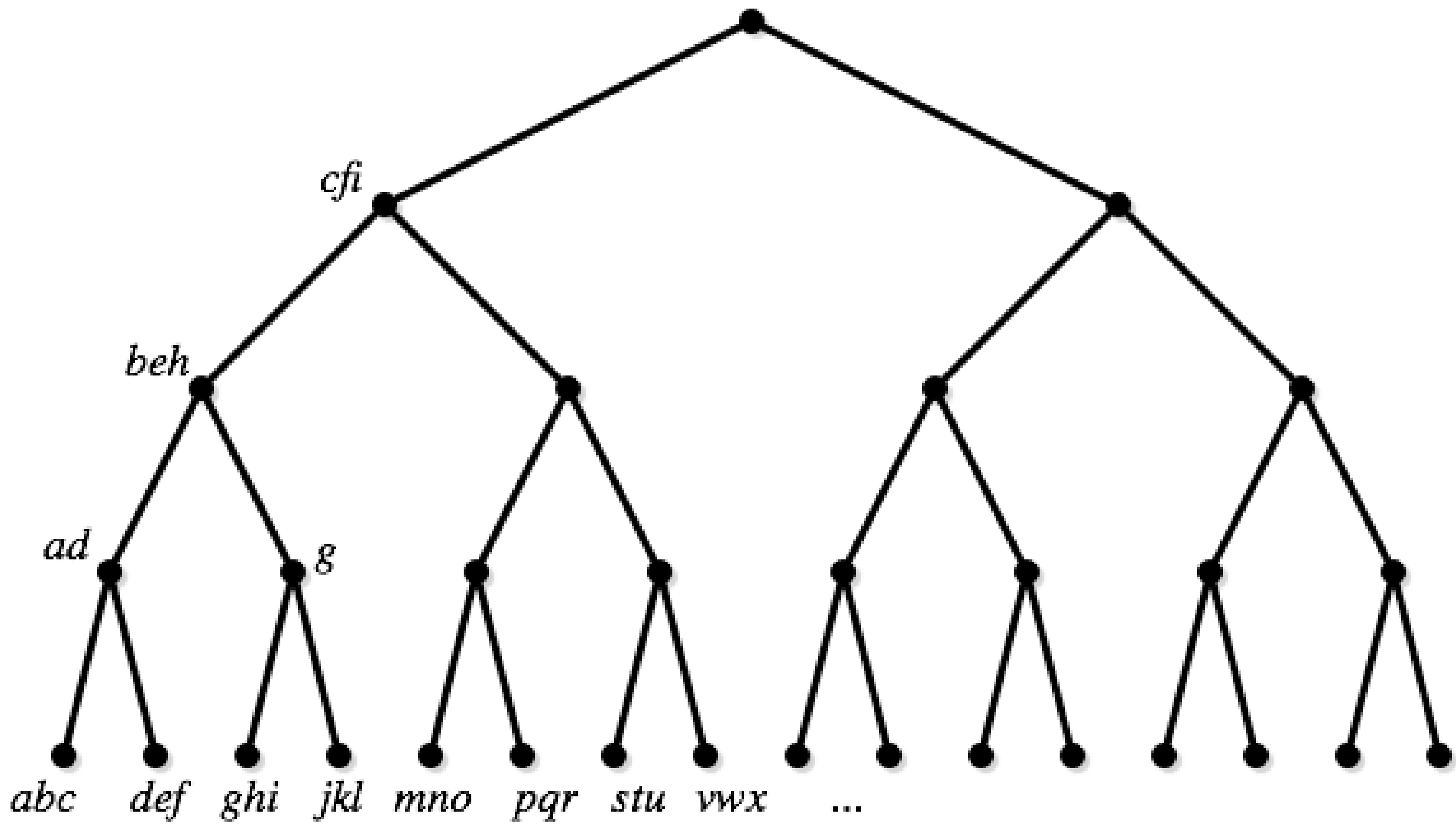

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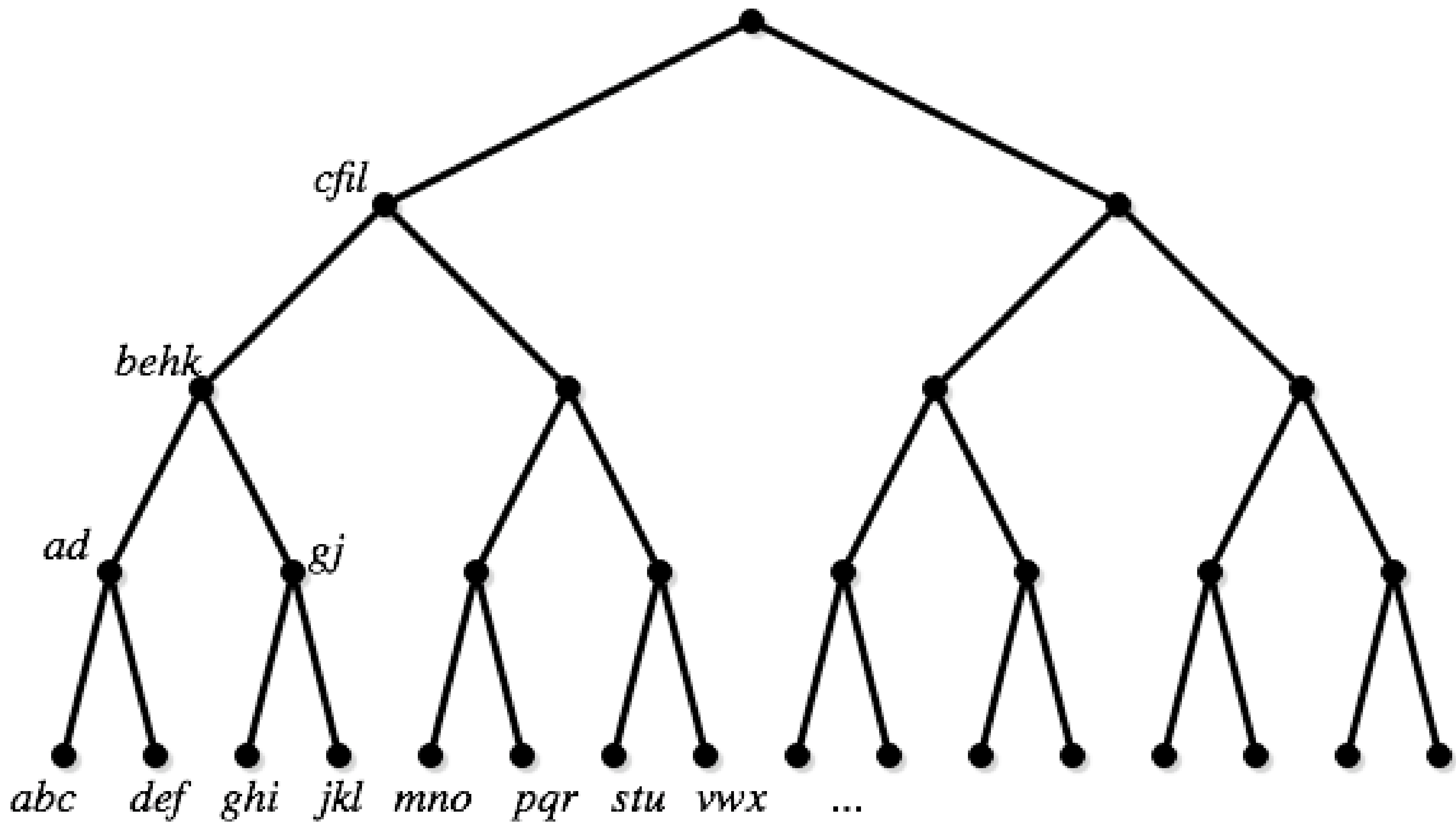

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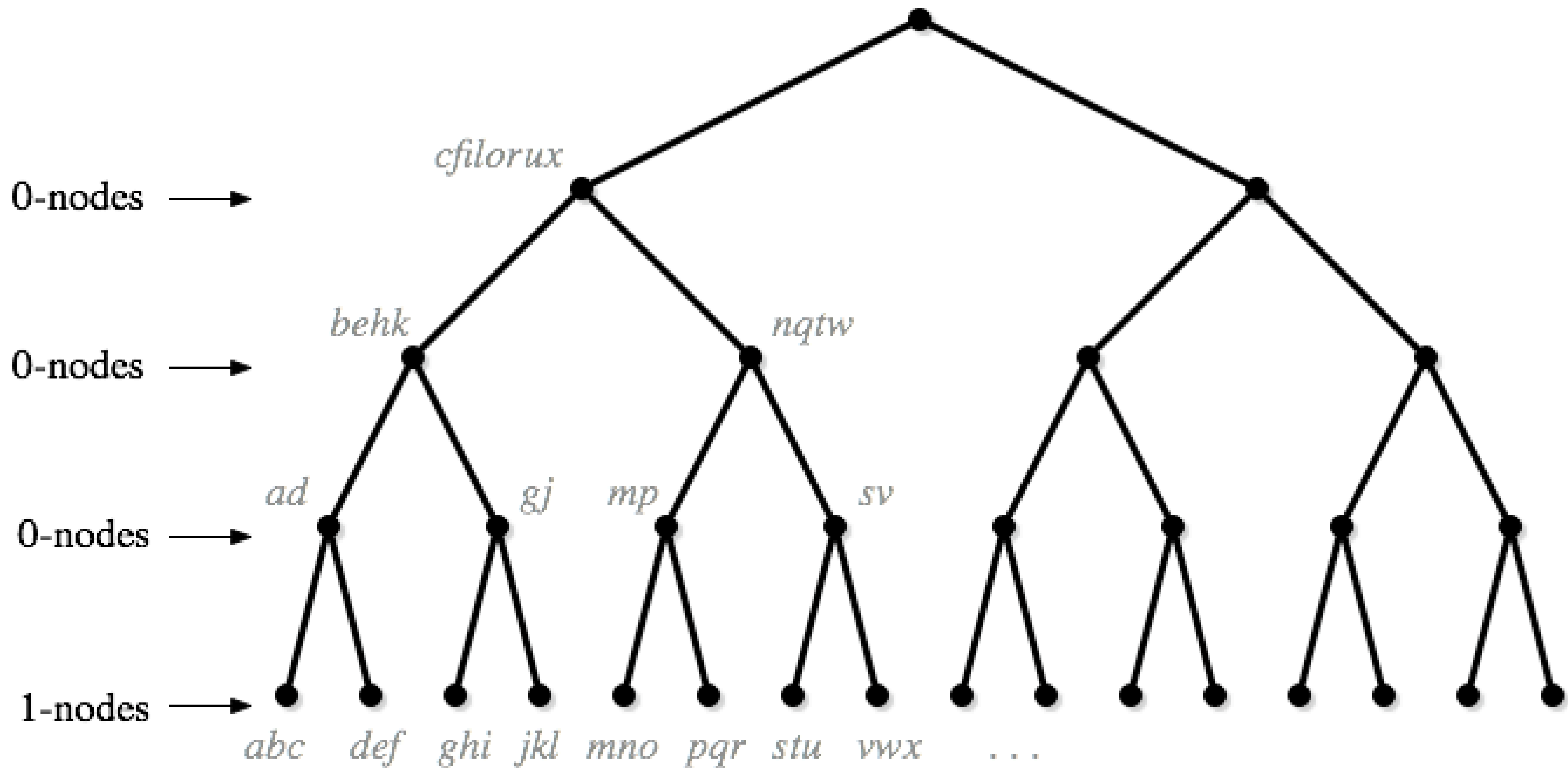

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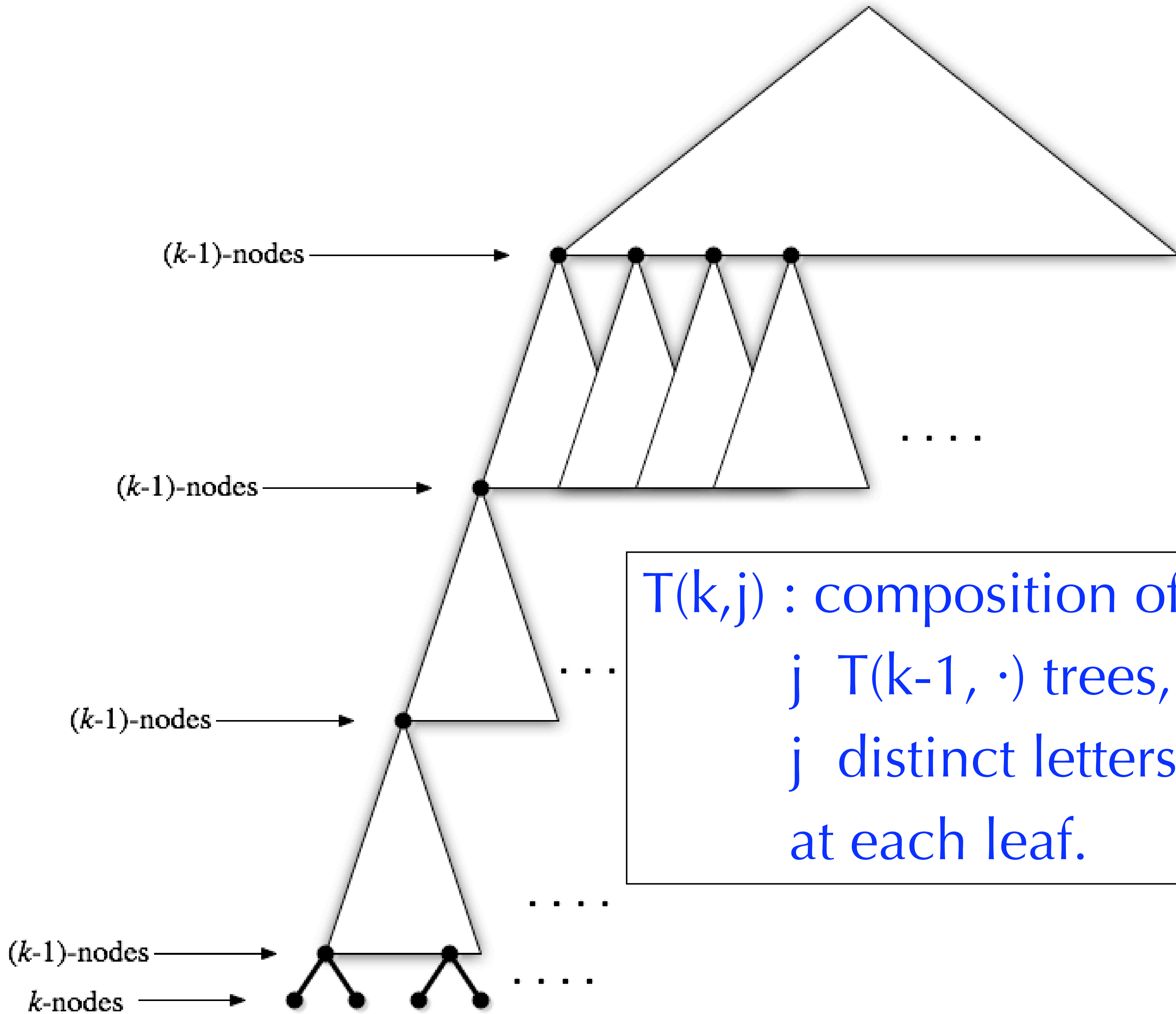

T(k,j) : composition of
j T(k-1, ·) trees,
j distinct letters
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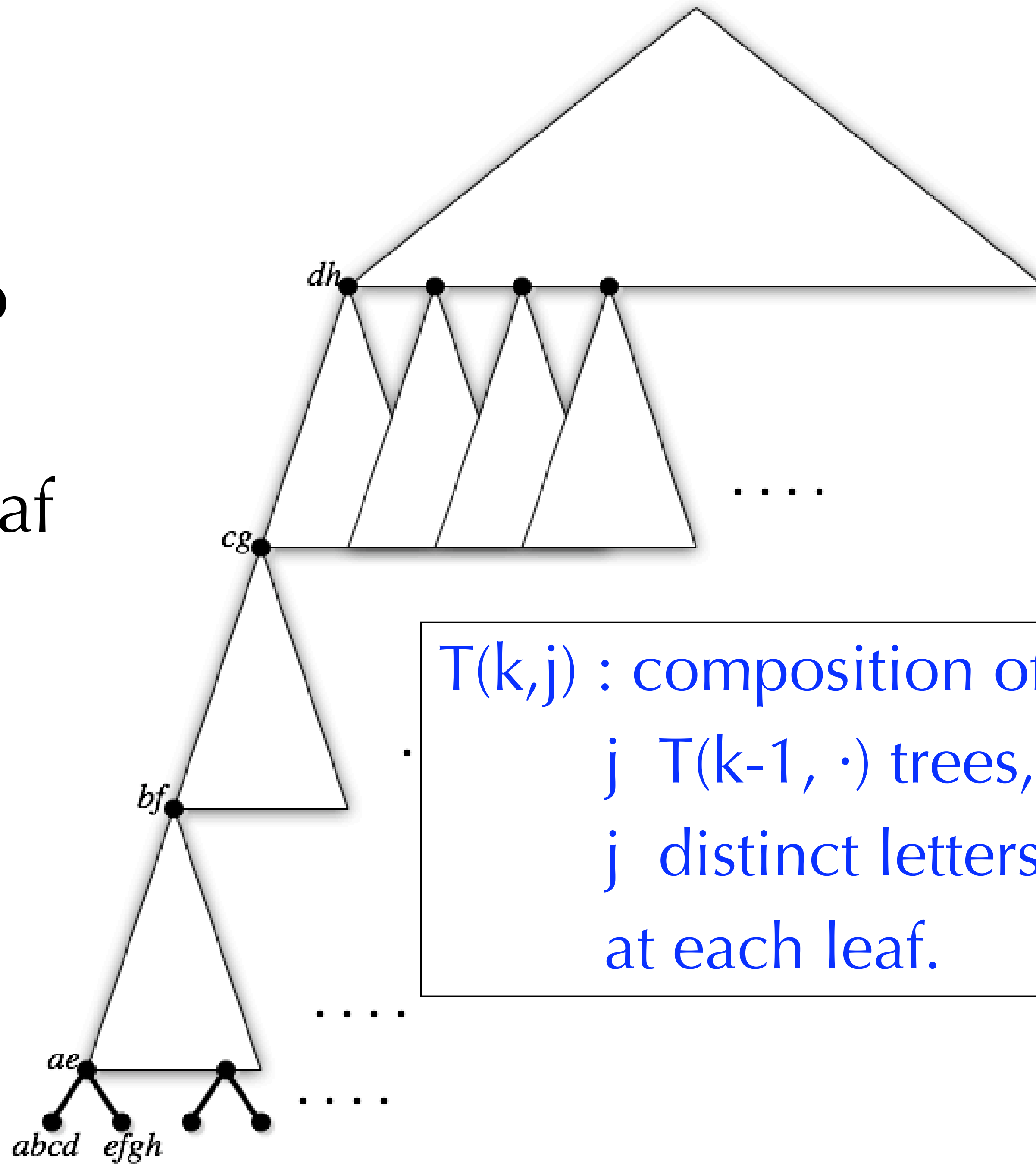
the ith letter at a leaf is assigned to the ith (k-1)-node ancestor of the leaf



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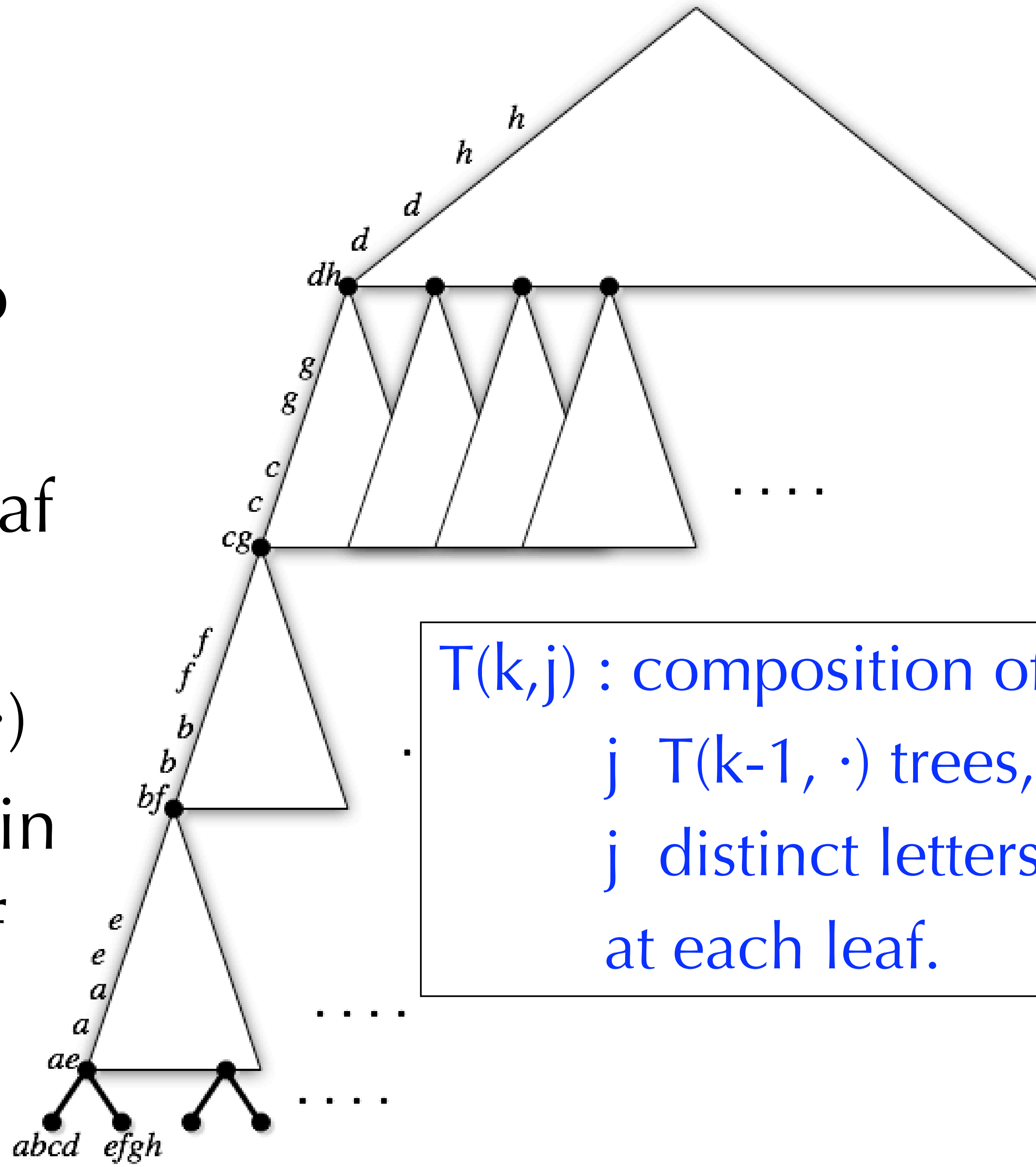
the i^{th} letter at a leaf is assigned to the i^{th} (k-1)-node ancestor of the leaf

…and the T(k-1, ·) trees are defined in terms of their leaf labels…



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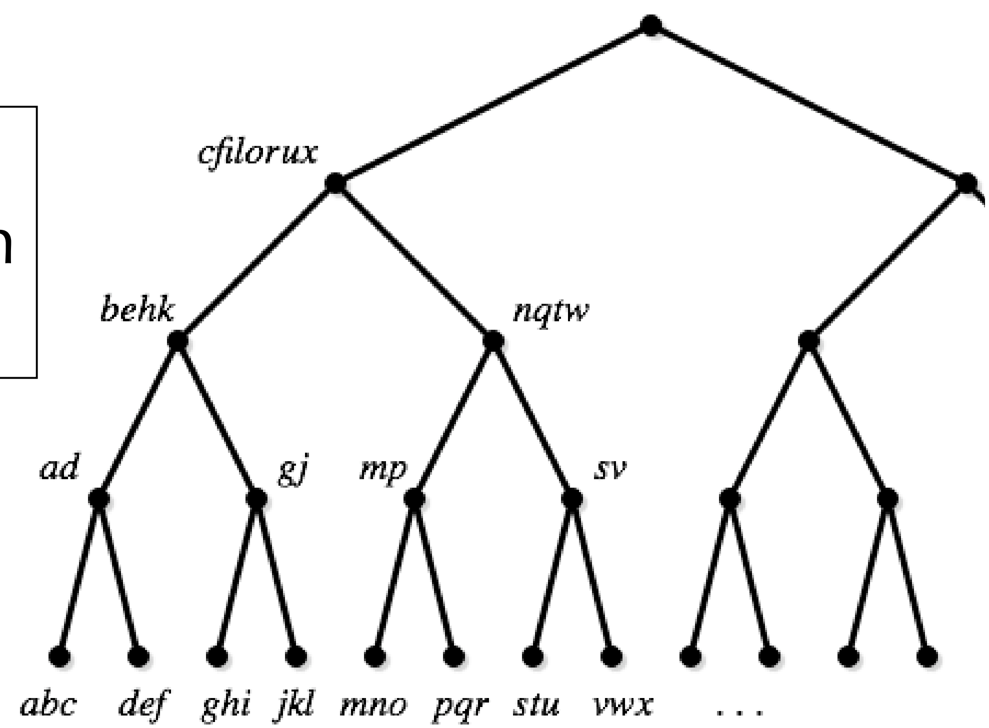
Constructing Sequences

- $v_1, v_2, \ldots, v_n$: nodes listed in ***postorder***
- L(v) : the label of v in ***reverse order***
- The final sequence: $\Sigma = L(v_1), L(v_2), \ldots, L(v_n)$

The sequence for T(1,4) :

cba fed da ihg lkj jg kheb onm rqp pm uts xwv vs wtqn xurolifx …



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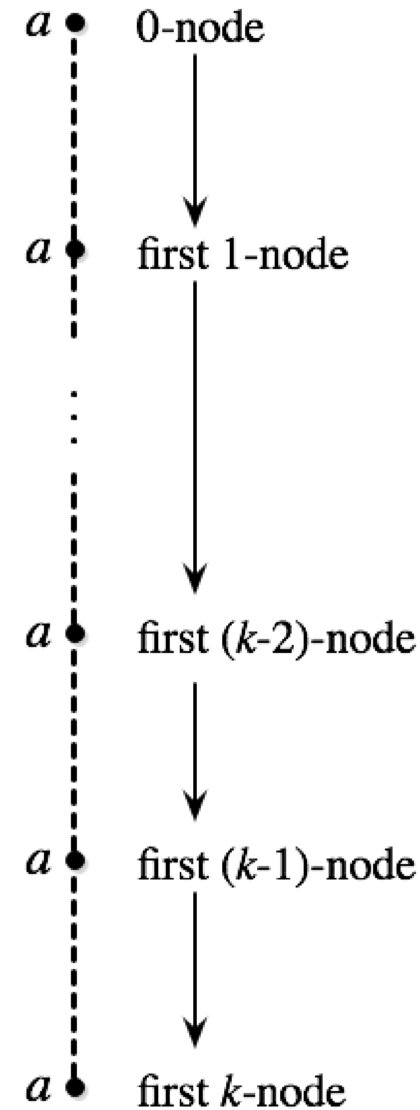
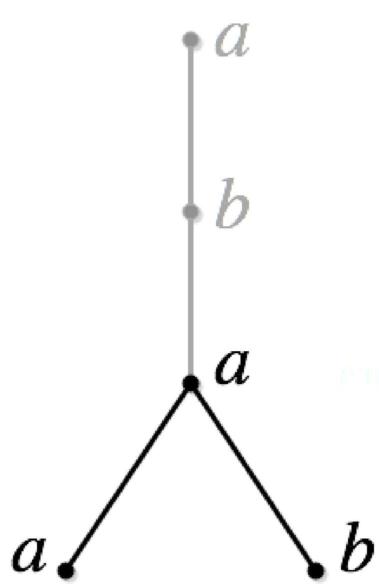
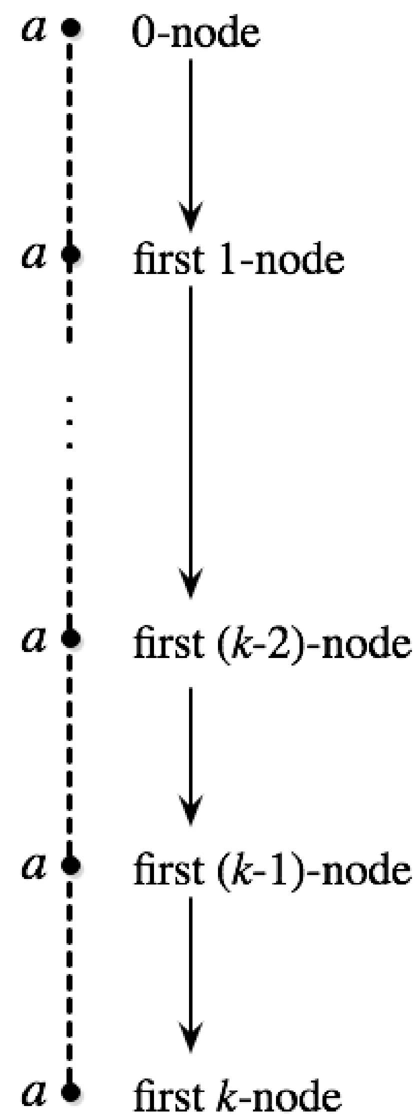

a
0-node
a
first 1-node
a
first (k-2)-node
a
first (k-1)-node
a
first k-node

Forbidden subseq: ***ababa***

- Σ is (ababa)-free:





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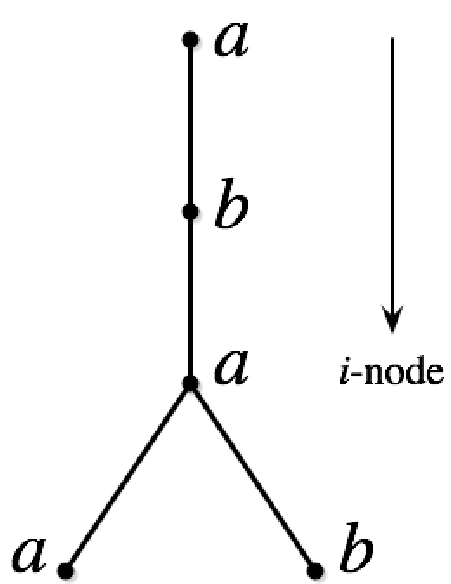

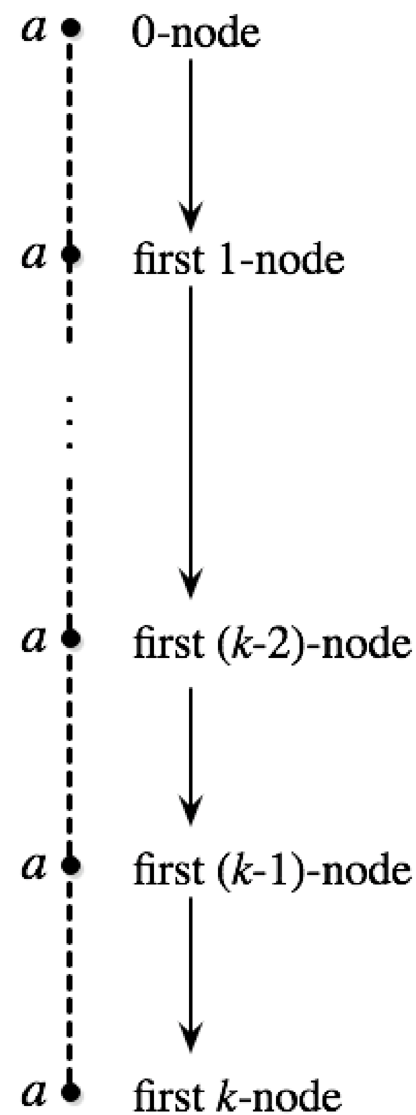

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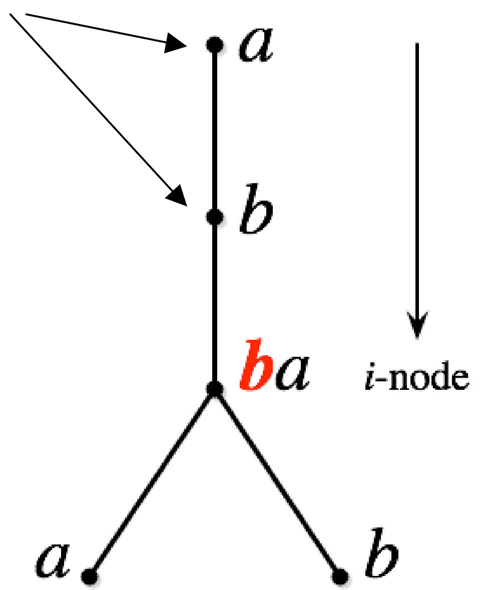
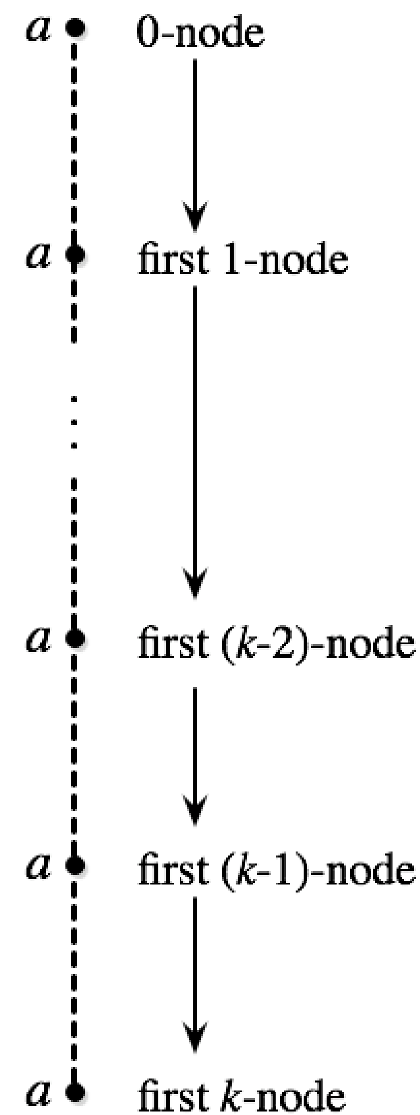
these are in the wrong order!





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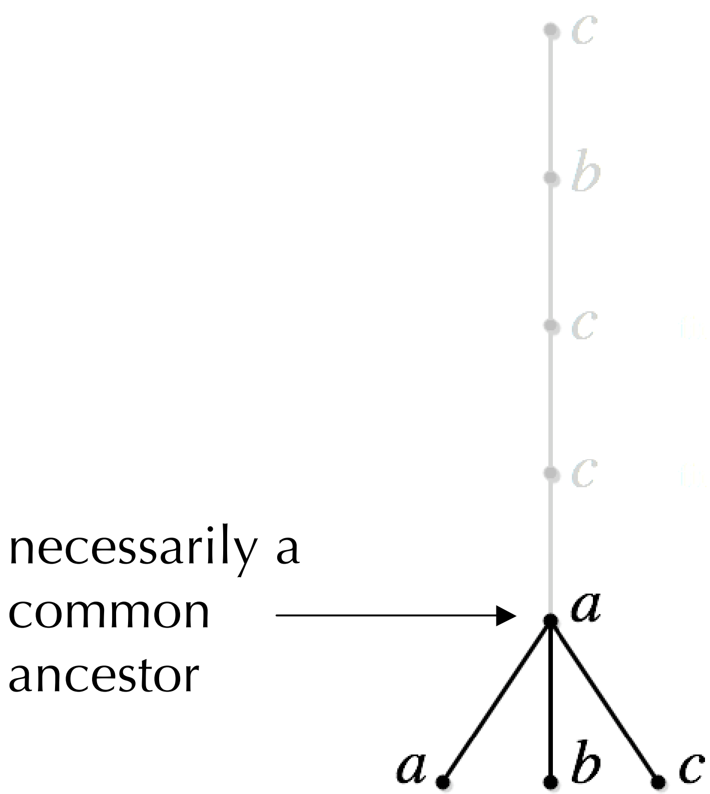
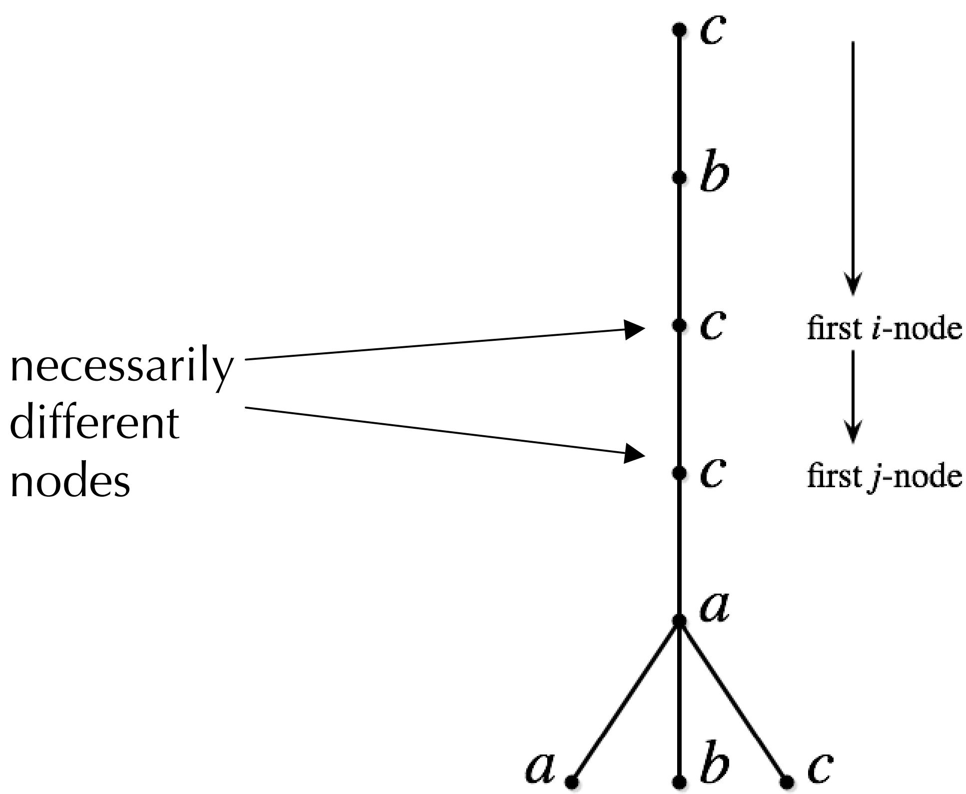
Forbidden subseq: ***abcaccbc***

- Σ is (abcaccbc)-free:



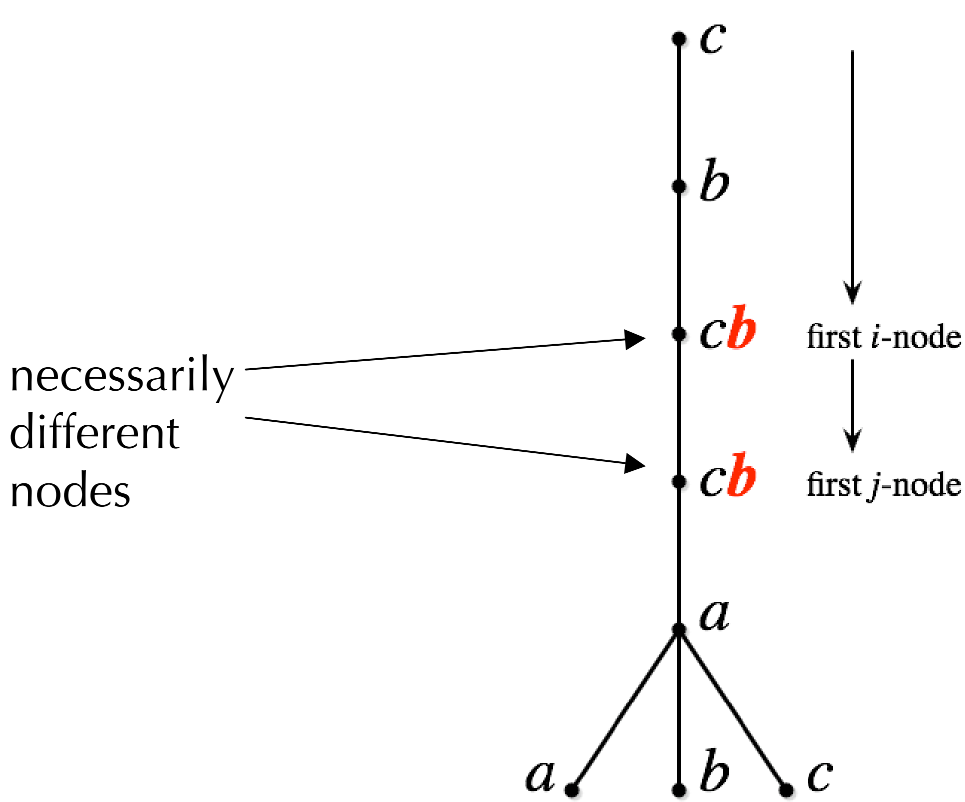
Forbidden subseq: ***abcaccbc***

- Σ is (abcaccbc)-free:



Forbidden subseq: ***abcaccbc***

- Σ is (abcaccbc)-free:

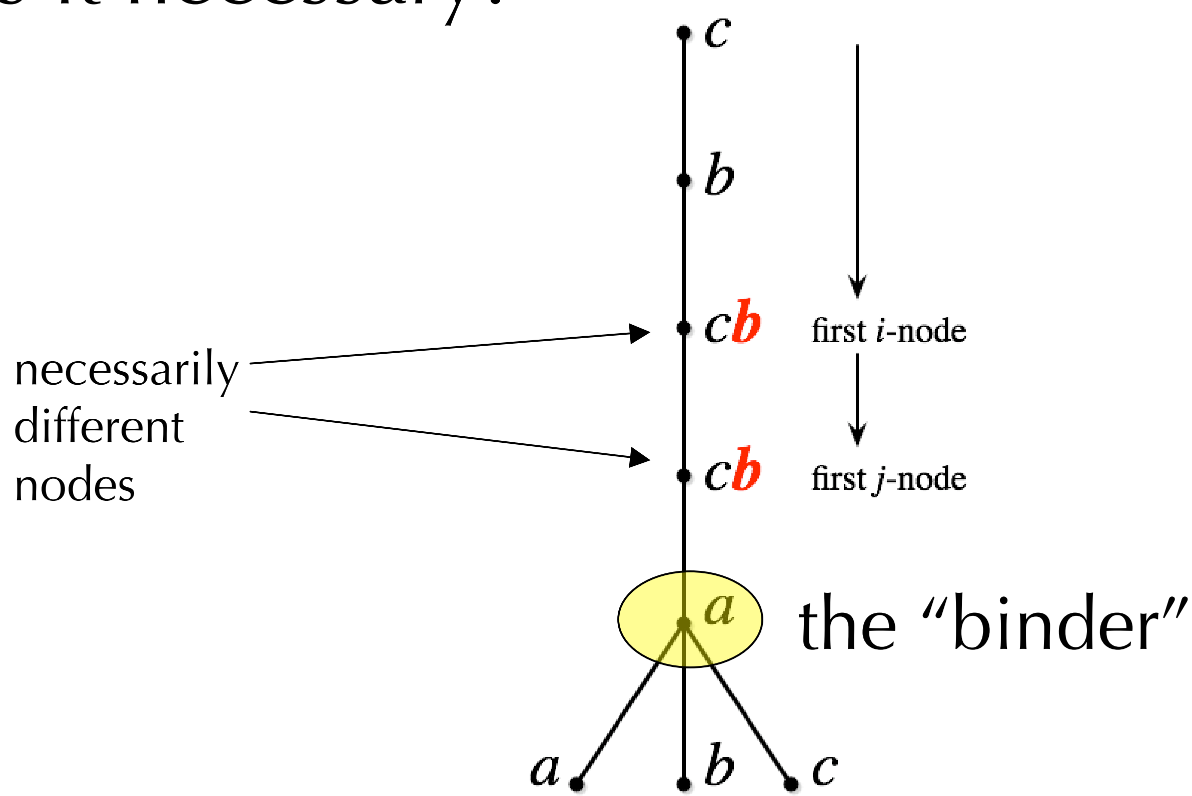


Forbidden subseq: ***abcaccbc***

"a" does not appear in the final contradiction

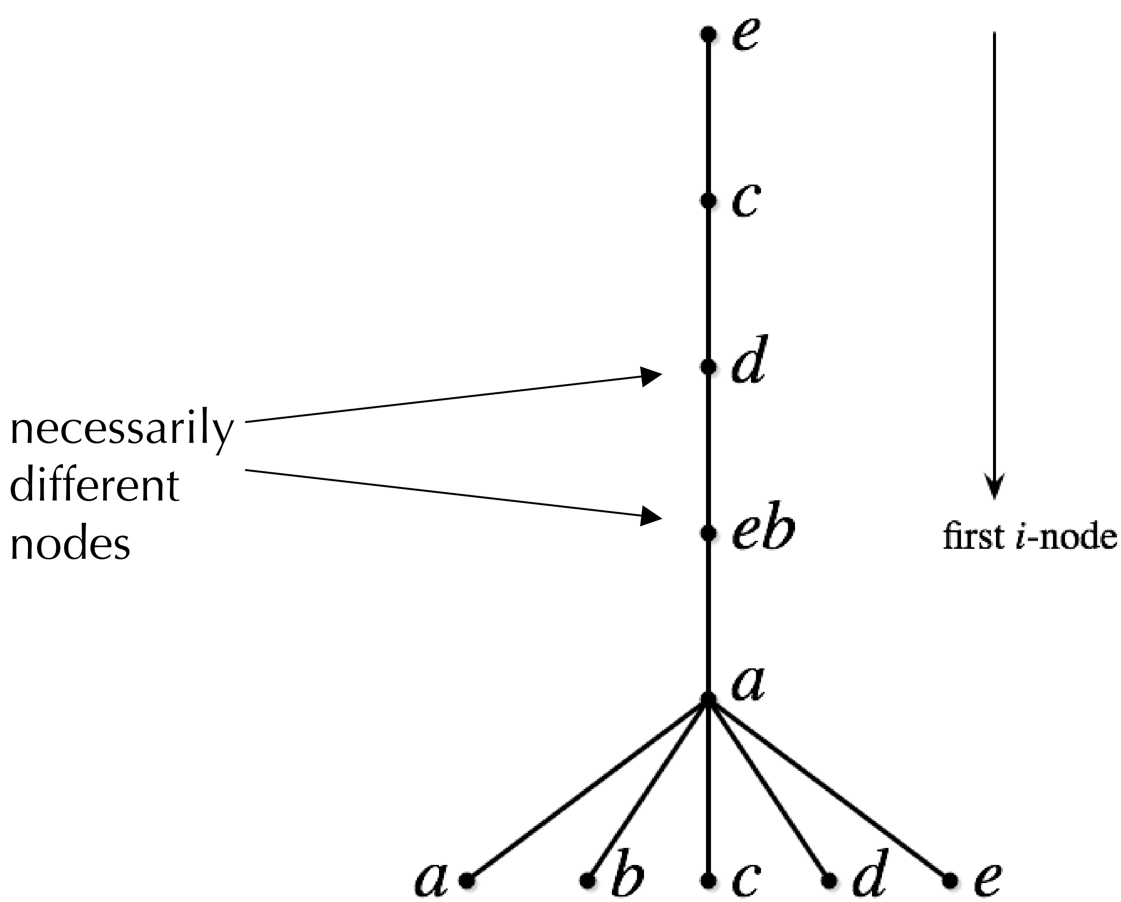
(an implied occurrence of ***bcbcbc***)

Why is it necessary?



Forbidden subseq: ***abcdeaebdce***

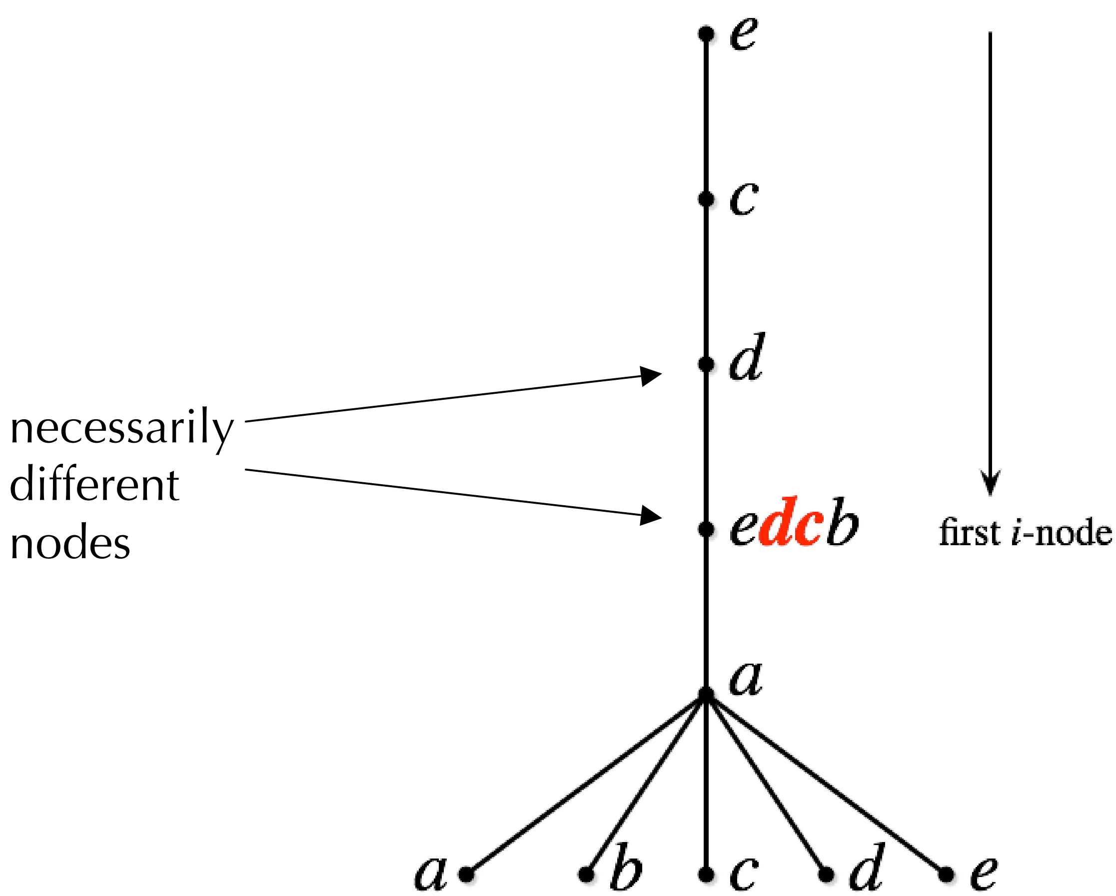
- Σ is (abcdeaebdce)-free:



Seth Pettie

Forbidden subseq: ***abcdeaebdce***

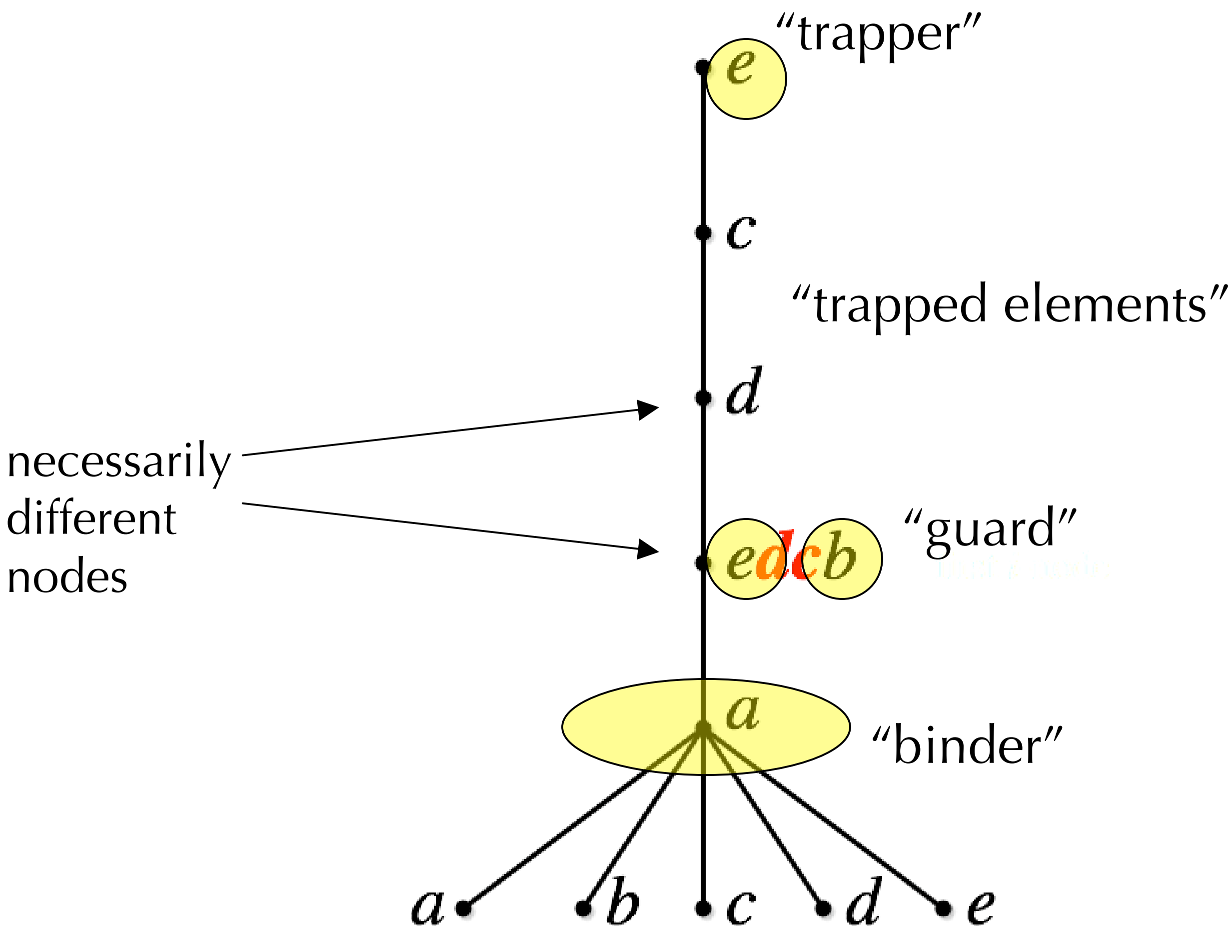
- Σ is (abcdeaebdce)-free:



Seth Pettie

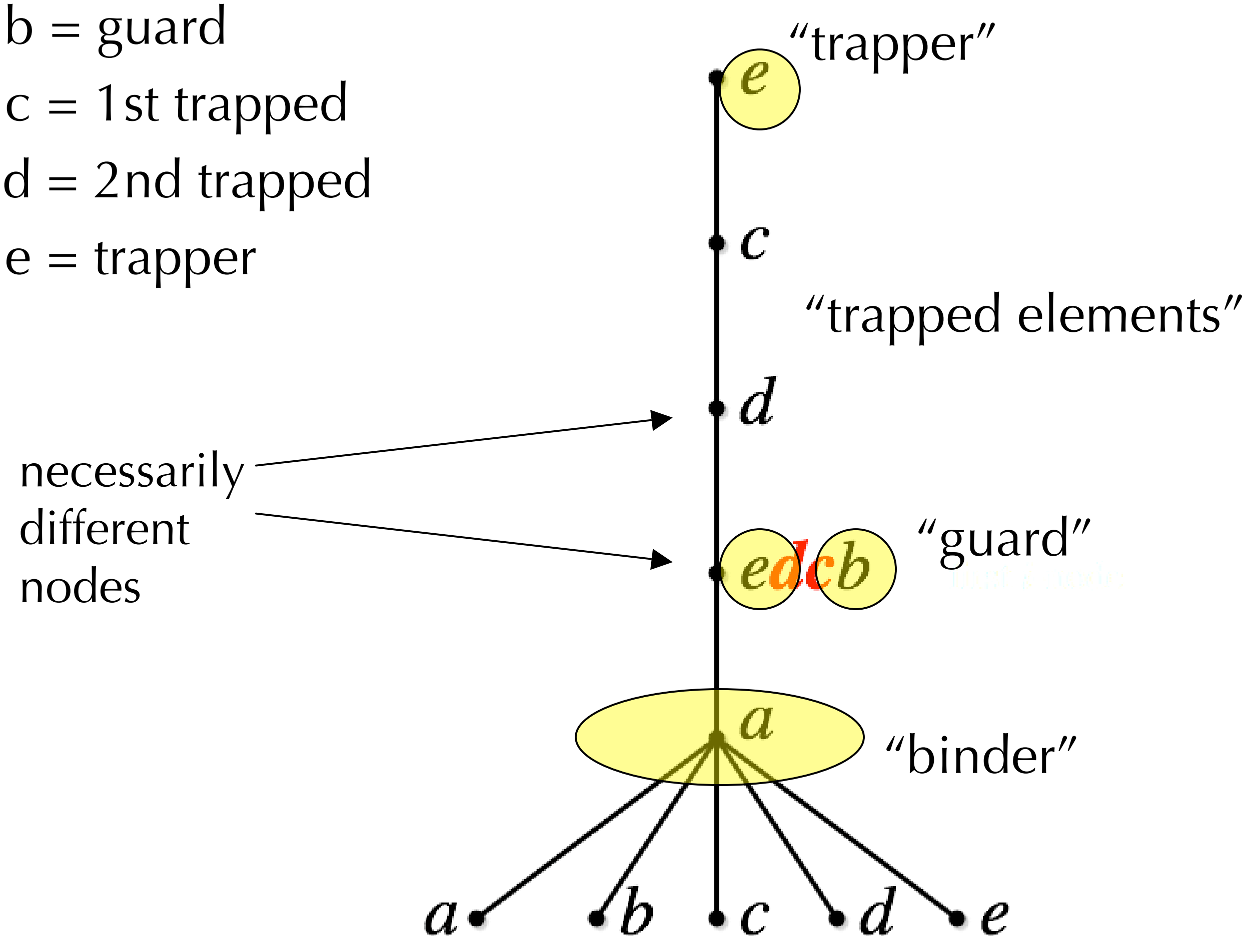
Forbidden subseq: ***abcdeaebdce***

- Σ is (abcdeaebdce)-free:



Forbidden subseq: ***abcdeaebdce***

- Succinct Encoding: ♥♦♠♠♣
  - ♥ : a = binder
  - ♦ : b = guard
  - ♠ : c = 1st trapped
  - ♠ : d = 2nd trapped
  - ♣ : e = trapper



All of these encodings make sense & work:

♥♦♠♠♣

♦♥♠♠♣

♥♠♦♠♣

These don’t:

♦♠♠♥♣ ⟵ the binder doesn’t bind (but this can be fixed!)

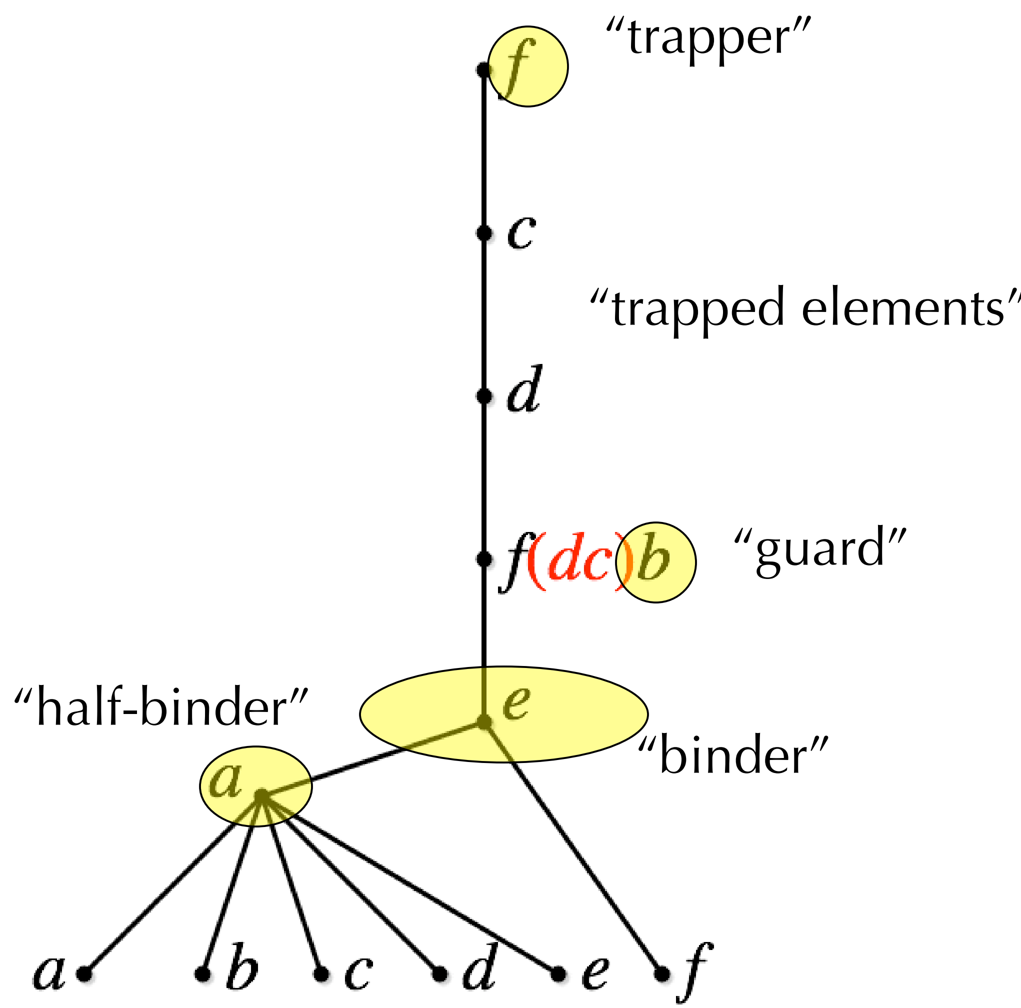
♥♠♠♦♣ ⟵ the guard doesn’t guard

♥♣♦♠♣ ⟵ this doesn’t make any sense

Seth Pettie

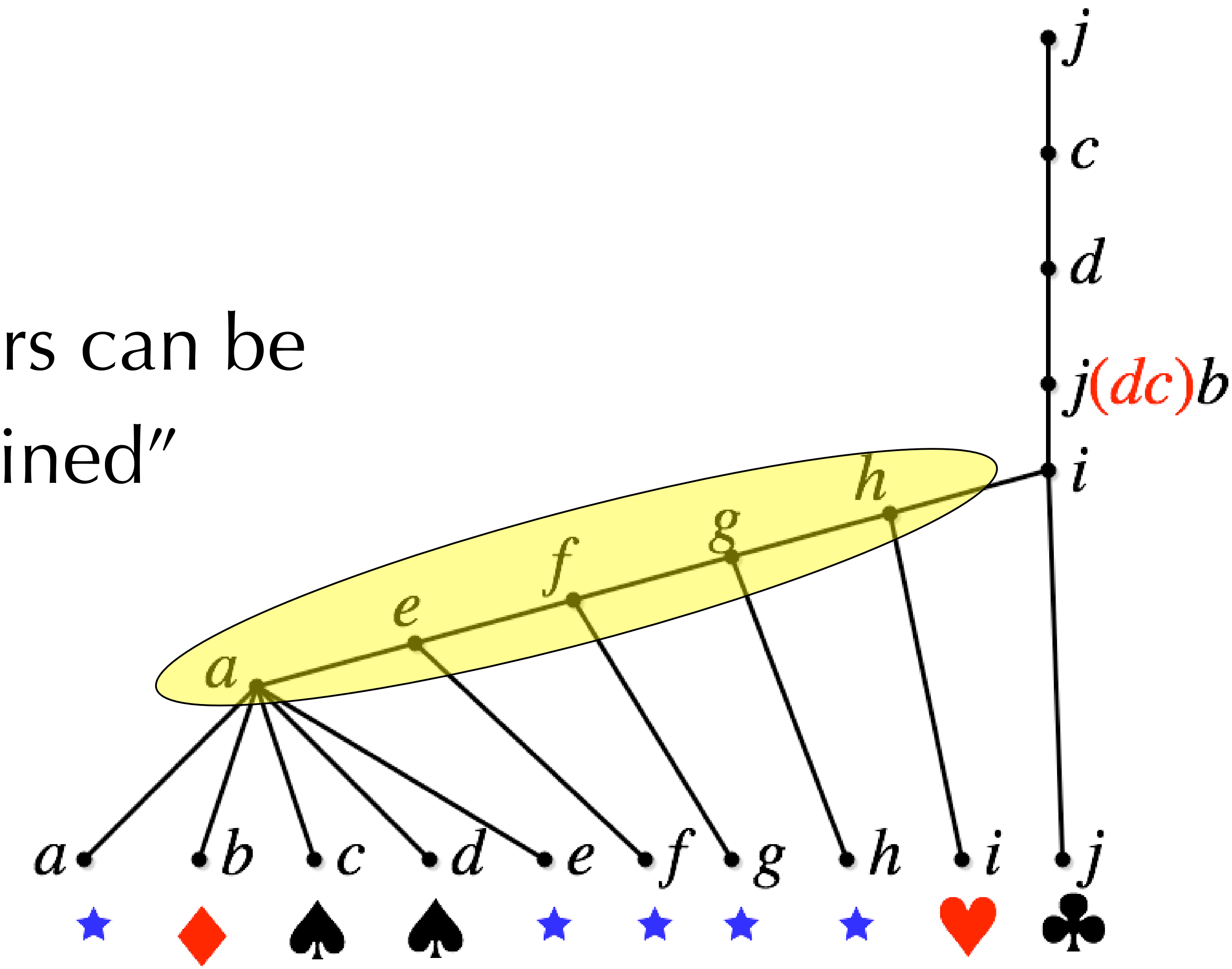
Forbidden subseq: ***abcdeafefbdcf***

- Encoding: ★♦♠♠♥♣
  - ★ : a = half-binder
  - ♦ : b = guard
  - ♠ : c = 1st trapped
  - ♠ : d = 2nd trapped
  - ♥ : e = binder
  - ♣ : f = trapper



Forbidden subseq: ***abcdeafegfhgihjijbdcj***

Half-binders can be
"daisy-chained"



Seventeen legal encodings

♥♠(♦♠♣)

★♣♠♥♣

★♠♥(♦♠♣)

★♦♠♠♥♣

♦♥♠♠♣

★♠♦♥♠♣

★♠♥(♦♠)♣

♥♣♠♣

♦★♠♥♠♣

♥♠♦♠♣

★♠♦♠♥♣

★♠♥♦♠♣

♥♠(♦♠)♣

♦★♠♠♥♣

♥♦♠♠♣

★♦♠♥♠♣

★♠(♦♠)♥

Seth Pettie

- Some open problems
  - Are there infinitely many “causes” of non-linearity?
  - Are there any more linear seqs. to be discovered?
  - For each c, is there an (ababa)-free σ such that:

$$Ex(\sigma,n) = n \exp(\alpha^c(n))$$