Optimal Block-Decodable Encoders for Constrained Systems

Panu Chaichanavong

Center for Magnetic Recording Research
University of California, San Diego

Brian Marcus

Department of Mathematics
University of British Columbia

March 23, 2004

Outline

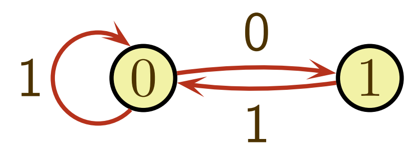
- Constrained systems and finite-state encoders
- Block-decodable encoder and its relatives
- Sets of principal states
- Complexity of determining the optimal rate
- Encoder construction
- Asymptotic analysis of optimal code rate and sets of principal states

Constrained Systems and Their Presentations

G: **labeled graph**

(with vertex set $V = V_G$)

G



$S = S(G)$: **constrained system,**

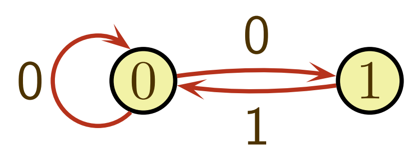
set of all words obtained from reading labels of paths of G

$S(G) =$ set of all words that do not contain 00

Say that G is a **presentation** of S

deterministic graph:

at each state, all outgoing edges have distinct labels



nondeterministic graph

$A = A_G$: **adjacency matrix,**

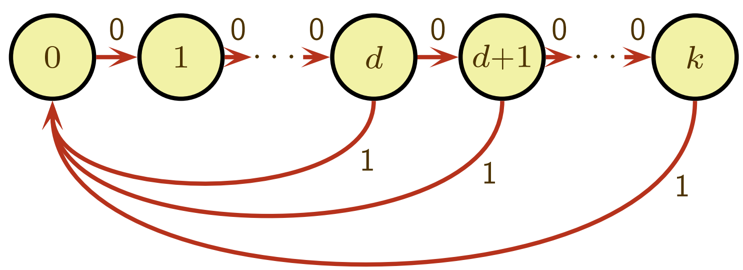
$|V| \times |V|$ matrix defined by

$A_{u,v} =$ number of edges from u to v

$$A_G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

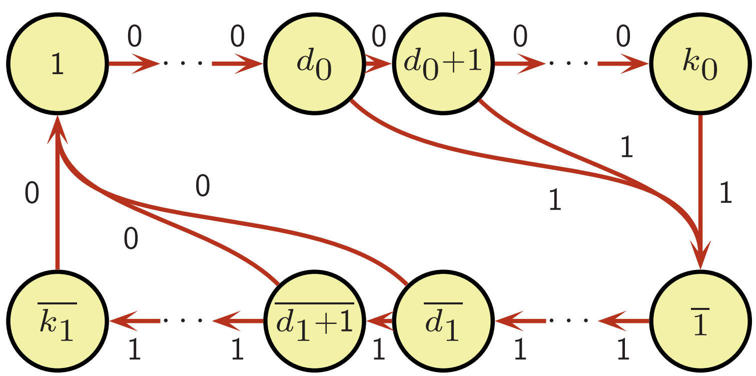
RLL(d, k) and Asymmetric-RLL(d_0, k_0, d_1, k_1)

Runlength limited RLL(d, k)



- $d \leq$ run of zeros $\leq k$
- employed in CDs, DVDs, and magnetic tapes

Asymmetric-RLL(d_0, k_0, d_1, k_1)

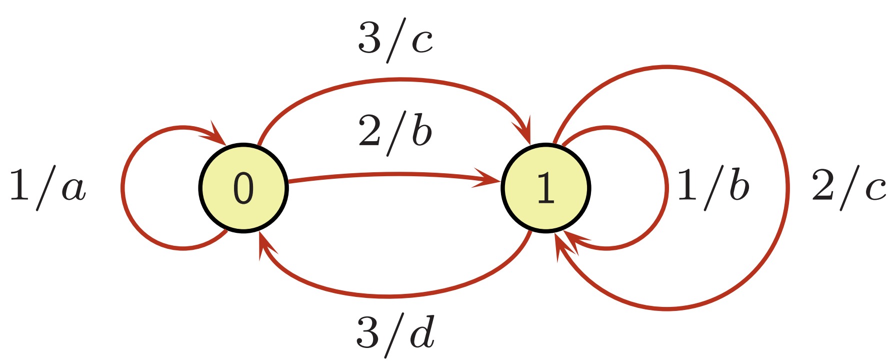


- $d_0 \leq$ run of zeros $\leq k_0$
- $d_1 \leq$ run of ones $\leq k_1$
- employed in optical recording systems

Finite-state encoders

An (S, n) **encoder** is a graph that

- has a constant out-degree n, i.e., each state has n outgoing edges
- has two types of labeling: *input* and *output*, where
  - the input alphabet size is n
  - at each state, the input labels of the outgoing edges are distinct
  - the output labeling satisfies the constraint S
- can be "decoded"

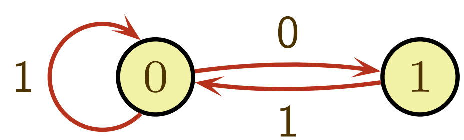


A finite-state encoder

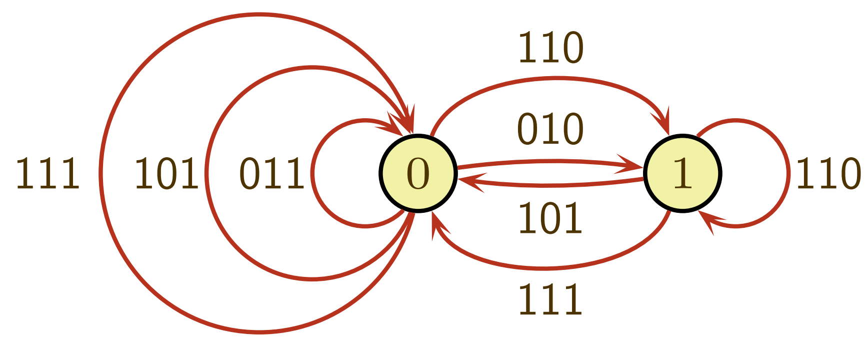
Higher power graphs

Usually we want to construct an encoder whose edge labels are words of some length q. These words are called **codewords** and the length is called **block length**.

Let G be a graph. The q **th power of** G**,** denoted G^q, is the labeled graph with the same set of states as G, but one edge for each path of length q in G



G

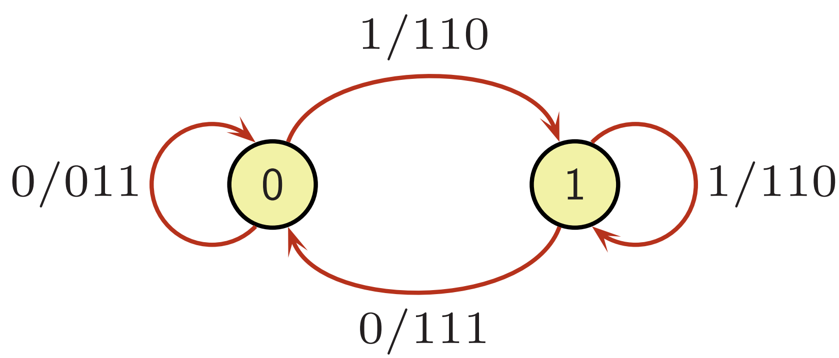


G^3

If $S = S(G)$, define $S^q = S(G^q)$. Then we construct an encoder for S^q.

If A is the adjacency matrix of G, then the adjacency matrix of G^q is A^q

An $(S^3, 2)$ encoder



An $(S^3, 2)$ encoder

The **code rate** of an (S^q, n) encoder is defined to be $\frac{\log n}{q}$

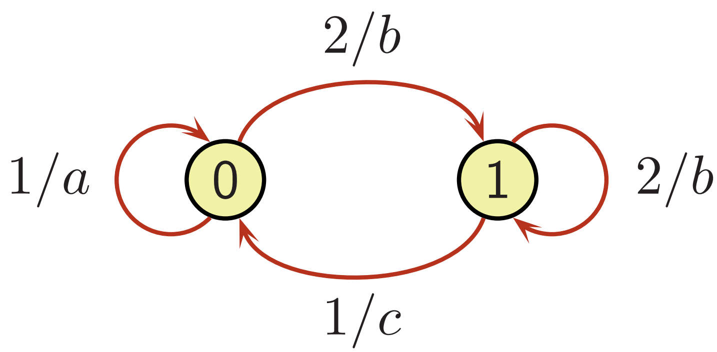
The **capacity** of a constraint S, denoted $\mathrm{cap}(S)$ is defined: $\lim_{n\to\infty}(1/q)\log_2(N(q;S))$.

Shannon: $\frac{\log n}{q} \leq \mathrm{cap}(S)$

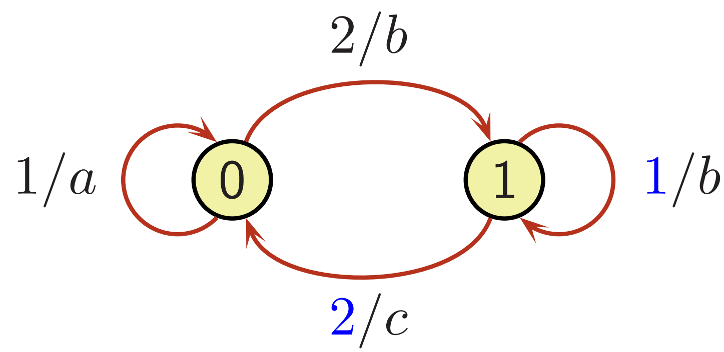
GOAL: For a given constraint, block length q *and encoder class,* determine optimal rate, equivalently optimal n.

Block-decodable encoders

A **block-decodable encoder** is a finite-state encoder such that the input label of any edge e can be uniquely determined from the output label of e



block decodable



not block decodable

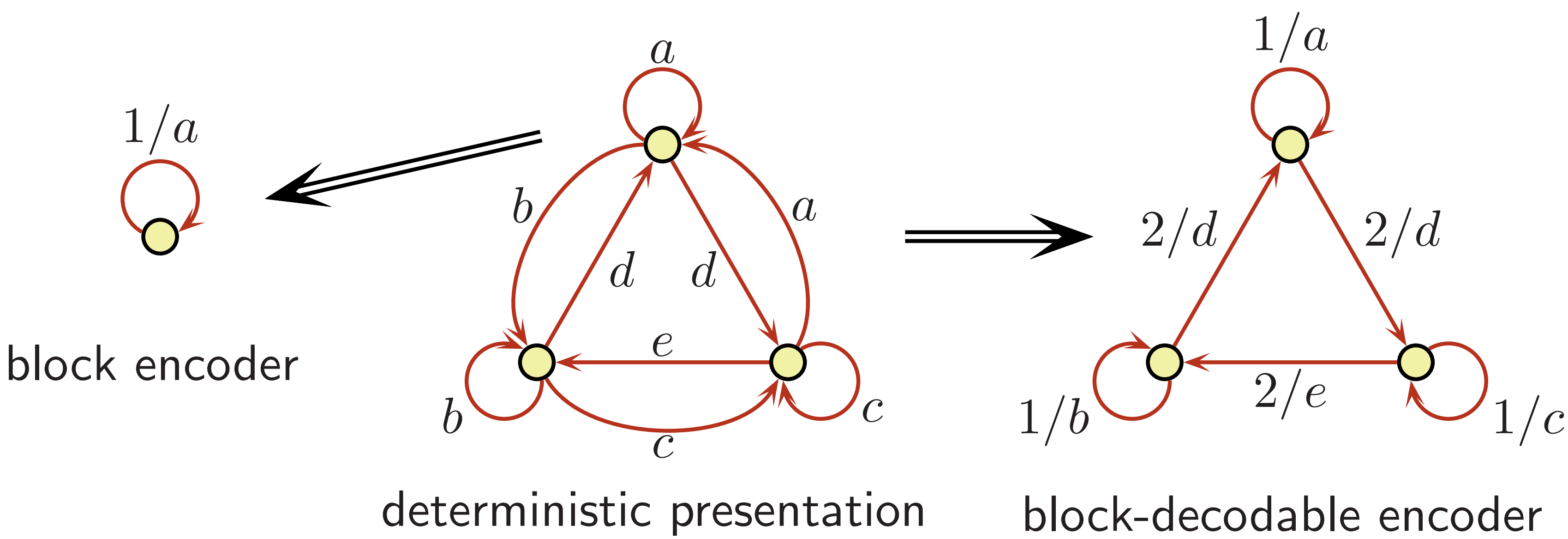
A **deterministic encoder** is a finite-state encoder whose output labeling is deterministic, i.e., at each state, all outgoing edges have distinct output labels

A **block encoder** is a finite-state encoder such that there is a 1-1 mapping between input labels and output labels

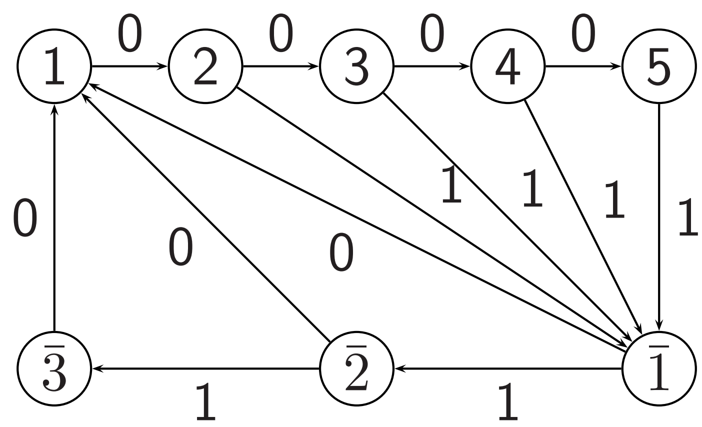
Basic facts about {blk, blkdec, det} encoders

Fact 1 Block $\Rightarrow$ Block-decodable $\Rightarrow$ Deterministic

Theorem 1 (**[Freiman and Wyner, 1964], [Franaszek, 1968], [Marcus et al., 1998]**) Let S be a constraint presented by a deterministic graph G. For each class of encoder $\mathcal{C} \in \{\mathsf{blk}, \mathsf{blkdec}, \mathsf{det}\}$, there exists an (S, n) encoder in class $\mathcal{C}$ if and only if there exists such an encoder which is a subgraph of G.



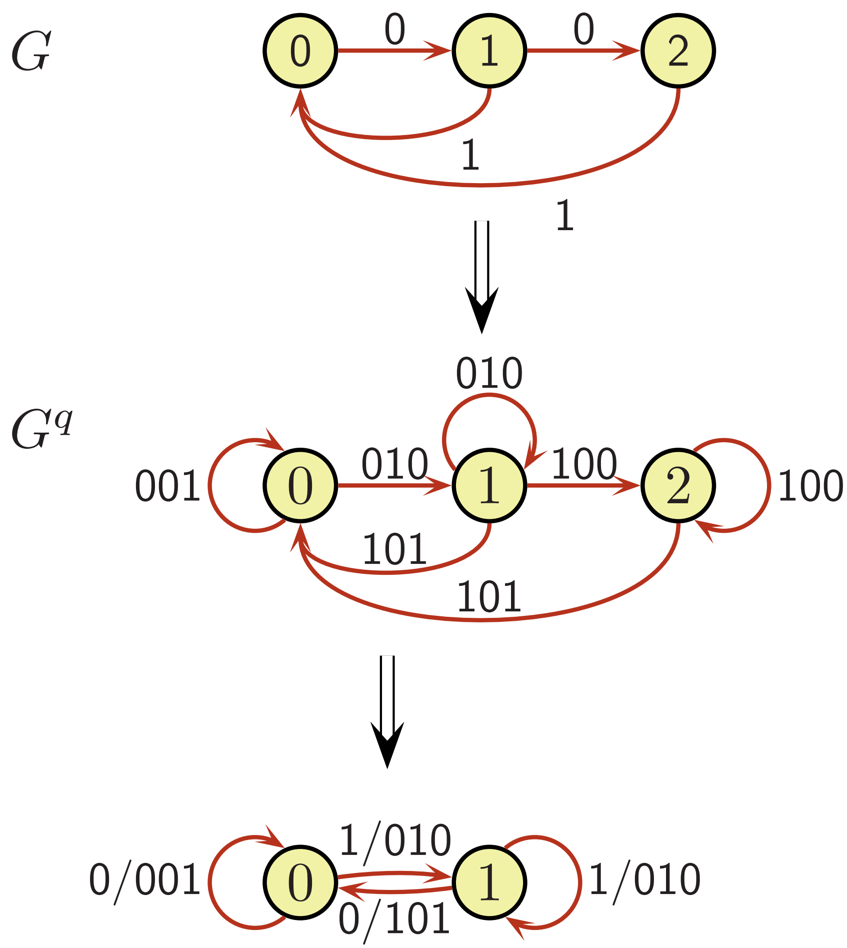
An example: the asymmetric RLL $(2, 5, 1, 3)$



The capacity is 0.7112

block length	optimal code rate			
	block	block-decodable (lower bound)	block-decodable (upper bound)	deterministic
6	0.4308	0.5283	0.6346	0.6346
10	0.5585	0.6524	0.6615	0.6714
15	0.6088	0.6774	0.6779	0.6844
20	0.6343	0.6862	0.6862	0.6910
30	0.6599	0.6946	0.6946	0.6978

Construction of $\{\mathrm{blk}, \mathrm{blkdec}, \mathrm{det}\}$ encoder



(S^q, n) encoder

(1) Start with a deterministic presentation G of the desired constraint S

(2) Compute the qth power of G, denoted G^q

(3) Choose a subgraph of G^q to be used as encoder

(4) Assign input labels

Set of principal states

Step (3) can be broken into two steps:

(a) First choose a set of states of G, called a **set of principal states**, to be used as encoder states.

(b) Then choose edges.

Prior work

Deterministic encoders:

Franaszek [Franaszek, 1968] gave a very efficient algorithm to choose an optimal set of principal states and a corresponding subgraph for the class of deterministic encoder.

Block encoders:

Given a set of principal states, Freiman and Wyner [Freiman and Wyner, 1964] presented an algorithm, based on generating functions, to find a optimal block encoder. They also gave a way to limit the choices of set of principal states that need to be considered. Marcus, Siegel, and Wolf [Marcus et al., 1992] further improved the efficiency.

Block-decodable encoders:

No general algorithm known. Must rely on heuristic and approximation.

For some classes of constraints, this problem coincides with the case of deterministic encoders [Franaszek, 1970, Chaichanavong and Marcus, 2003].

Complexity of determining the optimal code rate

S: constrained system with deterministic presentation G
n: integer

For each class $\mathcal{C}$ of encoder, consider complexity of three problems:

(1) Determining whether there exists (S, n) encoder in class $\mathcal{C}$ which is a subgraph of G
(2) Same as (1) but require that the set of principal states $P = V_G$
(3) $|V_G|$ fixed

The following table summarizes the results in [Franaszek, 1968, Ashley et al., 1996, Chaichanavong, 2003].

encoder class	subgraph encoder problem	$P = V_G$	$\lvert V_G \rvert$ fixed
deterministic	polynomial	polynomial	polynomial
block	NP-complete (polynomial for any fixed n)	polynomial	polynomial
block-decodable	NP-complete for fixed $n \geq 2$	NP-complete for fixed $n \geq 2$	polynomial

Some notations

S: a constrained system
G: a deterministic presentation of S
V_G: the set of states of G
$\mathcal{C} \in \{\text{blk}, \text{blkdec}, \text{det}\}$: a class of encoder

Define the following two quantities

$M_{\mathcal{C}}(q, P)$: maximum n such that there exists an (S^q, n) encoder in class $\mathcal{C}$ constructed from the set of principal states P

Thus the optimal code rate is

$$\max_{P \subseteq V_G} \frac{\log M_{\mathcal{C}}(q, P)}{q}$$

Note: $M_{\text{blk}}(q, P) \leq M_{\text{blkdec}}(q, P) \leq M_{\text{det}}(q, P)$

Deterministic Encoders

Computing $M_{\mathsf{det}}(q, P)$:

Let G be a deterministic graph with $V_G = \{a, b, c, d\}$; pick $P = \{a, b, d\}$

$$A^q = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 4 & 5 & \cancel{2} & 4 \\ 6 & 4 & \cancel{2} & 4 \\ \cancel{3} & \cancel{2} & \cancel{1} & \cancel{2} \\ 5 & 3 & \cancel{3} & 3 \end{bmatrix} \end{array} \begin{array}{ccc} & & \text{row sum} \\ a & \Rightarrow & 13 \\ b & \Rightarrow & 14 \\ c & & \\ d & \Rightarrow & 11 \end{array} \left.\vphantom{\begin{array}{c} 1\\1\\1\\1\\1 \end{array}}\right\} M_{\mathsf{det}}(q, P) = \min\{13, 14, 11\} = 11$$

In general,

$$M_{\mathsf{det}}(q, P) = \min_{u \in P} \sum_{v \in P} A^q_{u,v}$$

This is the same as multiplying A^q by the **characteristic vector** $\mathbf{x}$ of P:

$$A^q \mathbf{x} = \begin{bmatrix} 4 & 5 & 2 & 4 \\ 6 & 4 & 2 & 4 \\ 3 & 2 & 1 & 2 \\ 5 & 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \\ 7 \\ 11 \end{bmatrix} \geq \begin{bmatrix} 11 \\ 11 \\ 0 \\ 11 \end{bmatrix} = 11\mathbf{x}$$

Deterministic Encoders

Determining whether $M_{\text{det}}(q) \geq n$ is equivalent to determining whether there exists a 0-1 vector x, not all 0, such that $A^q x \geq nx$.

This can be solved by the Franaszek algorithm (Franaszek 1968):

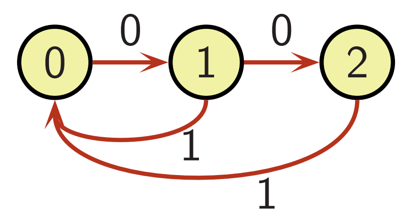
$$x^{(0)} = [1, \ldots, 1]^T$$

$$x^{(\ell+1)} = \min(x^{(\ell)}, \lfloor A^q x^{(\ell)}/n \rfloor)$$

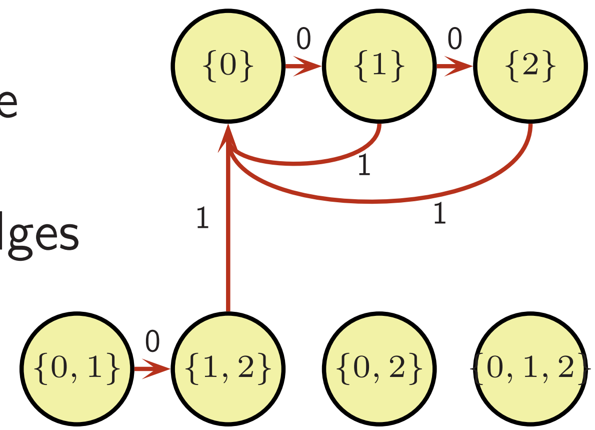
By varying n, we can determine $M_{\text{det}}(q)$

Block encoders

G: deterministic presentation of S



$\bar{G}$: labeled graph with $V_{\bar{G}} =$
set of all nonempty subsets of V_G
with an edge from U to V labeled by w if
1) For each $u \in U$, there is an outgoing edge
with label w - and -
2) V is the set of terminal states of these edges



$\bar{A}$: adjacency matrix of $\bar{G}$

Theorem 2

$$M_{\mathsf{blk}}(q,P) = \sum_{U \subseteq P} \bar{A}^q_{P,U}$$

Block-decodable encoder

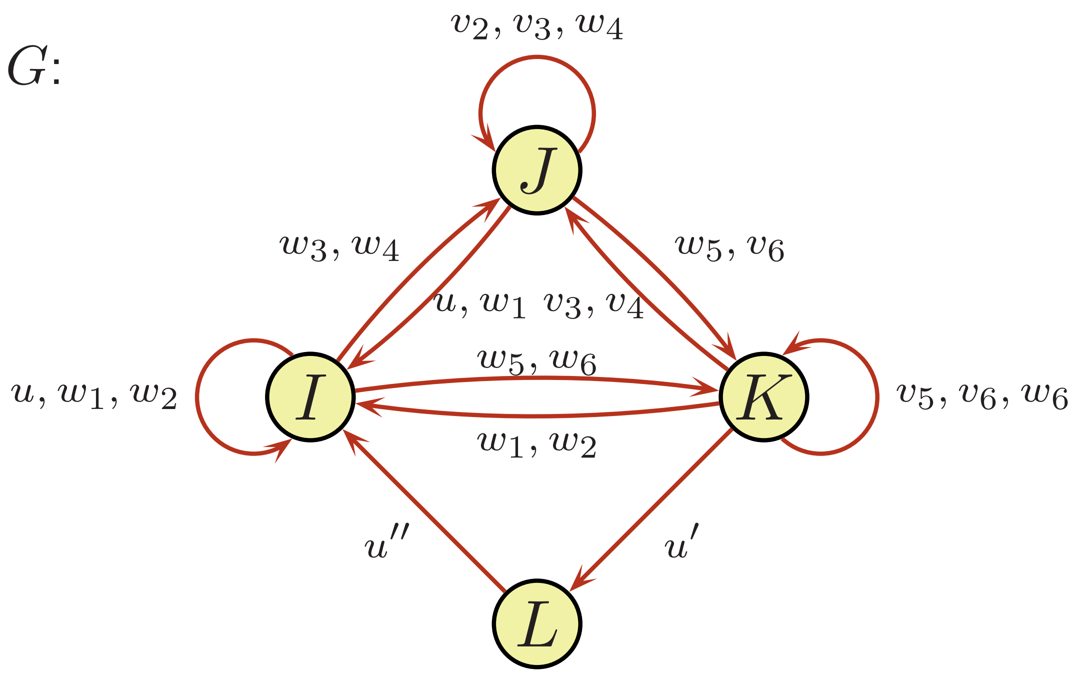
G: deterministic graph

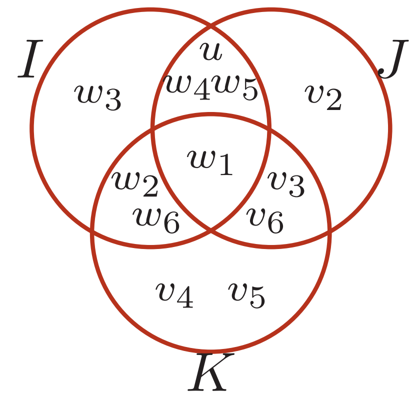
Find a block-decodable encoder that is a subgraph of G and has the same set of states as G

Input label assignment algorithm:

let $\Psi \leftarrow$ set of all codewords of G
set $\tau \leftarrow 1$
while (it is possible to choose a set of codewords $\psi = \{w_1, \ldots, w_k\} \subseteq \Psi$
  such that each state of G can generate at least one w_i)
  do assign input label τ to each word in ψ
    $\tau \leftarrow \tau + 1$
    $\Psi \leftarrow \Psi \setminus \psi$

An example of input label assignment





Partition of codewords for $P = \{I, J, K\}$

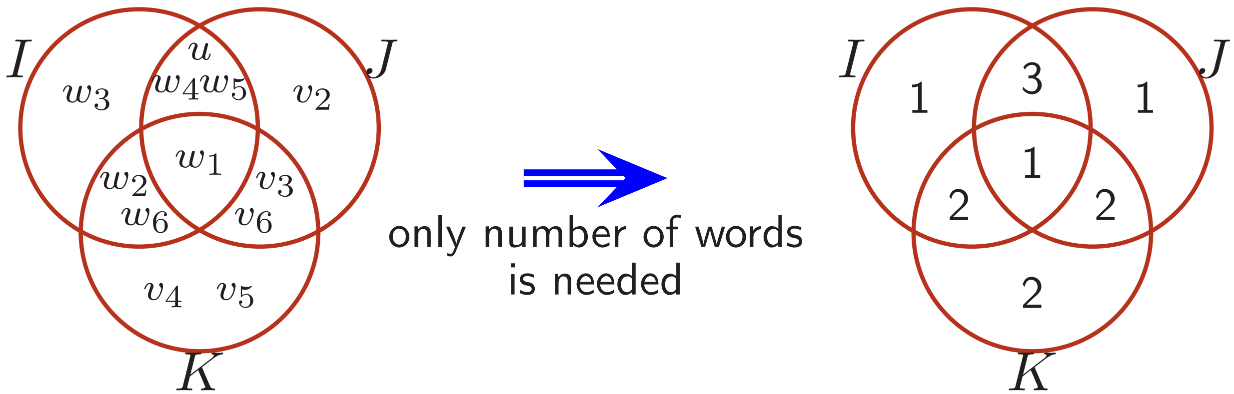
Pick $P = \{I, J, K\}$; Want to compute $M_{\text{blkdec}}(1, P)$

Partition codewords into classes according to initial states

For each input, choose a combination of regions such that their union is P

There are 8 such combinations; each combination is denoted by a 0-1 vector $\mathbf{z}$ of size $2^{|P|} - 1 = 7$, where $z_U = 1$ if U is in the combination and 0 otherwise

Input label assignment as an integer program



c_i: number of times that we choose combination i

maximize $c_1 + c_2 + \cdots + c_8$

subject to (1) $c_i \in \mathbb{Z}$, (2) $c_i \geq 0$,
(3)

$$\sum_{i=1}^{8} c_i \mathbf{z}_i = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \begin{array}{l} \Leftarrow \{I\} \\ \Leftarrow \{J\} \\ \Leftarrow \{K\} \\ \Leftarrow \{I, J\} \\ \Leftarrow \{I, K\} \\ \Leftarrow \{J, K\} \\ \Leftarrow \{I, J, K\} \end{array}$$

Input label assignment as an integer program

Rewrite the problem:

$$\text{maximize} \quad c_1 + c_2 + \cdots + c_t$$

subject to (1) $c_i \in \mathbb{Z}$,
(2) $c_i \geq 0$,
(3)

$$\sum_{i=1}^{t} c_i \mathbf{z}_i \leq \mathbf{x}(q, P)$$

t: number of combinations

$\mathbf{x}(q, P)$: sizes of regions, can be computed from $\bar{A}^q$

Indeed, $M_{\mathsf{blkdec}}(q, P) = \max c_1 + \cdots + c_t$

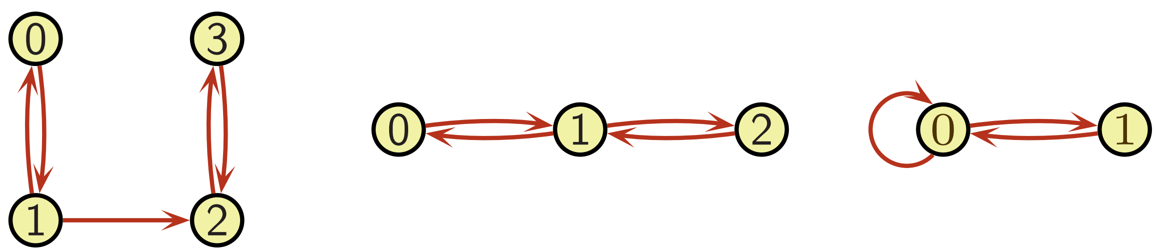
Remove condition (1) to get a linear programming problem. Solve the relaxed problem and round the solution to integers.

Irreducibility and Primitivity

Irreducible graph:
for any pair u, v of states, there is a path from u to v and v to u

Primitive graph: there exists an integer N such that
for all u and v, there are paths from u to v and v to u of length N

Fact 2 Primitive $\Rightarrow$ Irreducible



Reducible Irreducible Primitive

A constraint is **irreducible** if it has an irreducible presentation
A matrix is **irreducible** if it is the adjacency matrix of an irreducible graph

Primitive constraint and matrix are defined similarly

From now on, assume primitivity

Asymptotic analysis on the code rate

The capacity of a constraint S with a deterministic presentation G is

$$\mathrm{cap}(S) = \log \lambda,$$

where λ is the largest positive eigenvalue of A_G.

Theorem 3 ([Shannon, 1948]) Let S be a constrained system presented by a deterministic graph G. Let $P \subseteq V_G$. For any class $\mathcal{C}$ of encoder,

$$\lim_{q \to \infty} \frac{\log M_{\mathcal{C}}(q, P)}{q} = \mathrm{cap}(S).$$

Expect $M_{\mathcal{C}}(q, P)$ to grow as λ^q

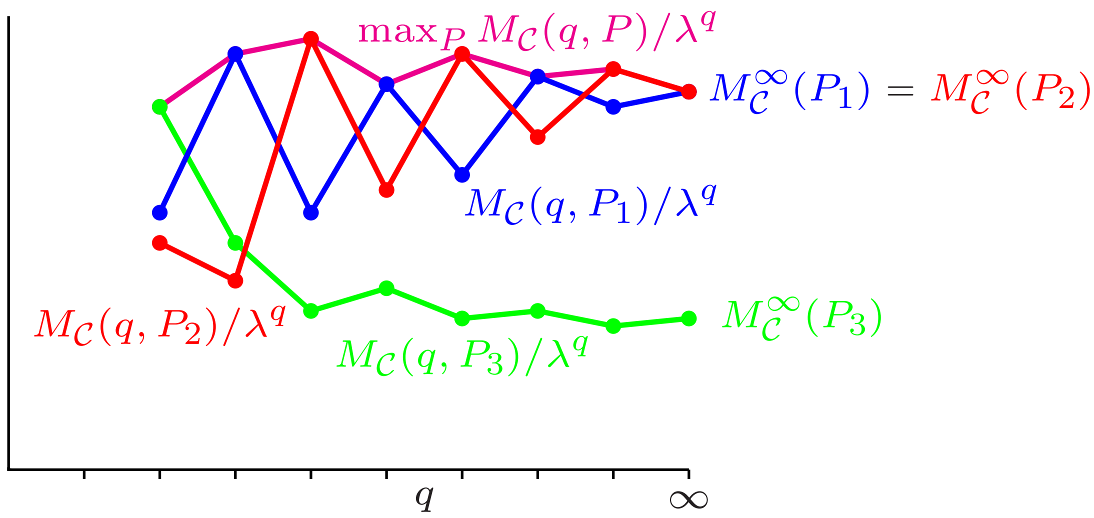
Asymptotic analysis on the code rate

Define the **asymptotic rate**

$$M_{\mathcal{C}}^{\infty}(P) = \lim_{q \to \infty} \frac{M_{\mathcal{C}}(q, P)}{\lambda^q}$$

A set of principal states that maximizes the asymptotic rate is called **asymptotically optimal.**

Proposition 1 For sufficiently large block length, every optimal set of principal states is asymptotically optimal.



Asymptotic Results for Deterministic Encoders

Perron-Frobenius Theory for primitive matrix A:

- A has a unique largest positive eigenvalue $\lambda = \lambda(A)$.
- The right ($\mathbf{r}$) and left ($\mathbf{l}$) eigenvectors associated with λ are positive.
- Suppose $\mathbf{r}$ and $\mathbf{l}$ are normalized so that $\mathbf{lr} = 1$, define $\Lambda = \mathbf{rl}$. Then

$$\lim_{q \to \infty} \frac{A^q}{\lambda^q} = \Lambda$$

Recall:

$$M_{\mathsf{det}}(q, P) = \min_{u \in P} \sum_{v \in P} A^q_{u,v}$$

From the Perron-Frobenius Theory, we have the following.

Theorem 4

$$M^{\infty}_{\mathsf{det}}(P) = \min_{u \in P} \sum_{v \in P} \Lambda_{u,v}$$

Asymptotic Results for Deterministic Encoders

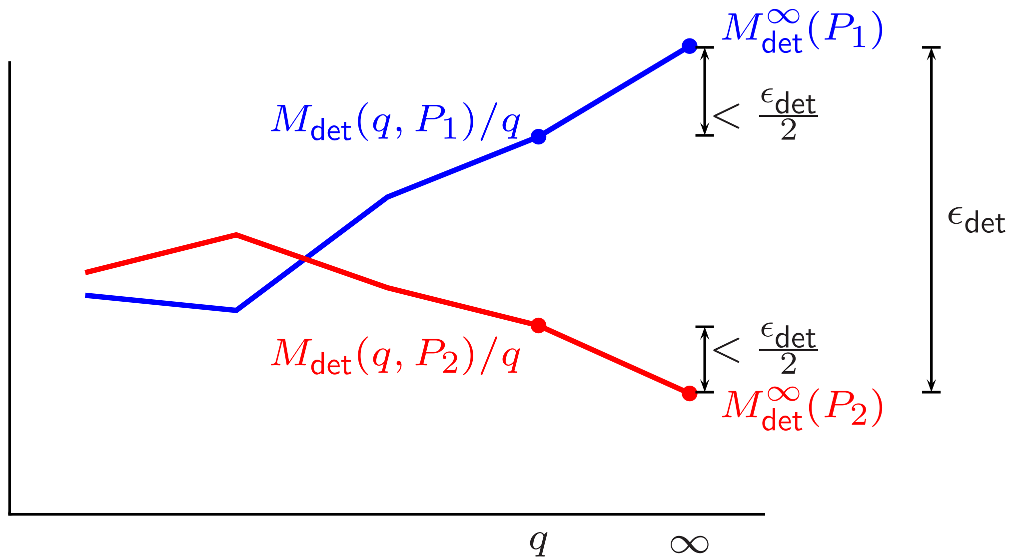
For each class $\mathcal{C}$, define

$$\epsilon_{\mathcal{C}} = \max_{P \subseteq V_G} M_{\mathcal{C}}^{\infty}(P) - \text{second largest } M_{\mathcal{C}}^{\infty}(P).$$

Theorem 5 If

$$\left\| \frac{A^q}{\lambda^q} - \Lambda \right\|_{\infty} < \frac{\epsilon_{\mathsf{det}}}{2}$$

then any optimal set of principal states at block length q is asymptotically optimal.



Asymptotic Results for Block Encoders

Recall

$$M_{\mathsf{blk}}(q, P) = \sum_{U \subseteq P} \bar{A}^q_{P,U}$$

Property of $\bar{A}$: λ is an eigenvalue of $\bar{A}$. Suppose the right ($\bar{\mathbf{r}}$) and left ($\bar{\mathbf{l}}$) eigenvectors of $\bar{A}$ associated with λ are normalized so that $\bar{\mathbf{l}}\bar{\mathbf{r}} = 1$, define $\bar{\Lambda} = \bar{\mathbf{r}}\bar{\mathbf{l}}$. Then

$$\lim_{q \to \infty} \frac{\bar{A}^q}{\lambda^q} = \bar{\Lambda}.$$

Theorem 6

$$M^{\infty}_{\mathsf{blk}}(P) = \bar{\mathbf{r}}_P \sum_{u \in P} \bar{\mathbf{l}}_{\{u\}}.$$

Theorem 7 If

$$\left\| \frac{\bar{A}^q}{\lambda^q} - \bar{\Lambda} \right\|_{\infty} < \frac{\epsilon_{\mathsf{blk}}}{2}$$

then any optimal set of principal states at block length q for block encoders is asymptotically optimal.

Asymptotic results for block-decodable encoders

Recall

$$M_{\mathsf{blkdec}}(q, P) = \max \quad c_1 + c_2 + \cdots + c_t$$

subject to (1) $c_i \in \mathbb{Z}$,
(2) $c_i \geq 0$,
(3)

$$\sum_{i=1}^{t} c_i \mathbf{z}_i \leq \mathbf{x}(q, P),$$

where $\mathbf{z}_i$ depends only on P and $\mathbf{x}(q, P)$ can be computed from $\bar{A}^q$.

Remove condition (1) to get a linear programming problem.

View the maximum of the objective function of the relaxed problem as a function $\mu(\mathbf{x}(q, P))$.

Lemma 1

$$M_{\mathsf{blkdec}}(q, P) \leq \mu(\mathbf{x}(q, P)) \leq M_{\mathsf{blkdec}}(q, P) + t$$

Asymptotic results for block-decodable encoders

Define $\mathbf{x}^\infty(P) = \lim_{q\to\infty} \frac{1}{\lambda^q}\mathbf{x}(q,P)$

From the convergence of $\frac{\bar{A}^q}{\lambda^q}$ to $\bar{\Lambda}$, we can show that $\mathbf{x}^\infty(P)$ exists and can be computed from $\bar{\Lambda}$

Theorem 8

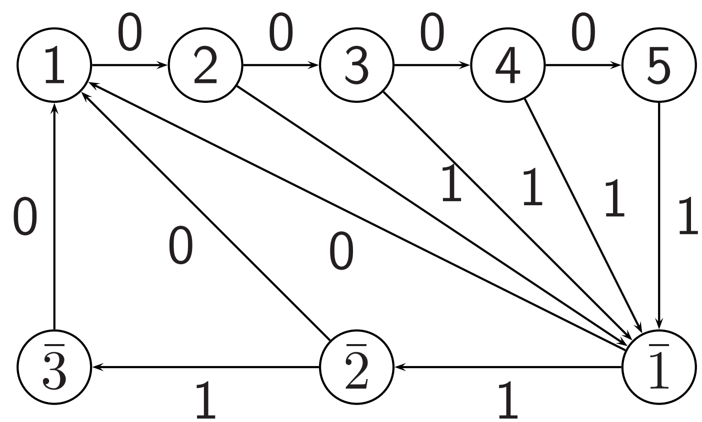
$$M^\infty_{\mathsf{blkdec}}(P) = \mu(\mathbf{x}^\infty(P)).$$

Let

$$\rho(G,q) = (2^{|V_G|}-1)\sum_{U,V}\left|\left(\frac{\bar{A}^q}{\lambda^q}\right)_{U,V} - \bar{\Lambda}_{U,V}\right| + \frac{\text{explicit constant}}{\lambda^q}$$

Theorem 9 If $\rho(G,q) < \frac{\epsilon_{\mathsf{blkdec}}}{2}$, then any optimal set of principal states at block length q for block-decodable encoders is asymptotically optimal.

An example: the asymmetric RLL $(2, 5, 1, 3)$



encoder class $\mathcal{C}$	optimal P	$\max_P M_{\mathcal{C}}^{\infty}(P)$	bound on q	known stable q
deterministic	$\{1, 2, 3, 4, \bar{1}, \bar{2}\}$	0.7563	17	1
block	$\{2, 3, \bar{1}\}$,$\{2, \bar{1}, \bar{2}\}$	0.3445	21	6
block decodable	$\{1, 2, 3, \bar{1}, \bar{2}\}$	0.7076	54	12

Conclusion

- We have investigated three classes of encoders: block, block-decodable, and deterministic.
- Finding an optimal deterministic encoder is easiest. Finding an optimal block-decodable encoder is most complex.
- Given a set of principal states, finding an optimal block-decodable encoder can be formulated as an integer program. The integer program can be relaxed to find a good bound on the rate of an optimal block-decodable encoder.
- We have established a relationship between optimal sets of principal states at finite and asymptotically large block length. The asymptotic results hold for all $q \geq q_0$ for some *small* q_0.
- Integer program can be relaxed to find the asymptotic rate of an optimal block-decodable encoder.
- *Coming Attraction:* Integer program can be adapted to bounded-delay-encodable block-decodable encoders (Chaichanavong, ISIT04)

References

[Ashley et al., 1996] Ashley, J. J., Karabed, R., and Siegel, P. H. (1996). Complexity and sliding-block decodability. *IEEE Trans. Inform. Theory*, 42(6):1925–1947.

[Chaichanavong, 2003] Chaichanavong, P. (2003). *Block-type-decodable encoders for constrained systems*. PhD thesis, Stanford University.

[Chaichanavong and Marcus, 2003] Chaichanavong, P. and Marcus, B. H. (2003). Optimal block-type-decodable encoders for constrained systems. *IEEE Trans. Inform. Theory*, 49(5):1231–1250.

[Franaszek, 1968] Franaszek, P. A. (1968). Sequence-state coding for digital transmission. *Bell Sys. Tech. J.*, 47:143–155.

[Franaszek, 1970] Franaszek, P. A. (1970). Sequence-state methods for run-length-limited coding. *IBM J. Res. Develop.*, 14(4):376–383.

[Freiman and Wyner, 1964] Freiman, C. V. and Wyner, A. D. (1964). Optimum block codes for noiseless input restricted channels. *Inform.*

Contr., 7:398–415.

[Marcus et al., 1998] Marcus, B. H., Roth, R. M., and Siegel, P. H. (1998). *Handbook of Coding Theory*, chapter 20. Elsevier, Amsterdam, The Netherlands.

[Marcus et al., 1992] Marcus, B. H., Siegel, P. H., and Wolf, J. K. (1992). Finite-state modulation codes for data storage. *IEEE J. Select. Areas Commun.*, 10:5–37.

[Shannon, 1948] Shannon, C. E. (1948). The mathematical theory of communication. *Bell Sys. Tech. J.*, 27:379–423.