Constrained Systems with Unconstrained Positions: Graph Constructions and Tradeoff Functions (Part II)

Lei Poo

Stanford University

Panu Chaichanavong

Center for Magnetic Recording Research, UCSD

Brian Marcus

University of British Columbia

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Objectives

Goals:

- Given an insertion rate, find the maximum possible code rate.
- Given an insertion rate, find a set of unconstrained positions that (nearly) achieve the maximum code rate.

Outline:

- Tradeoff functions
- More properties of $\hat{G}$
- Properties of the tradeoff functions
- Bounds for the tradeoff functions

Tradeoff Functions

Let $I \subseteq \mathbb{N}$ be a set of unconstrained positions.

$M(q, I)$: number of words w of length q in $\hat{S}$ such that $w_i = \Box$ if and only if $i \in I$.

Let $\rho \in [0, 1]$ be an insertion rate.

$\mathcal{I}(\rho)$: set of all sequences (I_q) such that $I_q \subseteq \{1, \ldots, q\}$ and $|I_q|/q \to \rho$.

Example: $\rho = 1/3$. $I_q = \{3n \,:\, n \geq 1,\ 3n \leq q\}$.

$$\begin{array}{ccccccc} I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & \cdots \\ \emptyset & \emptyset & \{3\} & \{3\} & \{3\} & \{3, 6\} & \cdots \end{array}$$

(I_q) corresponds to $\_\ \_\ \Box\ \_\ \_\ \Box\ \_\ \_\ \Box\ \ldots$

$(I_q) \in \mathcal{I}(1/3)$.

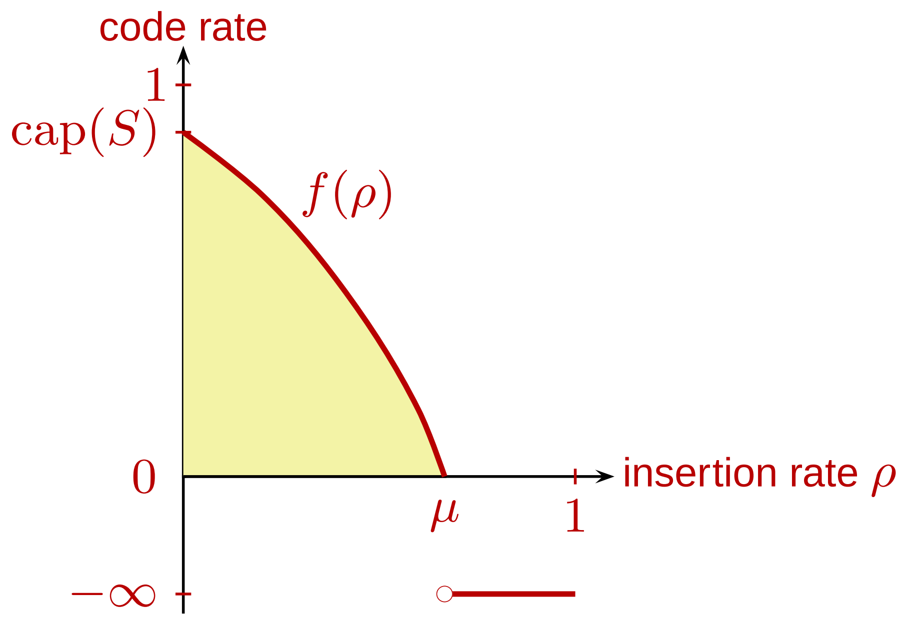
Tradeoff Functions

Tradeoff function:

$$f(\rho) = \sup_{(I_q) \in \mathcal{I}(\rho)} \limsup_{q \to \infty} \frac{\log M(q, I_q)}{q}.$$

Maximum insertion rate:

$$\mu = \sup_{f(\rho) > 0} \rho.$$

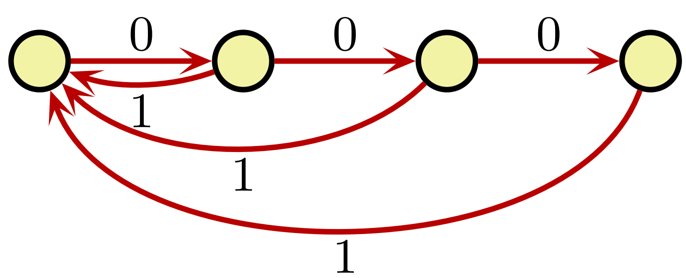


Finite-Type Constraints

A graph G has **finite memory** if there exists m so that all paths of length m with the same label end at the same state.

S is **finite-type** if it has a presentation with finite memory.

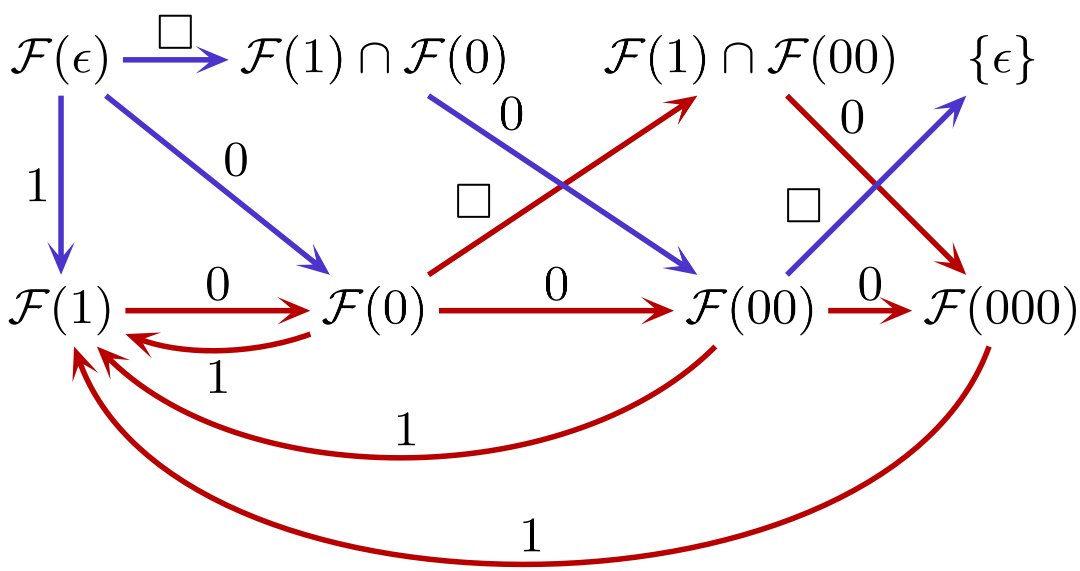
Example: RLL(1,3)



Tradeoff Function f for Finite-Type Constraints

Define G' to be the irreducible component of $\hat{G}$ that contains H.

Example: RLL(1,3)



Proposition 1: If S is finite-type, then G' is the only non-trivial irreducible component of $\hat{G}$.

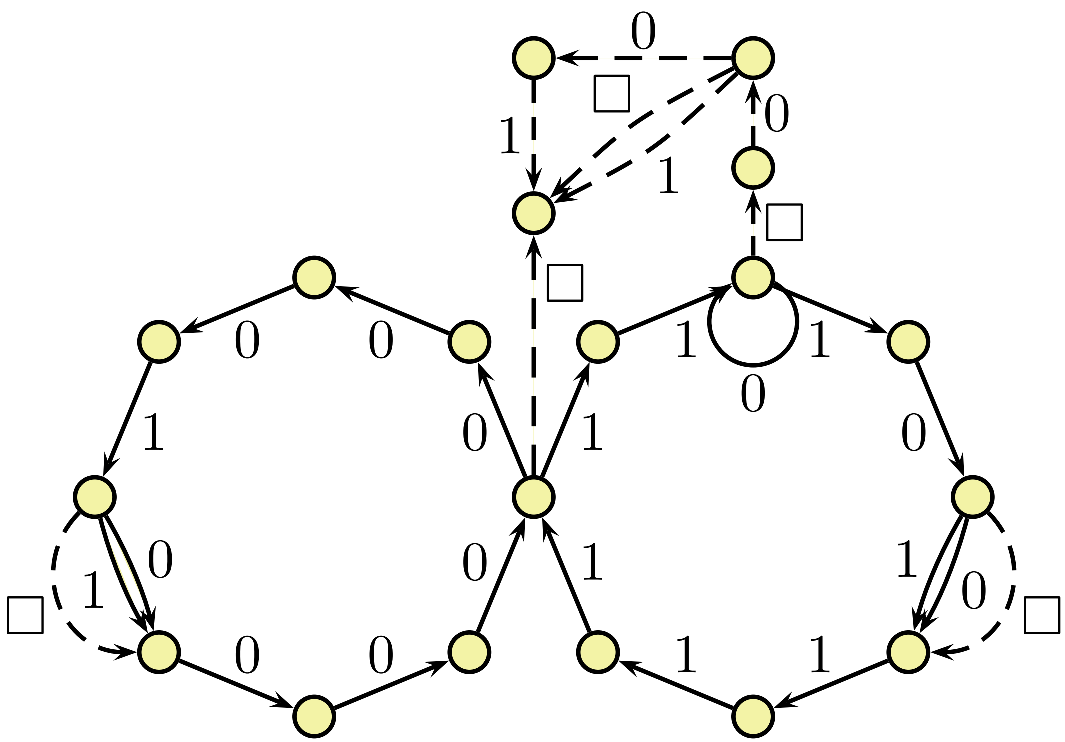
For any labeled graph G over $\{0, 1, \square\}$, denote the tradeoff function for G by f_G.

Corollary 2: Let S be a finite-type constrained system. Then $f(\rho) = f_{G'}(\rho)$.

An Example when $f(\mu) \neq 0$

G: graph below without dashed edges. Let $S = S(G)$. S is primitive since G is irreducible and aperiodic. The graph G has memory 7, so S is finite-type.

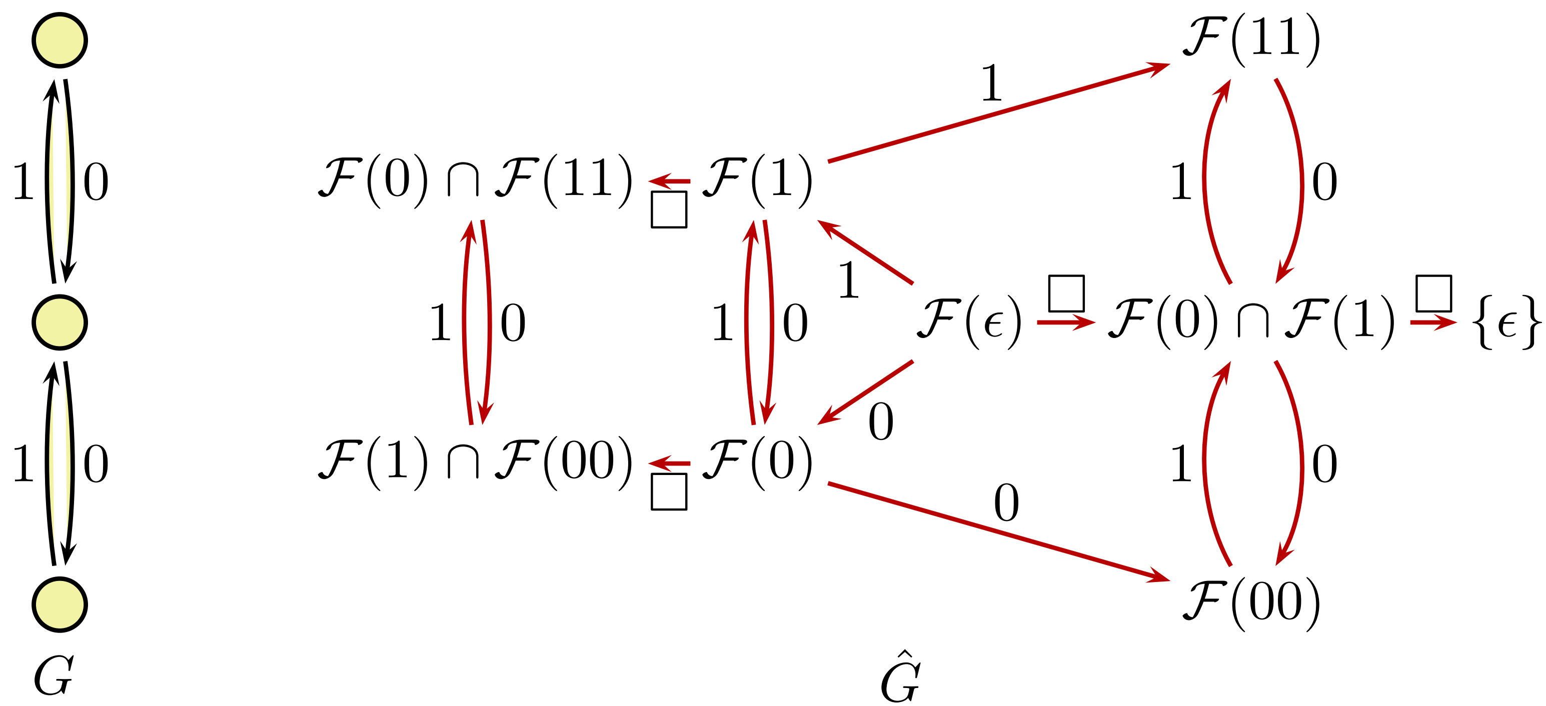
G': G + dashed edges. Then $f(\mu) = f(1/8) = 1/8 > 0$.



What about other Constraints?

2-Charge Constraint Example:

- The 2-charge constraint is non-finite-type.
- The capacity for this constraint is 0.5, and so $f(0) = 0.5$
- There is no word in $\hat{S}$ that has more than two $\square$. Therefore $f(\rho) = -\infty$ when $\rho > 0$



A More General Approach

So far, we considered

- Finite-Type Constraints,
- Specific Examples of Finite-Type and non-Finite-Type Constraints.

Need a more general approach in characterizing f for the case of n components in $\hat{G}$.

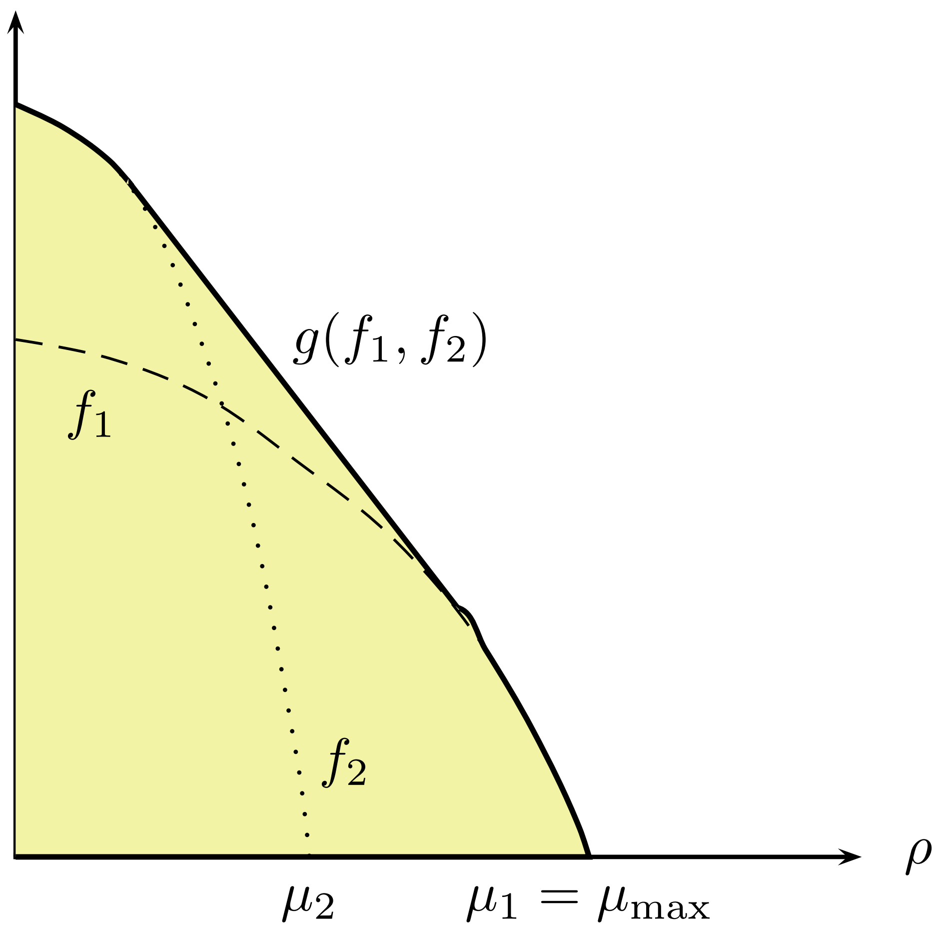
Intuition:

- f is non-increasing
- can apply timesharing concepts to achieve better code rates

Concave Function $g(f_1, \ldots, f_n)$

Let $f_i : [0,1] \to [-\infty, \infty)$, $i \in \{1, \ldots, n\}$, be functions.

Define $g(f_1, \ldots, f_n) : [0,1] \to [-\infty, \infty]$ to be the smallest concave function such that $g(f_1, \ldots, f_n)(\rho) \geq f_i(\rho)$ for all $i \in \{1, \ldots, n\}$ and $\rho \in [0,1]$.



Caratheodory's Theorem

To express $g(f_1, \ldots, f_n)(\rho)$ in terms of f_i, we apply a special case of **Caratheodory's Theorem** from convex analysis.

Proposition 3 [Rockafellar, 1970, Corollary 17.1.3]: Let $\{f_i \ : \ i \in I\}$ be an arbitrary collection of functions on $\mathbb{R}$, and let f be the convex hull of the collection. Then for any x,

$$f(x) = \inf \left\{ \sum_{1 \leq k \leq 2} \lambda_k f_{i_k}(x_k) \ : \ \sum_{1 \leq k \leq 2} \lambda_k x_k = x, \ i_k \in I \right\}.$$

where the infimum is taken over all expressions of x as a convex combination in which at most 2 of the coefficients λ_i are non-zero.

Applying Caratheodory's Theorem

Lemma 4: Let $f_i : [0,1] \to [-\infty, \infty)$, $i \in \{1, \ldots, n\}$, be functions. For any $\rho \in [0,1]$,

$$g(f_1, \ldots, f_n)(\rho) = \sup \; \theta f_i(x) + (1-\theta) f_j(y),$$

where the supremum is subject to

- $\theta, x, y \in [0,1]$,
- $i, j \in \{1, \ldots, n\}$,
- $\theta x + (1-\theta) y = \rho$.

Lemma 5: Let $f_i : [0,1] \to [-\infty, \infty)$, $i \in \{1, \ldots, n\}$, be functions. For any $\rho \in [0,1]$,

$$g(f_1, \ldots, f_n)(\rho) = \max_{i,j \in \{1,\ldots,n\}} g(f_i, f_j)(\rho).$$

Determining f in the Case of Many Components

- Let $G_1, \ldots, G_n$ be the irreducible components of $\hat{G}$.
- Let $P = \{(i,j) \in \{1,\ldots,n\}^2 : G_i \to G_j\}$.
- Denote f_{G_i} by f_i. For a fixed $0 \le \rho \le 1$,

$$g(f_1,\ldots,f_n) \overset{\text{(a)}}{\le} \max_{i,j\in\{1,\ldots,n\}} g(f_i,f_j) \overset{(b)}{\le} \max_{(i,j)\in P} g(f_i,f_j) \overset{(c)}{\le} f \overset{(d)}{\le} g(f_1,\ldots,f_n)$$

(a) Lemma 5, a consequence of Caratheodory's Theorem.

(b) Lemma 7, a property of $\hat{G}$.

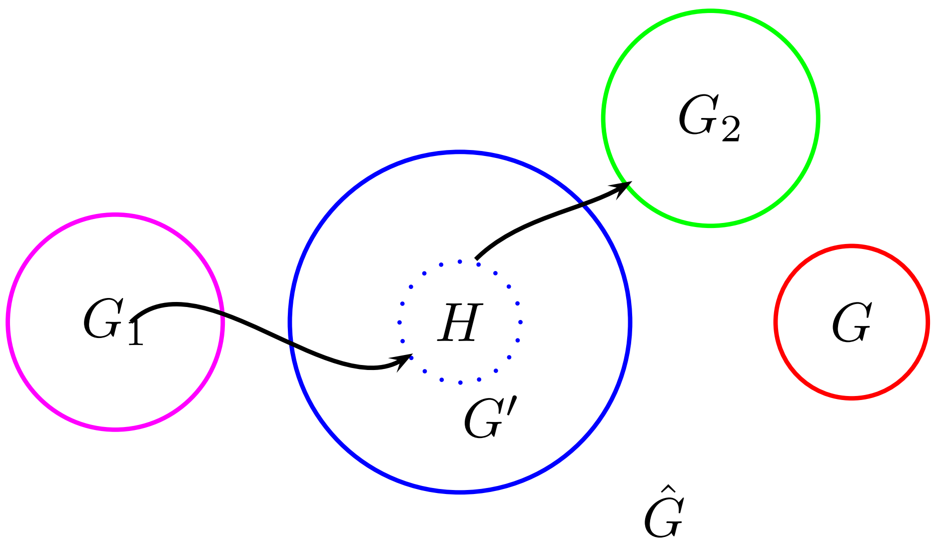
(c) Lemma 8, timesharing between $G_i \to G_j$, $i,j \in P$.

(d) Lemma 9

Inequality (b): Using a property of $\hat{G}$

Proposition 6: Let S be an irreducible constraint. Let G be an irreducible component of $\hat{G}$. There exist irreducible components G_1 and G_2 of $\hat{G}$ such that

- G_1 can reach H,
- H can reach G_2,
- $S(G) \subseteq S(G_1)$,
- $S(G) \subseteq S(G_2)$.



Proving Inequality (b)

Lemma 7: Let S be an irreducible constraint. Then

$$\max_{i,j \in \{1,\ldots,n\}} g(f_i, f_j)(\rho) \leq \max_{(i,j) \in P} g(f_i, f_j)(\rho).$$

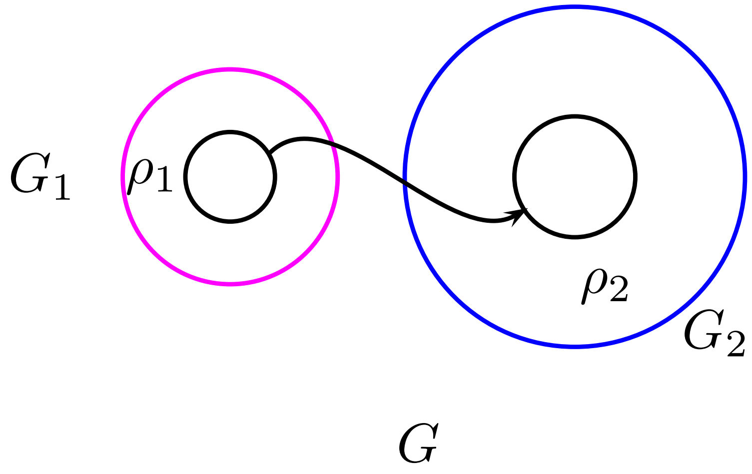
Proof: Let G_i and G_j be irreducible components of $\hat{G}$. By Proposition 6, there exist irreducible components $G_{i'}$ and $G_{j'}$ such that $G_{i'}$ can reach H, H can reach $G_{j'}$, $S(G_i) \subseteq S(G_{i'})$, and $S(G_j) \subseteq S(G_{j'})$. Thus $(i', j') \in P$ and $g(f_i, f_j)(\rho) \leq g(f_{i'}, f_{j'})(\rho)$. ■

Inequality (c): Timesharing between Components

Lemma 8 [Timesharing]: Let G be a graph over alphabet $\{0, 1, \square\}$. Let G_1 and G_2 be irreducible components of G such that G_2 can be reached from G_1. Then

$$f_G(\rho) \geq g(f_1, f_2)(\rho).$$

Proof idea: Suppose that $\rho_1 < \rho < \rho_2$. We concatenate the sequences in G_1 with insertion rate ρ_1 and the sequences in G_2 with insertion rate ρ_2 to obtain sequences with insertion rate ρ. Then we show that $f_G(\rho)$ must be at least the weighted average of $f_1(\rho_1)$ and $f_2(\rho_2)$.



Inequality (d)

Lemma 9: Let $G_1, \ldots, G_n$ be the irreducible components of $\hat{G}$. Then

$$f(\rho) \leq g(f_1, \ldots, f_n)(\rho).$$

Proof idea: Show for a given ρ, the existence of a chain of components $G_1, \ldots, G_c$ such that

(1) $G_1 \to G_2 \to \cdots \to G_c$

(2) There exists θ_i, ρ_i, for $i = 1, \ldots, c$ such that

2(i) $\sum_{i=1}^{c} \theta_i \rho_i = \rho$,

2(ii) $\sum_{i=1}^{c} \theta_i f_i(\rho_i) \geq f(\rho)$.

Then apply $\sum_{i=1}^{c} \theta_i f_i(\rho_i) \leq g(f_1, \ldots, f_c)(\rho) \leq g(f_1, \ldots, f_n)(\rho)$.

Main Results

From inequalities (a) to (d),

$$f = g(f_1, \ldots, f_n).$$

Theorem 10: Let S be an irreducible constrained system. Let G' be the irreducible component of $\hat{G}$ that contains H. Let $G_1, \ldots, G_k = G'$ be the irreducible components of $\hat{G}$ that can reach H. Let $G' = G_k, \ldots, G_m$ be the irreducible components of $\hat{G}$ that can be reached from H. Then

$$f(\rho) = g(f_1, \ldots, f_k)(\rho) = g(f_k, \ldots, f_m)(\rho).$$

- Important computationally as it is easier to construct the set of components reachable from H than the entire graph $\hat{G}$.

Concavity and Continuity of f

Let S be a constrained system.

Proposition 11: f is **non-increasing** on $[0, 1]$.

Proposition 12: f is **left-continuous** on $[0, \mu]$.

Corollary 13: The trade-off function f for an irreducible constraint S is **concave**. The restriction of f to the domain $[0, \mu]$ is **continuous**.

Computing f exactly?

Problems:

- Still do not know how to compute f exactly for a given constraint.
- Is there an algorithm that computes f exactly from $\hat{G}$?

Bounds for f

- For $0 \leq \rho \leq \mu$,

$$f(\rho) \leq \mathrm{cap}(S) - \rho.$$

- Greedy Lower Bound
- Dynamic Programming Lower Bound (DPLB)
- Approximate Dynamic Programming Upper Bound (Appox. DPUP)
- For constraints with more structure, it is possible to construct lower bounds by considering specific parity insertion schemes, e.g. Bit-stuffing for MTR constraints.
- Take the convex hull of all the lower bounds to obtain a better lower bound.

Bit-Stuffing Lower Bound for MTR(j,k)

Bit-stuffing for MTR(j,k): WLOG that $j \le k$. Let $b \le \min(j,k) - 1$. Begin with a string s that satisfies the MTR$(j-b, k-b)$ constraint. Subdivide s into intervals of length $k-b+1$. In between each of these intervals, insert a string of b ones. The resulting string satisfies MTR(j,k) and has parity insertion rate $\frac{b}{k+1}$.

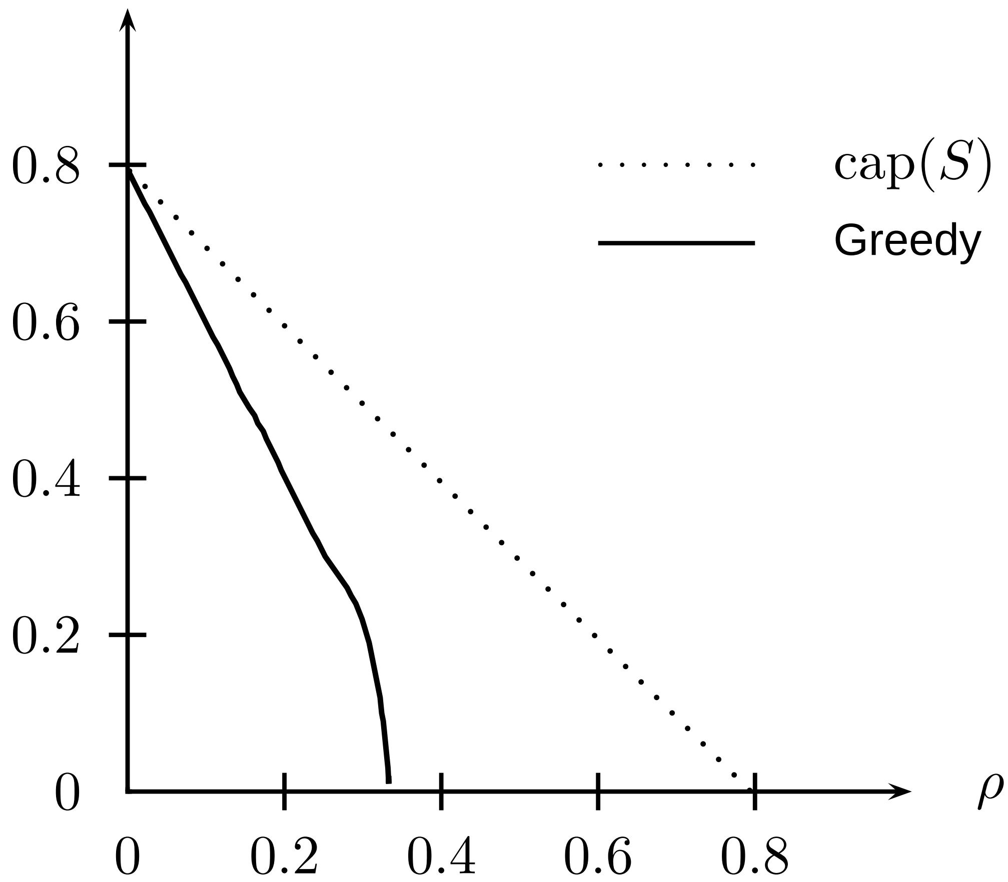
The piecewise-linear curve connecting the following $j+1$ points:

- $(0, \log \lambda_{j,k})$
- $(\frac{1}{k+1}, \frac{k}{k+1} \log \lambda_{j-1,k-1})$
- $(\frac{2}{k+1}, \frac{k-1}{k+1} \log \lambda_{j-2,k-2})$
- $\cdots$
- $(\frac{j-1}{k+1}, \frac{k-j+2}{k+1} \log \lambda_{1,k-(j-1)})$
- $(\mu, 0)$,

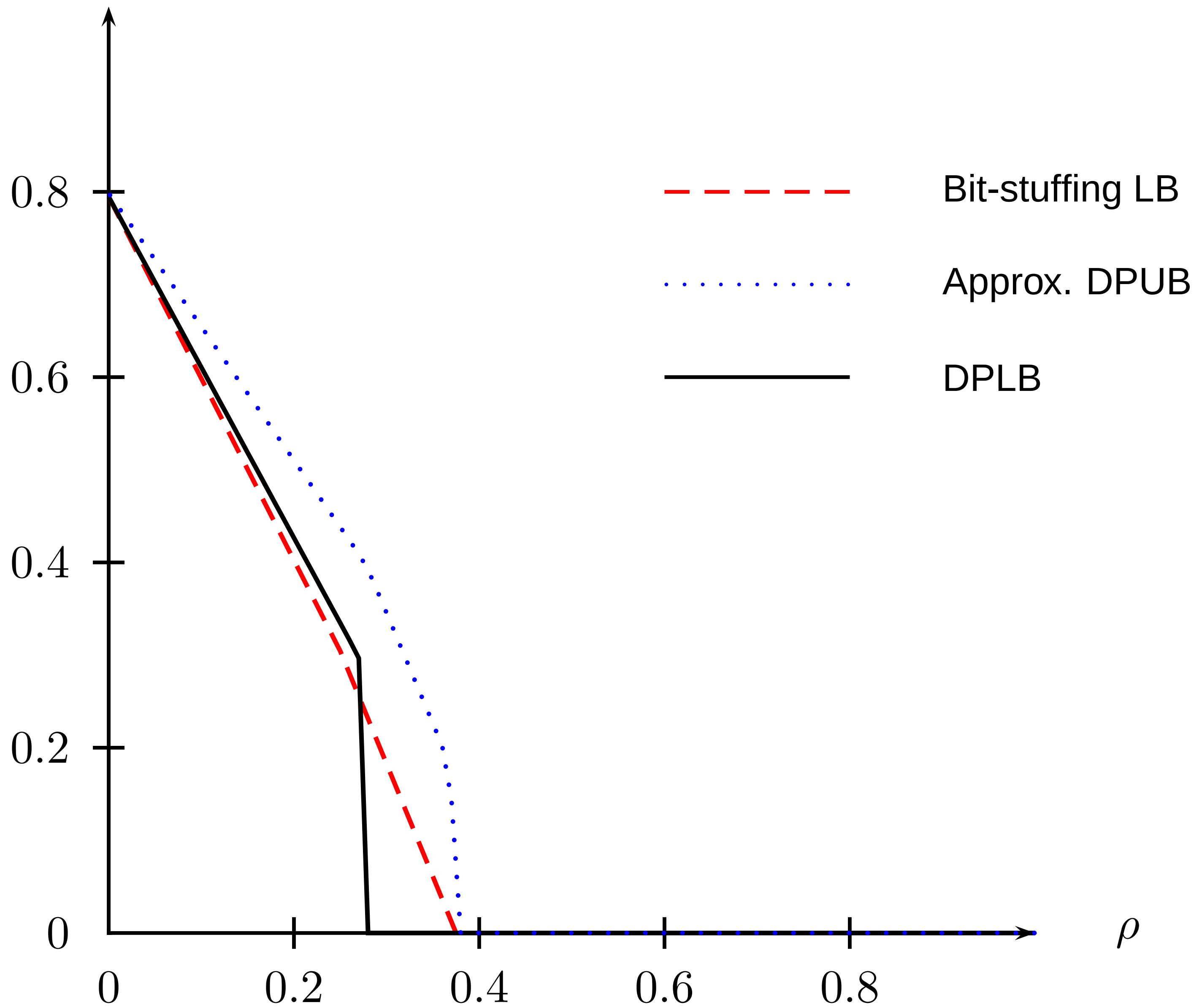
is a lower bound to $f_{\mathrm{MTR}(j,k)}$.

Example using MTR$(2,3)$

Let S be MTR$(2,3)$ constraint. Then $\mathrm{cap}(S) = 0.7947$, $\mu = 0.3750$. Take period to be 1000.

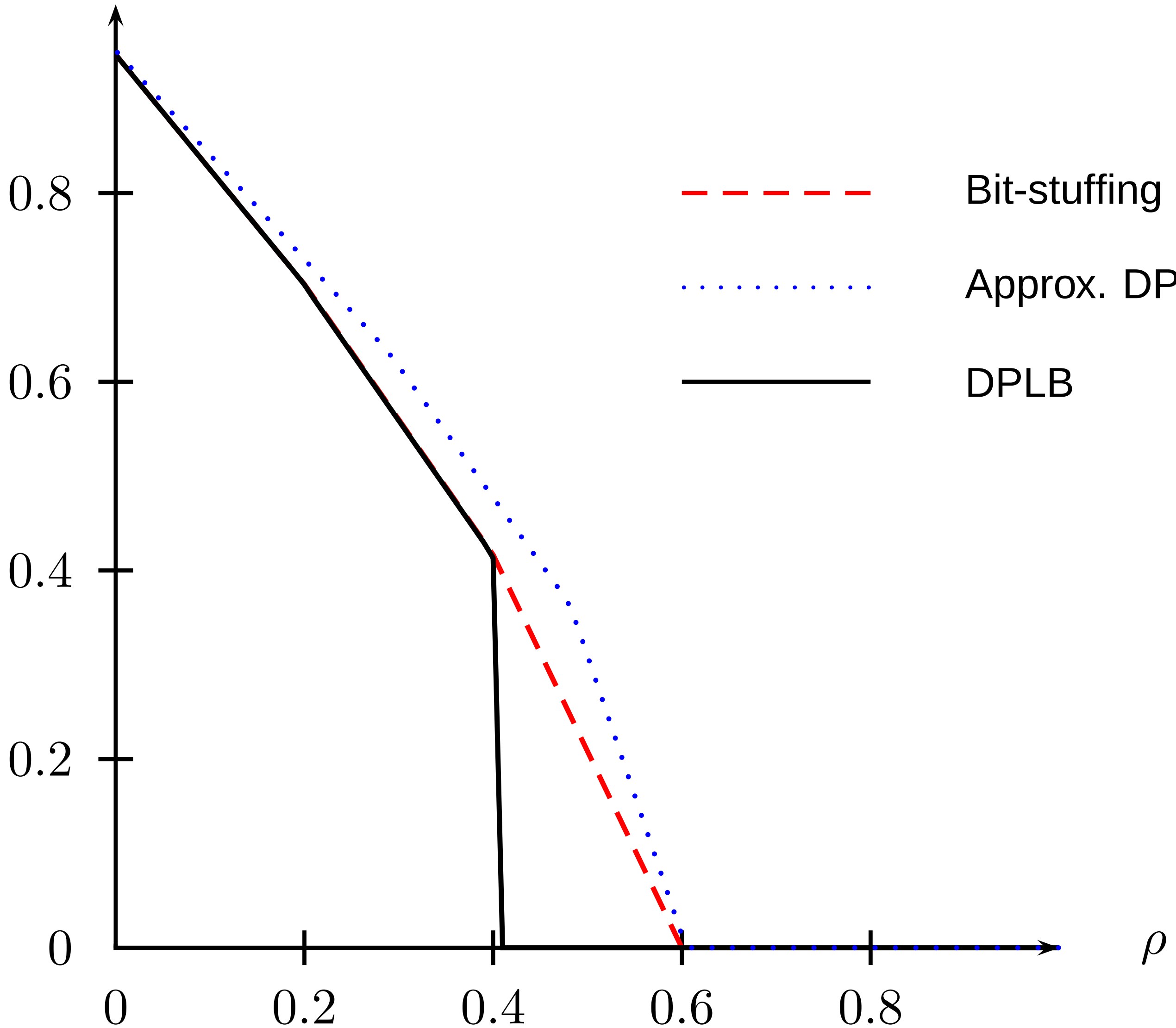


More bounds for MTR$(2, 3)$



Example using MTR$(4,4)$

Let S be MTR$(4,4)$ constraint. Then $\mathrm{cap}(S) = 0.9468$, $\mu = 0.6$.



Conclusions

- Constrained systems with unconstrained positions
- Introduce a constrained system $\hat{S}$ and a presentation $\hat{G}$ with unconstrained symbol
- Define tradeoff function and maximum insertion rate
- Introduced the notion of timesharing between components of $\hat{G}$
- Established using results from convex analysis that for irreducible constraints, f is equal to the concave hull of the code rate of all components in $\hat{G}$.
- In particular, we showed a stronger result that f is determined by components reachable from H.
- Determined that f is concave and continuous for an irreducible constraint.
- Showed some upper and lower bounds on f.

References

[de Souza et al., 2002] de Souza, J. C., Marcus, B. H., New, R., and Wilson, B. A. (2002). Constrained systems with unconstrained positions. *IEEE Trans. Inform. Theory*, 48(4):866–879.

[Rockafellar, 1970] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press, Princeton.