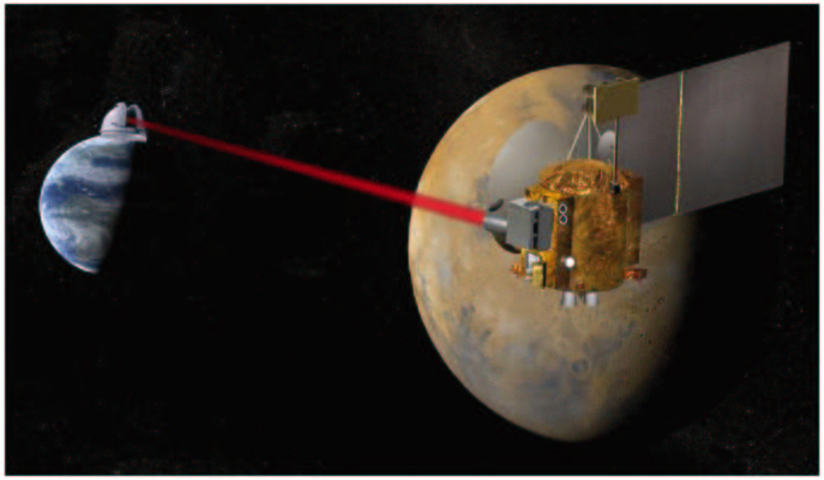
Modulation codes for the deep-space optical channel



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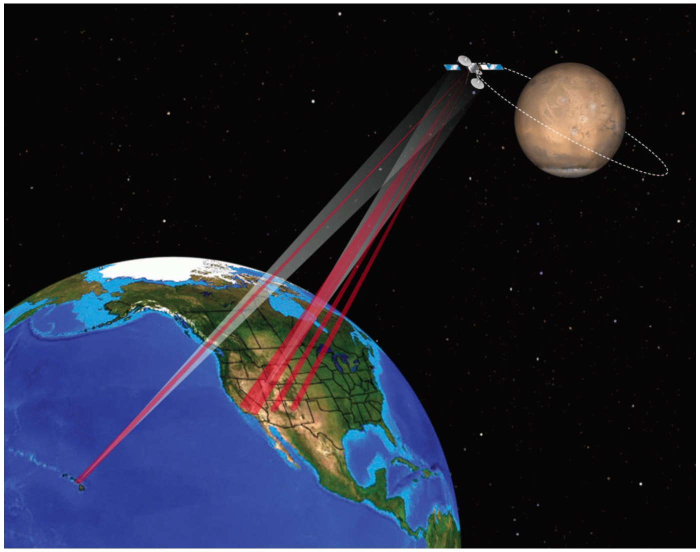
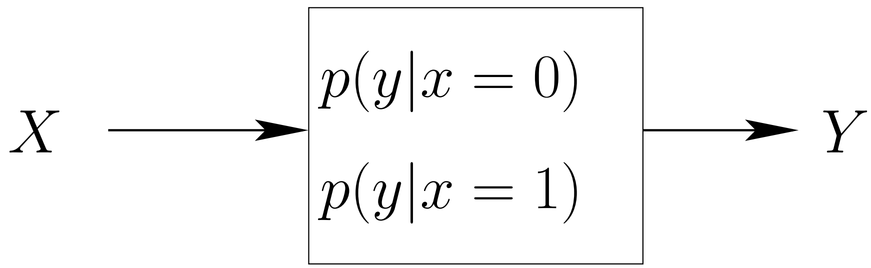
The deep-space optical channel

- Mars Telesat, scheduled to launch in 2009
- $5W$, 10–100 Mbps optical link demonstration
- 100W, 1.1 Mbps X-band
- 35W, 1.5 Mbps Ka-band



Deep-space optical communications channel	
Constraints	non-coherent, direct detection
	T_s = slot duration (pulse-width) $\geq$ 2 ns
	P_{av} = average signal photons/slot
	P_{pk} = maximum signal photons/pulse
Model	Memoryless Poisson

Poisson channel



Deep space optical channel modeled as binary-input, memoryless, Poisson.

$$p_0(k) = p(y = k|x = 0) = \frac{n_b^k e^{-n_b}}{k!}$$

$$p_1(k) = p(y = k|x = 1) = \frac{(n_b + n_s)^k e^{-(n_b+n_s)}}{k!}$$

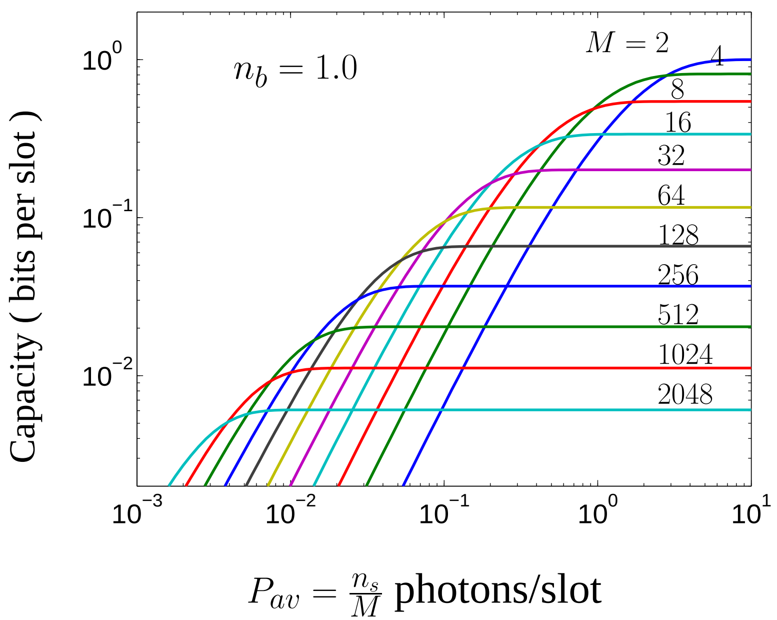
$$P(x = 1) = \frac{1}{M} = \text{ duty cycle (mean pulses per slot)}$$

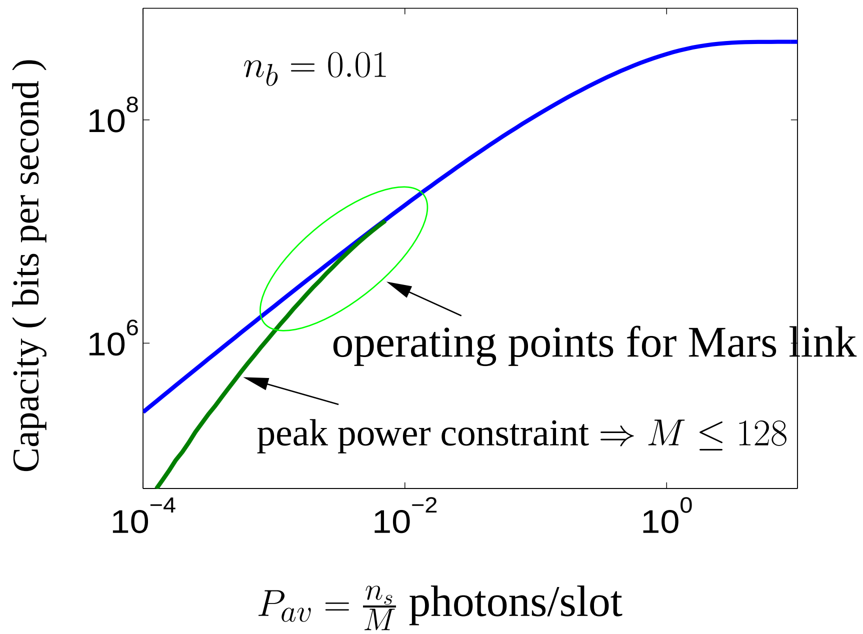
Peak power	$n_s \leq P_{pk}$ photons/pulse
Average power	$n_s/M \leq P_{av}$ photons/slot
	$\Rightarrow n_s \leq \min\{MP_{av}, P_{pk}\}$

Poisson channel

Capacity parameterized by P_{av}, optimized over M.

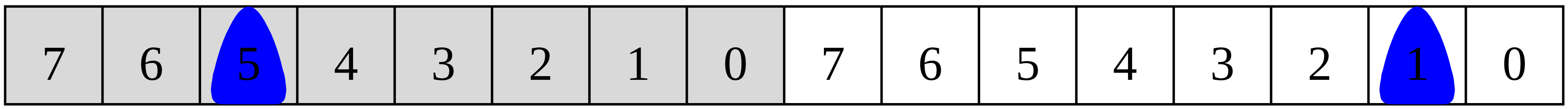
$$C(M) = \frac{1}{M} E_{Y|1} \log \frac{p_1(Y)}{p(Y)} + \frac{M-1}{M} E_{Y|0} \log \frac{p_0(Y)}{p(Y)}$$





Pulse-position-modulation

We can achieve low duty cycles and high peak to average power ratios by using PPM. M-PPM maps a binary $\log_2 M$ tuple to a M-ary binary vector with a single one in the *slot* indicated by the input. Example: $M = 8$, mapping of 101001.

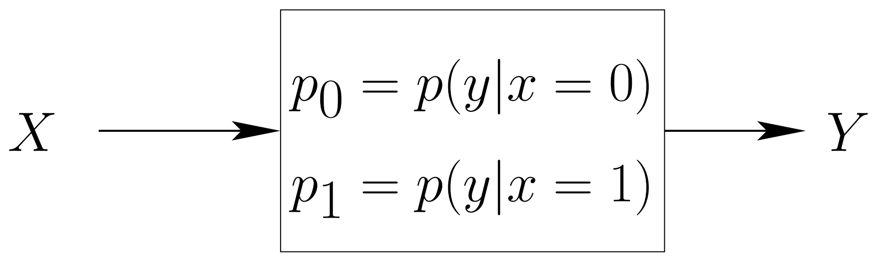


- PPM achieves a duty cycle of $1/M$
- Straight-forward to implement and analyze
- Known to be an efficient modulation for the Poisson channel *[Pierce, 78], [McEliece, Welch, 79], [Butman et. al., 80], [Lipes, 80],[Wyner, 88]*
- PPM satisfies the property that each symbol is a coordinate permutation of another
- *Generalized PPM*: a set of vectors S such that there is a group of coordinate permutations that fix* the set (a transitive set), e.g., PPM, multipulse PPM.

*a group of permutations G such that for each $g \in G$, $gS = S$ and for each $\mathbf{x}_i, \mathbf{x}_j \in S$ there exists $g \in G$ such that $\mathbf{x}_i = \sigma_g(\mathbf{x}_j)$, where σ_g is the mapping imposed by g.

Capacity of Generalized PPM

binary DMC



Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be a set of length n vectors and $p_{\mathbf{X}}(\cdot)$ a probability distribution on S.

$$C = \max_{p_{\mathbf{X}}} I(\mathbf{X};\mathbf{Y})$$

Theorem 1 *If S is a transitive set, then C_S if achieved by a uniform distribution on S.*

Theorem 2 *On a binary input channel with $p_1(y)/p_0(y) < \infty$,*

$$C = d_H D\left(p_1 || p_0\right) - D\left(p(\mathbf{y}) || p(\mathbf{y}|\mathbf{0})\right) \ \ \textit{bits/symbol}$$

where $D(\cdot||\cdot)$ is the Kullback-Liebler distance, d_H is the symbol Hamming weight, $p(\mathbf{y})$ is the density of n-vector $\mathbf{Y}$, $p(\mathbf{y}|\mathbf{0})$ the density of n-vector of noise slots.

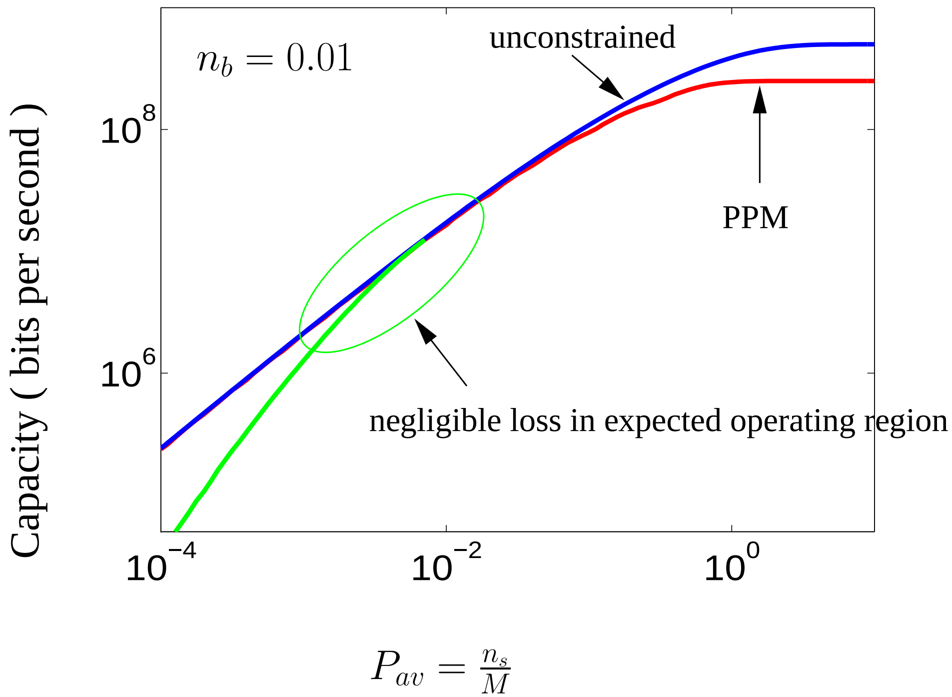
Capacity of PPM

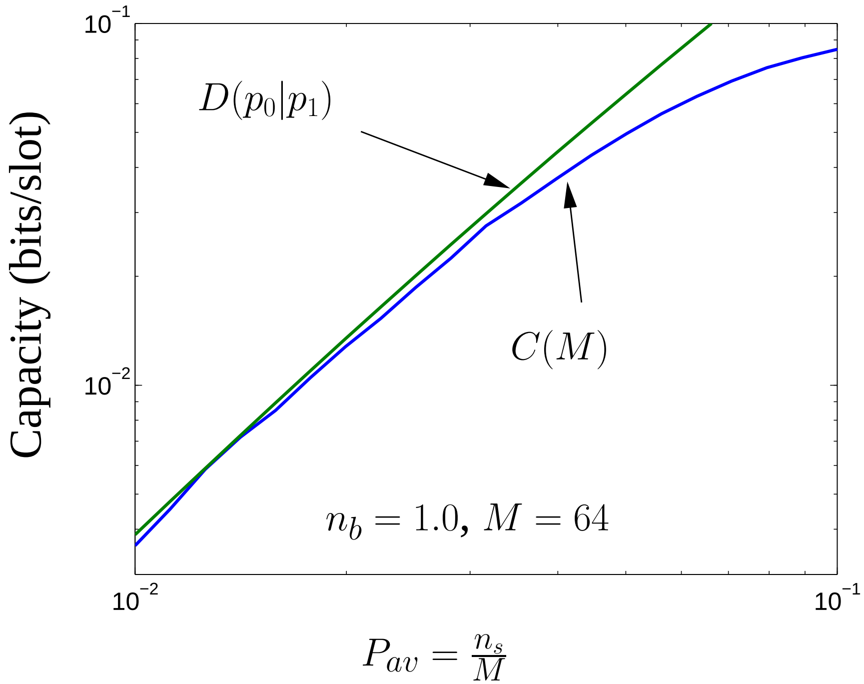
Corollary 1 *For the binary M-ary PPM channel,*

$$C(M) = D(p_1||p_0) - D(p(\mathbf{y})||p(\mathbf{y}|\mathbf{0})) \le D(p_1||p_0)$$

Theorem 3 *For fixed n_s, n_b,* $\lim_{M\to\infty} C(M) = D(p_1||p_0)$.

Poisson channel: $D(p_1||p_0) = (n_s + n_b)\log(1 + n_s/n_b) - n_s$. This term is also tight for small n_s.

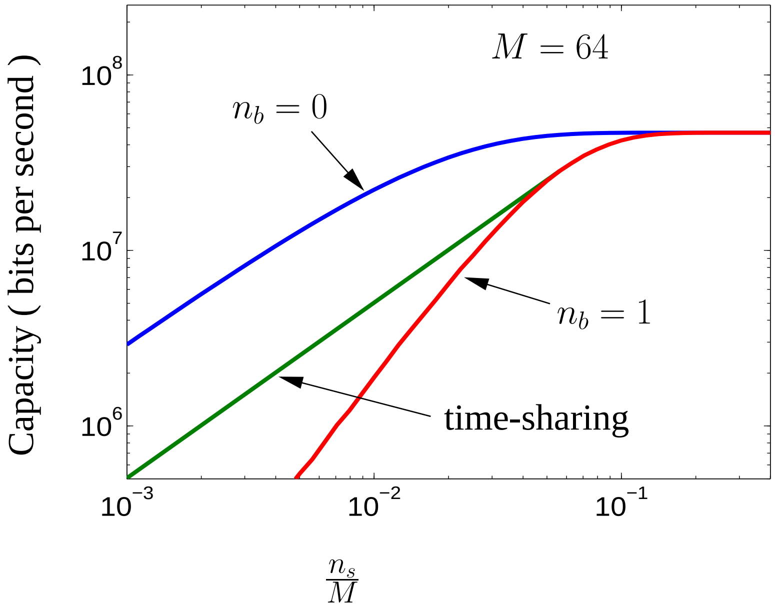




Poisson PPM Capacity: small n_s asymptotes, concavity in n_s

- $$\frac{C(M)}{M} = \begin{cases} \frac{M-1}{2M\log 2}\frac{n_s^2}{n_b} + O(n_s^3) & , n_b > 0 \\ n_s \log_2 + O(n_s^2) & , n_b = 0 \end{cases}$$

- for fixed order M, asymptotic slope in log-log domain is 1 for $n_b = 0$, 2 for $n_b > 0$
- implies 1 dB increase in signal power compensates for 2 dB increase in noise power (for small n_s)
- C is concave in n_s for $n_b = 0$ but not for $n_b > 0$ (single inflection point)
- time-sharing (using pairs $n_{s,1}, n_{s,2}$) is advantageous (up to peak power constraint)



Poisson PPM Capacity: convexity in M?

Theorem 4 *For* $n \leq m$,

$$C(km) + C(n) \leq C(kn) + C(m)$$
$$C(km) \leq C(k) + C(m)$$

□

This is essentially a *subadditivity* property. Let $f(x) = C(e^x)$. Then

$$f(x+y) \leq f(x) + f(y) \qquad \text{subadditive}$$

$$f(\alpha x + (1-\alpha)y) \overset{?}{\leq} \alpha f(x) + (1-\alpha) f(y) \qquad \text{convex } \cap$$

In practice, M chosen to be a power of 2.

Corollary 2 *For* $M = 2^j$, *(take* $k = 2, m = M, n = M/2$ *in above Theorem)*

$$C(2M) - C(M) \leq C(M) - C(M/2) \qquad \text{convex } \cap$$

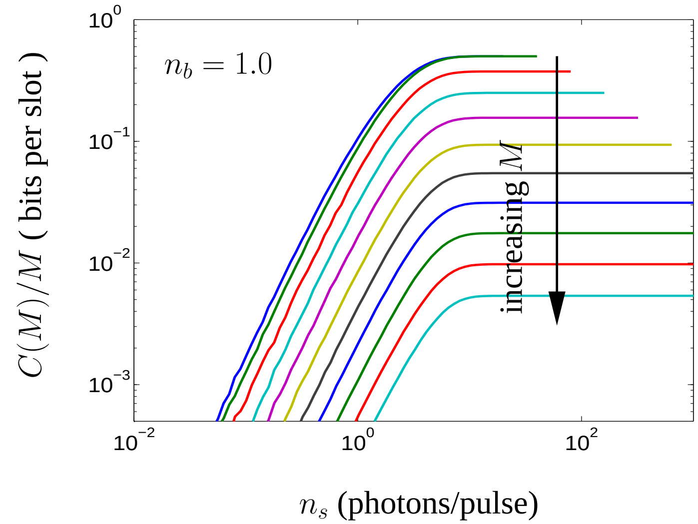
$$\frac{C(M)}{M} \text{ is decreasing in } M$$

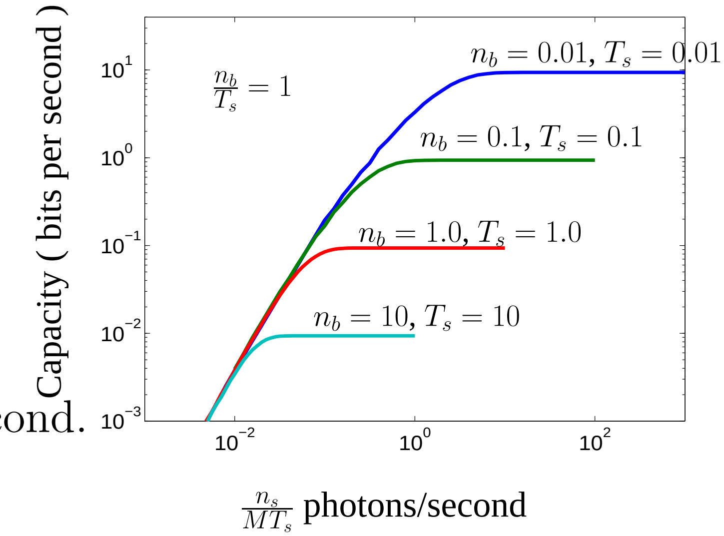
□

Poisson PPM Capacity: invariance to slot width

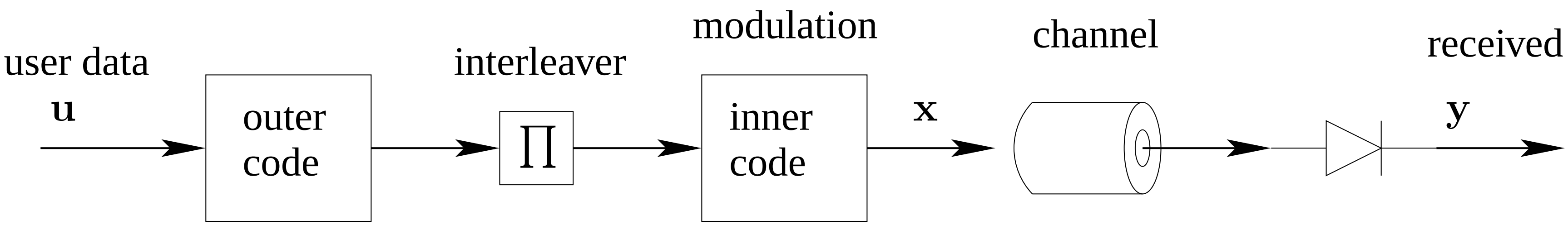
- For M a power of two, and fixed n_s, $C(M)/M$ is monotonically decreasing in M.
- Suppose P_{pk}/P_{av} is a power of two. Then optimum order satisfies $M \leq P_{pk}/P_{av}$.
- Let T_s be the slot width. Normalize photon arrival rates and capacity by the slot width. Let $\lambda_s = n_s T_s$ photons/second, $\lambda_b = n_b T_s$ photons/second. For small n_s,

$$\frac{C(M)}{MT_s} \approx \frac{M(M-1)}{2\ln 2}\left(\frac{\lambda_s^2}{\lambda_b}\right) \text{ bits/second.}$$





Achieving capacity: Coding and Modulation



	outer code	inner code
RSPPM	Reed-Solomon $(n, k) = (M^{\alpha} - 1, k)$, $\alpha = 1$, *[McEliece, 81]*,$\alpha > 1$, *[Hamkins, Moision, 03]*	M-PPM
SCPPM	convolutional code (w/o accumulate)*[Massey, 81]*, (iterate with PPM) *[Hamkins, Moision, 02]*	accumulate-M-PPM
PCPPM	parallel concatenated convolutional code *[Kiasaleh, 98]*,*[Hamkins, 99]*,(DTMRF, iterate with PPM) *[Peleg, Shamai, 00]*	M-PPM

Predicting iterative decoding performance

$$\text{Prob(bit error)} = \frac{1}{2^k} \sum_{\mathbf{u},\hat{\mathbf{u}}} \frac{d(\mathbf{u},\hat{\mathbf{u}})}{k} P(\hat{\mathbf{u}}|\mathbf{u})$$

The Bhattacharrya bound is commonly used to bound the pairwise error probability

$$P(\hat{\mathbf{u}}|\mathbf{u}) \leq P_2(\hat{\mathbf{x}}|\mathbf{x}) < \left(\sum_k \sqrt{p_0(k)p_1(k)} \right)^{d(\mathbf{x},\hat{\mathbf{x}})} =: z^{d(\mathbf{x},\hat{\mathbf{x}})}$$

For constant Hamming weight coded sequences (such as generalized binary PPM) on any channel with a monotonic likelihood ratio $p_1(k)/p_0(k)$ (Gaussian, Poisson, Webb-McIntyre-Conradi), we have

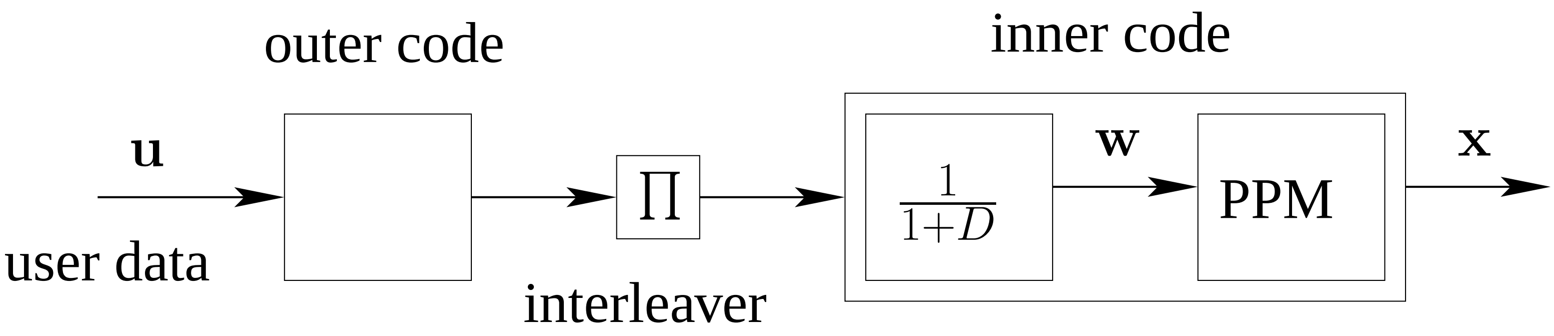
$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \sum_{k:x_k=1} y_k$$

Hence the ML pairwise codeword error may be bounded as

$$\begin{aligned} P(\hat{\mathbf{u}}|\mathbf{u}) \leq P_2(\hat{\mathbf{x}}|\mathbf{x}) &= P(S < N) + \frac{1}{2} P(S = N) \\ &= P_2(d(\hat{\mathbf{x}},\mathbf{x})) \leq z^{d(\hat{\mathbf{x}},\mathbf{x})} \end{aligned}$$

Where S is the sum of $d/2$ signal slots, N is the sum of $d/2$ noise slots.

IOWEF PPM bounds



PPM is a non-linear mapping, however, we can bound the distance in terms of the codeword weights

$$2\left\lceil \frac{d(\mathbf{w},\hat{\mathbf{w}})}{\log_2 M} \right\rceil \le d(\mathbf{x},\hat{\mathbf{x}}) \le 2\min\left\{ \frac{n}{\log_2 M}, d(\mathbf{w},\hat{\mathbf{w}}) \right\}.$$

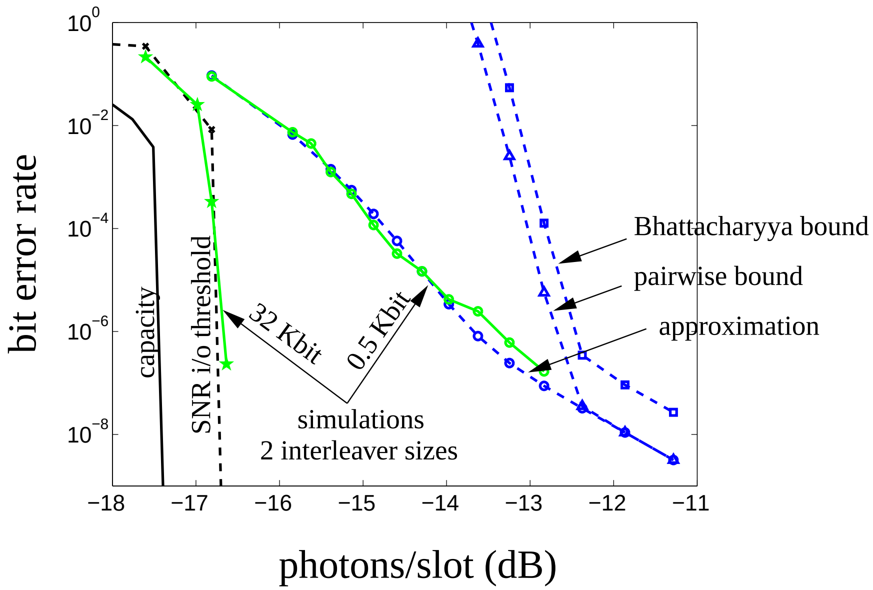
Now we have

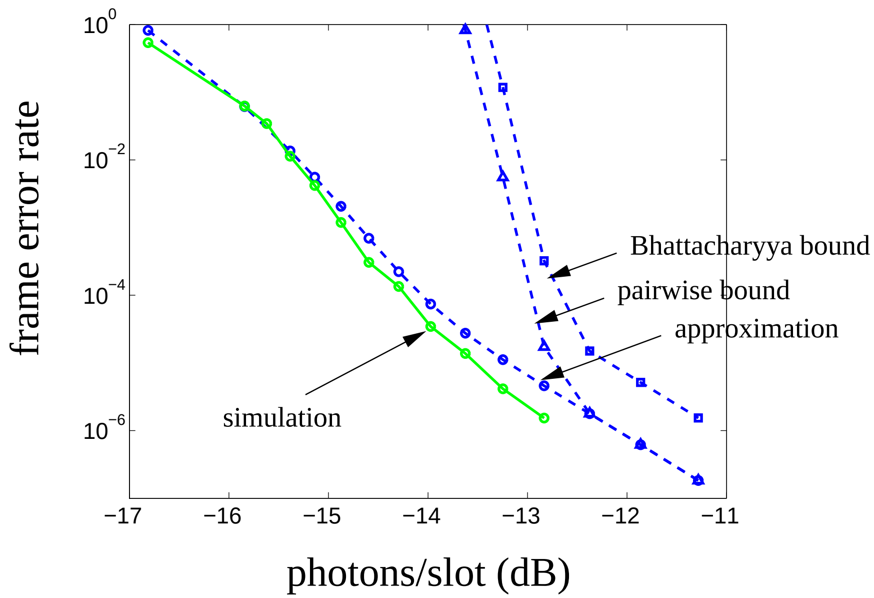
$$P_b \le \sum_{\mathbf{u}\neq\mathbf{0}} \frac{d(\mathbf{u})}{k} P_2\left(2\left\lceil \frac{d(\mathbf{x})}{\log_2 M}\right\rceil\right) = \sum_{w=1}^{k}\sum_{h=1}^{n} \frac{w}{k} A_{w,h} P_2\left(2\left\lceil \frac{h}{\log_2 M}\right\rceil\right)$$

where $A_{w,h}$ is the input-output-weight-enumerating-function (IOWEF)

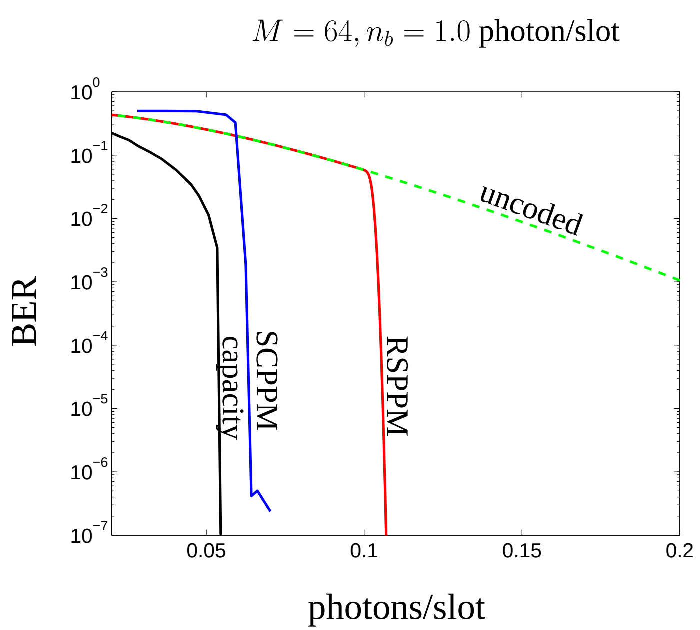
BER and FER bounds

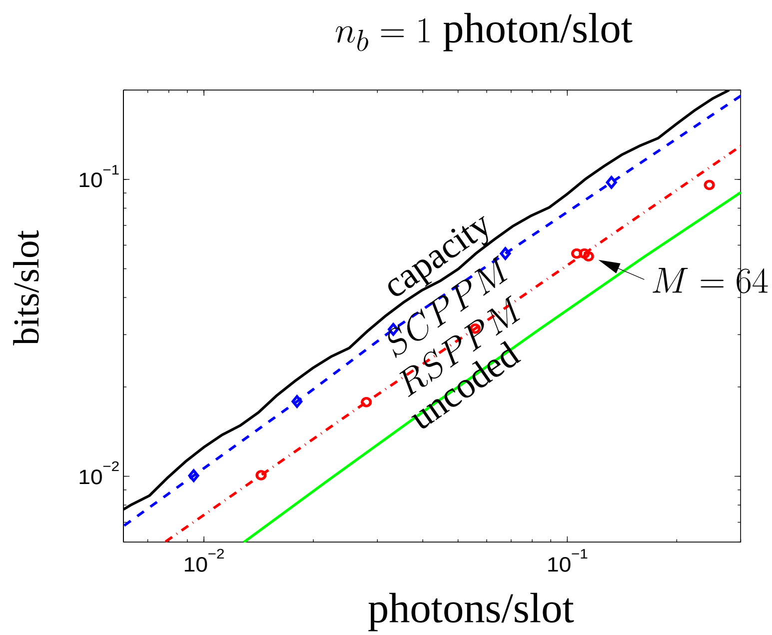
repeat-9 $\Rightarrow$ accumulate $\Rightarrow$ $M = 64$ PPM. Interleaver lengths 0.5 Kbit, 32 Kbit.





Performance





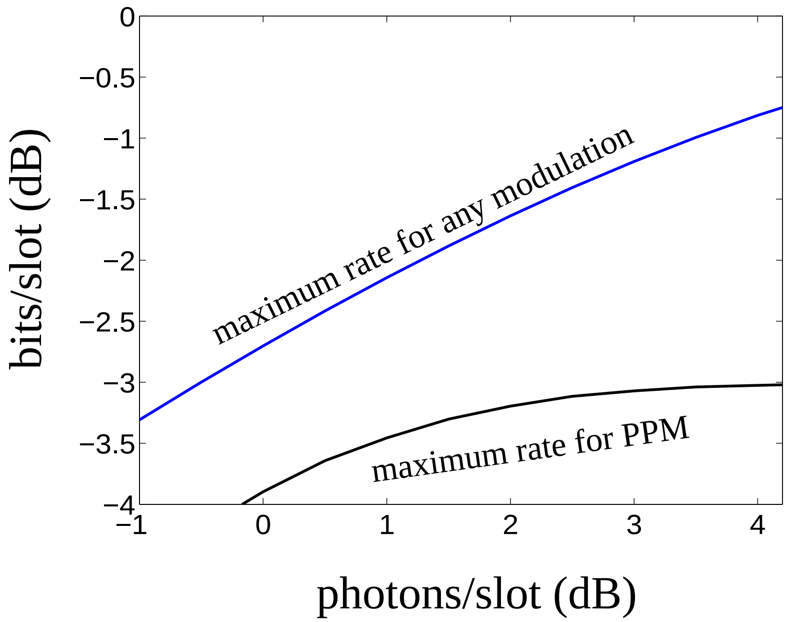
Gaps to capacity

BER=10^{-6}, Poisson channel	
SCPPM	0.75 dB
RSPPM	2.75 dB
uncoded	4.7 dB

(SCPPM: $|\Pi| = 16384$, stopping rule, max 32 operations)

High average power, bandwidth constraints

- Can continue to use PPM at high average powers with no loss by decreasing the slot width T_s up to the Bandwidth constraints of the system.
- Past that point, we see increasing losses by restricting modulation to PPM.
- For example, uplink has high average power and low Bandwidth.
- How to populate this region?



Variable-pulse modulation

Allow variable pulses per symbol. Now symbol mapping may be an issue.

input						symbol			
Gray			anti-Gray						
0	0	0	0	0	0	1	0	0	0
0	1	0	1	1	1	1	0	0	1
1	1	0	0	1	0	0	0	0	1
1	0	0	1	0	1	0	0	1	1
1	0	1	1	1	0	0	0	1	0
1	1	1	0	0	1	0	1	1	1
0	1	1	1	0	0	0	1	0	0
0	0	1	0	1	1	1	1	0	0