Ghostbusting:
Constrained Coding for Optical Communications

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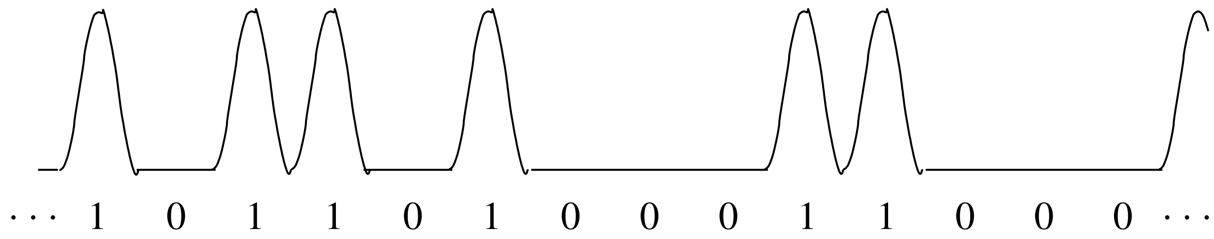
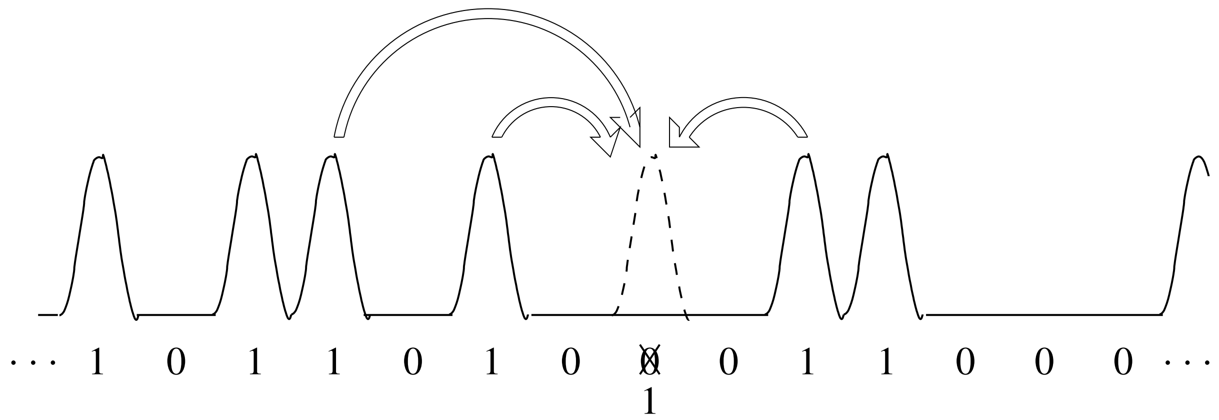
Motivation: Optical Communications



- Scenario: High data-rate ($\sim$ 40 Gbps) communications over long-haul fibre optic link.
- A train of pulses, corresponding to bits, is sent across an optical fibre.
- Each bit is allocated a time slot of some duration T, and a ‘1’ or a ‘0’ is marked by the presence or absence of a pulse.

4-Wave Mixing (The Ghost Pulse Effect)



- Interaction between pulses in the kth, lth and mth time slots (k, l, m need not all be distinct) pumps energy into the $(k + l - m)$th time slot.
- If the $(k + l - m)$th time slot did not originally contain a pulse, the transfer of energy creates a spurious "ghost" pulse there, thus changing the original '0' to a '1'.
- This effect may propagate — ghost pulses may interact with other pulses to create more ghost pulses. . .

Ghostbusting: Can Coding Help?

Some phase modulation schemes have been proposed in the optics literature to deal with the ghost pulse effect — *eg.*, Liu, Wei *et al* (2002), Alic & Fainman (2002).

Ghost pulse formation is phase-sensitive. Phase modulation schemes work by removing phase coherence in the pulses.

Is it possible to use codes to alleviate the ghost pulse effect?

Can such codes be efficient in terms of rate and complexity?

Related work — Vasic, Rao *et al* (2004).

Basis for a Coding Scheme

Binary data sequence, $b_0 b_1 \ldots b_{M-1}$.

Coded sequence (also binary), $c_0 c_1 \ldots c_{N-1}$.

Suppose the coded sequence can be constructed in such a way that no ghost pulses can be created at positions j such that $c_j = 0$.

Example of such coded sequences are the all-ones sequence or sequences of alternating zeros and ones.

Such sequences can be transmitted across the optical fibre without being corrupted by ghost pulse formation.

The Binary Ghost Pulse (BGP) Constraint

Definition [BGP constraint]:

A sequence, $\mathbf{x} = (x_0 x_1 \dots x_{n-1}) \in \{0,1\}^n$, $n \in \mathbb{Z}^+$,
is *BGP-constrained* if $\forall\ k, l, m \in [0, n-1]$
such that $x_k = x_l = x_m = 1$,

$$k + l - m \in [0, n-1] \quad \Longrightarrow \quad x_{k+l-m} = 1.$$

- $S_2(n) \ \triangleq\ \{\mathbf{x} \in \{0,1\}^n : \mathbf{x} \text{ is BGP-constrained}\}$.
- $S_2 \triangleq \bigcup_{n=1}^{\infty} S_2(n)$

Ghostbusting Scheme #1

Any $\mathbf{x} \in S_2$ can be transmitted across an optical fibre without being affected by ghost pulses.

To send binary data sequence $b_0 b_1 \ldots b_{M-1}$, encode it into a sequence $c_0 c_1 \ldots c_{N-1} \in S_2$.

The efficiency (coding rate) of such a scheme is limited by the *capacity* of S_2, which is defined to be

$$H_2 = \lim_{n \to \infty} \frac{\log_2 |S_2(n)|}{n}$$

Unfortunately . . .

Fact: (easily proved by an inductive argument)

$\mathbf{x} \in S_2$ iff supp($\mathbf{x}$) forms an arithmetic progression (A.P.)

(supp($\mathbf{x}$) $= \{k : x_k \neq 0\}$).

Consequently, $|S_2(n)| = O(n^2)$, and hence,

$$H_2 = \lim_{n\to\infty} \frac{1}{n} \log_2 |S_2(n)| = 0.$$

Thus, the BGP constraint is too strong.

Coding schemes using BGP-constrained sequences are doomed to be inefficient.

Weakening the Constraint

Pulse triples that are relatively close together in the transmitted pulse train cause the most severe problems. The interaction between pulses that are sufficiently far apart in the pulse train is weak.

In a typical optical communication scenario, pulses that are more than 10–12 time slots apart do not contribute significantly to ghost pulse formation.

So, we weaken the BGP constraint by disregarding the interactions between 1's that are separated by more than some fixed distance, t.

The BGP(t) Constraint

Let t be a positive integer.

Definition [BGP(t) constraint]:

A sequence, $\mathbf{x} = (x_0 x_1 \ldots x_{n-1}) \in \{0,1\}^n$, $n \in \mathbb{Z}^+$, satisfies the *BGP(t) constraint* if $\forall\ k, l, m \in [0, n-1]$ such that $x_k = x_l = x_m = 1$ and $\max\{|k-l|, |l-m|, |m-k|\} \leq t$,

$$k + l - m \in [0, n-1] \quad \Longrightarrow \quad x_{k+l-m} = 1.$$

In the BGP(t) constraint, only indices that are within a distance of t from each other play a role in the constraint.

The BGP(t) Constraint

- $S_{2,t}(n) \;\triangleq\; \{\mathbf{x} \in \{0,1\}^n : \mathbf{x} \text{ is BGP}(t)\text{-constrained}\}$.

- $S_{2,t} \triangleq \bigcup_{n=1}^{\infty} S_{2,t}(n)$.

  For moderate values of t (t's of 10–12 or higher), sequences in $S_{2,t}$ are only weakly affected by ghost pulses, and so can be used as codewords to transmit data across the fibre optic link.

Ghostbusting Scheme #2

To send binary data sequence $b_0 b_1 \ldots b_{M-1}$, encode it into a sequence $c_0 c_1 \ldots c_{N-1} \in S_{2,t}$.

The efficiency (coding rate) of such a scheme is limited by the *capacity* of $S_{2,t}$, which is defined to be

$$H_{2,t} = \lim_{n \to \infty} \frac{\log_2 |S_{2,t}(n)|}{n}$$

Determining $H_{2,t}$: The Structure of $S_{2,t}$

$S_{2,t}$ consists of two kinds of sequences —

(i) (t, ∞)-constrained sequences: sequences in which any two successive ones are separated by at least t zeros.

Such sequences are unaffected by the BGP(t) constraint.

(ii) sequences $\mathbf{x}$ such that supp($\mathbf{x}$) forms an arithmetic progression with common difference $d \leq t$.

This is forced by the BGP(t) constraint, for the same reasons as in the case of BGP-constrained sequences.

For any n, there are at most $(t+1)^2$ such sequences of length n.

Determining $H_{2,t}$

It thus follows that $H_{2,t} = C_{t,\infty}$, where $C_{t,\infty}$ is the capacity of the well known (t,∞) constraint, defined in the usual manner:

$$C_{t,\infty} = \lim_{n\to\infty} \frac{1}{n} \log_2 |S_{t,\infty}(n)|$$

Hence, for any $t \in \mathbb{Z}^+$,

$$H_{2,t} = \log_2 \rho_t,$$

where ρ_t is the largest-magnitude root of the polynomial $z^{t+1} - z^t - 1$.

$H_{2,t}$ for $1 \le t \le 20$

t	$H_{2,t}$	t	$H_{2,t}$
1	0.6942	11	0.2301
2	0.5515	12	0.2180
3	0.4650	13	0.2073
4	0.4057	14	0.1977
5	0.3620	15	0.1891
6	0.3282	16	0.1813
7	0.3011	17	0.1742
8	0.2788	18	0.1678
9	0.2600	19	0.1618
10	0.2440	20	0.1564

Encoding & Decoding

BGP constraint:

Enumerative coding based on the fact that

$$|S_2(n)| = \begin{cases} \frac{1}{4}(n+2)^2 & n \text{ even} \\ \frac{1}{4}(n+1)(n+3) & n \text{ odd} \end{cases}$$

BGP(t) constraint:

Essentially nothing lost by encoding into the (t, ∞) constraint instead of the BGP(t) constraint.

Using constrained coding techniques, we can design efficient finite-state encoders and sliding-block decoders for the (t, ∞) constraint.

A Different Approach: Phase Modulation

At transmitter end, apply phase shift of π to certain pulses.

Interaction of pulses with opposite phases suppresses ghost pulse formation.
[Liu, Wei *et al.* (2002), Alic & Fainman (2002)]

Effectively, this phase modulation converts a binary sequence $b_0 b_1 \ldots b_{M-1}$ into a ternary sequence $c_0 c_1 \ldots c_{M-1}$, $c_i \in \{-1, 0, 1\}$, such that $b_i = |c_i|$.

We shall assume that the most severe problems are created by situations when there are indices k, l, m such that $c_k = c_l = c_m = \pm 1$ and $c_{k+l-m} = 0$.

The Ternary Ghost Pulse (TGP) Constraint

Definition [TGP constraint]:

A sequence, $\mathbf{x} = (x_0 x_1 \ldots x_{n-1}) \in \{-1, 0, 1\}^n$, $n \in \mathbb{Z}^+$,

is *TGP-constrained* if $\forall\ k, l, m \in [0, n-1]$

such that $x_k = x_l = x_m = \pm 1$,

$$k + l - m \in [0, n-1] \implies x_{k+l-m} \neq 0.$$

- $T_3(n) \triangleq \{\mathbf{x} \in \{-1, 0, 1\}^n : \mathbf{x} \text{ is BGP-constrained}\}$.
- $T_3 \triangleq \bigcup_{n=1}^{\infty} T_3(n)$.

Ghostbusting Scheme #3: Combining Coding & Phase Modulation

Any $\mathbf{x} \in T_3$ can be transmitted across an optical fibre without being severely affected by the ghost pulse effect.

To send binary data sequence $b_0 b_1 \ldots b_{M-1}$, encode it into a sequence $c_0 c_1 \ldots c_{N-1} \in T_3$.

Well, Not Quite . . .

In reality, an optical receiver can only detect the magnitude of the received signal, <u>not its phase</u>.

In other words, the receiver cannot distinguish between a 1 and a -1.

So, if the transmitted ternary sequence was $c_0 c_1 \ldots c_{N-1}$, then the receiver only sees the sequence $|c_0||c_1|\ldots|c_{N-1}|$.

As a result, cannot use two ternary sequences differing only in sign (phase) to encode two different binary sequences.

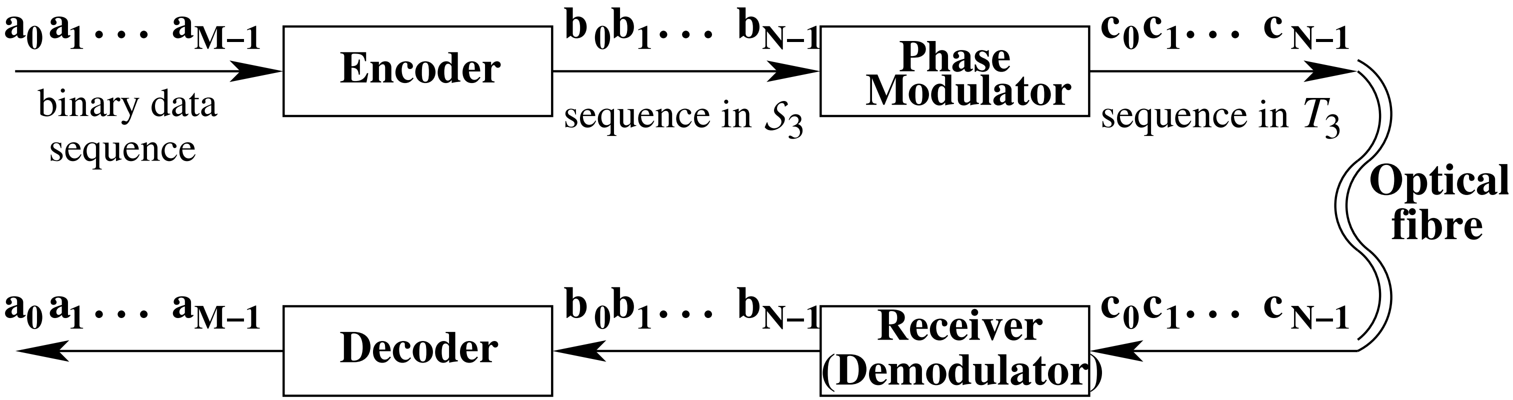
This means that coding and phase modulation must be done separately.

Ghostbusting Scheme #3
(Coding + Phase Modulation)

$$S_3(n) \triangleq \{\mathbf{y} \in \{0,1\}^n : \mathbf{y} = |\mathbf{x}| \text{ for some } \mathbf{x} \in T_3(n)\}.$$

$S_3 \triangleq \bigcup_{n=1}^{\infty} S_3(n)$.

Schematic of proper encoding procedure for scheme using TGP-constrained sequences:



Capacity of S_3

The efficiency (coding rate) of Scheme 3 is limited by the *capacity* of S_3, which is defined to be

$$H_3 = \lim_{n \to \infty} \frac{\log_2 |S_3(n)|}{n}$$

A classification of sequences in S_3, similar to that obtained for sequences in S_2 and $S_{2,t}$, would help us determine H_3.

Unfortunately, this is a much harder problem, which we have not been able to solve completely.

More on this later.

Relaxing the TGP constraint

It makes sense to relax the TGP constraint as was done for the BGP constraint.

As before, only indices that are within a distance of t from each other are allowed to play a role in the constraint.

Definition [TGP(t) constraint]:

A sequence, $\mathbf{x} = (x_0 x_1 \ldots x_{n-1}) \in \{-1, 0, 1\}^n$, $n \in \mathbb{Z}^+$, satisfies the *TGP(t) constraint* if $\forall\ k, l, m \in [0, n-1]$ such that $x_k = x_l = x_m = \pm 1$ and $\max\{|k-l|, |l-m|, |m-k|\} \leq t$,

$$k + l - m \in [0, n-1] \implies x_{k+l-m} \neq 0.$$

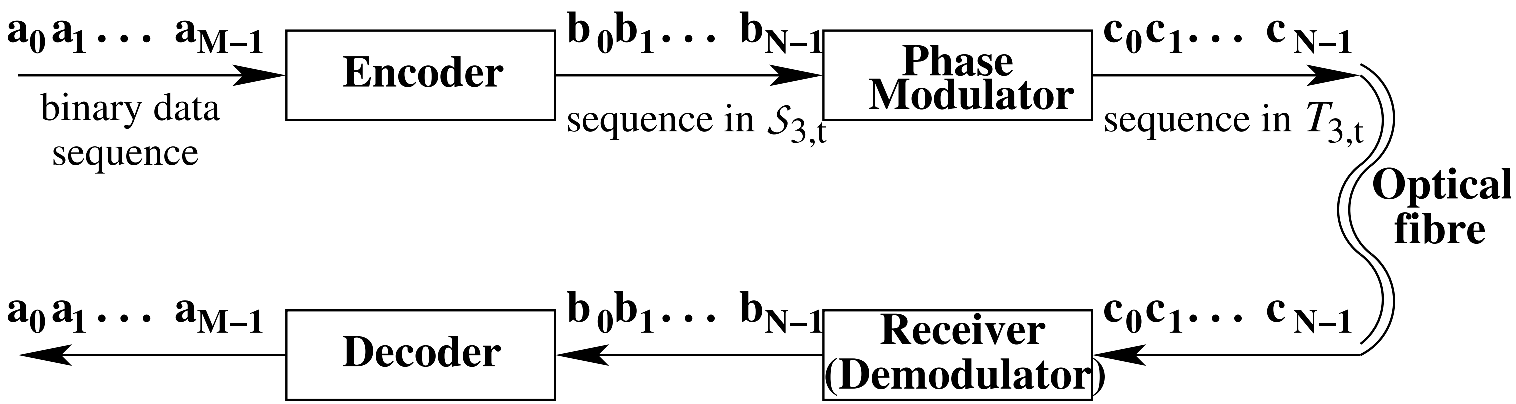
Ghostbusting Scheme #4

Define $T_{3,t}(n)$ and $T_{3,t} = \bigcup_{n=1}^{\infty} T_{3,t}(n)$ as usual.

$$S_{3,t}(n) \triangleq \{\mathbf{y} \in \{0,1\}^n : \mathbf{y} = |\mathbf{x}| \text{ for some } \mathbf{x} \in T_{3,t}(n)\}.$$

$$S_{3,t} \triangleq \bigcup_{n=1}^{\infty} S_3(n).$$

Schematic of encoding procedure:



The Capacity of $S_{3,t}$

The *capacity* of $S_{3,t}$ is defined to be

$$H_{3,t} = \lim_{n\to\infty} \frac{\log_2 |S_{3,t}(n)|}{n}$$

This capacity is useful for two reasons:

(i) It indicates how efficient coding schemes using TGP(t)-constrained sequences can be.

(ii) It is an upper bound on H_3, the capacity of the TGP constraint. In fact,

$$H_3 = \inf_{t\geq 1} H_{3,t} = \lim_{t\to\infty} H_{3,t}.$$

So, the $H_{3,t}$'s form a sequence of increasingly tight upper bounds on H_3.

Classification of Sequences in $S_{3,t}$

As was done for the BGP-constrained systems of S_2 and $S_{2,t}$, we attempt to classify the sequences in $S_{3,t}$.

The classification can then be used to determine $H_{3,t}$.

Also, such a classification can provide us with the tools to design efficient encoders and decoders for the TGP(t) constraint.

This also turns out to be a difficult problem in general. The cases $t = 1, 2$ have been completely analyzed, but higher t's remain open.

$\mathbf{t = 1}$

The TGP(1) constraint on a ternary sequence $\mathbf{x} = (x_0 x_1 \ldots x_{n-1})$ is equivalent to the following condition:

$$x_k = x_{k+1} \neq 0 \implies x_{k-1} \neq 0,\ x_{k-2} \neq 0$$

This is seen to be equivalent to the condition that $\mathbf{x}$ does not contain 011, 110, $0\overline{1}\overline{1}$ or $\overline{1}\overline{1}0$ as a subblock. ($\overline{1} = -1$.)

We can easily construct a "phase modulation" function that maps an arbitrary binary sequence to a ternary sequence satisfying the above.

This shows that $S_{3,1}(n) = \{0,1\}^n$, and hence,

$$H_{3,1} = \lim_{n\to\infty} \frac{\log_2 |S_{3,1}(n)|}{n} = 1.$$

$\mathbf{t = 2}$

<u>Theorem</u>:

Let $\mathcal{F}_2 = \{011100, 001110, 001111100\}$.

A finite-length binary sequence, $\mathbf{y}$, is in $S_{3,2}$ if and only if $\mathbf{y}$ contains no member of $\mathcal{F}_2$ as a subblock.

Remarks on the proof:

Necessity is relatively easy to show.

Sufficiency is shown by an explicit construction of a "phase modulation" mapping that takes the binary sequence $\mathbf{y}$ to a ternary sequence satisfying the TGP(2) constraint.

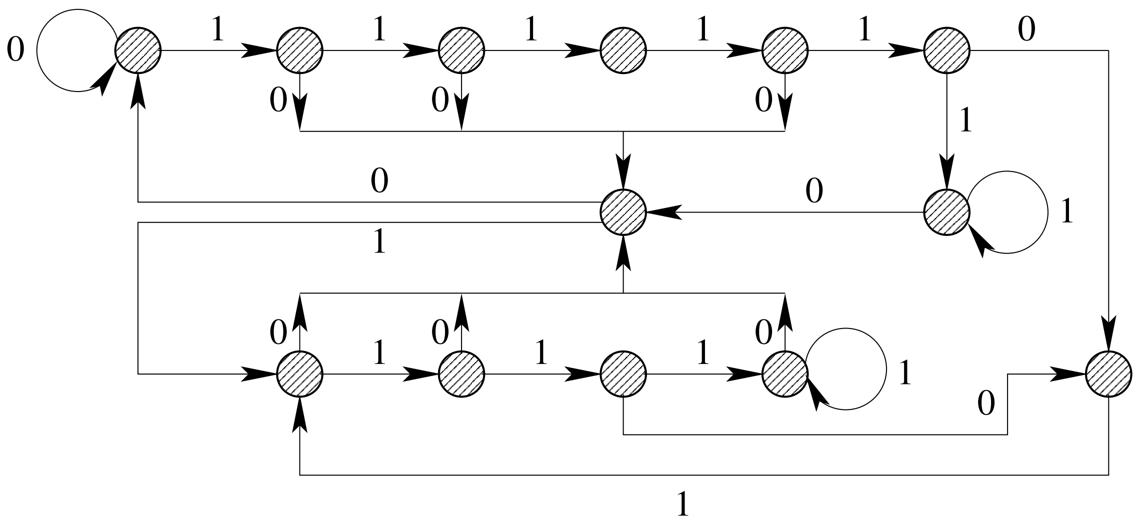
$\mathbf{t = 2}$

Corollary:

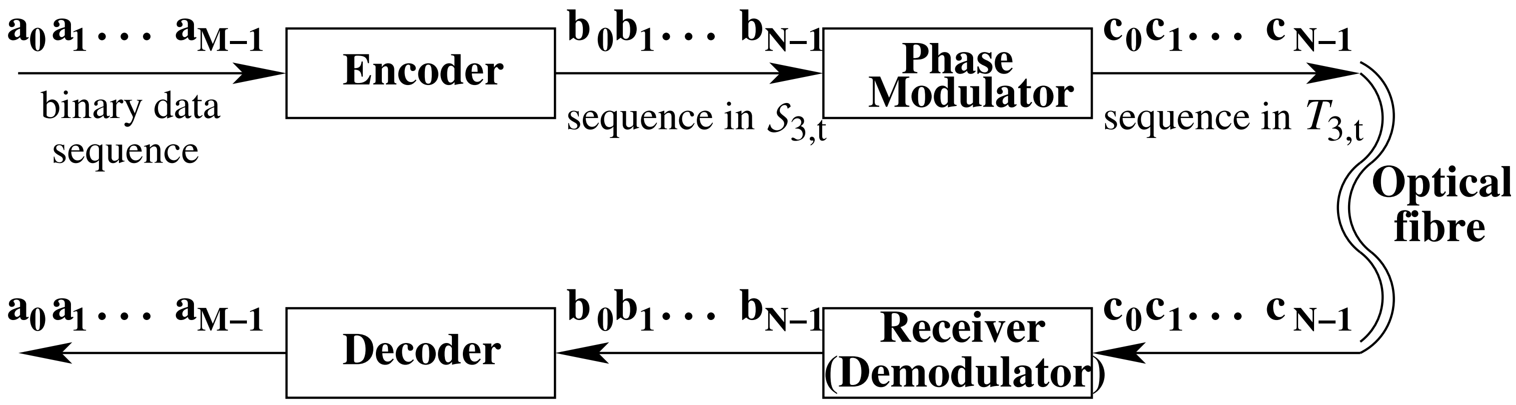
$H_{3,2} = \log_2 \rho$, where ρ is the largest-magnitude root of the polynomial $z^{10} - 2z^9 + z^5 - z^4 + 2z^3 - z^2 - 2z + 1$.

Numerically, $H_{3,2} \approx 0.96048$.

Proof:



Encoding & Decoding



Phase modulator: Explicit mappings given by proofs.

Encoder/Decoder: not needed for $t = 1$;

for $t = 2$, can be derived from graph on previous slide.

Comparing BGP(t) and TGP(t)

$t = 1$:

$$
\begin{aligned}
H_{2,1} &= 0.6942 \\
H_{3,1} &= 1
\end{aligned}
$$

$t = 2$:

$$
\begin{aligned}
H_{2,2} &= 0.5515 \\
H_{3,2} &= 0.9605
\end{aligned}
$$

$S_{3,t}$ for Arbitrary t

The analysis for $t = 1, 2$ does not easily generalize to arbitrary t.

A systematic approach:

We construct a *deterministic presentation* of the constrained system $S_{3,t}$.

A presentation of $S_{3,t}$ is a finite, directed, labeled graph, $\mathcal{G}$, such that $S_{3,t}$ is precisely the set of sequences obtained from reading the labels of finite paths in $\mathcal{G}$.

A deterministic presentation is one in which the outgoing edges at each vertex have distinct labels.

$S_{3,t}$ for Arbitrary t

Given a deterministic presentation, $\mathcal{G}_{3,t}$, of $S_{3,t}$, we can directly compute its capacity, $H_{3,t}$, and design encoders/decoders.

The capacity of $S_{3,t}$ is given by

$$H_{3,t} = \log_2 \lambda(A_{3,t}),$$

where $\lambda(A_{3,t})$ is the largest eigenvalue of the adjacency matrix, $A_{3,t}$, of $\mathcal{G}_{3,t}$.

Systematic procedures for constructing finite-state encoders and corresponding decoders can be found in the constrained coding literature (see *e.g.*, Marcus, Roth & Siegel in HCT).

Constructing $\mathcal{G}_{3,t}$

Define the finite, directed, labeled graph, $\Gamma_{3,t}$ as follows:

Vertices: All ternary sequences $(x_{-t}x_{-t+1}\ldots x_0\ldots x_{2t})$ such that for all $k,l,m \in [0,t]$,

$$x_k = x_l = x_m = \pm 1 \implies x_{k+l-m} \neq 0$$

Edges: An edge goes from $(x_{-t}x_{-t+1}\ldots x_{2t})$ to $(\hat{x}_{-t}\hat{x}_{-t+1}\ldots \hat{x}_{2t})$ iff

$$(x_{-t+1}x_{-t+2}\ldots x_{2t}) = (\hat{x}_{-t}\hat{x}_{-t+1}\ldots \hat{x}_{2t-1})$$

Edge labels: Above edge labeled by $\hat{x}_{2t} \in \{-1,0,1\}$.

This is a deterministic presentation of the constrained system, $T_{3,t}$, of TGP(t)-constrained ternary sequences.

Constructing $\mathcal{G}_{3,t}$

From $\Gamma_{3,t}$, derive the graph $\widehat{\mathcal{G}}_{3,t}$ by replacing each edge label, $x \in \{-1, 0, 1\}$, in $\Gamma_{3,t}$, with its absolute value, $|x| \in \{0, 1\}$.

$\hat{\mathcal{G}}_{3,t}$ is a presentation of $S_{3,t}$, but it is no longer deterministic.

Apply the *subset construction* method to $\widehat{\mathcal{G}}_{3,t}$ to obtain a deterministic presentation, $\mathcal{G}_{3,t}$, of $S_{3,t}$.

Main Drawback:

The resultant $\mathcal{G}_{3,t}$ has *at least* 2^{9^t} vertices (and this is a vast underestimate).

Some Notes on the Full-Blown TGP constraint

We extend the definition of the constraint to bi-infinite sequences, to gain some insight into the finite-length case.

Definition [TGP constraint]:

A ternary sequence, $\mathbf{x} = (x_k)_{k \in \mathbb{Z}} \in \{-1, 0, 1\}^{\mathbb{Z}}$, is *TGP-constrained* if $\forall\ k, l, m \in \text{supp}(\mathbf{x})$,

$$x_k = x_l = x_m \quad \Longrightarrow \quad x_{k+l-m} \neq 0.$$

[Note: $\text{supp}(\mathbf{x}) = \{k : x_k \neq 0\}$.]

- $T_3^* \ \triangleq\ \{\mathbf{x} \in \{-1, 0, 1\}^{\mathbb{Z}} : \mathbf{x} \text{ is TGP-constrained}\}$.
- $S_3^* \triangleq \{\mathbf{y} \in \{0, 1\}^{\mathbb{Z}} : \text{supp}(\mathbf{y}) = \text{supp}(\mathbf{x}) \text{ for some } \mathbf{x} \in T_3^*\}$

Classification of Sequences in S_3^*

We use results from the branch of mathematics known as Ramsey theory to analyze the structure of sequences in S_3^*.

Theorem 3:

Let $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ be aperiodic. Then, $\mathbf{y} \in S_3^*$ if and only if $\mathrm{supp}(\mathbf{y})$ is one of the following:

(i) a set with either 1 or 2 elements;

(ii) $(k\mathbb{Z} + i) \cup \{j\}$ for some $k \in \mathbb{Z}^+$, $i \in [0, k-1]$, and $j \in \mathbb{Z}$, $j \not\equiv i \pmod{k}$;

(iii) $(3t\mathbb{Z} + i) \cup V$ for some $t \in \mathbb{Z}^+$, $i \in [0, 3t-1]$, and $V \subset \mathbb{Z}$, $|V| = 2$, $V \equiv \{t+i, 2t+i\} \pmod{3t}$.

Consequently, any binary sequence $\mathbf{y} \in S_3^*$ can be made periodic by changing at most *two* 1's to 0's.

Periodic Sequences in S_3^*

This is still work in progress.

We have a complete classification of sequences of *prime* period in S_3^*.

<u>Theorem</u>:

Let $\mathbf{y} \in \{0,1\}^{\mathbb{Z}}$ be periodic with fundamental period p prime. Then, $\mathbf{y} \in S_3^*$ iff one of the following two conditions holds:

(i) the fundamental period of $\mathbf{y}$ contains at most two 1's;

(ii) $p = 5$ and $\mathbf{y}$ is $(01111)^{\mathbb{Z}}$ or one of its shifts.

The finite-length case: $|S_3(n)|$ for $1 \leq n \leq 18$

n	$\|S_3(n)\|$	n	$\|S_3(n)\|$
1	2	11	501
2	4	12	705
3	8	13	937
4	16	14	1248
5	32	15	1609
6	60	16	2078
7	100	17	2591
8	162	18	3245
9	240		
10	358		

Conjecture

$$H_3 = \lim_{n \to \infty} \frac{\log_2 |S_3(n)|}{n} = 0$$

Open Problems

- How effective are these codes in actually mitigating the ghost pulse effect?
- Determining H_3, $H_{3,t}$.
- Designing practical and efficient encoders/decoders for these codes.