Constrained Systems with Unconstrained Positions: Graph Constructions and Tradeoff Functions

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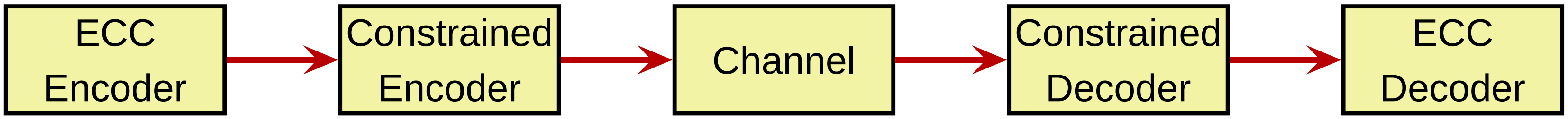
Constrained Codes and Error-Correcting Codes

Constrained Code: transforms data into constrained sequences that are suitable for the channel

Error-Correcting Code (ECC): transforms data into sequences with large distance

Standard Concatenation:



Problem: error propagation from constrained decoder

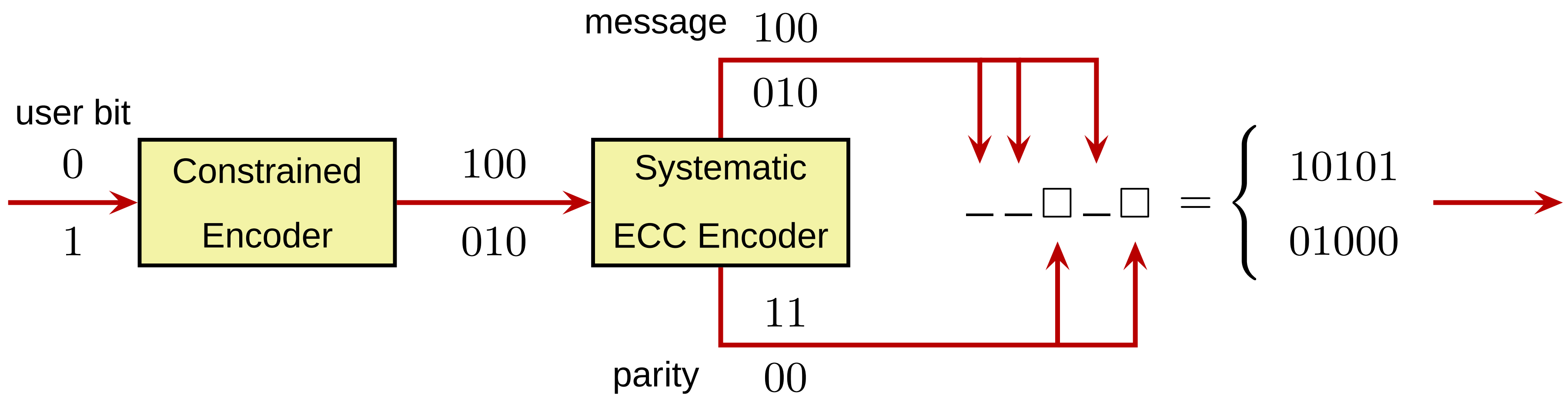
Constrained Systems with Unconstrained Positions

Example [van Wijngaarden and Immink, 2001]

The MTR(2) constraint requires every runlength of 1 to be ≤ 2.

Consider the constrained block code $\{10101, 01101\}$ for MTR(2).

No violation if bits 3 and/or 5 are flipped.



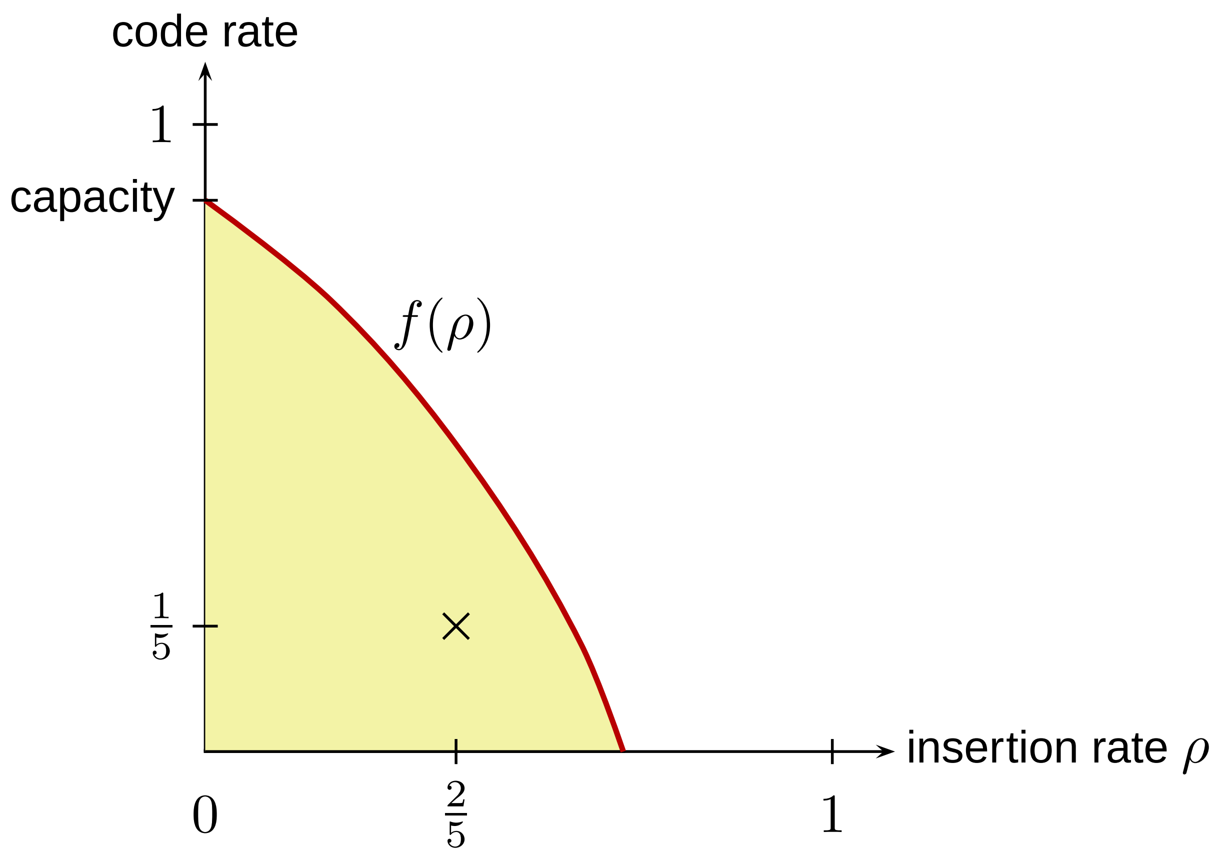
We say that the **code rate** is 1/5 and the **insertion rate** is 2/5.

Bottom line: Some positions in the code are left *unconstrained*.

Constrained Systems with Unconstrained Positions

Questions:

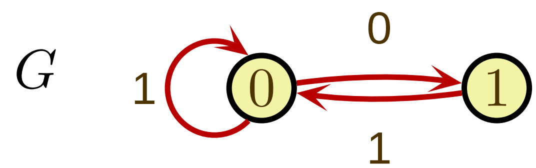
- Given an insertion rate, what is the maximum possible code rate?
- Given an insertion rate, what are the unconstrained positions that (nearly) achieve the maximum code rate?



Constrained Systems and Their Presentations

G: **labeled graph**

(with vertex set $V = V_G$)



$S = S(G)$: **constrained system,**

set of all words obtained from

reading labels of paths of G

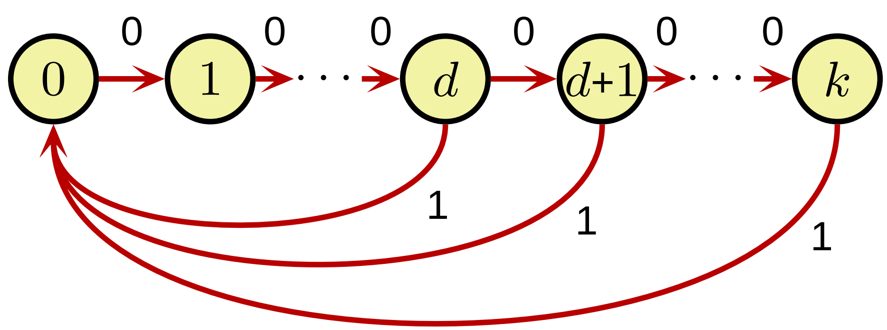
$S(G) =$ set of all words that do not contain 00

Say that G is a **presentation** of S

Note: We consider the **empty word** ϵ to be in S

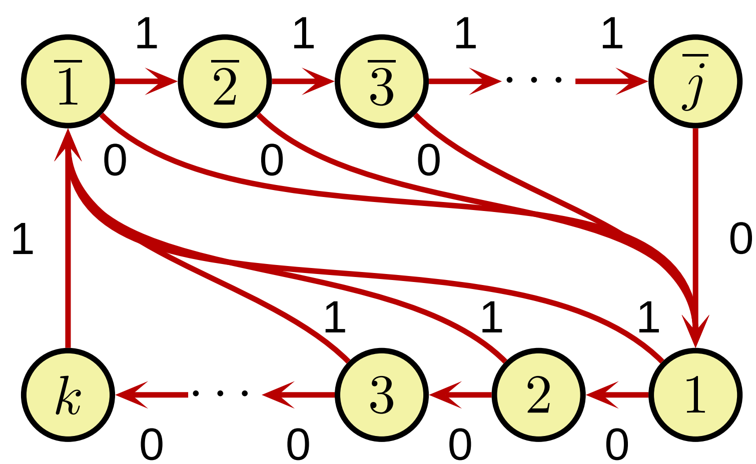
Examples of Constrained Systems

Runlength Limited RLL(d, k)



- $d \leq$ run of zeros $\leq k$

Maximum Transition Run MTR(j, k)



- run of ones $\leq j$
- run of zeros $\leq k$

Capacity

S: a constrained system

Suppose that the insertion rate is zero. What is the maximum code rate?

We need to count the number of words in S.

The **capacity** of a constrained system S is

$$\mathrm{cap}(S) = \lim_{q \to \infty} \frac{\log M(q)}{q},$$

where $M(q)$ is the number of words of length q in S.

Introducing the Unconstrained Symbol

Suppose that the insertion rate is not zero. What is the maximum code rate?

Fix a word length, say 5. Fix the unconstrained positions, say $\{3, 5\}$, that yield the desired insertion rate. We need to count the number of words of the form

$$\_\ \_\ \Box\ \_\ \Box,$$

where $\Box$ can be replaced by 0 and 1 and the constraint is still satisfied.

For this reason, we are interested in words over $\{0, 1, \Box\}$.

Let w be a word over $\{0, 1, \Box\}$. Define $\Phi(w)$ to be the set of binary words obtained from w by replacing every $\Box$ independently with 0 or 1.

Example: If $w = 0\Box1\Box$, then $\Phi(w) = \{0010, 0011, 0110, 0111\}$.

Let S be a constrained system. Define

$$\hat{S} = \{w\ :\ \Phi(w) \subseteq S\}.$$

Tradeoff Functions

Let $I \subseteq \mathbb{N}$ be a set of unconstrained positions.

$M(q, I)$: number of words w of length q in $\hat{S}$ such that $w_i = \Box$ if and only if $i \in I$.

Let $\rho \in [0, 1]$ be an insertion rate.

$\mathcal{I}(\rho)$: set of all sequences (I_q) such that $I_q \subseteq \{1, \ldots, q\}$ and $|I_q|/q \to \rho$.

Example: $\rho = 1/3$. $I_q = \{3n \,:\, n \geq 1,\ 3n \leq q\}$.

$$\begin{array}{ccccccc} I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & \cdots \\ \emptyset & \emptyset & \{3\} & \{3\} & \{3\} & \{3,6\} & \cdots \end{array}$$

(I_q) corresponds to $\_\ \_\ \Box\ \_\ \_\ \Box\ \_\ \_\ \Box\ \ldots$

$(I_q) \in \mathcal{I}(1/3)$.

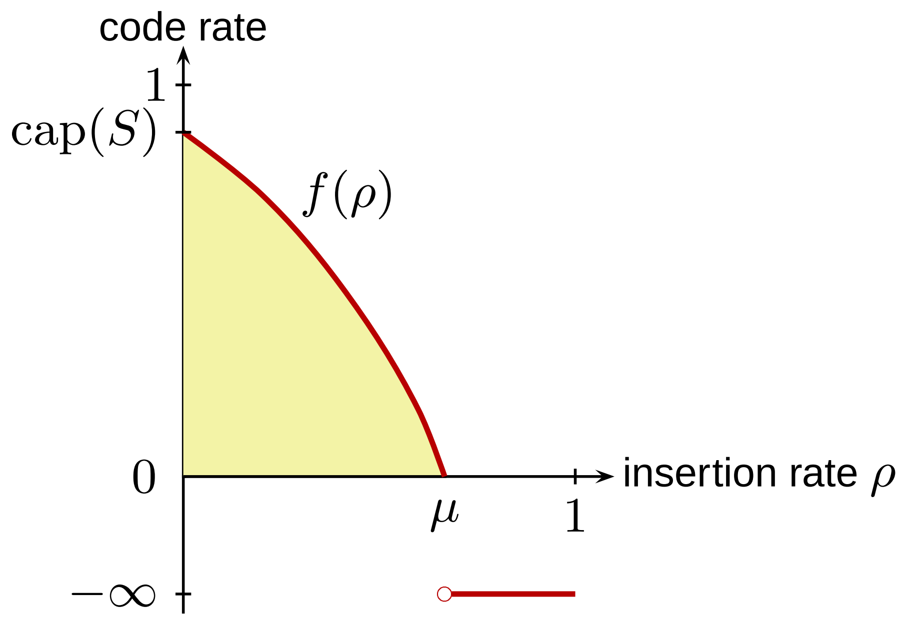
Tradeoff Functions

Tradeoff function:

$$f(\rho) = \sup_{(I_q) \in \mathcal{I}(\rho)} \limsup_{q \to \infty} \frac{\log M(q, I_q)}{q}.$$

Maximum insertion rate:

$$\mu = \sup_{f(\rho) > 0} \rho.$$



Follower Sets and Follower Set Graphs

$\mathcal{F}(x) = \mathcal{F}_S(x) = \{y \in S \ : \ xy \in S\}$: set of all words that can follow a word $x \in S$.

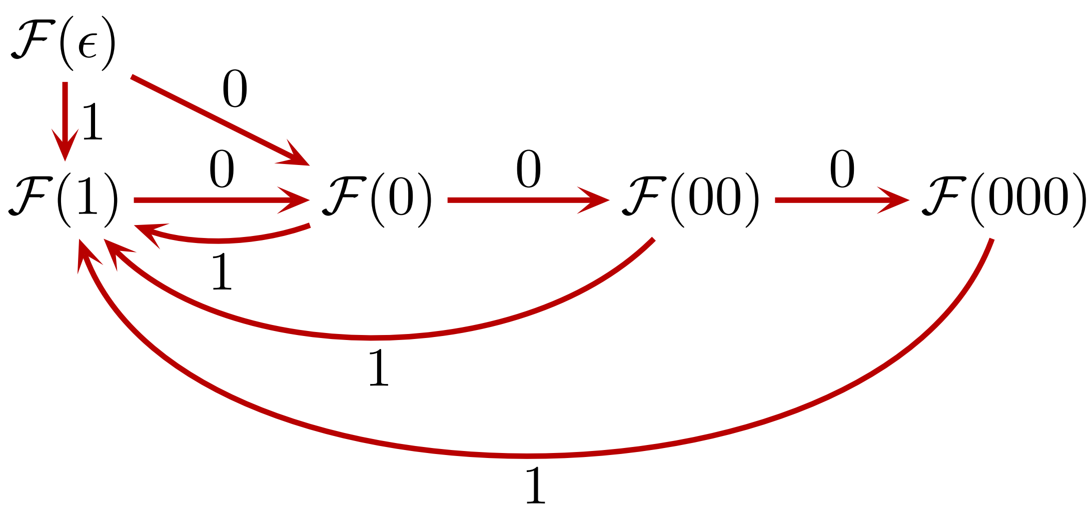
If x is the empty word ϵ, then $\mathcal{F}(\epsilon) = S$.

Fact: S has finitely many follower sets since it has a finite-state presentation.

Follower set graph:

- states: $\mathcal{F}(x)$ for all $x \in S$
- transitions: $\mathcal{F}(x) \xrightarrow{a} \mathcal{F}(xa)$, where $a \in \{0, 1\}$ and $xa \in S$

Example: RLL$(1, 3)$



The Graph $\hat{G}$

States: All intersections of the follower sets of words in S

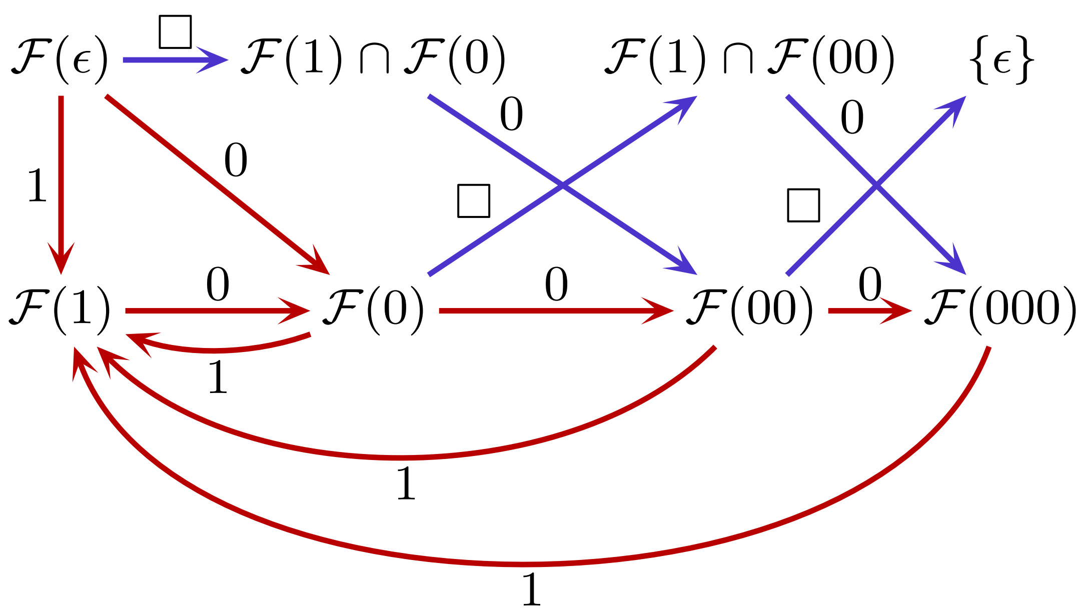
Transitions:

$$\bigcap_{i=1}^{k} \mathcal{F}(x_i) \xrightarrow{0} \bigcap_{i=1}^{k} \mathcal{F}(x_i 0) \qquad \text{if } x_i 0 \in S \text{ for all } 1 \le i \le k$$

$$\bigcap_{i=1}^{k} \mathcal{F}(x_i) \xrightarrow{1} \bigcap_{i=1}^{k} \mathcal{F}(x_i 1) \qquad \text{if } x_i 1 \in S \text{ for all } 1 \le i \le k$$

$$\bigcap_{i=1}^{k} \mathcal{F}(x_i) \xrightarrow{\square} \bigcap_{b=0}^{1} \bigcap_{i=1}^{k} \mathcal{F}(x_i b) \qquad \text{if } x_i 0, x_i 1 \in S \text{ for all } 1 \le i \le k$$

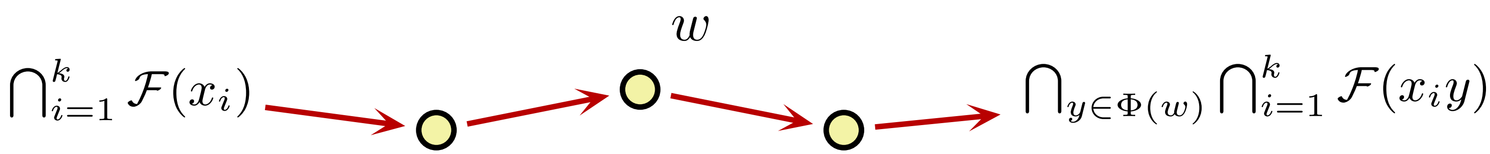
Example: RLL$(1, 3)$



The Graph $\hat{G}$

Theorem: $\hat{S}$ is the constrained system presented by $\hat{G}$.

Proof: Suppose $w \in S(\hat{G})$.

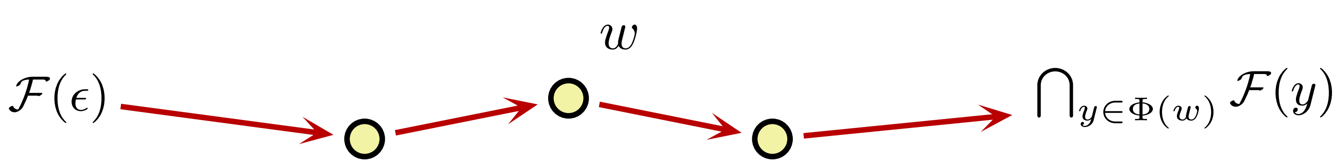


$$\Longrightarrow \quad x_i y \in S \text{ for all } i \text{ and } y \in \Phi(w)$$

$$\Longrightarrow \quad y \in S \text{ for all } y \in \Phi(w)$$

$$\Longrightarrow \quad w \in \hat{S}$$

Conversely, suppose $w \in \hat{S}$.



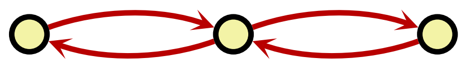
$\blacksquare$

For RLL(d, k), $\hat{G}$ has $dk + k + 2d + 1 - d^2$ states.

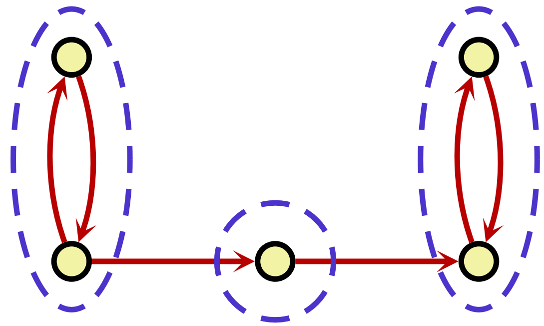
For MTR(j, k), $\hat{G}$ has $(j + 1)(k + 1)$ states.

Irreducibility and Shannon Cover

Irreducible graph: For any states u and v, there is a path from u to v and v to u.



Irreducible



Reducible

A reducible graph can be decomposed into **irreducible components** with transitional edges between them.

An irreducible component is called **trivial** if it consists of a single state and no edge.

A constrained system is **irreducible** if it has an irreducible presentation.

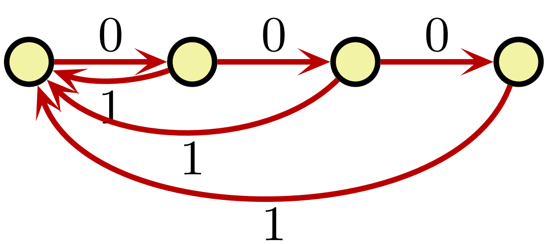
Fact: Every irreducible constrained system has a unique minimal presentation called the **Shannon cover**.

Embedding of Shannon Cover in $\hat{G}$

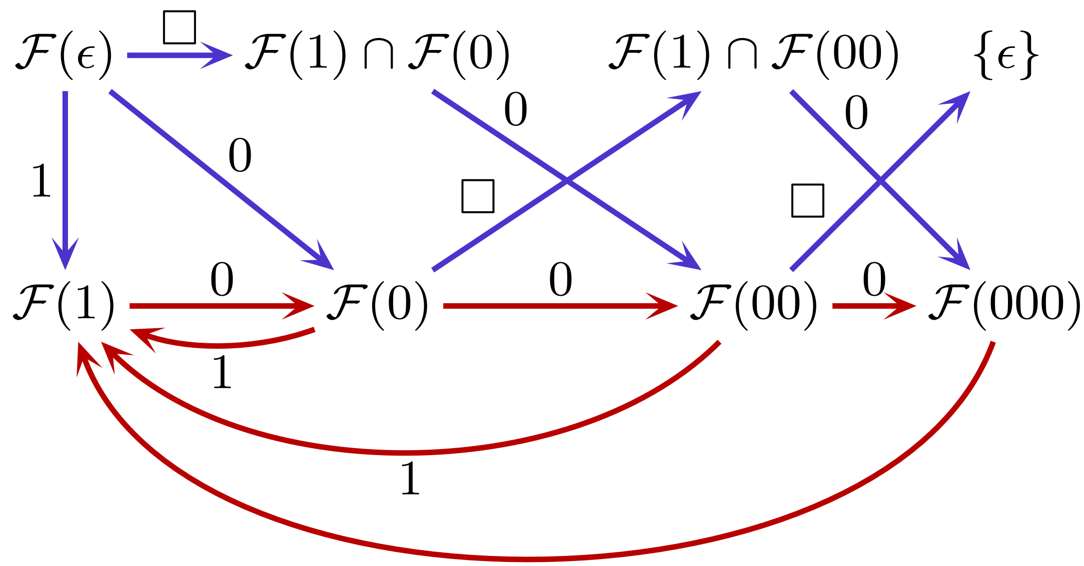
S: irreducible constrained system

Proposition: There is a unique subgraph H of $\hat{G}$ that is isomorphic to the Shannon cover for S.

Example: RLL$(1, 3)$



Shannon cover



$\hat{G}$

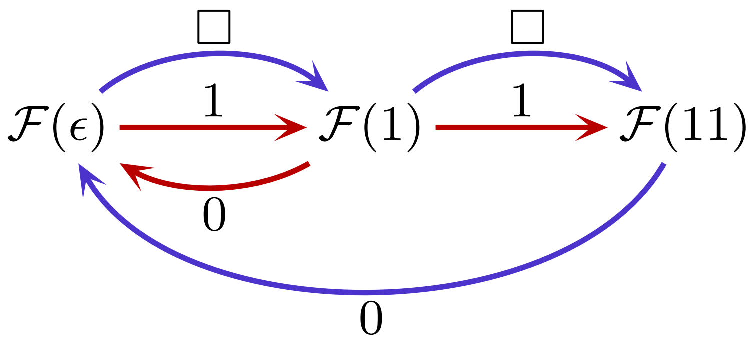
Maximum Insertion Rates

γ: path in $\hat{G}$

$\nu(\gamma)$: ratio of number of $\square$ in the label of π to its length

A cycle that maximizes ν is called a **max-insertion-rate cycle**.

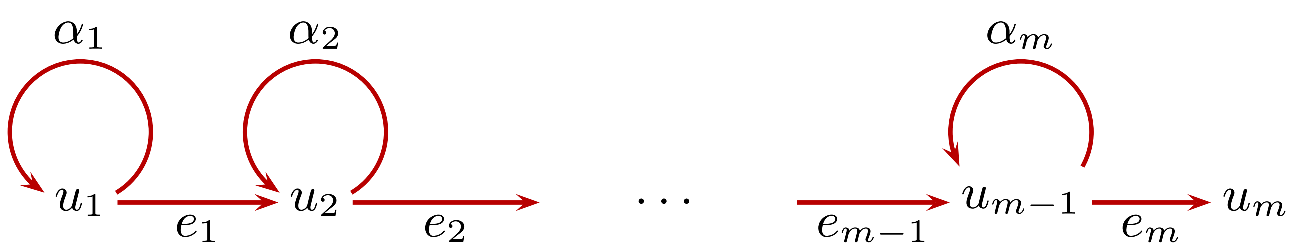
Example: MTR(2)



Maximum Insertion Rates

Proposition: Let γ be a max-insertion-rate cycle. Then $\mu = \nu(\gamma)$.

Proof (sketch): Any path π in $\hat{G}$ can be written as



where $m \leq |V_{\hat{G}}|$ and u_i are distinct.

$$\begin{aligned}
\text{number of } \square \text{ in label of } \pi \quad &\leq \quad \nu(\alpha_1)|\alpha_1| + \cdots + \nu(\alpha_m)|\alpha_m| + |V_{\hat{G}}| \\
&\leq \quad \nu(\gamma)(|\alpha_1| + \cdots + |\alpha_m|) + |V_{\hat{G}}| \\
&\leq \quad \nu(\gamma)|\pi| + |V_{\hat{G}}| \\
\text{ratio of } \square \text{ in label of } \pi \quad &\leq \quad \nu(\gamma) + \frac{|V_{\hat{G}}|}{|\pi|} \longrightarrow \nu, \ \text{ as } |\pi| \longrightarrow \infty.
\end{aligned}$$

Therefore $\mu \leq \nu(\gamma)$.

Maximum Insertion Rates

Conversely, periodically replace some $\square$ in the label of π with 0 and 1 to obtain insertion rate ρ slightly below $\nu(\gamma)$ such that $f(\rho) > 0$.

$$(\blacksquare\ \square\ 0)\ (\square\ \square\ 0)\ (\square\ \square\ 0)\ (\blacksquare\ \square\ 0)\ldots$$

Therefore $\nu(\gamma) \leq \mu$. $\blacksquare$

With this result, we can apply the **Karp's algorithm** [Karp, 1978] to $\hat{G}$ to find the maximum insertion rate.

Maximum Insertion Rates for RLL(d,k)

For RLL(d,k), $k < \infty$,

$$\mu = \frac{\left\lfloor \frac{k-d}{d+1} \right\rfloor}{\left\lfloor \frac{k+1}{d+1} \right\rfloor (d+1)}.$$

This is achieved by the sequence

$$1\ \underbrace{\overbrace{0\,0\,0}^{d}\ \square\ \overbrace{0\,0\,0}^{d}\ \square\ \overbrace{0\,0\,0}^{d}}_{\leq k} 1\ \ldots$$

For RLL(d,∞),

$$\mu = \frac{1}{d+1}.$$

This is achieved by the sequence

$$\square\ \overbrace{0\,0\,0}^{d}\ \square\ \overbrace{0\,0\,0}^{d}\ \square\ \ldots$$

Maximum Insertion Rates for MTR(j, k)

For MTR(j, k), if $\gcd(j+1, k+1) \neq 1$,

$$\mu = 1 - \frac{1}{j+1} - \frac{1}{k+1}.$$

If $\gcd(j+1, k+1) = 1$,

let m be the smallest positive integer such that $m(j+1) = k \mod (k+1)$,

let n be the smallest positive integer such that $n(j+1) = 1 \mod (k+1)$.

Then

$$\mu = \begin{cases} L_1 & \text{if } m > n, \\ \max\{L_0, L_1\} & \text{if } m < n, \end{cases}$$

where

$$\begin{aligned} L_0 &= 1 - \frac{n}{n(j+1)-1} - \frac{1}{k+1}, \\ L_1 &= 1 - \frac{1}{j+1} - \frac{m(j+1)+1}{m(j+1)(k+1)}. \end{aligned}$$

Maximum Insertion Rates for Higher-Dimensional Constraints

S: a constrained system

S_n: the n-dimensional constrained system such that every coordinate satisfies S

μ_n: maximum insertion rate for S_n, defined similarly to the one-dimensional case

Proposition: $\mu = \mu_2 = \mu_3 = \cdots$.

Proof (sketch):

$$\begin{array}{ccc} \square & \square & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} 0 & \square & \square \\ \square & 0 & \square \\ \square & \square & 0 \end{array}$$

Therefore $\mu \leq \mu_2 \leq \mu_3 \leq \cdots$.

Conversely, let P be a pattern of size $q \times q$ in $\hat{S}_2$.

$$\begin{array}{rcl} \text{number of } \square \text{ in each row} & \leq & \mu q + c \\ \text{number of } \square \text{ in } P & \leq & \mu q^2 + cq \\ \text{ratio of } \square & \leq & \mu + \dfrac{c}{q} \to \mu, \ \text{ as } q \to \infty \end{array}$$

Therefore $\mu \geq \mu_n$.

Maximum Insertion Rate and Capacity

Proposition: $\mathrm{cap}(S_n) \geq \mu$.

Proof: Let P be a $q \times q \times \cdots \times q$ pattern in $\hat{S}_n$ such that

- ratio of $\square$ equals maximum insertion rate,
- P can be freely concatenated.

Fill every $\square$ with 0 and 1 to obtain $2^{\mu q^n}$ patterns.

Therefore

$$\mathrm{cap}(S_n) \geq \frac{\log 2^{\mu q^n}}{q^n} = \mu.$$

Maximum Insertion Rate and Capacity

Corollary:

$$C_\infty = \lim_{n \to \infty} \operatorname{cap}(S_n) \geq \mu.$$

Recall for RLL(d, k),

$$\mu = \frac{\left\lfloor \frac{k-d}{d+1} \right\rfloor}{\left\lfloor \frac{k+1}{d+1} \right\rfloor (d+1)}.$$

[Ito et al., 1999]: $C_\infty = 0$ if and only if $k \leq 2d$.

Recall for RLL(d, ∞),

$$\mu = \frac{1}{d+1}.$$

[Ordentlich and Roth, 2002]:

$$C_\infty = \frac{1}{d+1}.$$

Conclusion

- Constrained systems with unconstrained positions
- Introduce a constrained system $\hat{S}$ and a presentation $\hat{G}$ with unconstrained symbol
- Define tradeoff function and maximum insertion rate
- maximum insertion rate is rational and represented by certain cycles in $\hat{G}$
- maximum insertion rate for higher-dimensional constraints

To be continued...

- More properties of $\hat{G}$
- Properties of the tradeoff function
- Bounds for the tradeoff function

References

[Ito et al., 1999] Ito, H., Kato, A., Nagy, Z., and Zeger, K. (1999). Zero capacity region of multidimensional run length constraints. *Electr. J. Combinatorics*, 6(R33).

[Karp, 1978] Karp, R. M. (1978). A characterization of the minimum cycle mean in a digraph. *Disc.Math*, 23:309–311.

[Ordentlich and Roth, 2002] Ordentlich, E. and Roth, R. M. (2002). The asymptotic capacity of multi-dimensional runlength-limited constraints and independent sets in hypergraphs. Technical report, HP Labs. HPL-2002-348.

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