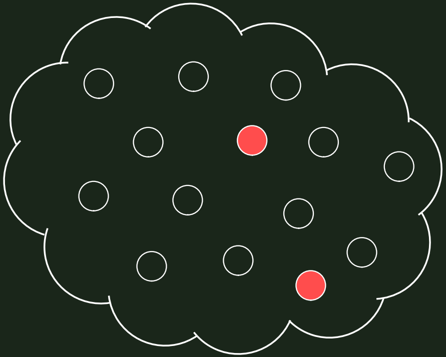
Polytope Codes in Networks, Storage, and Multiple Descriptions

Oliver Kosut

Joint work with Lang Tong, David Tse, Aaron Wagner, and Xiaoqing Fan

April 1, 2015

Networks with Active Adversaries



Distributed system in the presence of active omniscient adversaries

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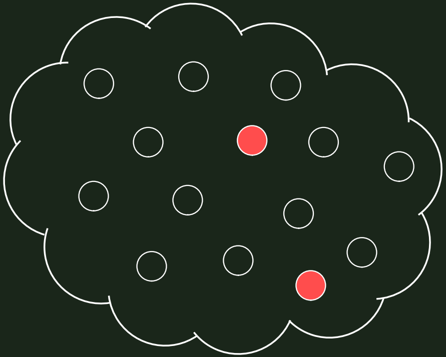

Distributed system in the presence of active omniscient adversaries

Applications:

- Man-in-the-middle attacks
- Wireless jamming attacks
- Distributed storage systems

Polytope Codes

A new-ish coding paradigm using:

- linear constructions on the integers
- covariance matrices as checksums

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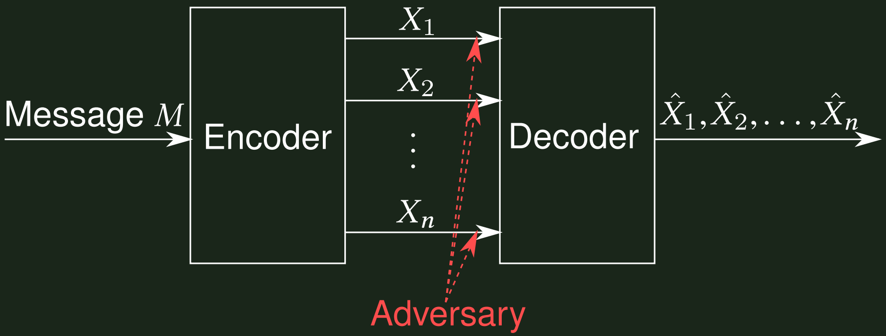
Advantages:

- Partial decoding
- Distributed detection and correction of adversarial errors

Classical Coding Formulation



- X_i in finite field $\mathbb{F}$
- Adversary may replace any z packets (min. distance $d \geq 2z + 1$)
- Decoder must output all packets without error
- Fundamental limit: Singleton bound $k \leq n - 2z$ where k is dimension of message — achievable by MDS codes

Classical setting

Must decode all information

Network setting

Partial information may do

— any partial information

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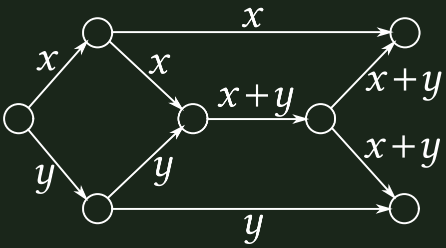
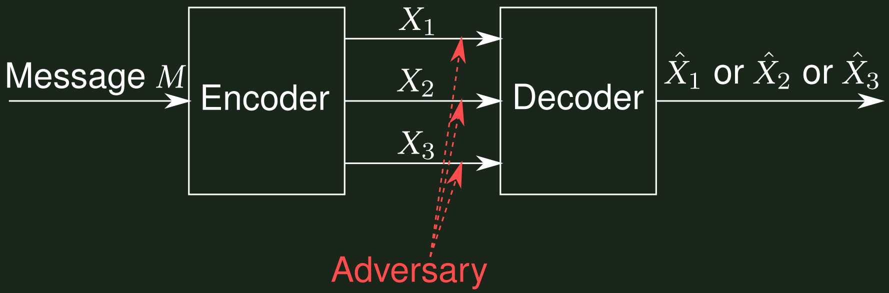


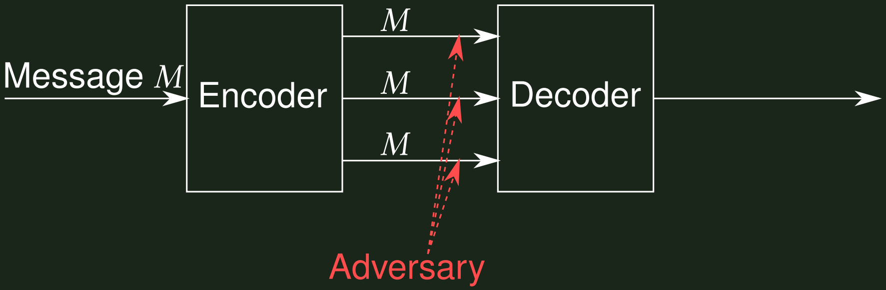
Motivating Toy Problem



- $M \in \{1, 2, \ldots, 2^{qR}\}$
- $X_i \in \{1, 2, \ldots, 2^q\}$
- M must be recoverable from any two of X_1, X_2, X_3
- Adversary may replace one of the three packets
- Decoder must output one packet without error

Finite Field Constructions

(3,1) MDS code: Let $M \in \mathbb{F}$



Achieves $R = 1$

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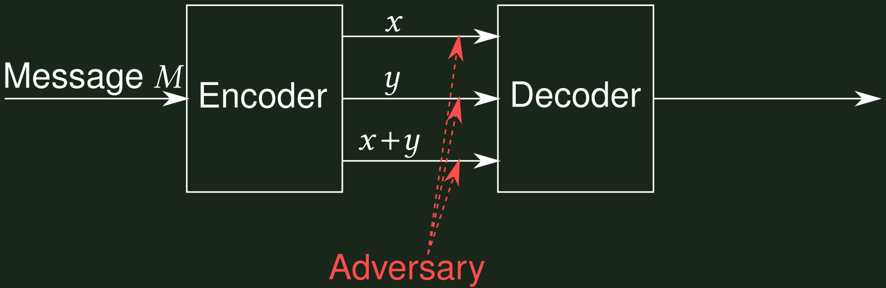
(3,2) MDS code: Let $M = (x, y)$, $x, y \in \mathbb{F}$



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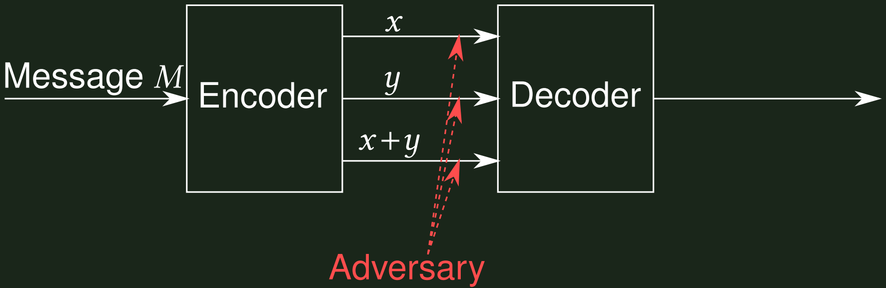

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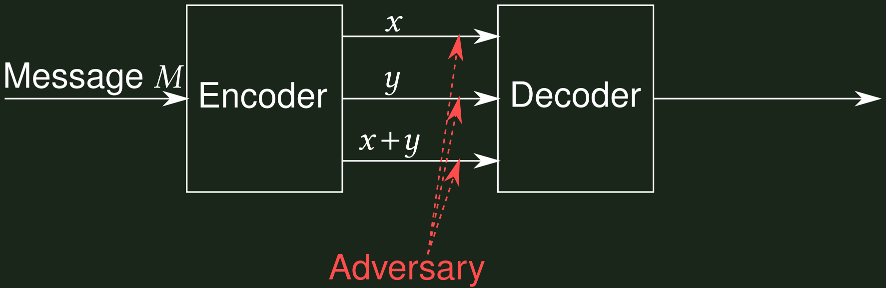

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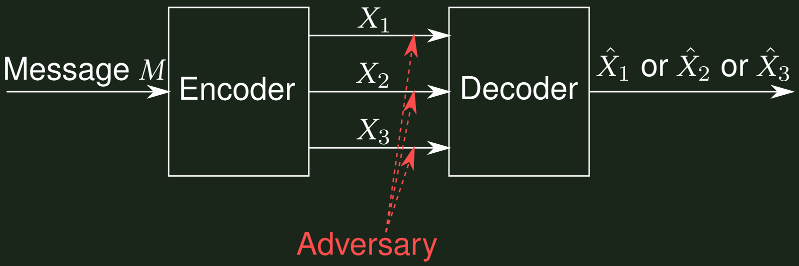
- Finite field code cannot do better than $R = 1$

What would it take to achieve $R = 2$?



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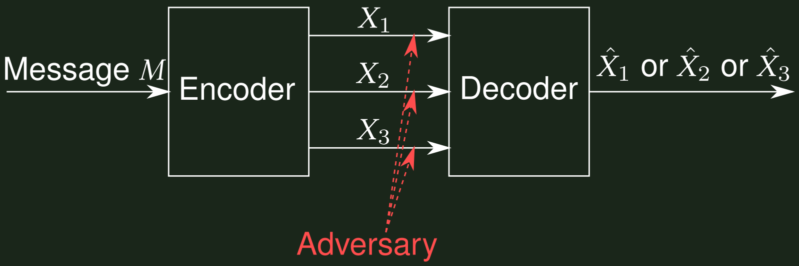

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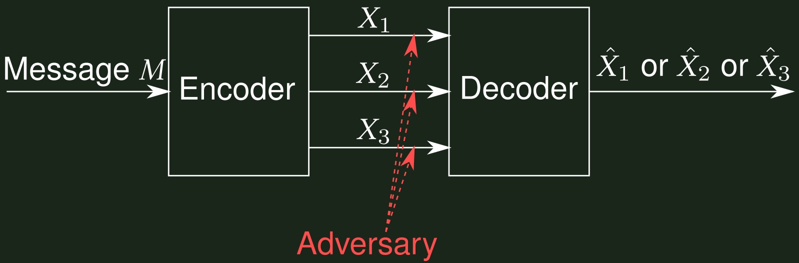

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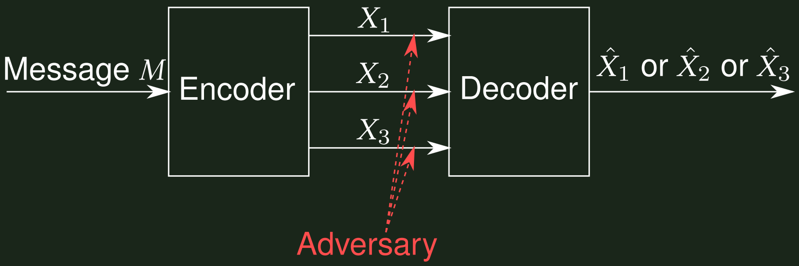

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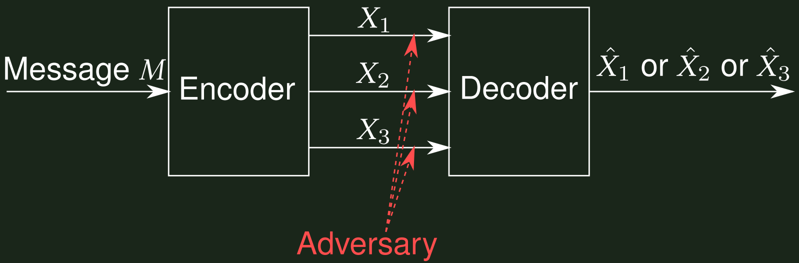

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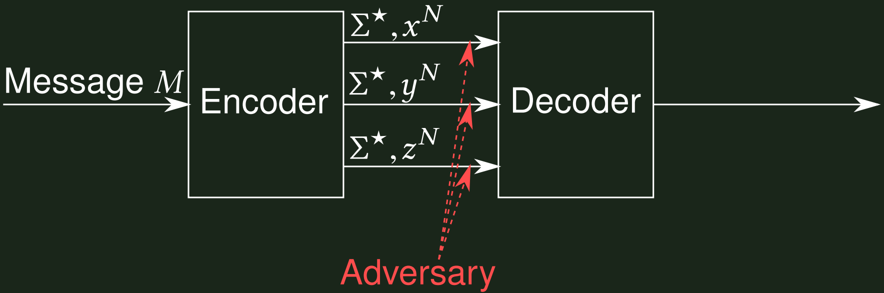
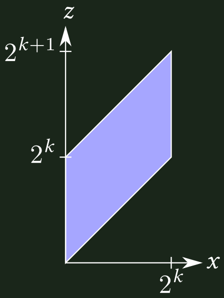


MDS structure



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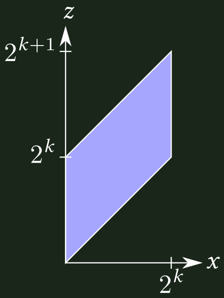

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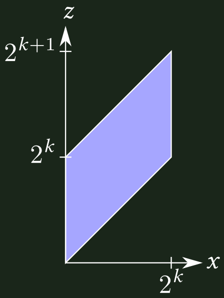

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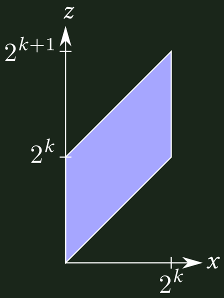

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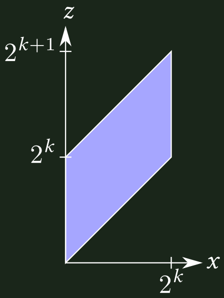

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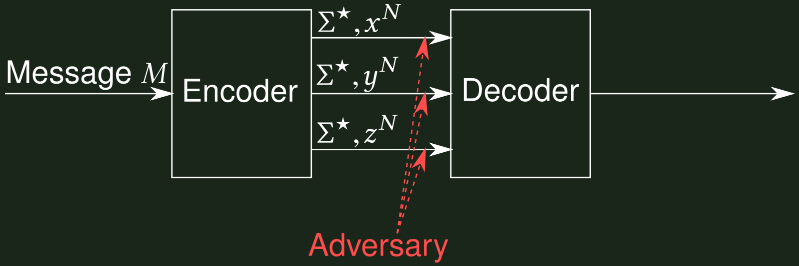
- (x^N, y^N, z^N) form a $(3, 2)$ MDS polytope code

Decoding


Message M
Encoder
$\Sigma^\star, x^N$
$\Sigma^\star, y^N$
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Decoder
Adversary

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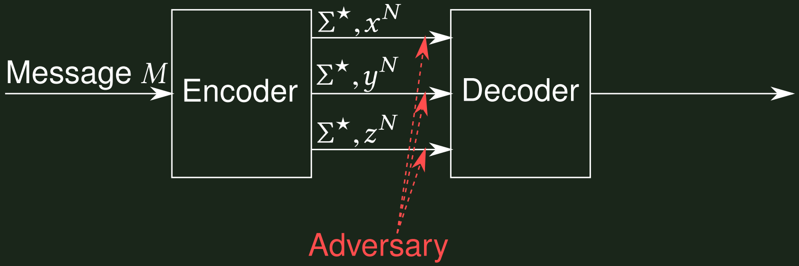

- Recover the should-be covariance $\Sigma^\star$ using majority rule

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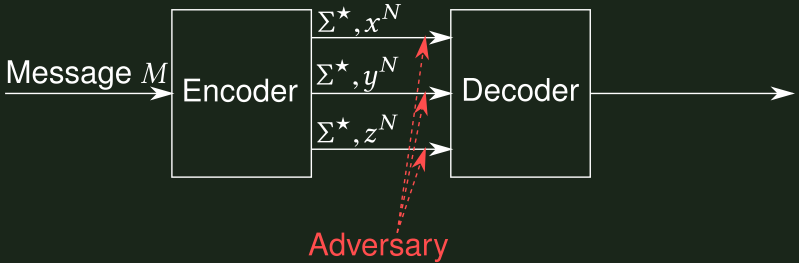

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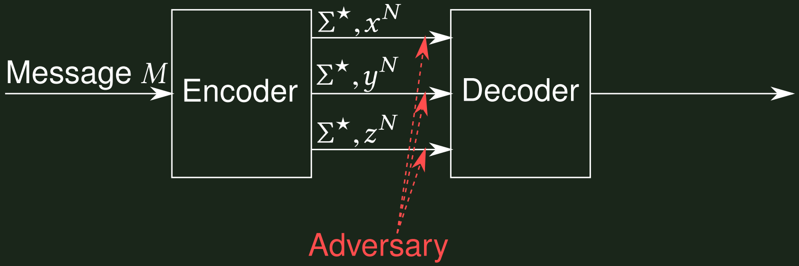

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- By comparing $\Sigma^\star$ with Σ, the decoder can always find a trustworthy packet

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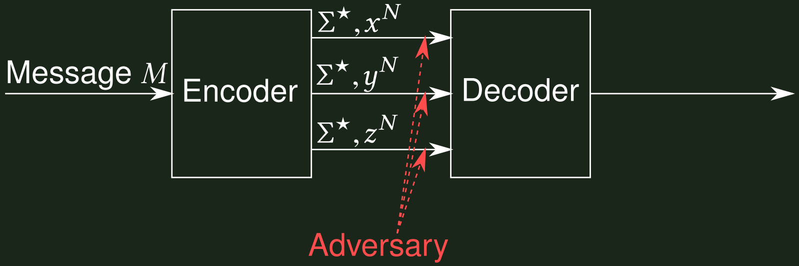

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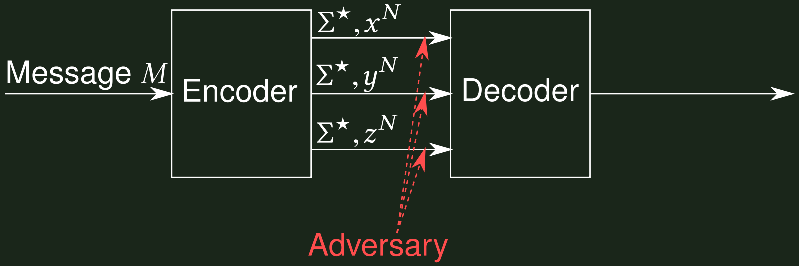

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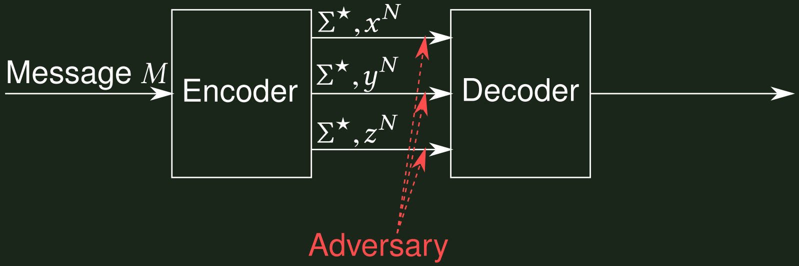

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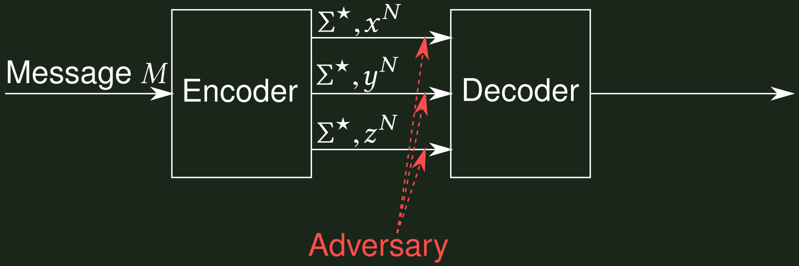

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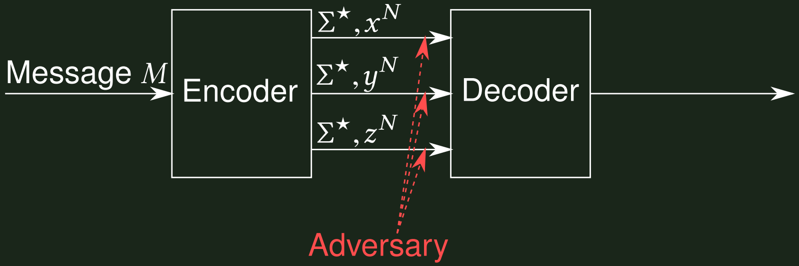

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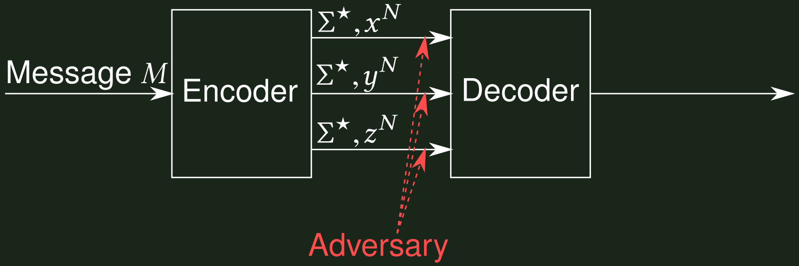

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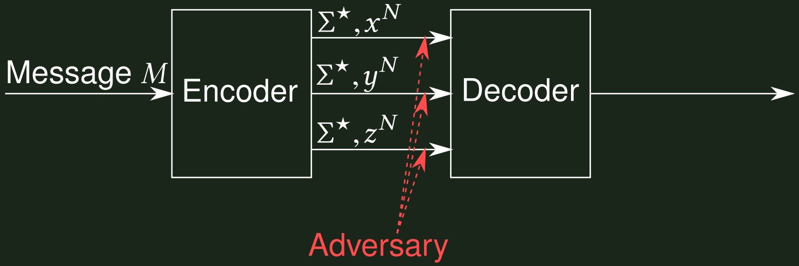

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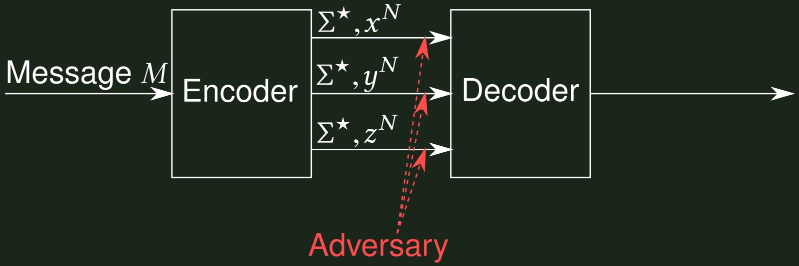

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- Therefore all packets are trustworthy

Outline

- Polytope codes in general
- Polytope codes in network coding
- Polytope codes in distributed storage systems
- Polytope codes in multiple descriptions

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These constructions can mimic most finite field linear codes

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  For C satisfying $CA = 0$, $\|Cy^N\|^2 = 0$, so $Cy^N = 0$, i.e. all linear constraints match

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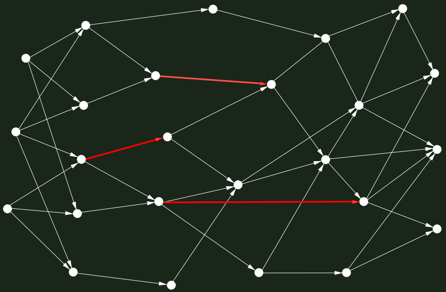
Network Error Correction



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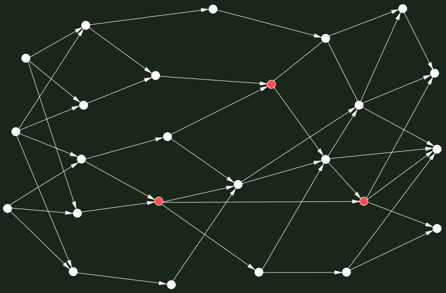

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- Converse via network version of the Singleton bound
- Achievability via network version of (linear) MDS codes

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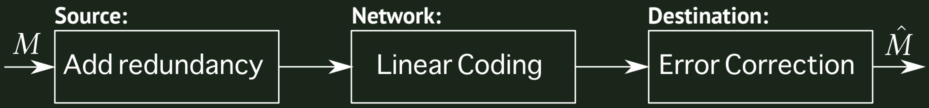
- Achievability via network version of (linear) MDS codes

Can be viewed as a separation theorem:



Theorem (Cai-Yeung (2006))

For a single multicast, and an adversary that controls any z unit-capacity edges:

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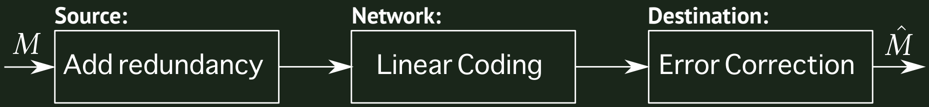
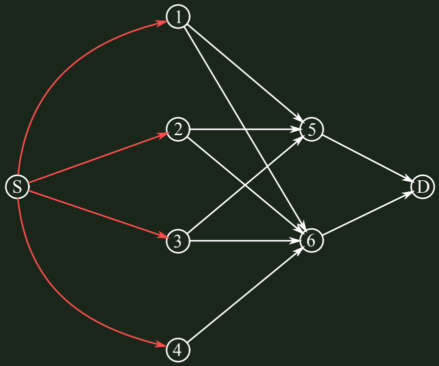


Polytope codes allow error detection/correction inside the network

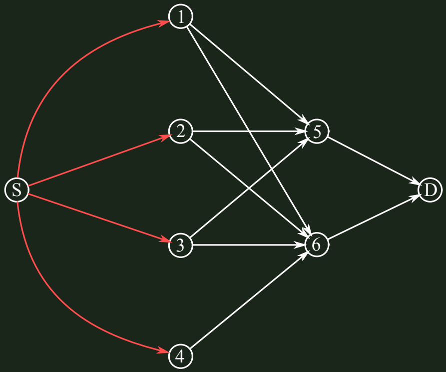
The Caterpillar Network



- Single unicast from S to D
- All links have unit capacity
- Adversary may control any one of the red edges
- Simple upper bound: $C \leq 2$

Polytope Code Achievability

Let message $m = (x^N, y^N)$, where $x^N, y^N \in \{1, \ldots, 2^k\}^N$



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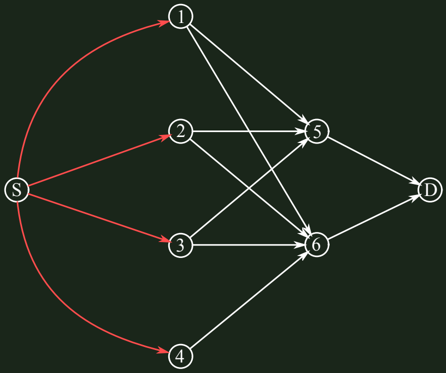

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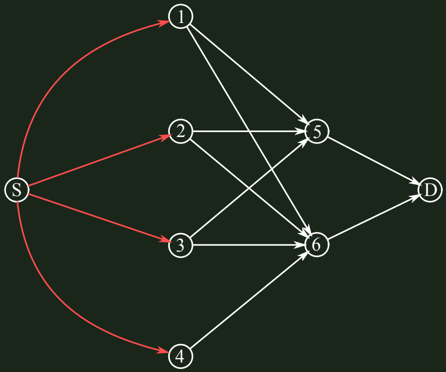
(x^N, y^N, z^N, w^N) form a $(4,2)$ MDS polytope code



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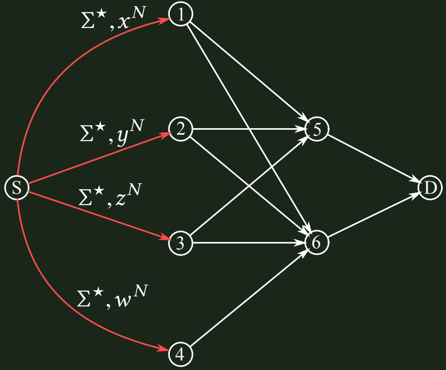

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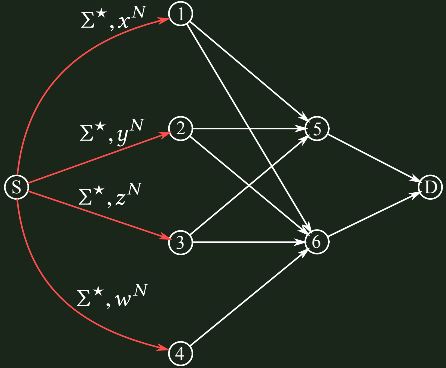

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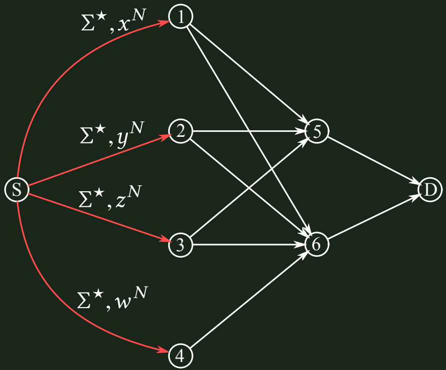

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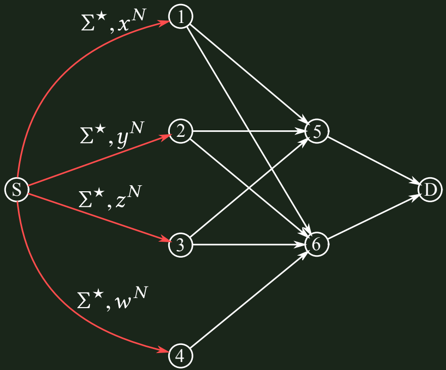

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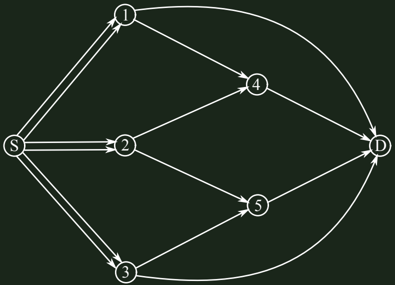
No finite field linear code achieves this rate

Cockroach Network



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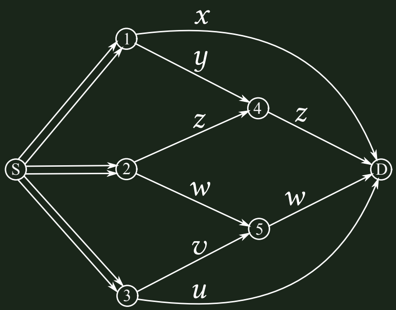

- One node is controlled by the adversary — controls all outgoing messages
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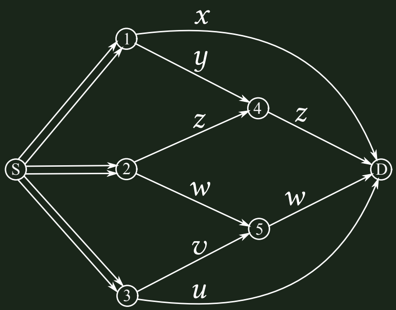

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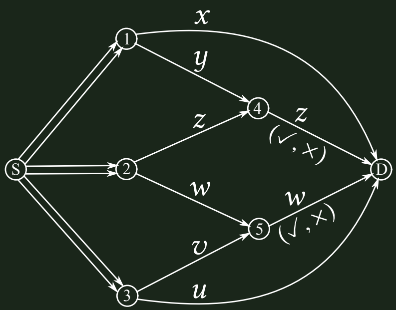

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- Let $(x^N, y^N, z^N, w^N, v^N, u^N)$ be a $(6, 2)$ MDS polytope code
- $\Sigma^\star$ included in all packets
- Nodes 4 and 5 compare covariance of incoming pair of packets — transmit outcome of comparison

A Class of Networks Solved by Polytope Codes

Theorem (Kosut-Tong-Tse (2014))

Polytope codes achieve the cut-set bound if

- *Network is planar*
- *1 adversary node*
- *No node has more than 2 unit-capacity output edges*
- *No node has more outputs than inputs*

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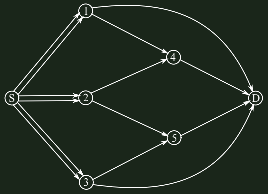
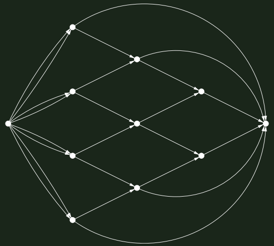
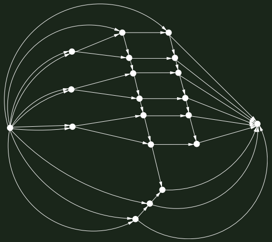


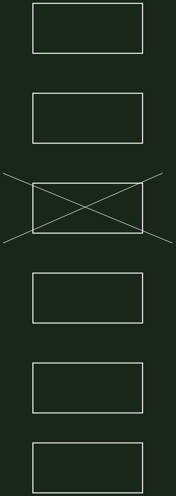




Outline

- Polytope codes in general
- Polytope codes in network coding
- Polytope codes in distributed storage systems
- Polytope codes in multiple descriptions

Distributed Storage Systems



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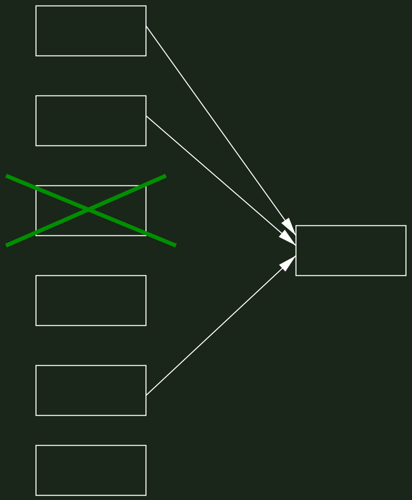

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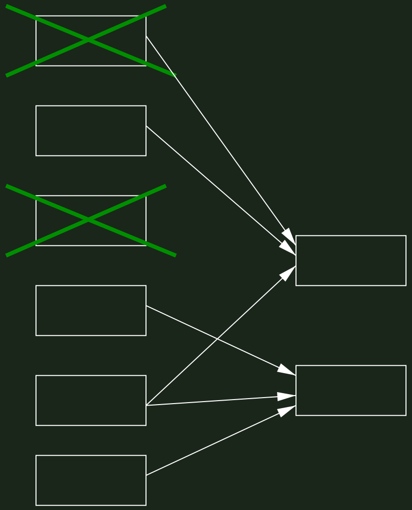

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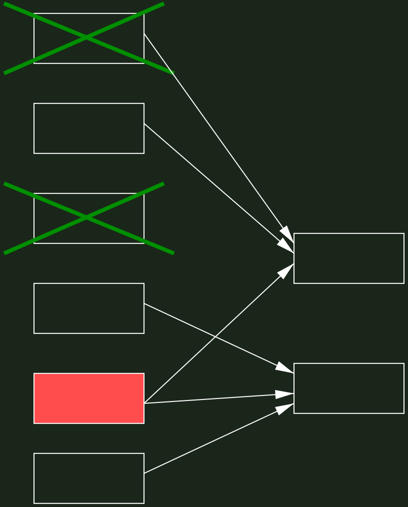

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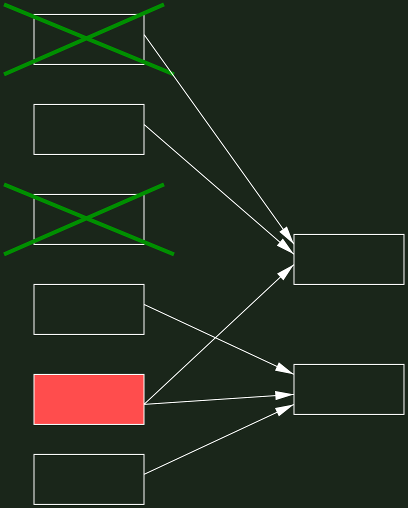

- Single adversarial node may transmit many times

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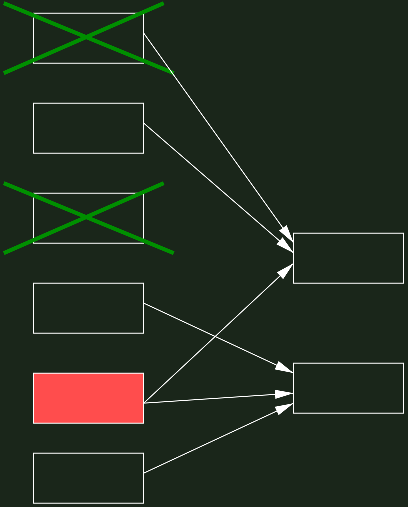

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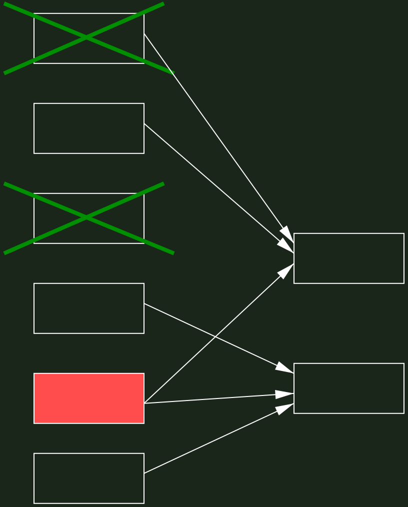


- Single adversarial node may transmit many times
- Naturally suited to the node-based adversary model
- Functional repair rather than exact repair

Parameters

- α: Storage capacity of single node
- β: Download bandwidth when forming new node
- n: Number of active storage nodes
- k: Number of nodes contacted by data collector (DC) to recover file
- d: Number of nodes contacted to construct new node
- z: Number of (simultaneous) adversarial nodes

Existing Bounds

- Pawar-El Rouayheb-Ramchandran (2011): Storage capacity is upper bounded by

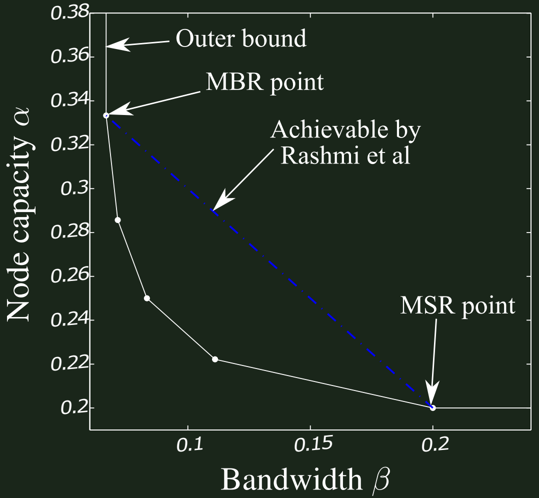
$$C \leq \sum_{i=0}^{k-2z-1} \min\{(d-2z-i)\beta, \alpha\}$$

  Identical to bound without adversaries where $k \to k-2z$ and $d \to d-2z$

- Rashmi et al (2012): The Minimum Storage Regeneration (MSR) and Minimum Bandwidth Regeneration (MBR) points are achievable with exact repair

Existing Bounds, Ctd.

Parameters: $n = 8$, $k = d = 7$, $z = 1$



Structure of Polytope Code for DSS

- Initial file to store $m \in \{1, 2, \ldots, 2^k\}^{R \times N}$

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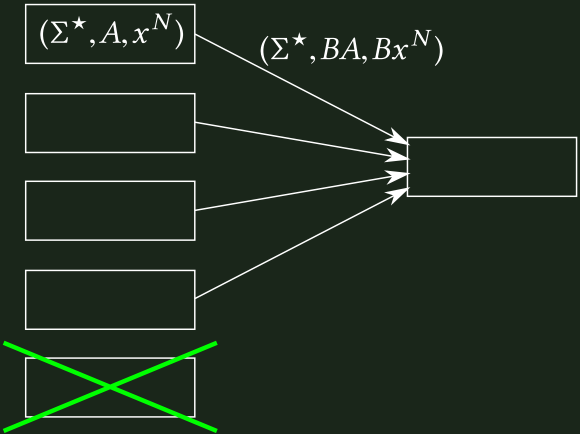
- For storage packet $x^N \in \{1, 2, \ldots, 2^k\}^{\alpha \times N}$
  For transmission packet $x^N \in \{1, 2, \ldots, 2^k\}^{\beta \times N}$

Messages for new node

Choose linear transformation $B \in \mathbb{Z}^{\beta \times \alpha}$



New Node Construction

Given $(\Sigma^\star, A_i, y_i^N)$ for $i = 1, 2, \ldots, d$

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- Goal: Find trustworthy packets from which to form stored data

Syndrome Graphs

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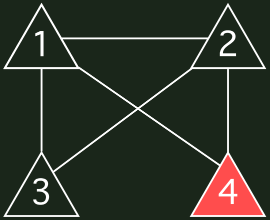
Example: $d = 4$ and $z = 1$:



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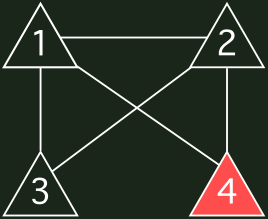

- Use packets 1 and 2 to form stored data
- This is the typical case where $d - 2z$ trustworthy packets can be identified

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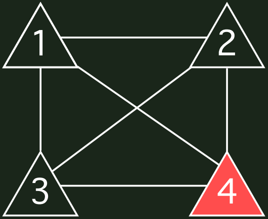

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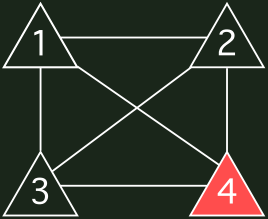

- Use all packets to form stored data
- Linear constraints (because covariances match) mean the adversary data is uncorrupted

Syndrome Graphs

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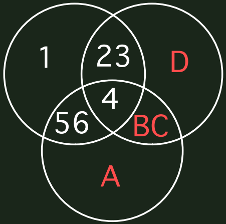
Example: $d = 10$ and $z = 4$

- Call honest nodes 1,2,3,4,5,6 and adversary nodes A,B,C,D
- Three cliques of size 6:

123456
456ABC
234BCD



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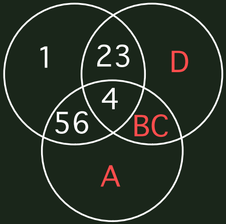

- Use packet 4 to form stored data
- Less than $d - 2z$ trustworthy packets!

Algorithm to find trustworthy packets

1. Discard all packets not in a clique of size $d - z$
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- Any chosen adversarial packet must match covariances with all $d - z$ honest nodes
- If $R \leq (d - z)\beta$, then linear constraints ensure all stored data is uncorrupted
- This procedure always finds at least $d - v_z$ packets where

$$v_z = (\lfloor \tfrac{z}{2} \rfloor + 1)(\lceil \tfrac{z}{2} \rceil + 1)$$

z	1	2	3	4	5	6
v_z	2	4	6	9	12	16

Note $v_z = 2z$ only for $z \leq 3$

Resulting Achievability Bound

Theorem (Kosut (2013))

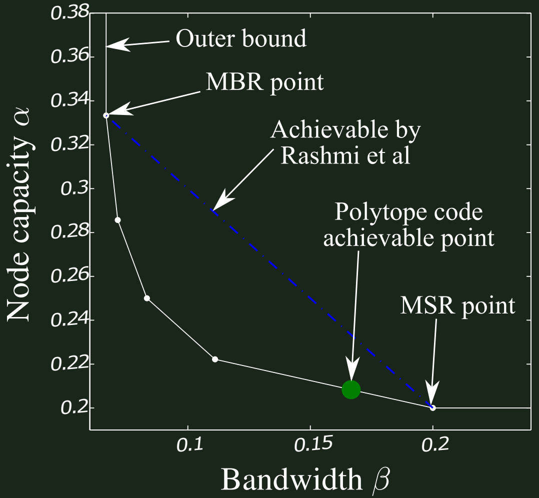
There exists a distributed storage code achieving rate

$$\min\left\{\sum_{i=0}^{k-v_z-1}\min\{(d-v_z-i)\beta,\alpha\},\ (d-z)\beta,\ (k-z)\alpha\right\}.$$

where $v_z = (\lfloor \frac{z}{2} \rfloor + 1)(\lceil \frac{z}{2} \rceil + 1)$.

Achievability Plot

Parameters: $n = 8$, $k = d = 7$, $z = 1$



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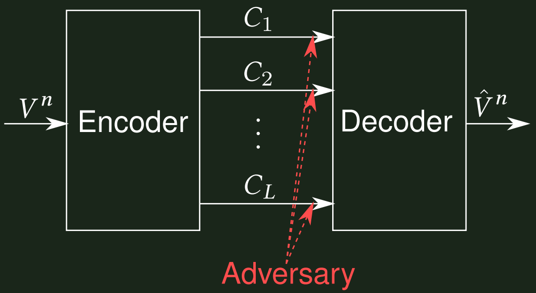
Adversarial Multiple Descriptions

Problem formulated in Fan-Wagner-Ahmed (2013)



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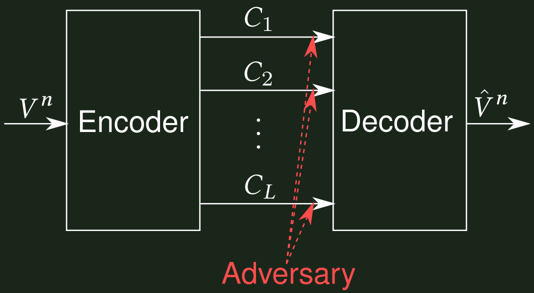
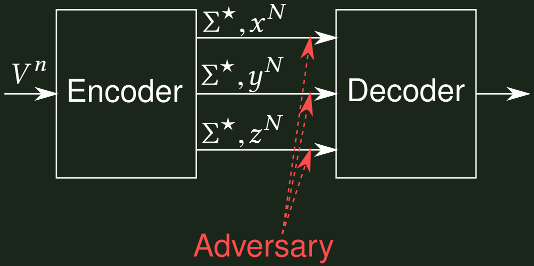


Construct a single code that fails gracefully — fewer adversarial packets gives smaller distortion

- $V^n \in \{0,1\}^n$
- $C_i \in \{1, 2, \ldots, 2^{nR}\}$
- Adversary controls z packets
- Distortion: $D = \sum_{i=1}^{n} d(X_i, \hat{X}_i)$ where d is the erasure distortion

3-Description Example

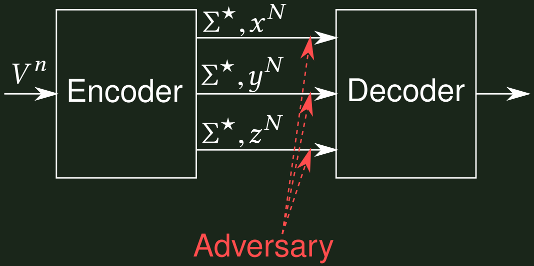
- $R = 1/2$
- Write $V^n = (x^N, y^N)$ where $x^N, y^N \in \{1, 2, \ldots, 2^k\}^N$
- $z^N = x^N + y^N$



- If $z = 0$, then entire source sequence can be decoded, so $D = 0$
- If $z = 1$, then one trustworthy packet (half the message) can be identified, so $D = 1/2$

3-Description Example

- $R = 1/2$
- Write $V^n = (x^N, y^N)$ where $x^N, y^N \in \{1, 2, \ldots, 2^k\}^N$
- $z^N = x^N + y^N$



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- If $z = 1$, then one trustworthy packet (half the message) can be identified, so ~~$D = 1/2$~~

  Problem: z^N is not a systematic part of source V^n

3-Description Example

- $V^n = (V_1^{n/3}, V_2^{n/3}, V_3^{n/3})$, and write $V_i^{n/3} = (x_i^N, y_i^N)$

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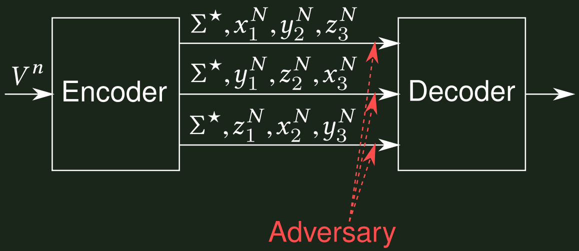

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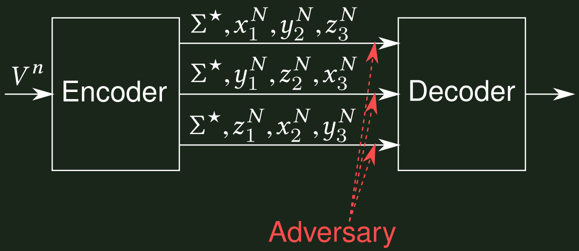

- Decoder can always identify one trustworthy packet, containing two systematic parts of V^n
- Thus $D = 2/3$

Conclusions

- Polytope codes operate on the integers and can mimic most finite field codes
- Covariances are used as checksums, allowing for:
  - Partial decoding
  - Distributed error detection/correction
- Polytope codes outperform finite field codes, but many achievable results have no matching converse
  — seems to be very hard to find the best polytope code
- All results for omniscient adversary — weaker adversary models require different techniques