On the Connection Between Multiple-Unicast Network Coding and Single-Source Single-Sink Network Error Correction

Jörg Kliewer
NJIT

Joint work with Wentao Huang and Michael Langberg

Network Error Correction

Problem:

- Adversary has control of some edges in a network
- Objective: Design coding scheme that is resilient against adversary
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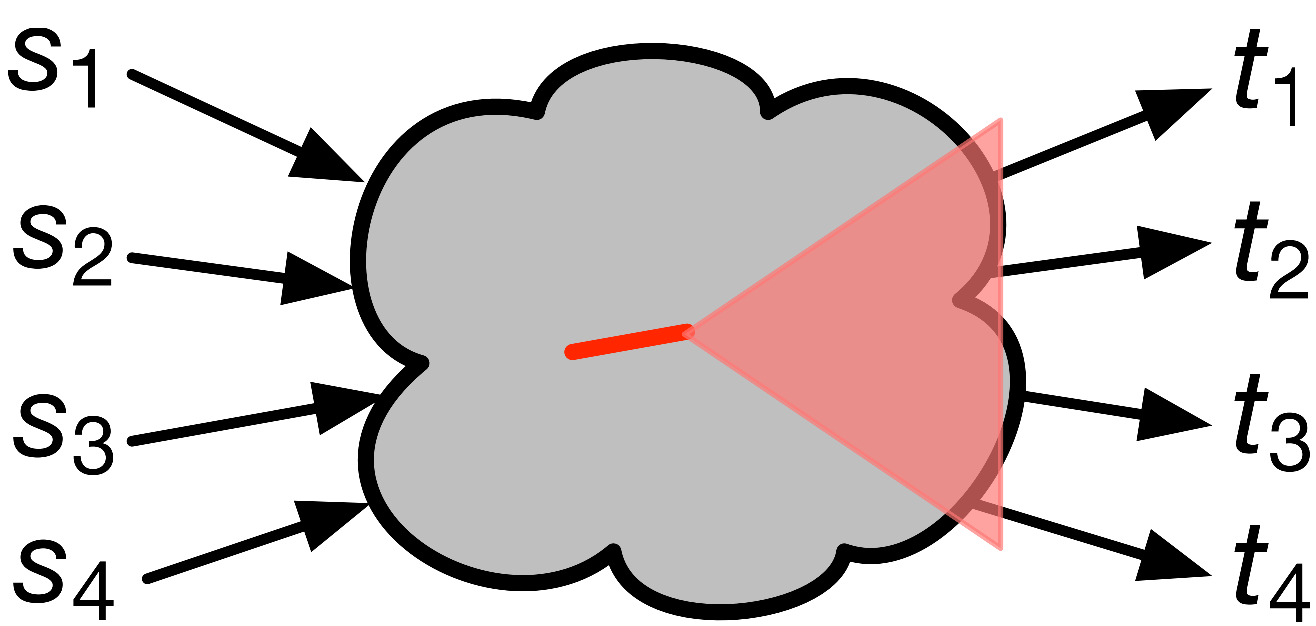
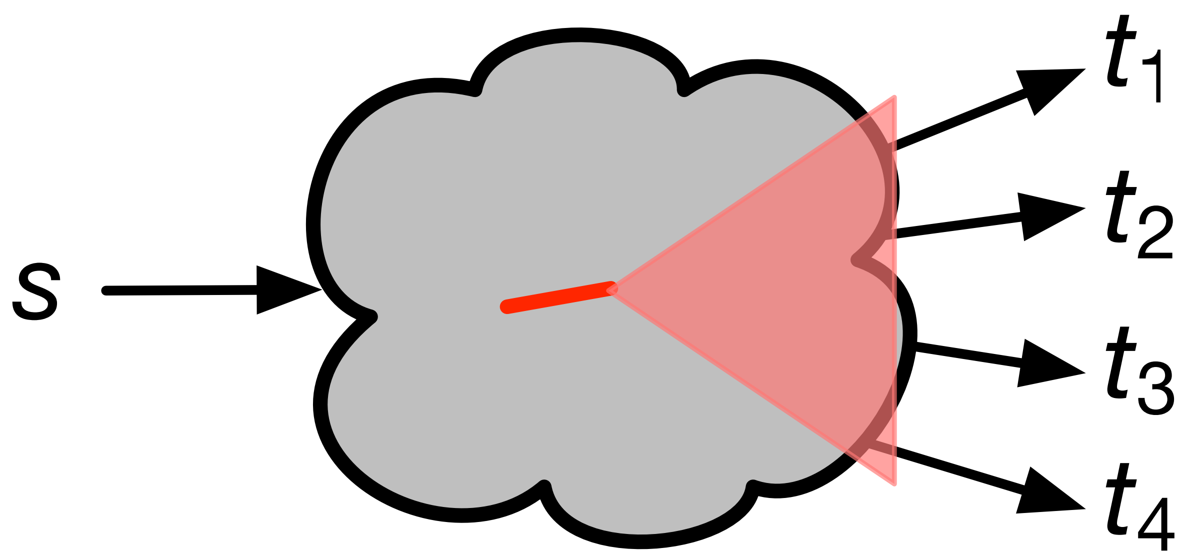


Known Cases

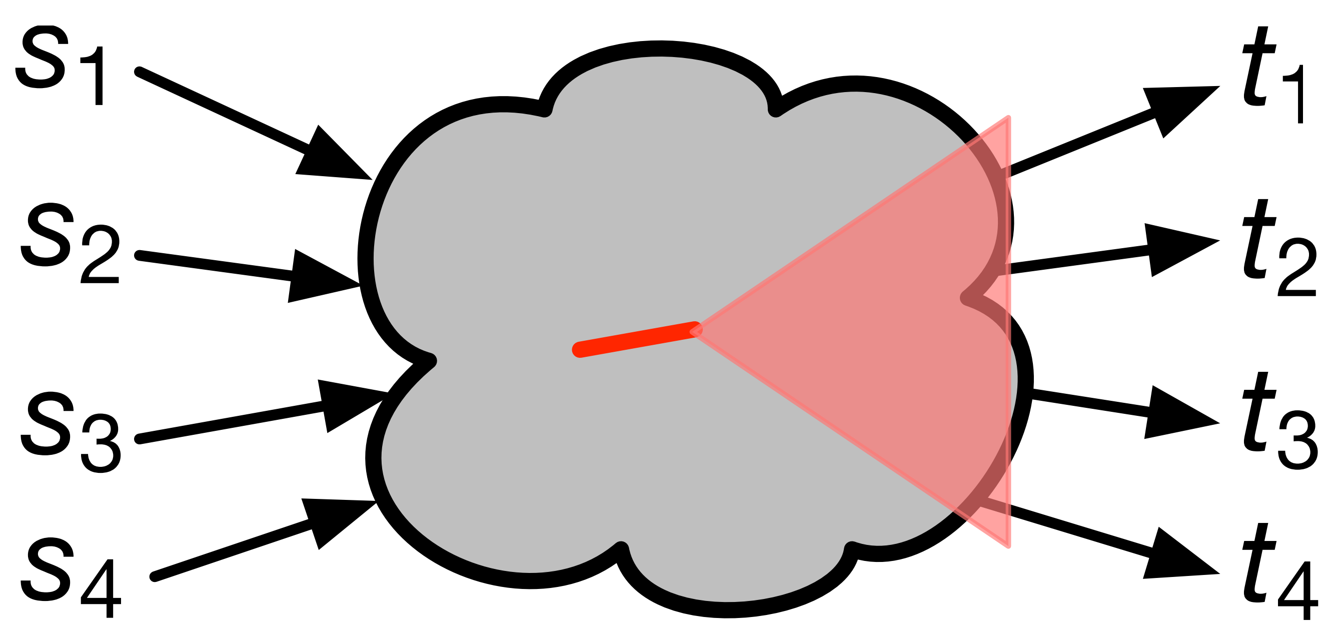
- Adversary controls z links in the network
- **Single source multicast**, equal capacity links
  - Capacity: mincut – $2z$
  - Code design, e.g., in [Cai & Yeung 06], [Koetter & Kschischang 08], [Jaggi et al. 08], [Silva et al. 08], [Brito, Kliewer 13]



Less Known and Studied Cases

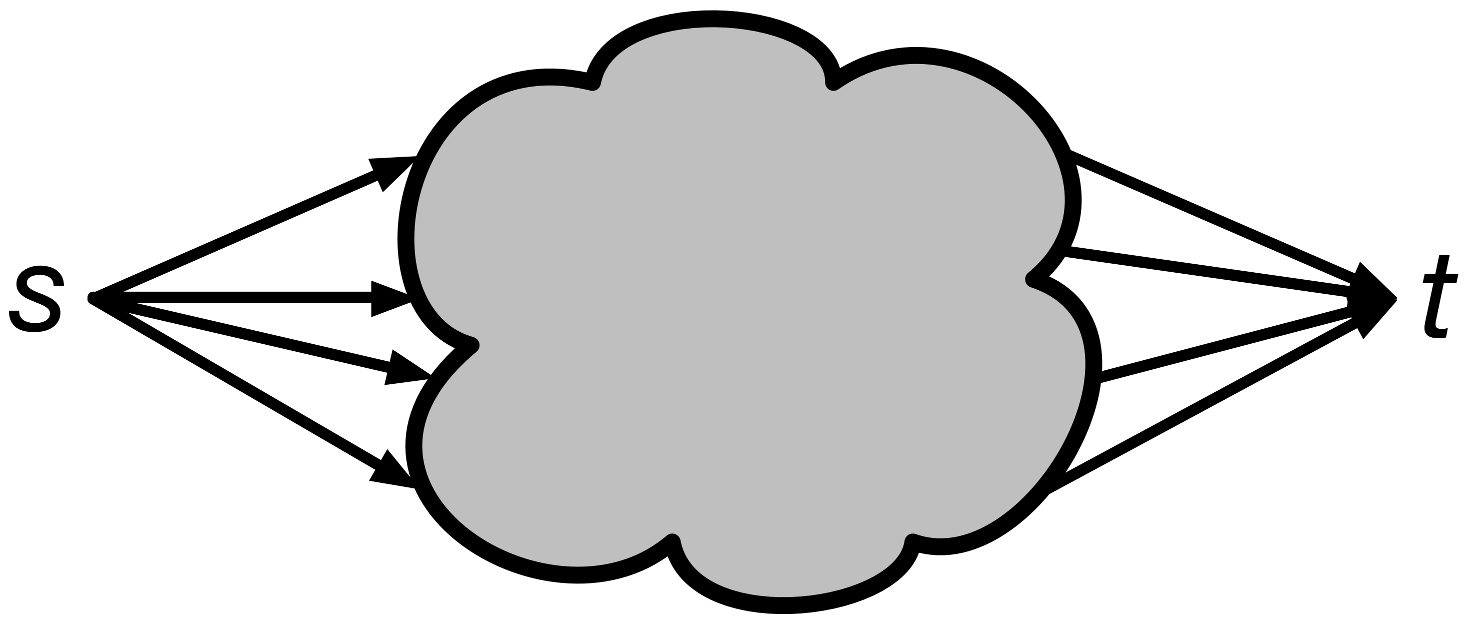
- **Single source multicast:** different edge capacities, node adversaries, restricted adversaries (e.g., [Kosut, Tong, Tse 09], [Kim et al. 11], [Wang, Silva, Kschischang 08])

- **Multiple sources and terminals:** Upper and lower capacity bounds [Vyetrenko, Ho, Dikaliotis 10], [Liang, Agrawal, Vaidya 10]



This Work

- Single-source single-sink
- Acyclic network
- Edges may not have unit capacity



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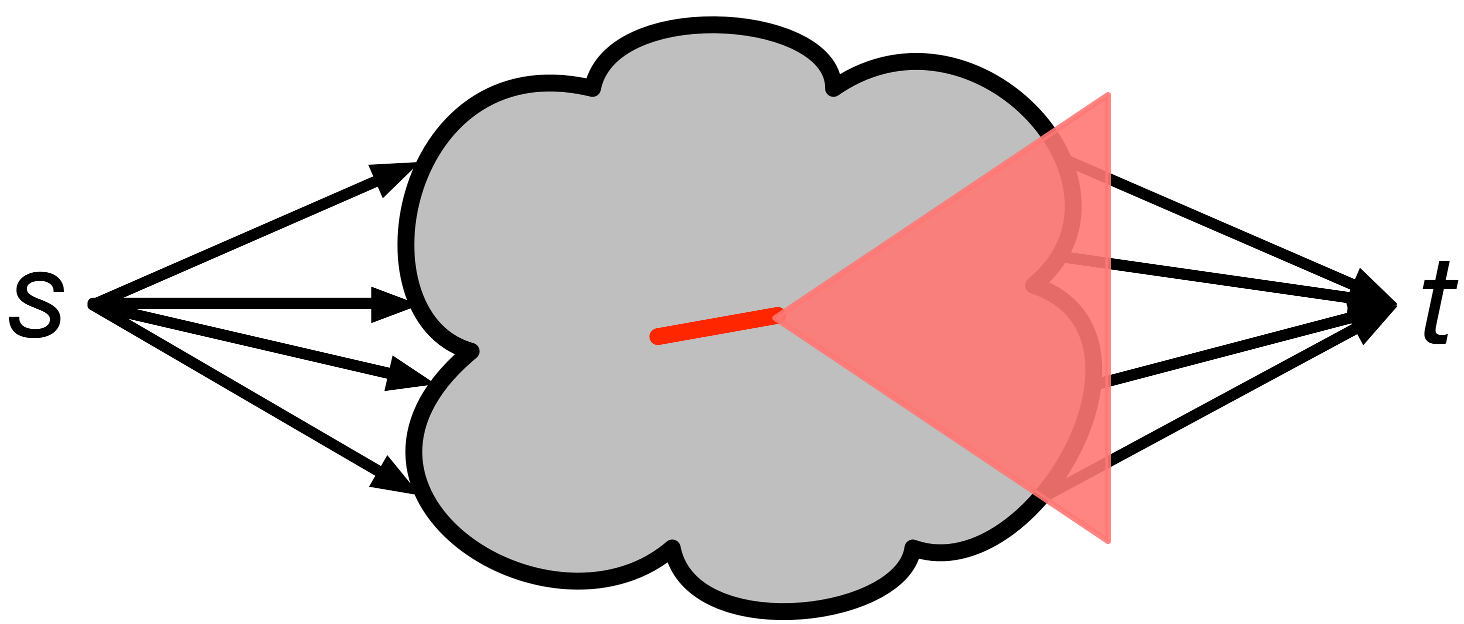

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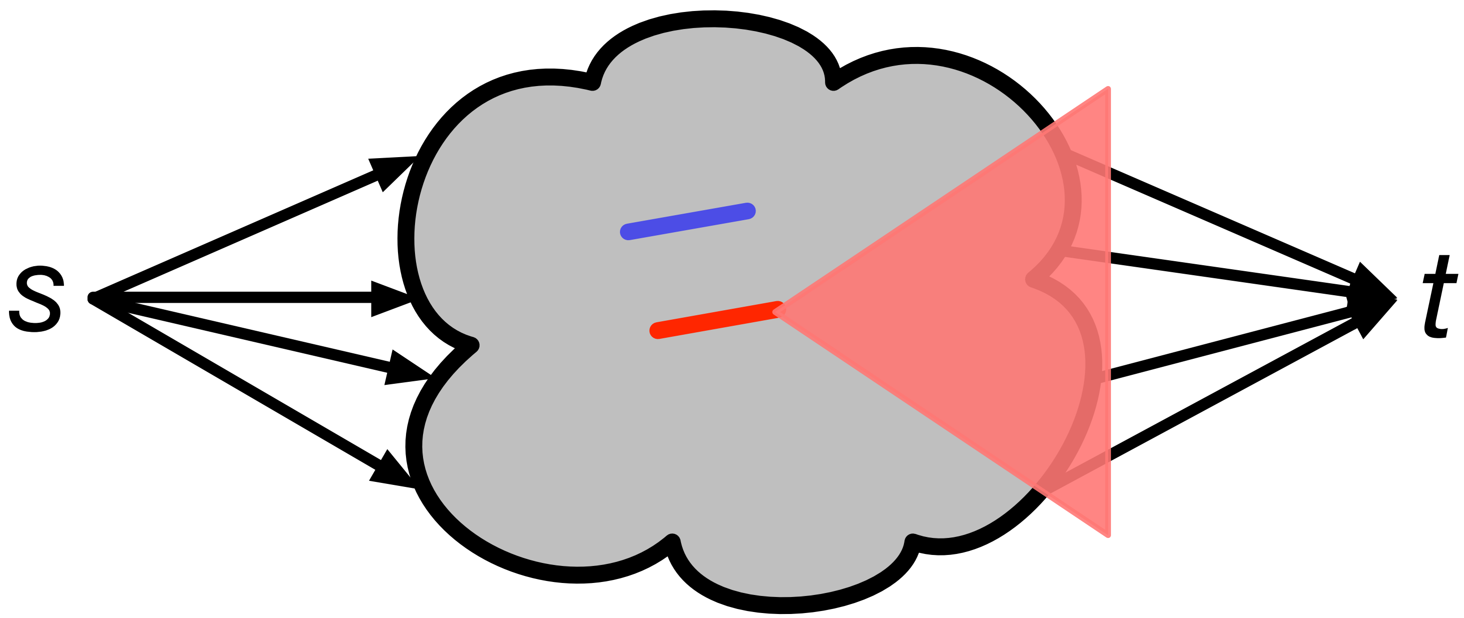

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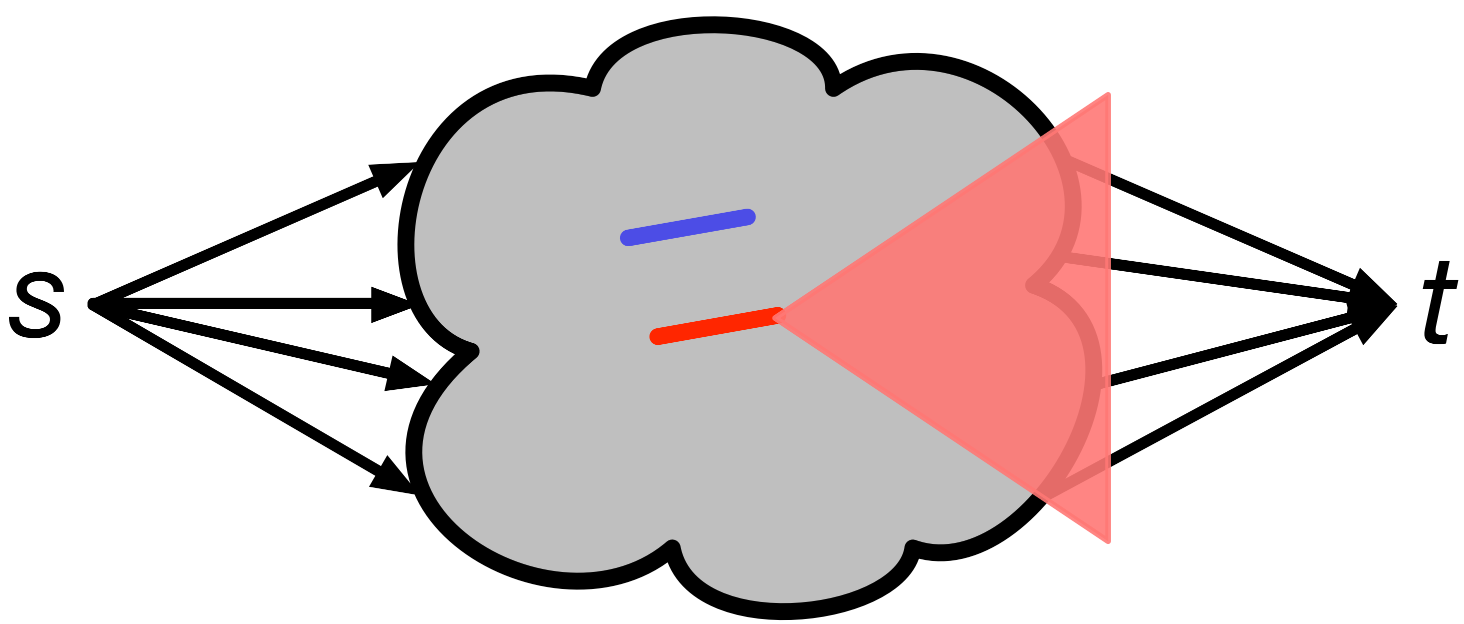
- **Reliable communication rate?**



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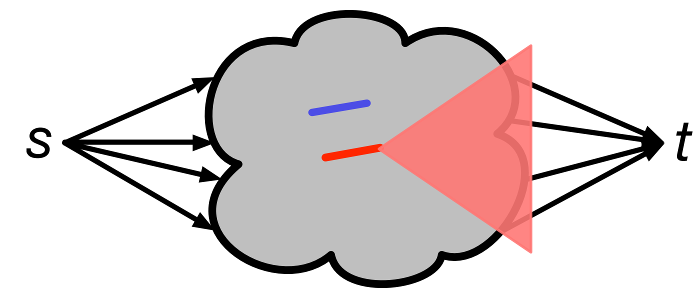

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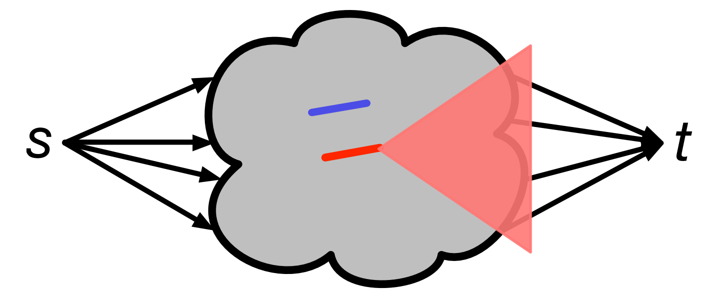

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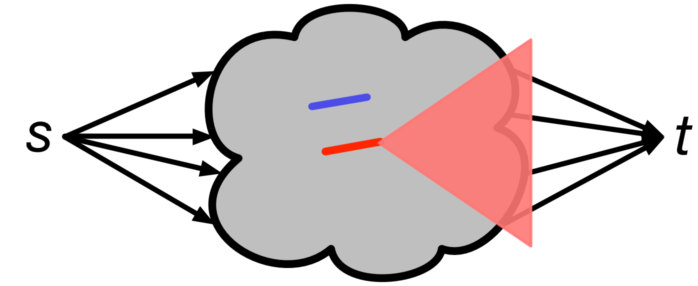

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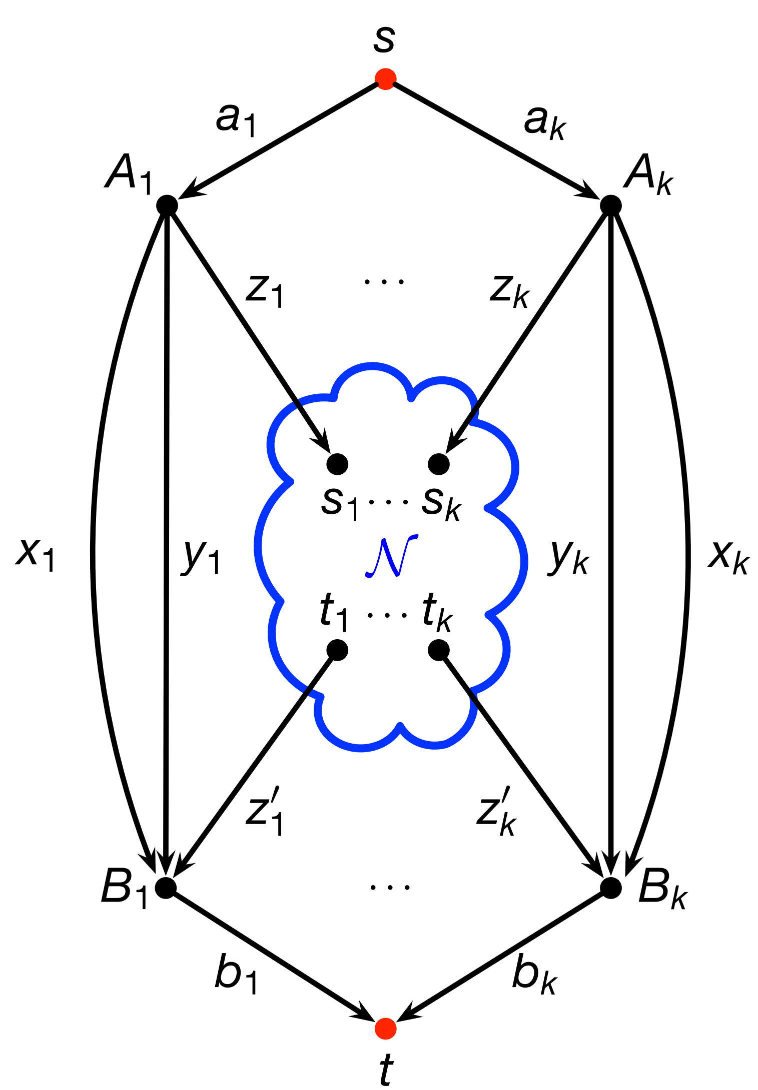

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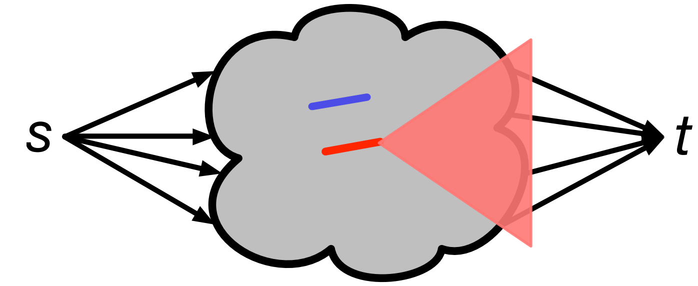

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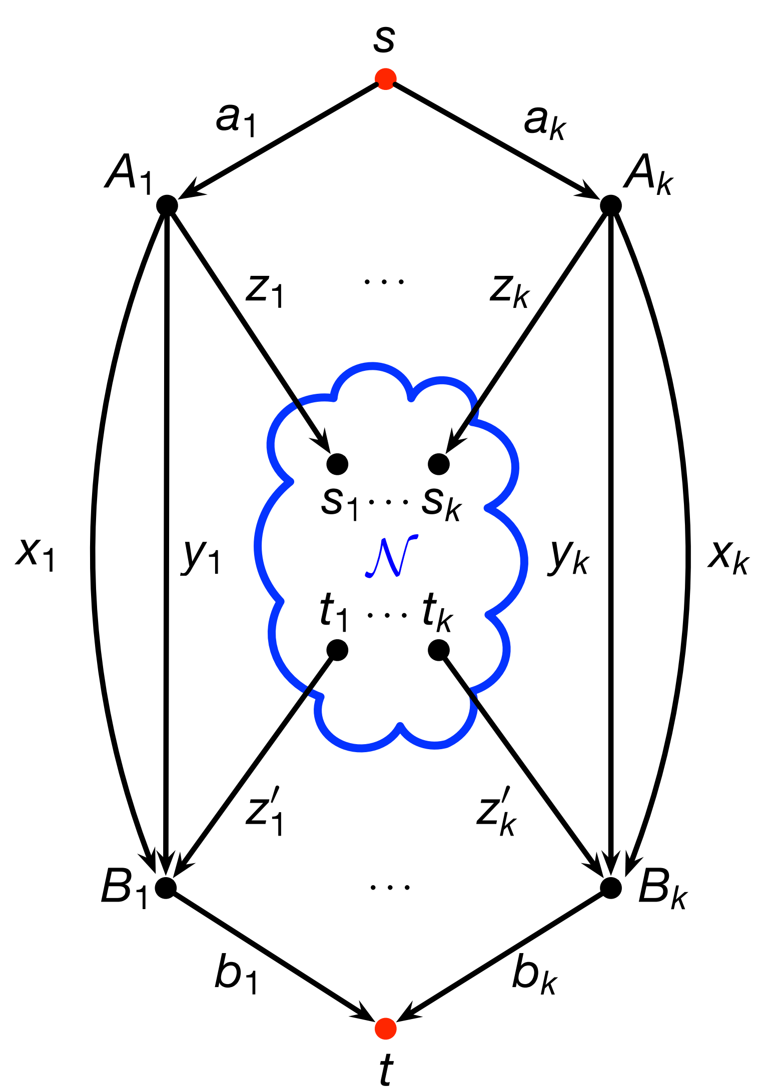

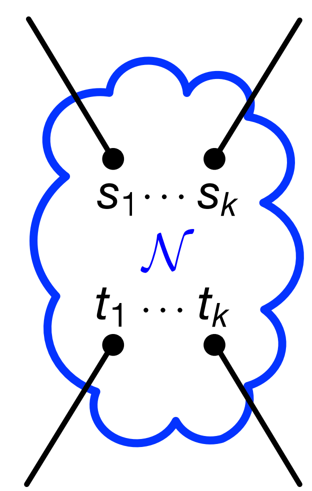

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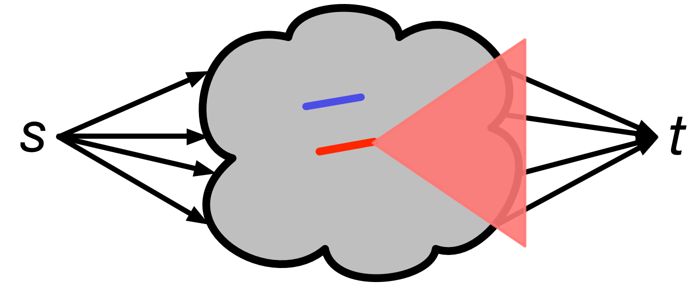
Network error correction problem: 

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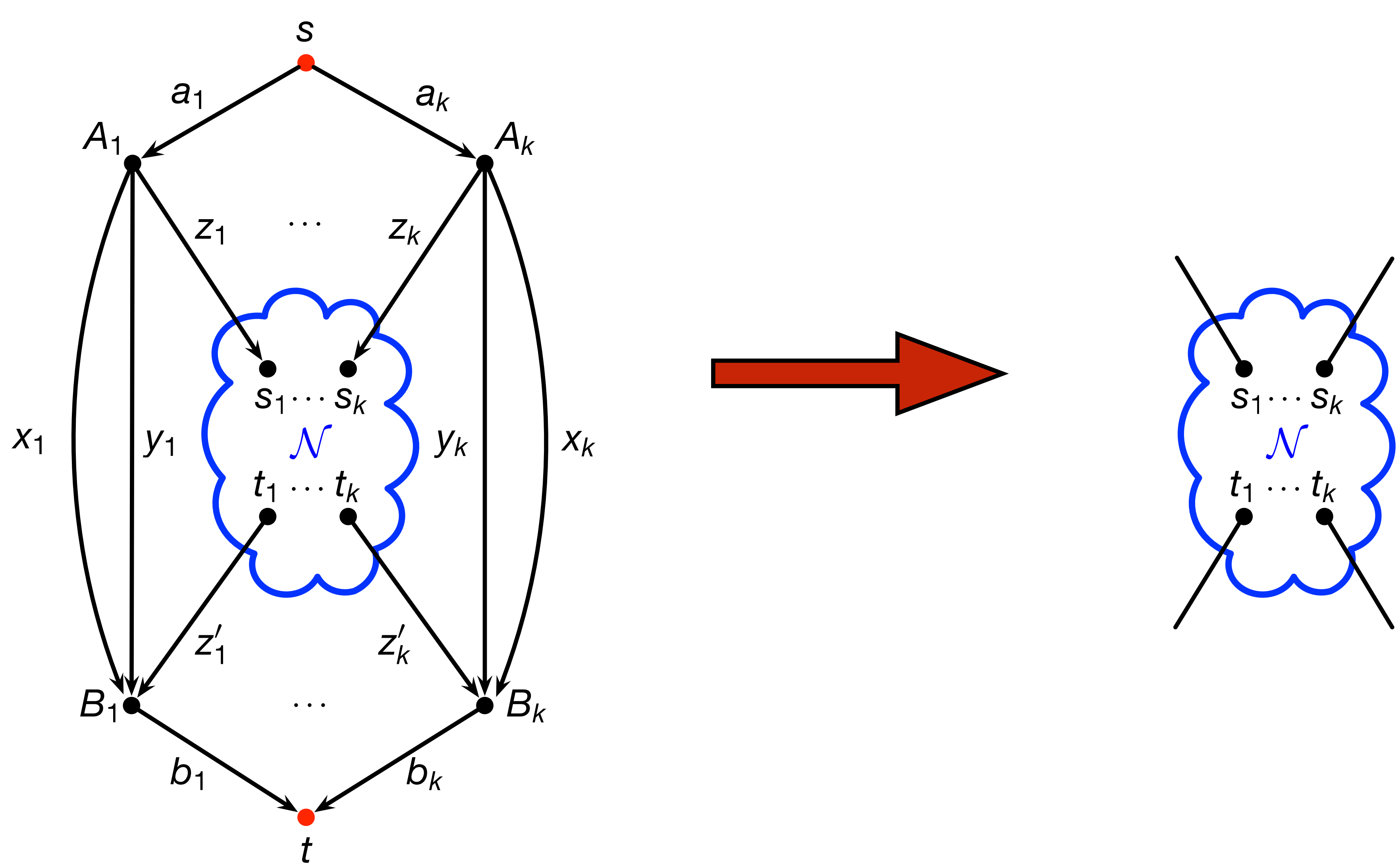

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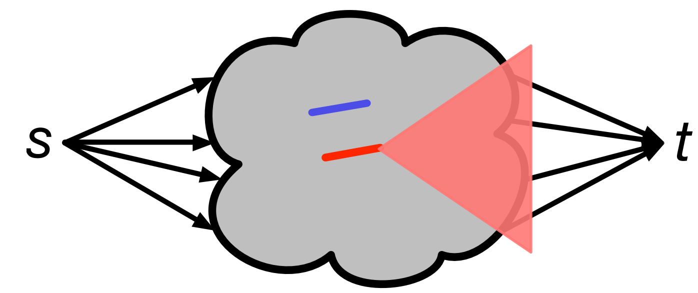

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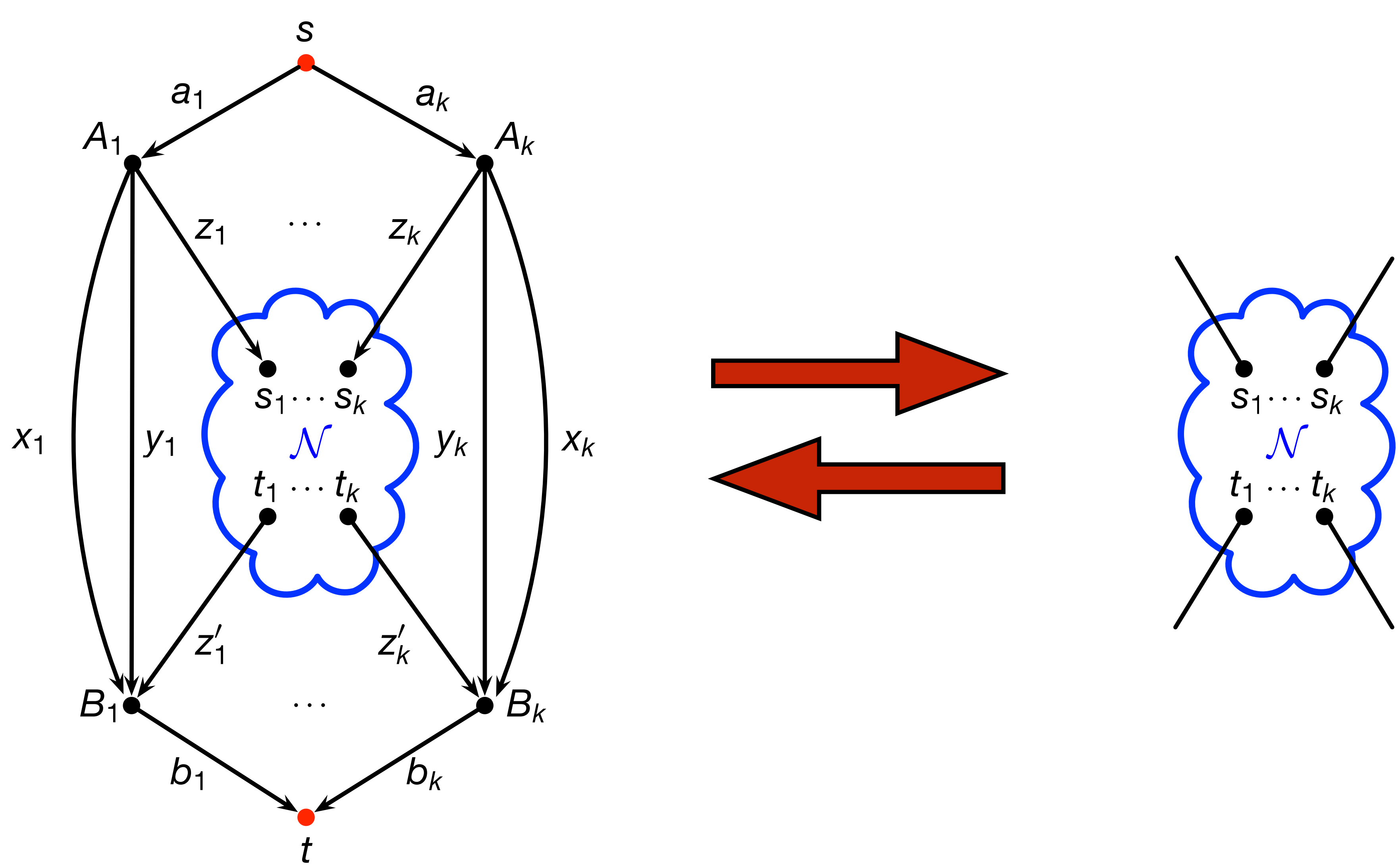

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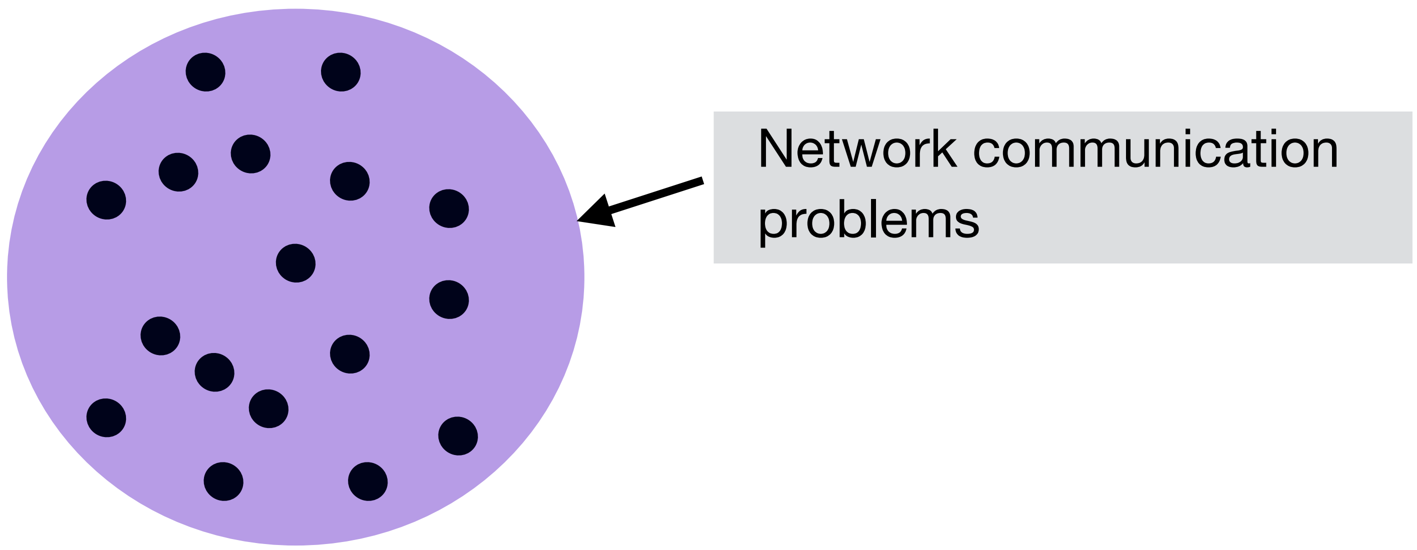

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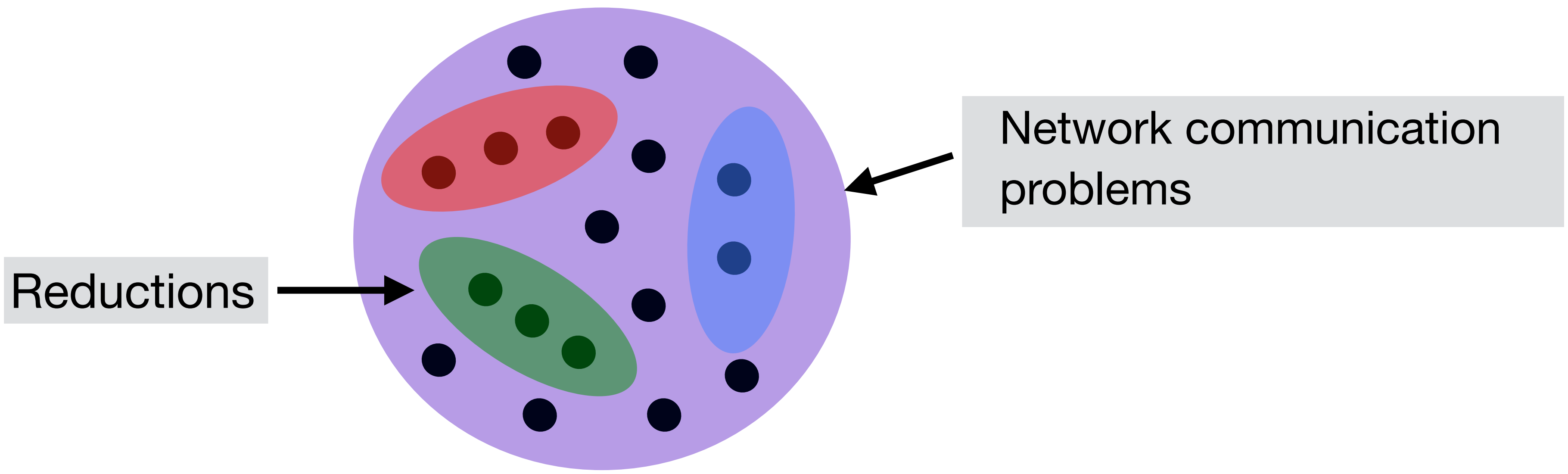

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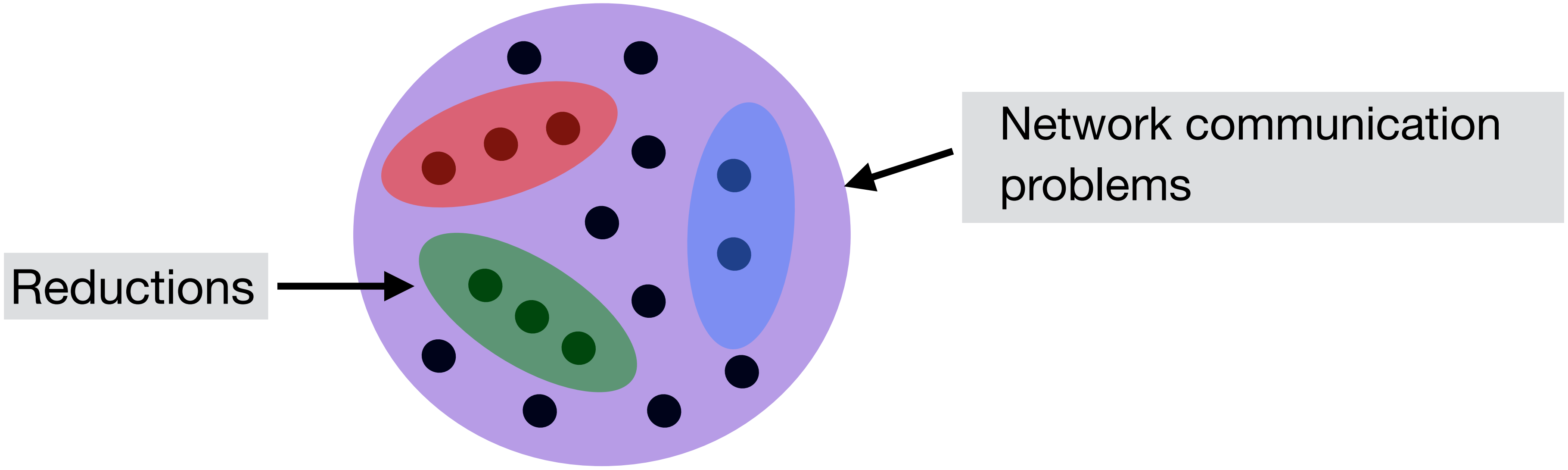

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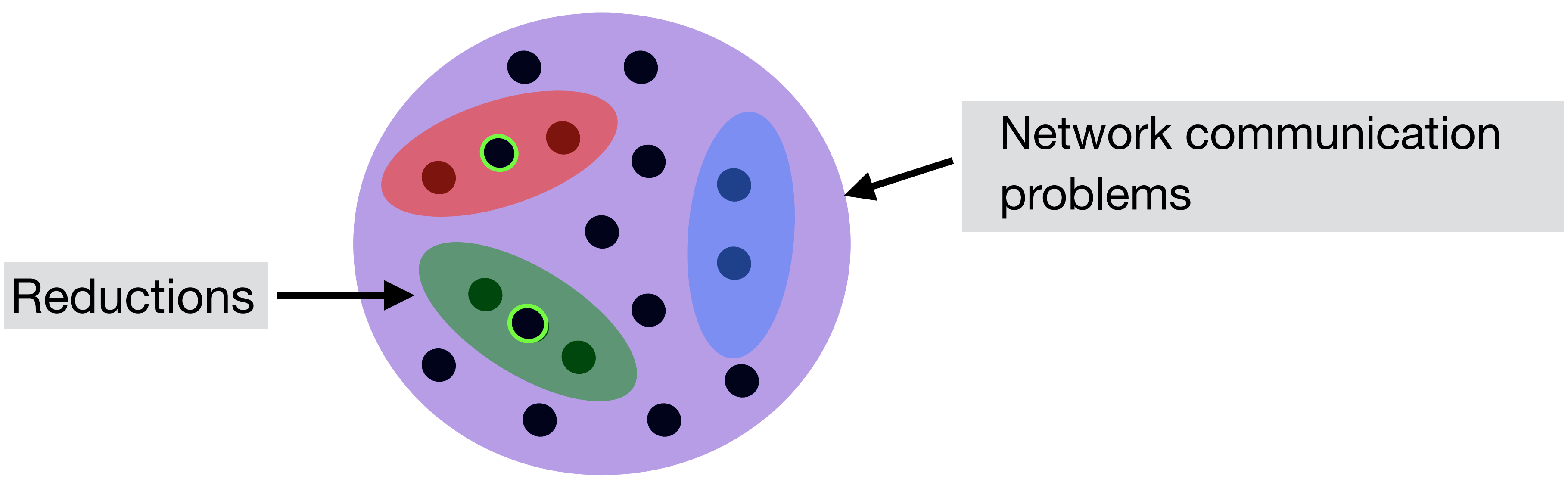

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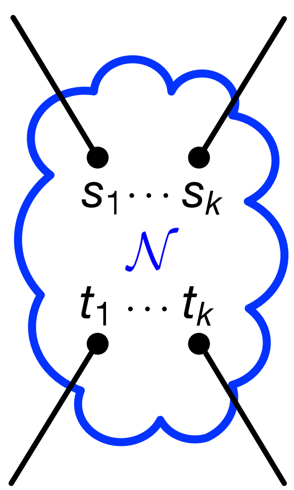

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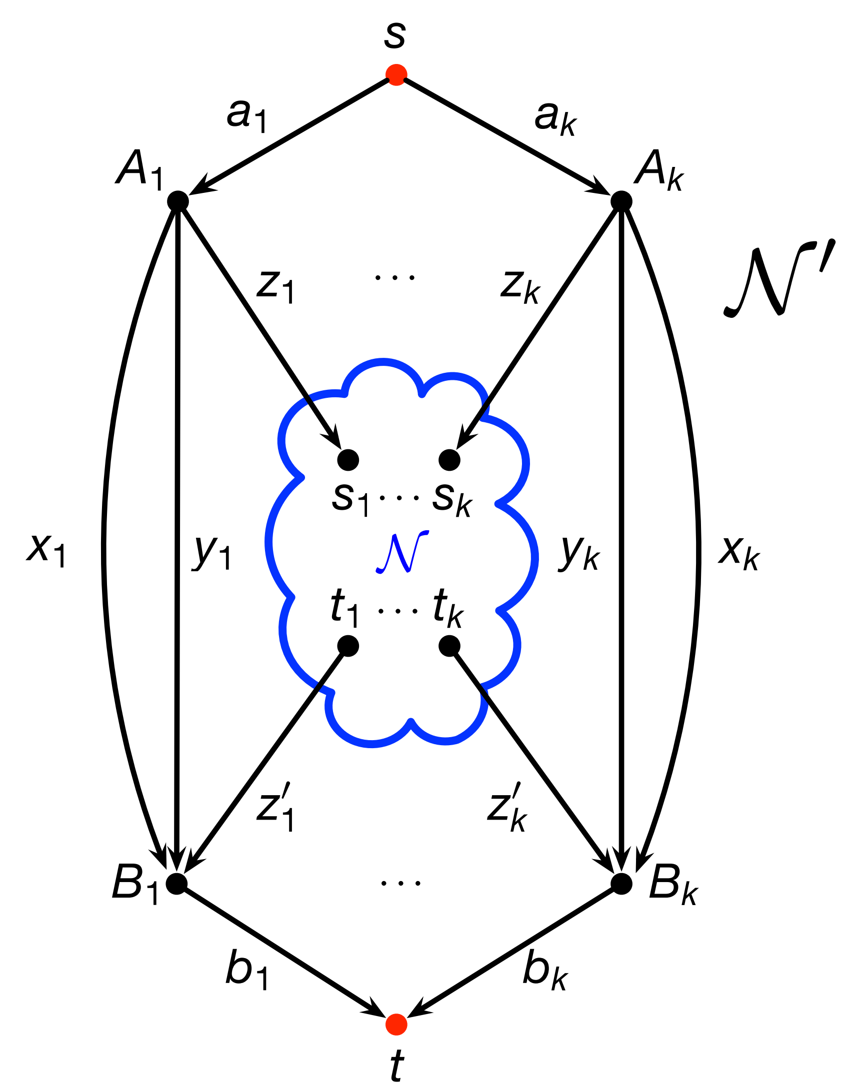

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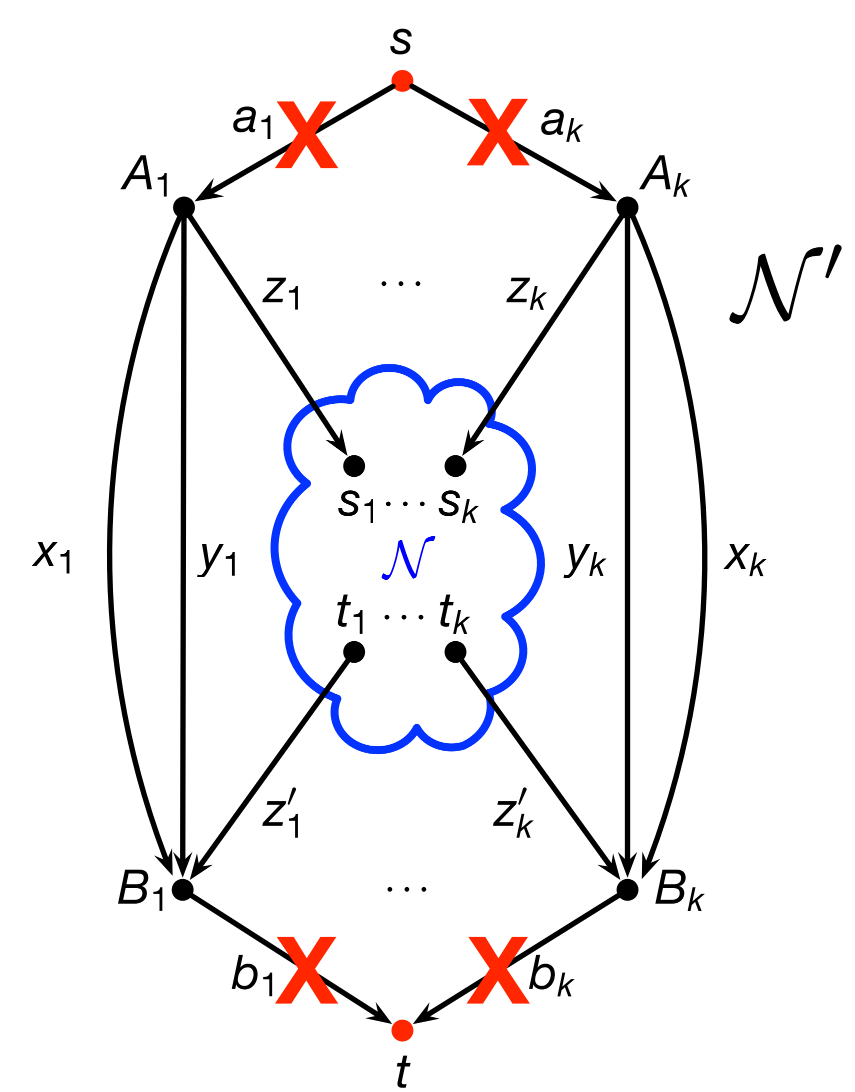
- Adversary can access any single link except links leaving *s* and *t*



Zero Error Case

Theorem

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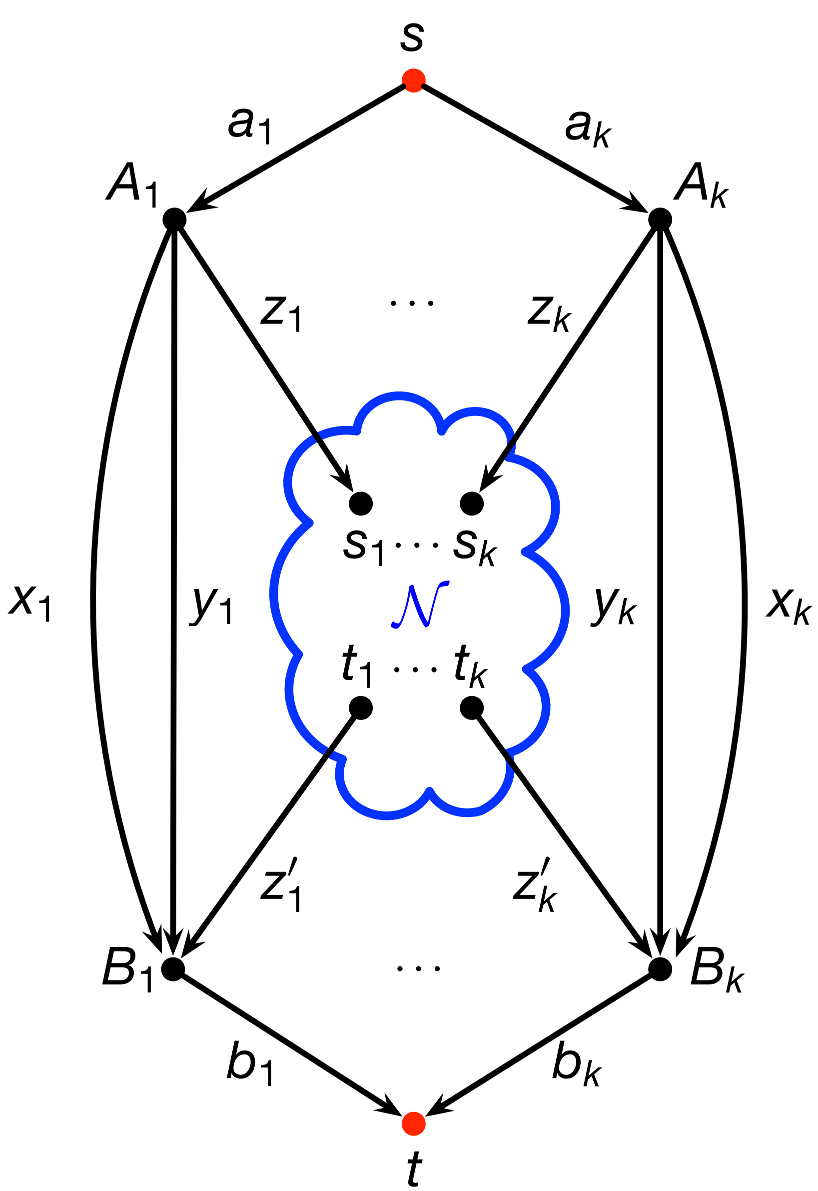

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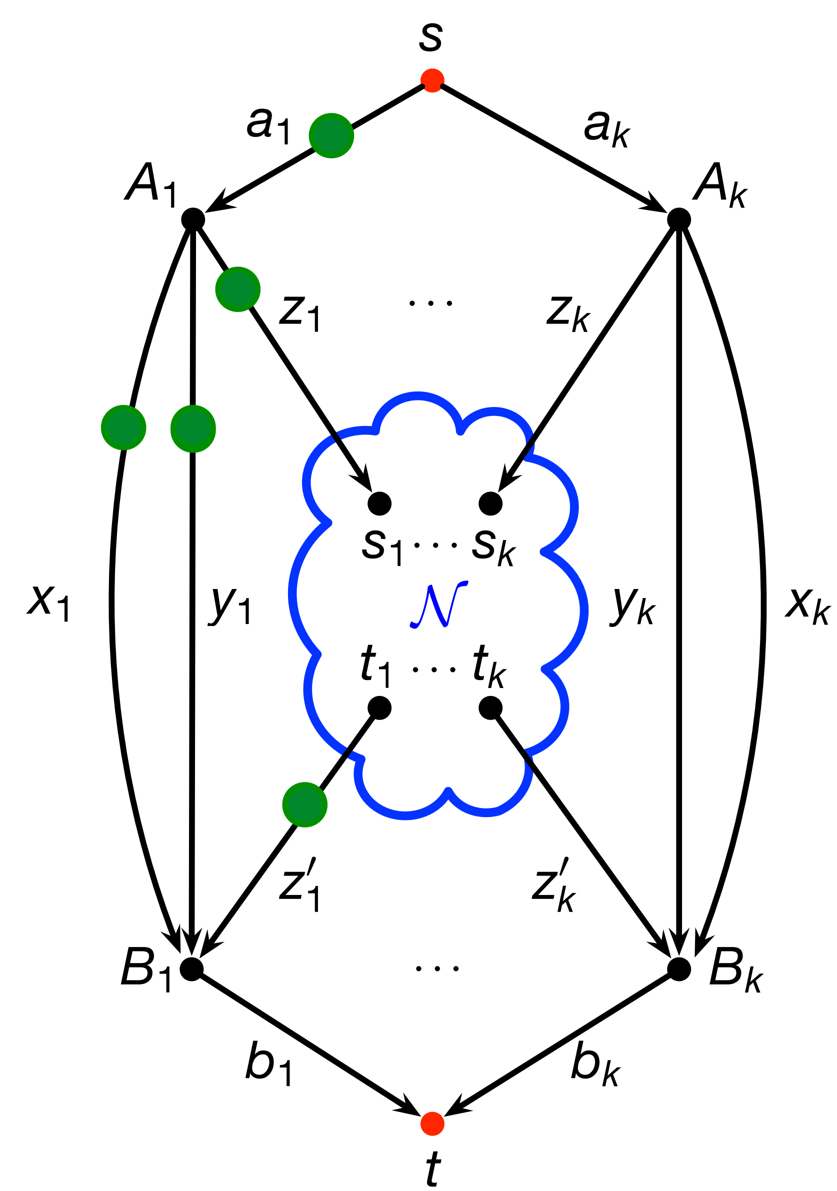

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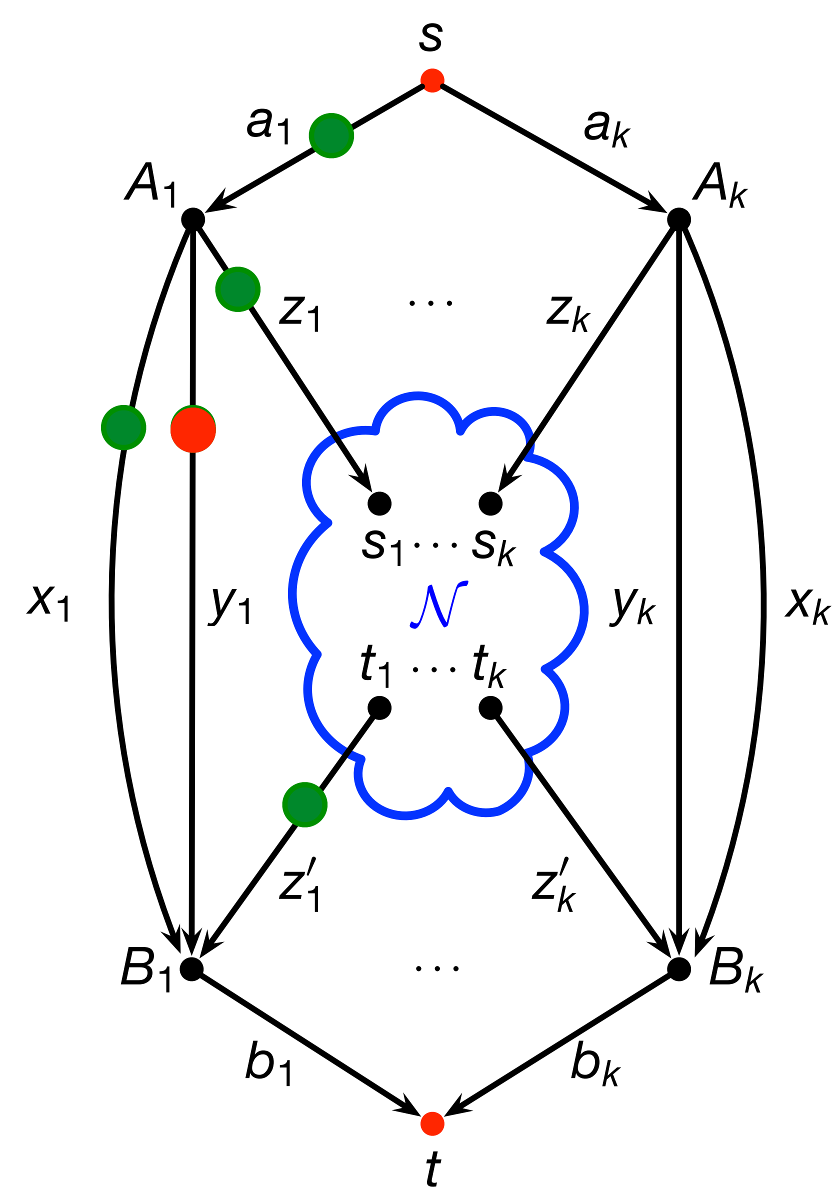

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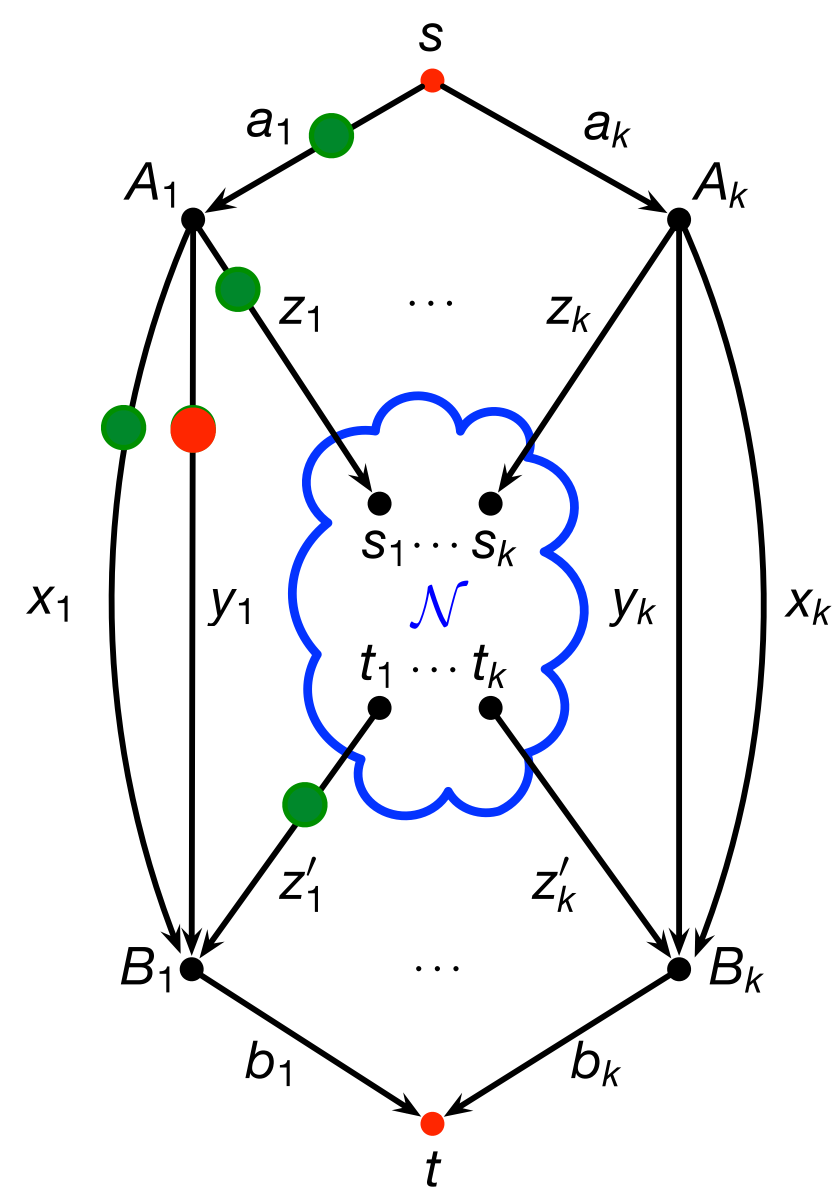
- B_i performs majority decoding
- Rate k is possible on $\mathcal{N}'$



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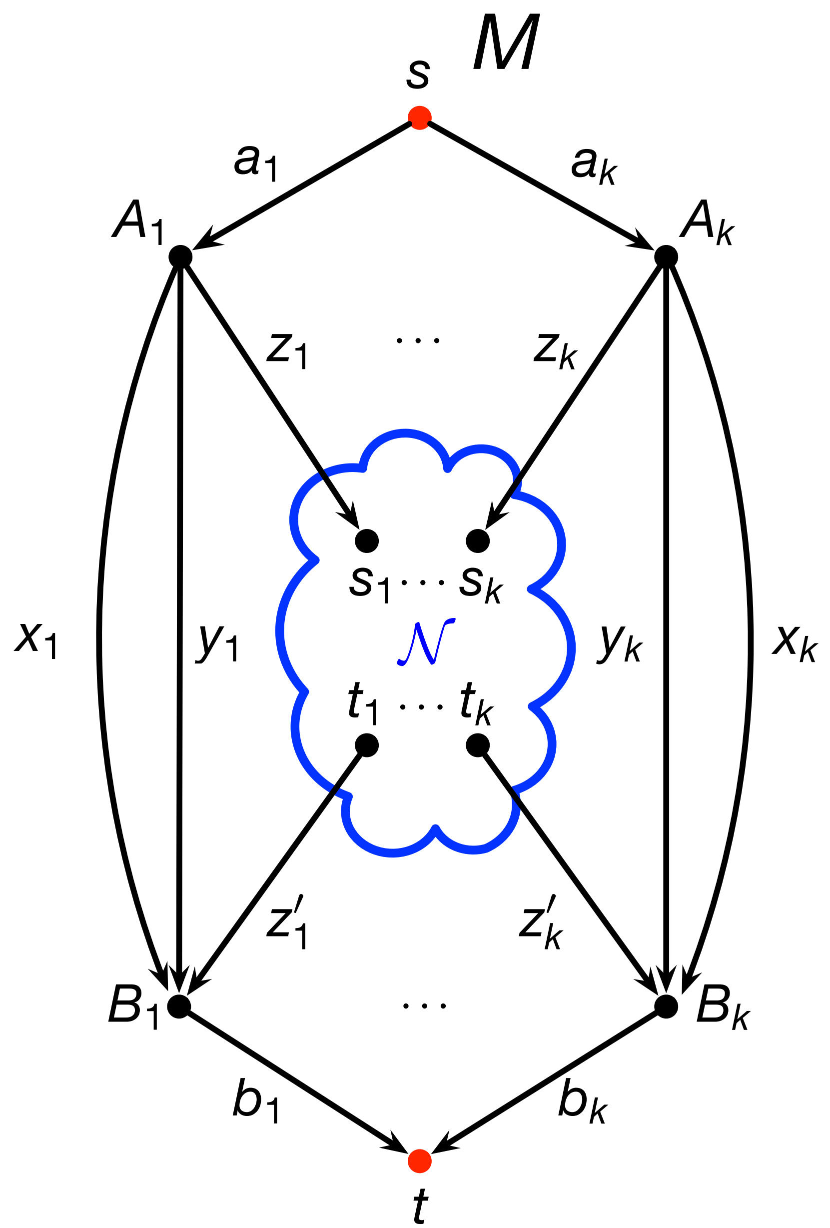

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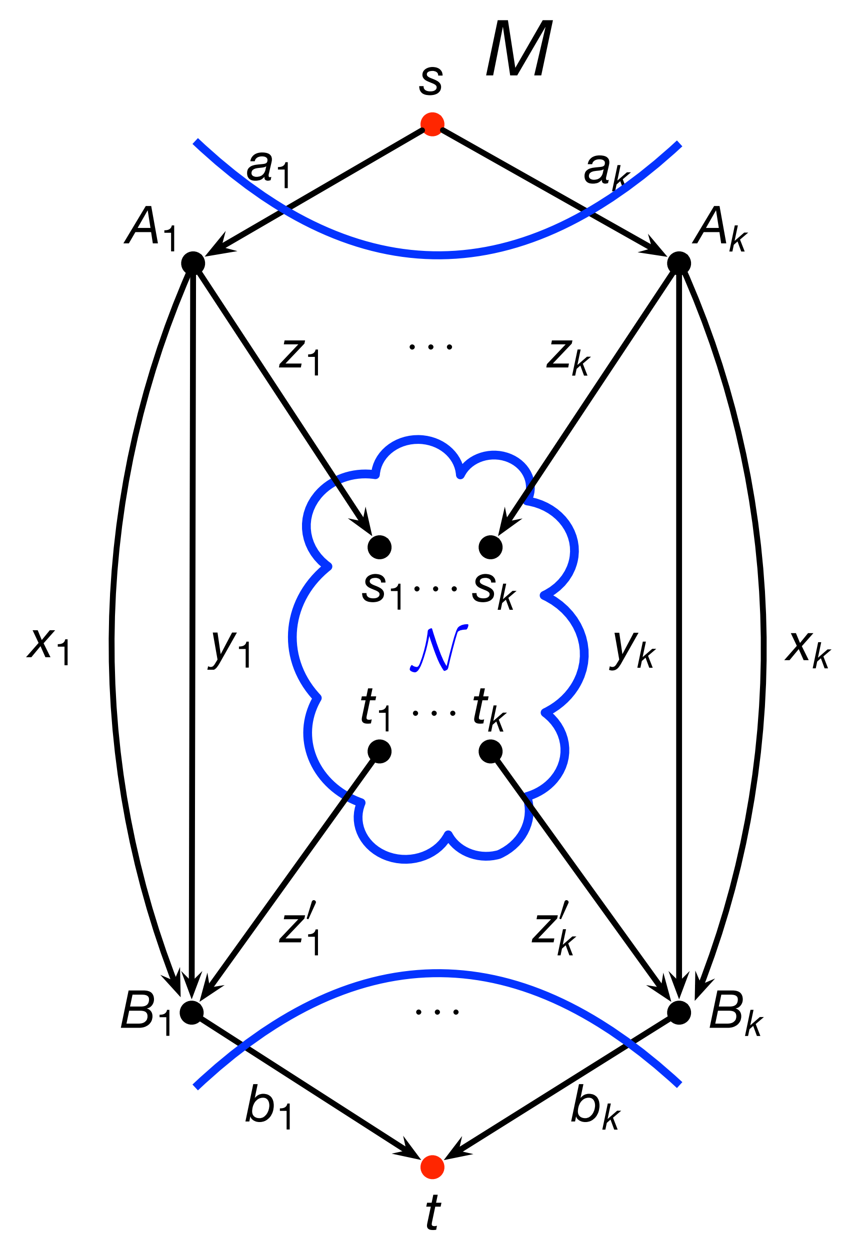

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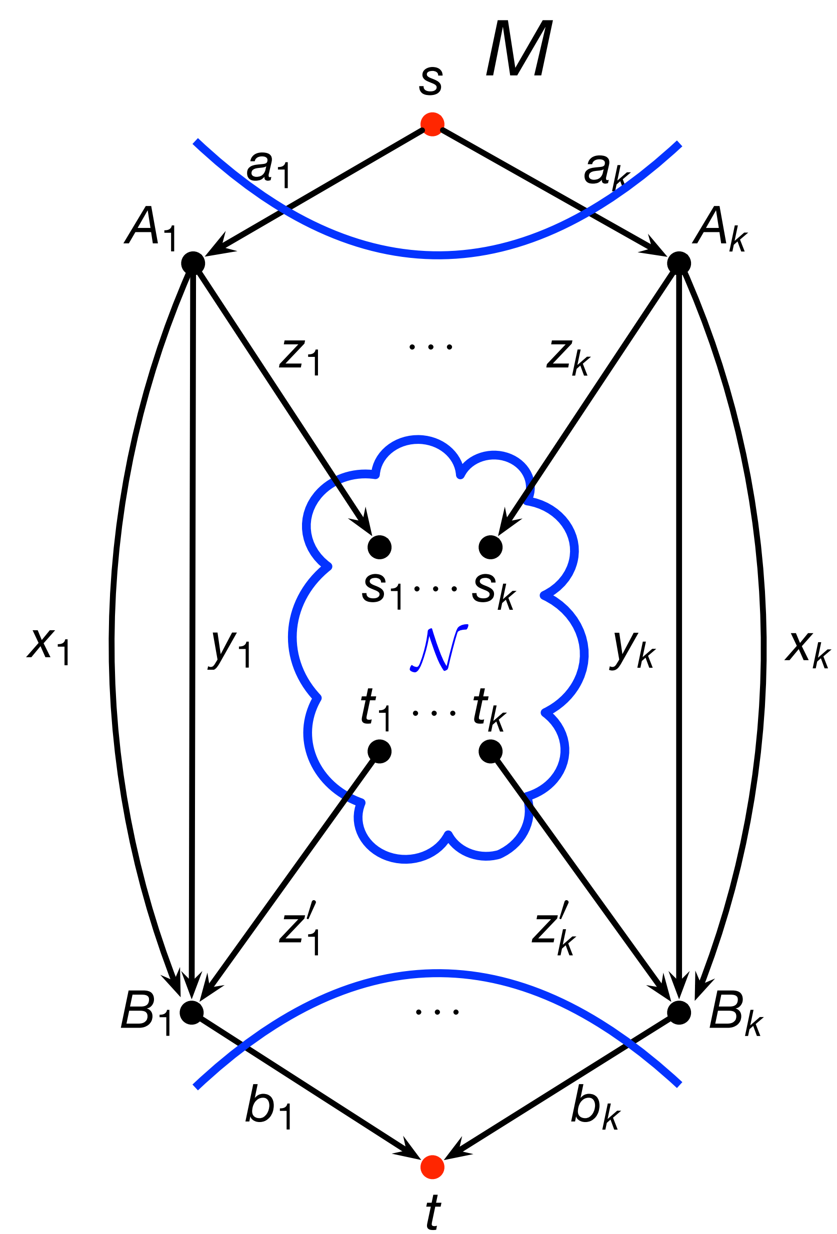

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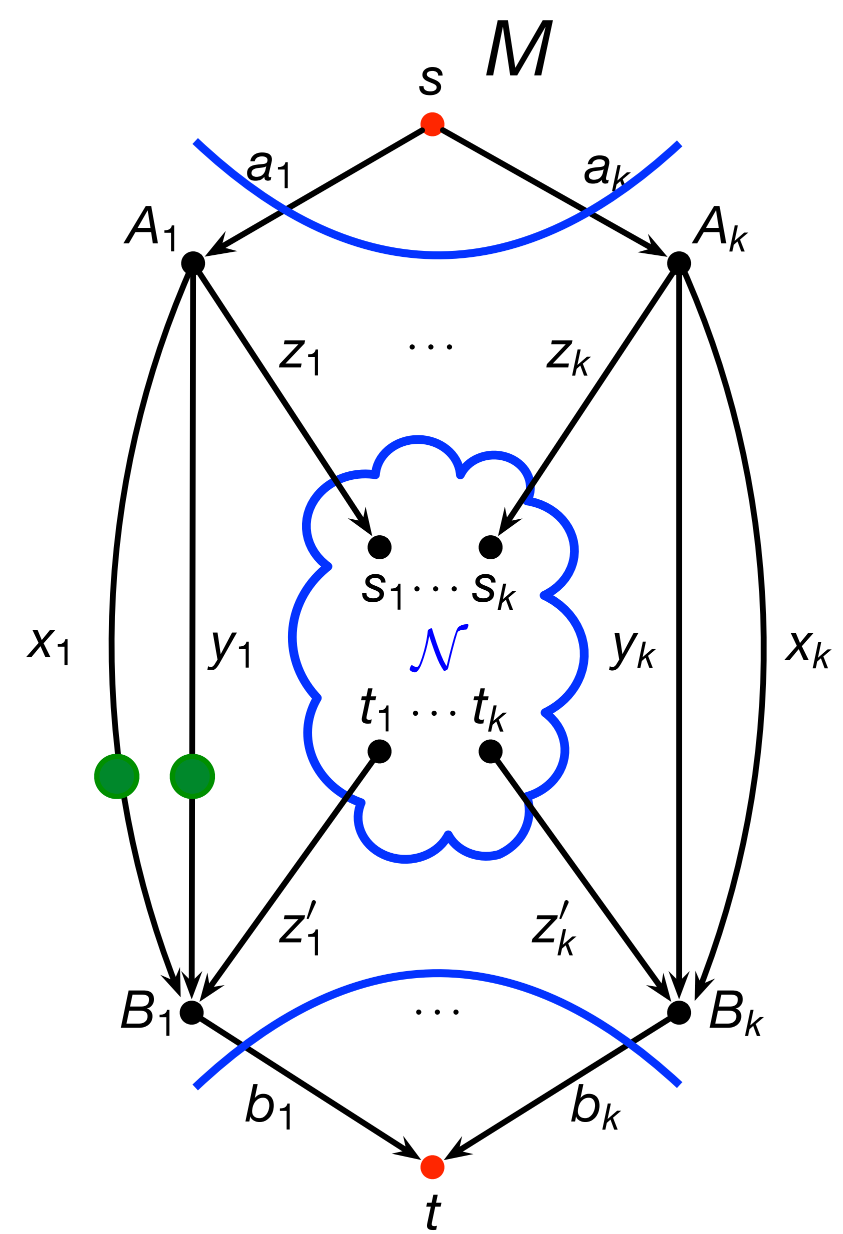

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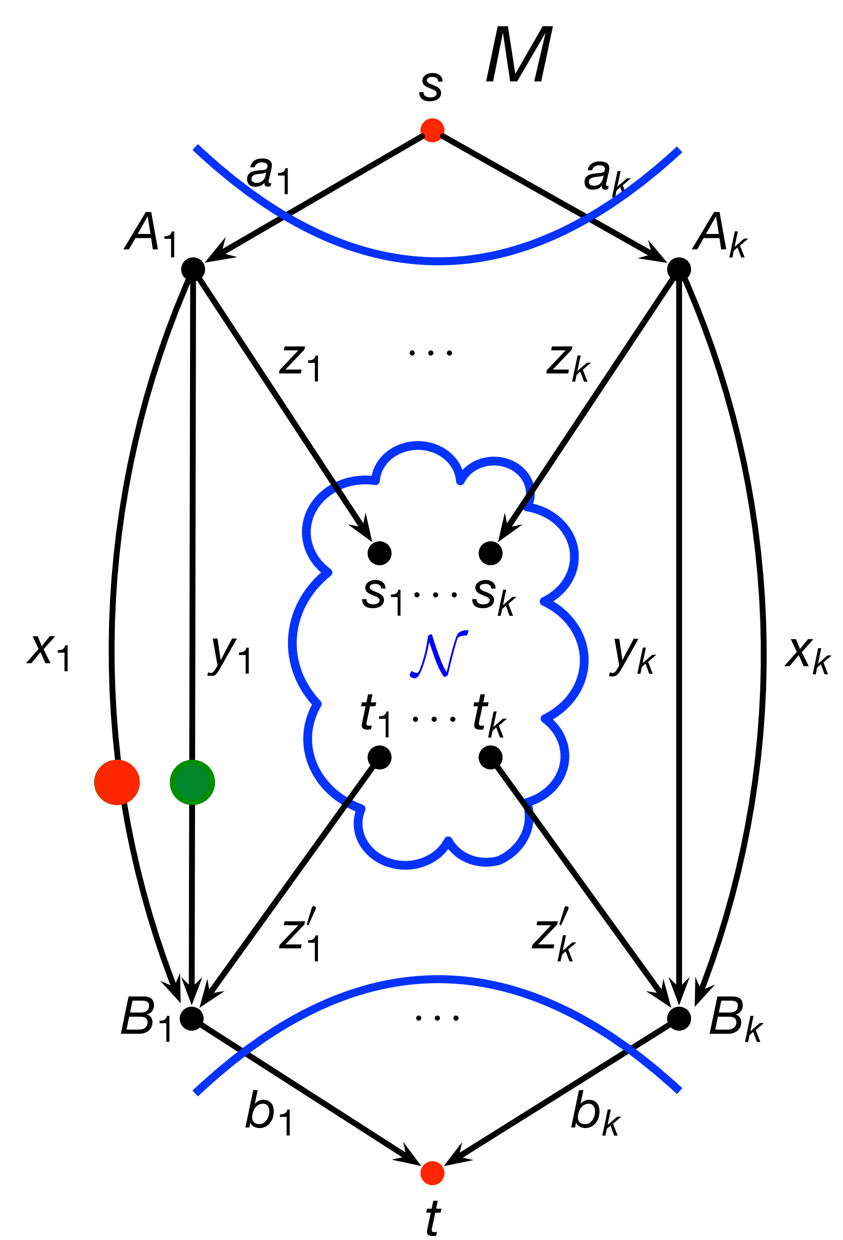

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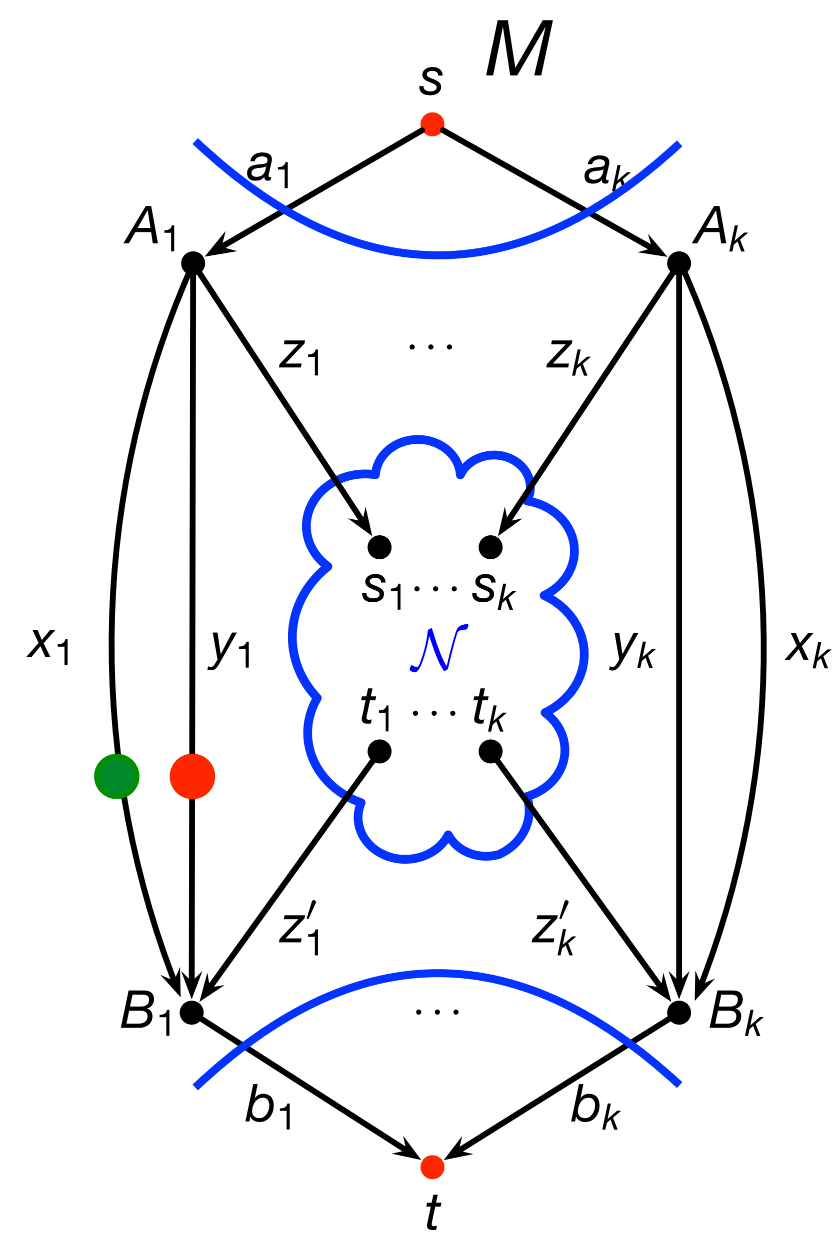

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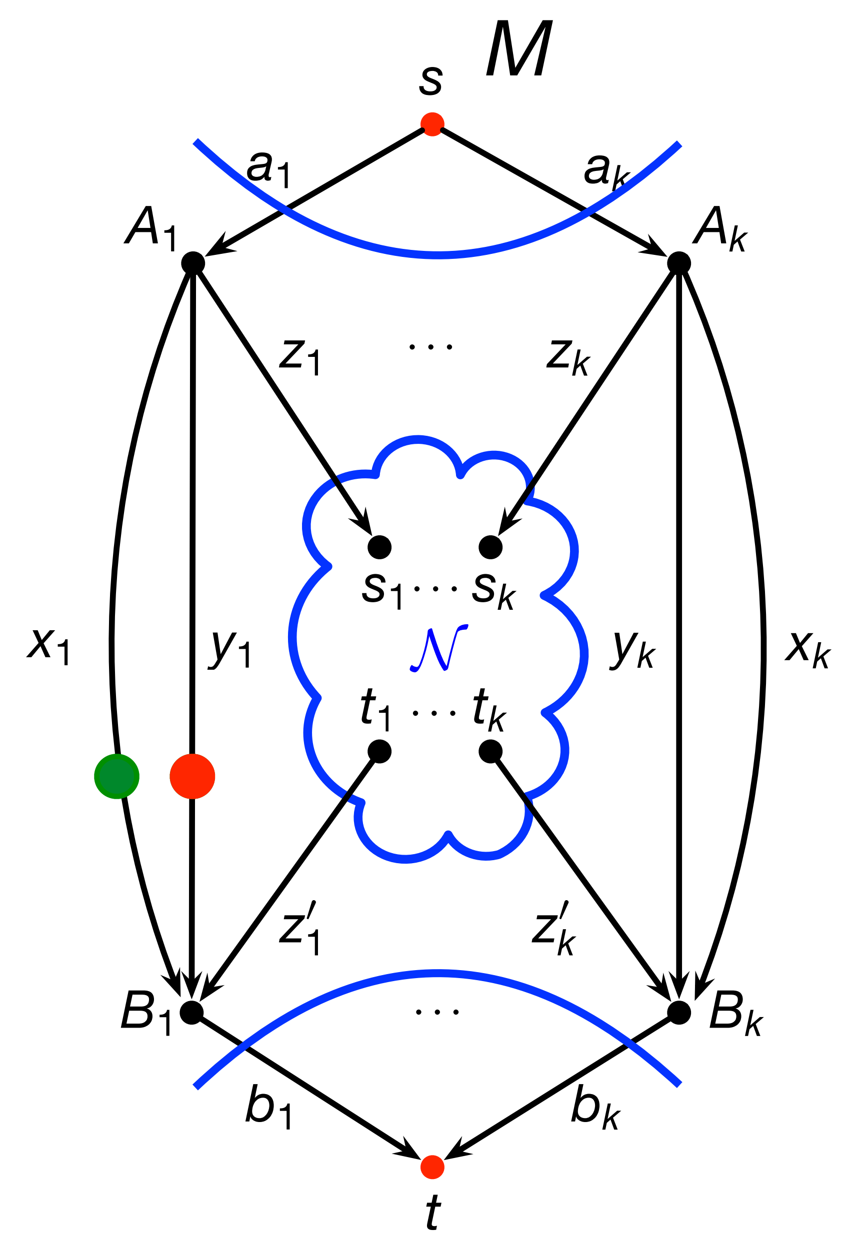

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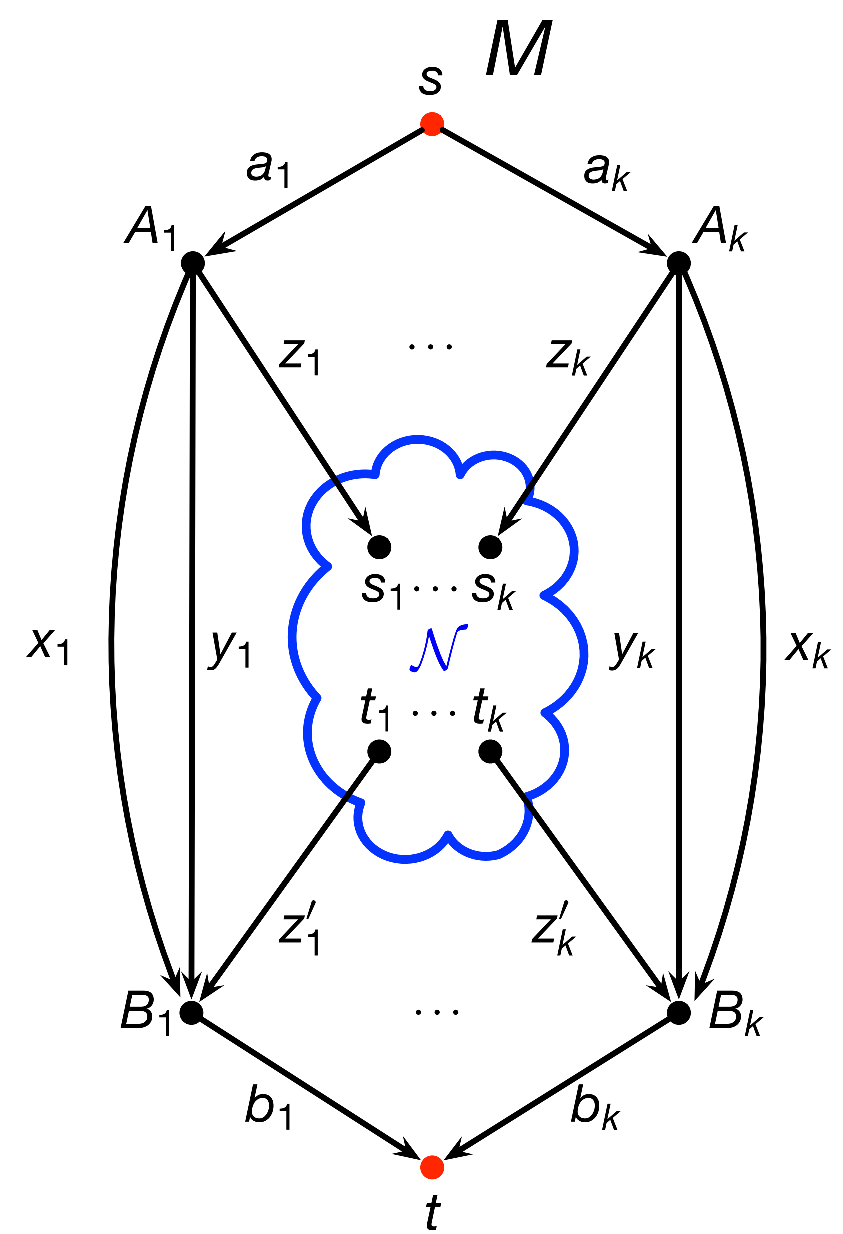

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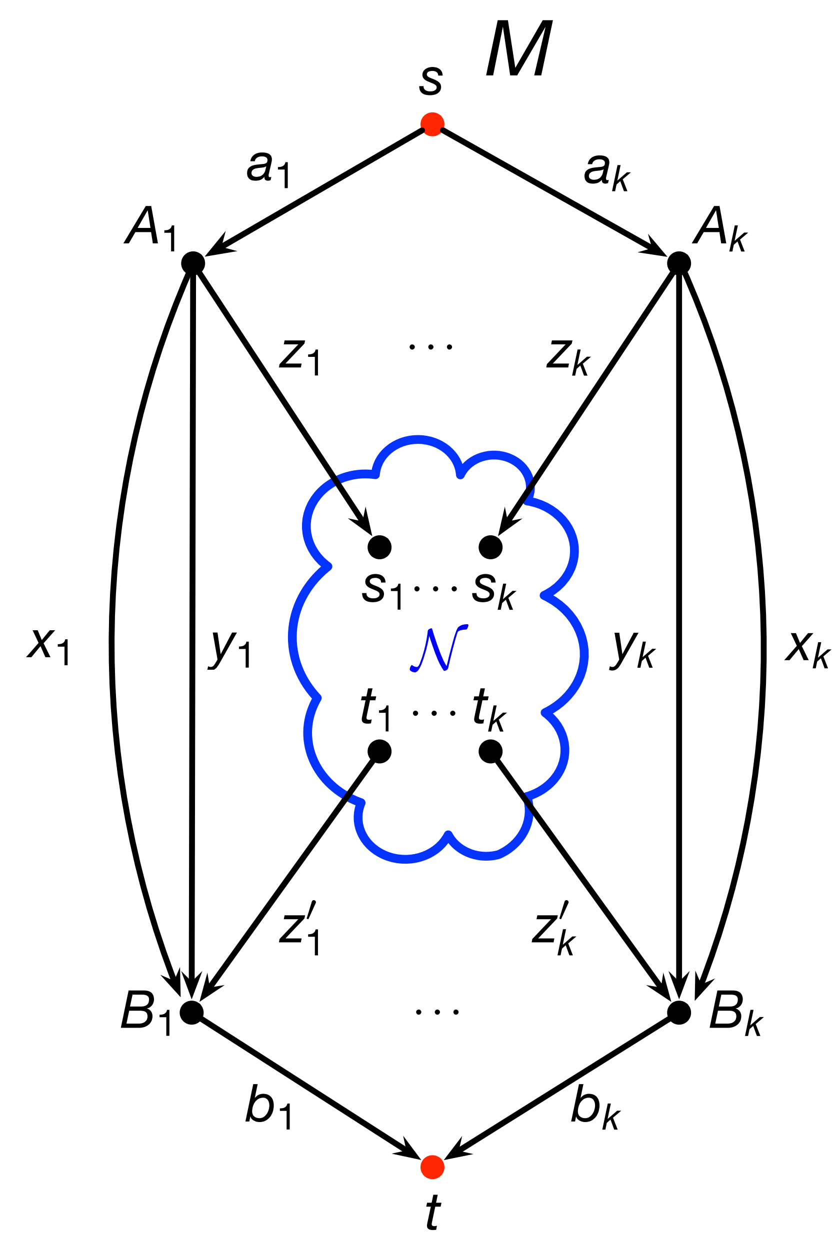

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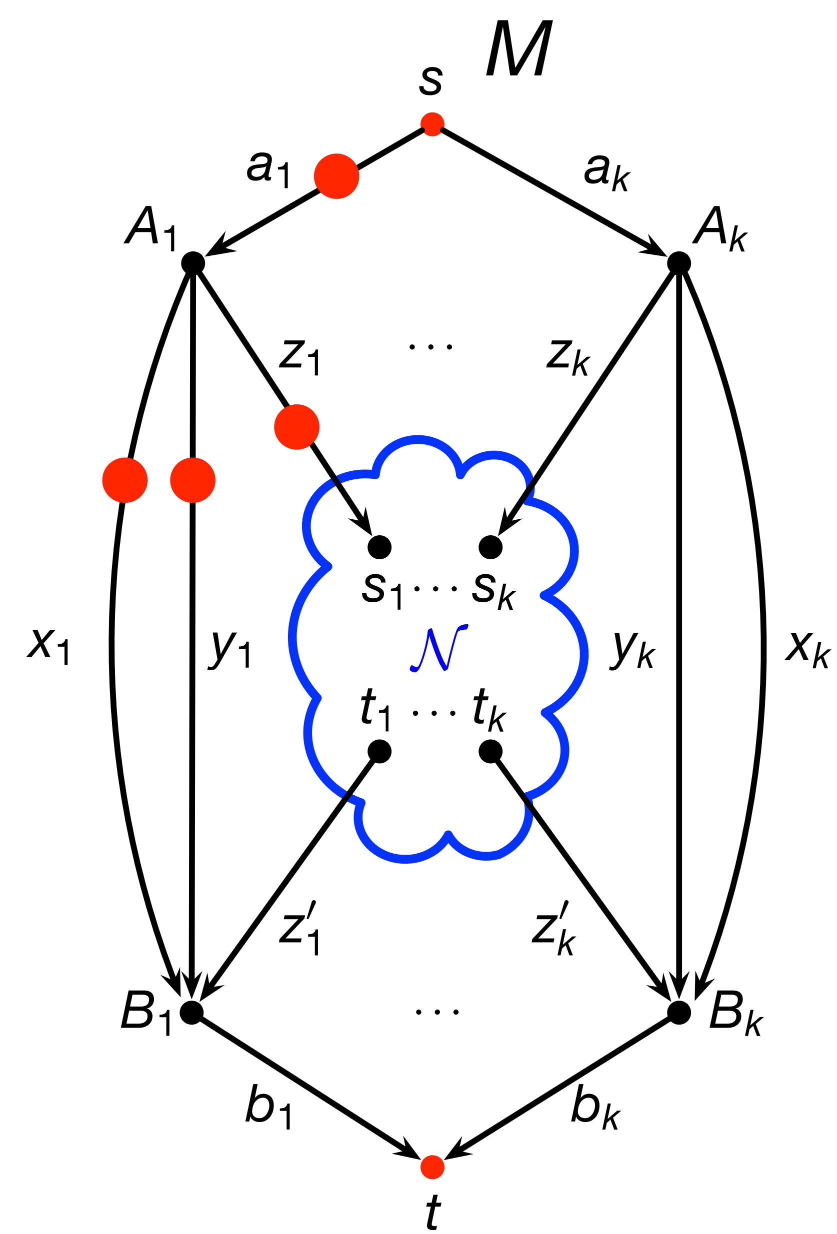

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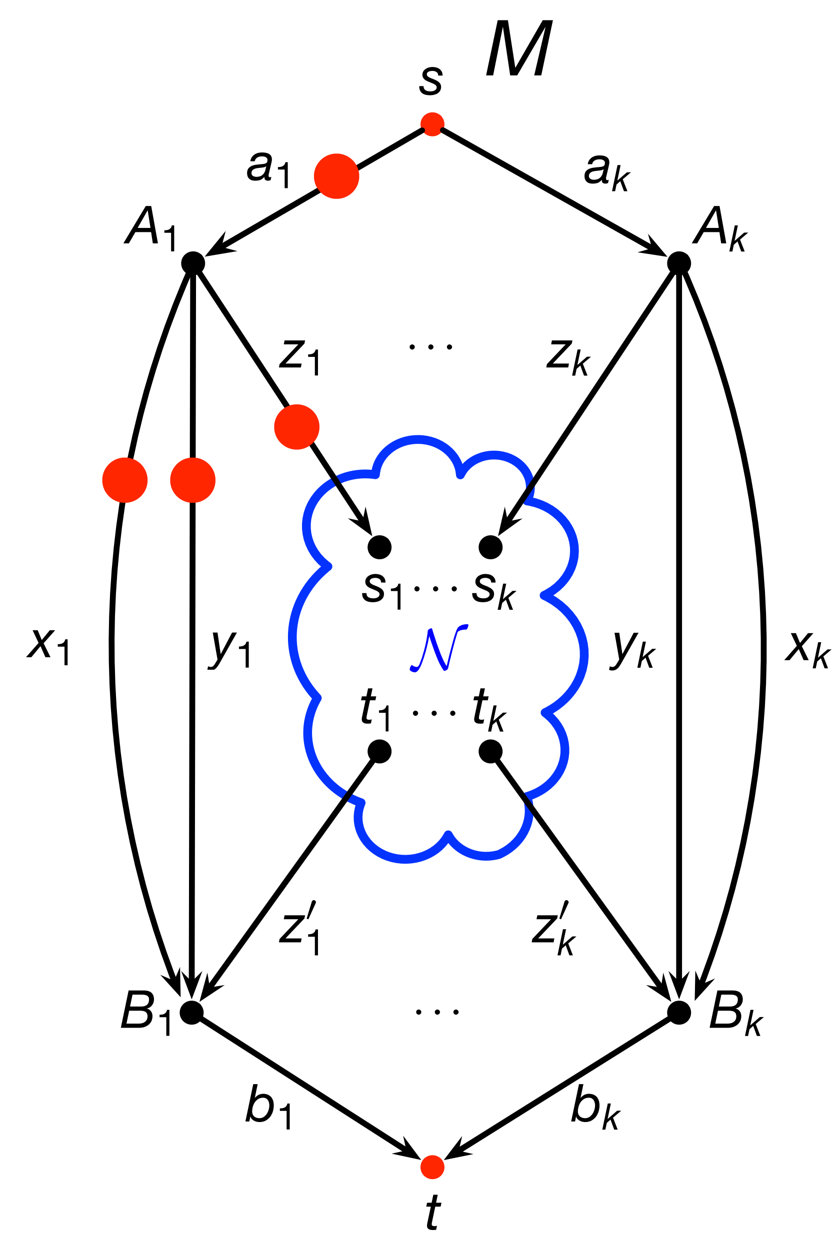

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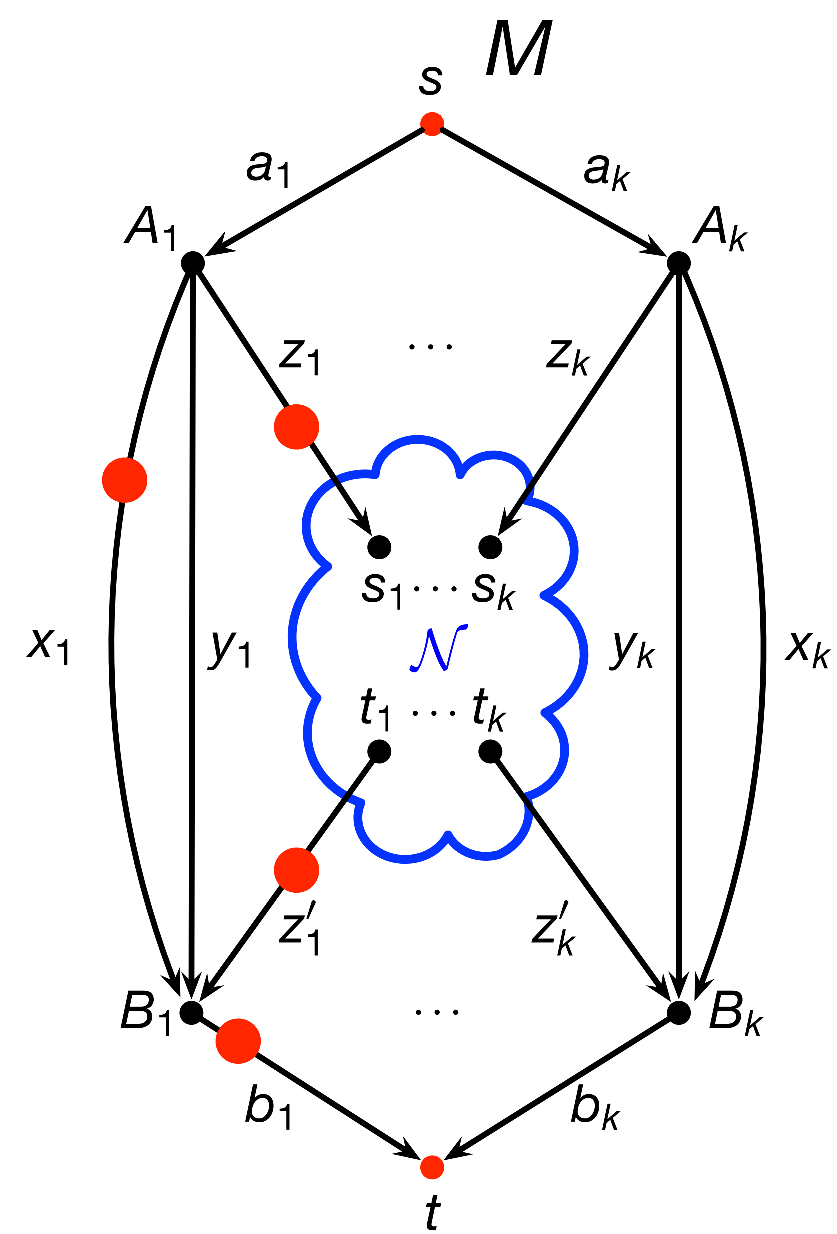

Asymptotic Error Case

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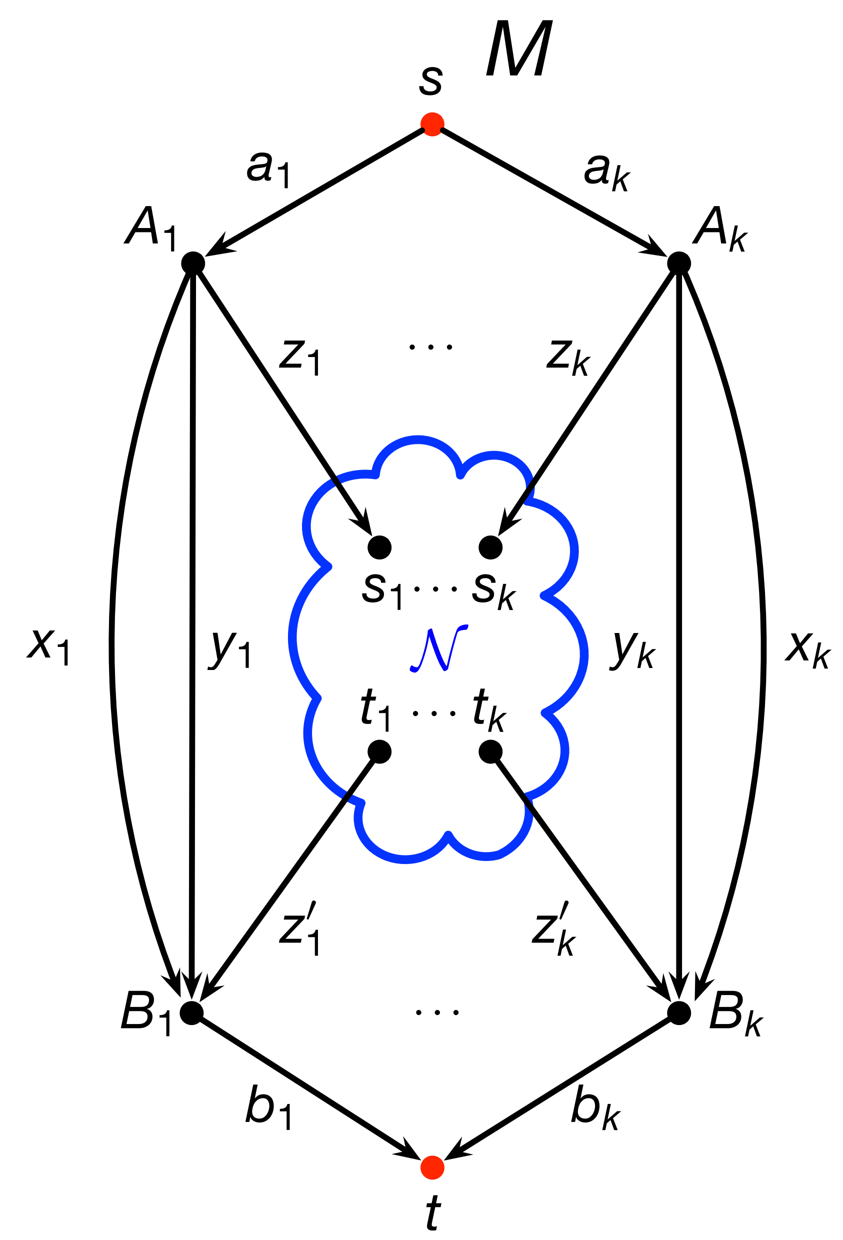

Asymptotic Error Case

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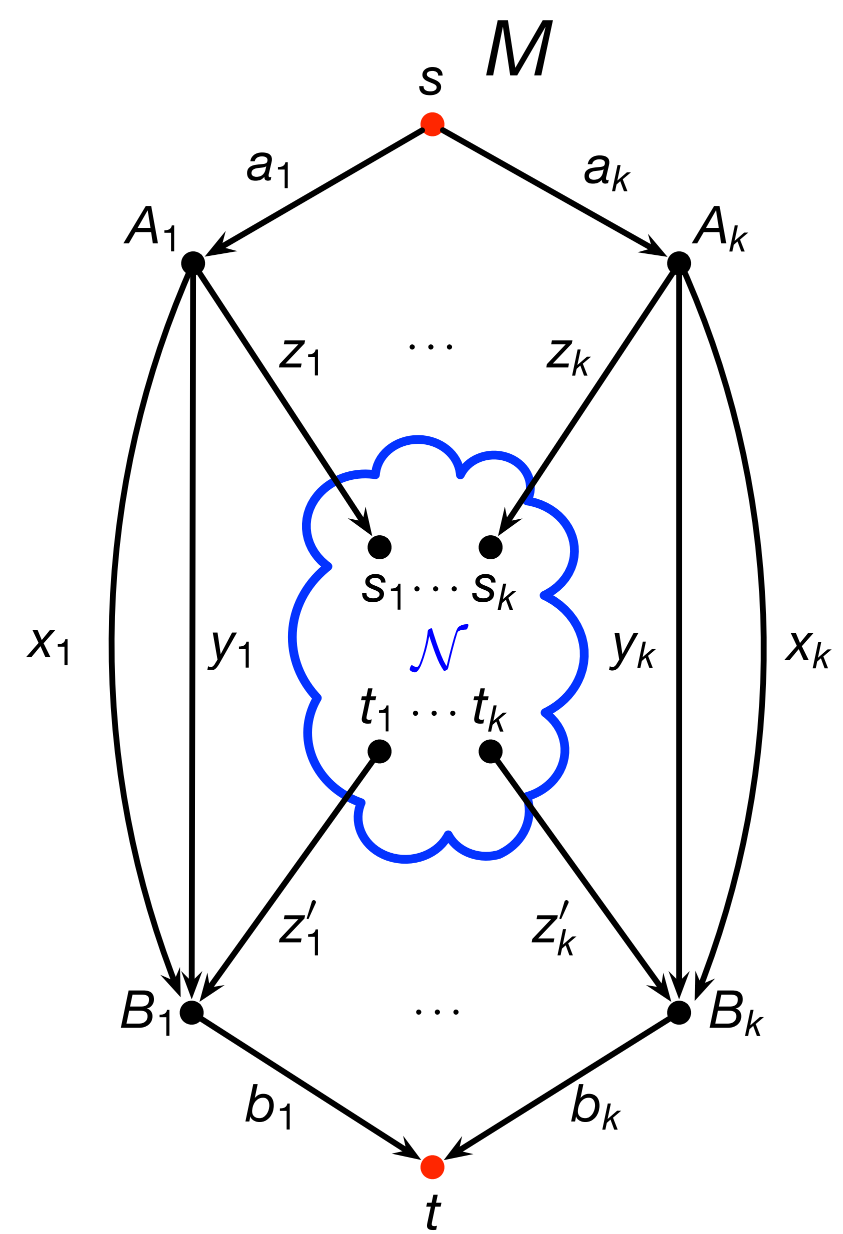

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- ⇒ We show:
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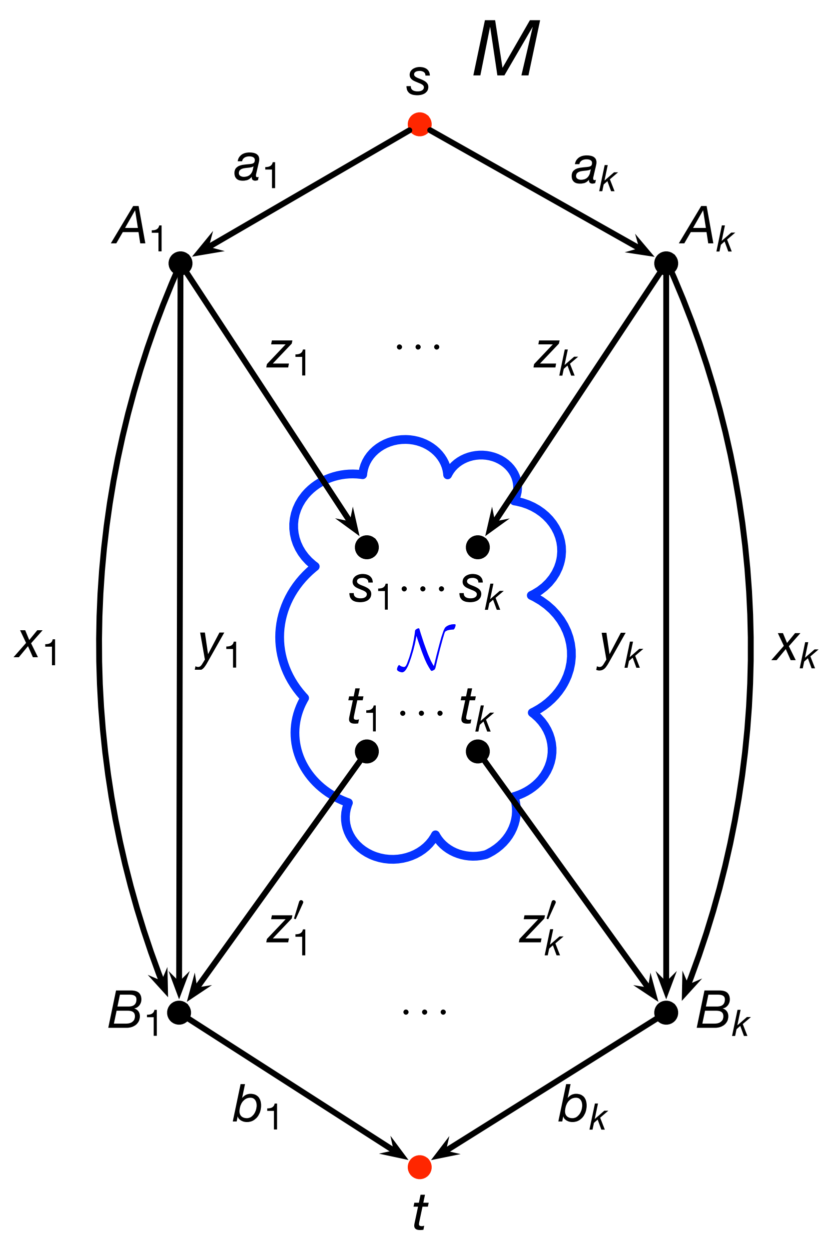

Asymptotic Error Case

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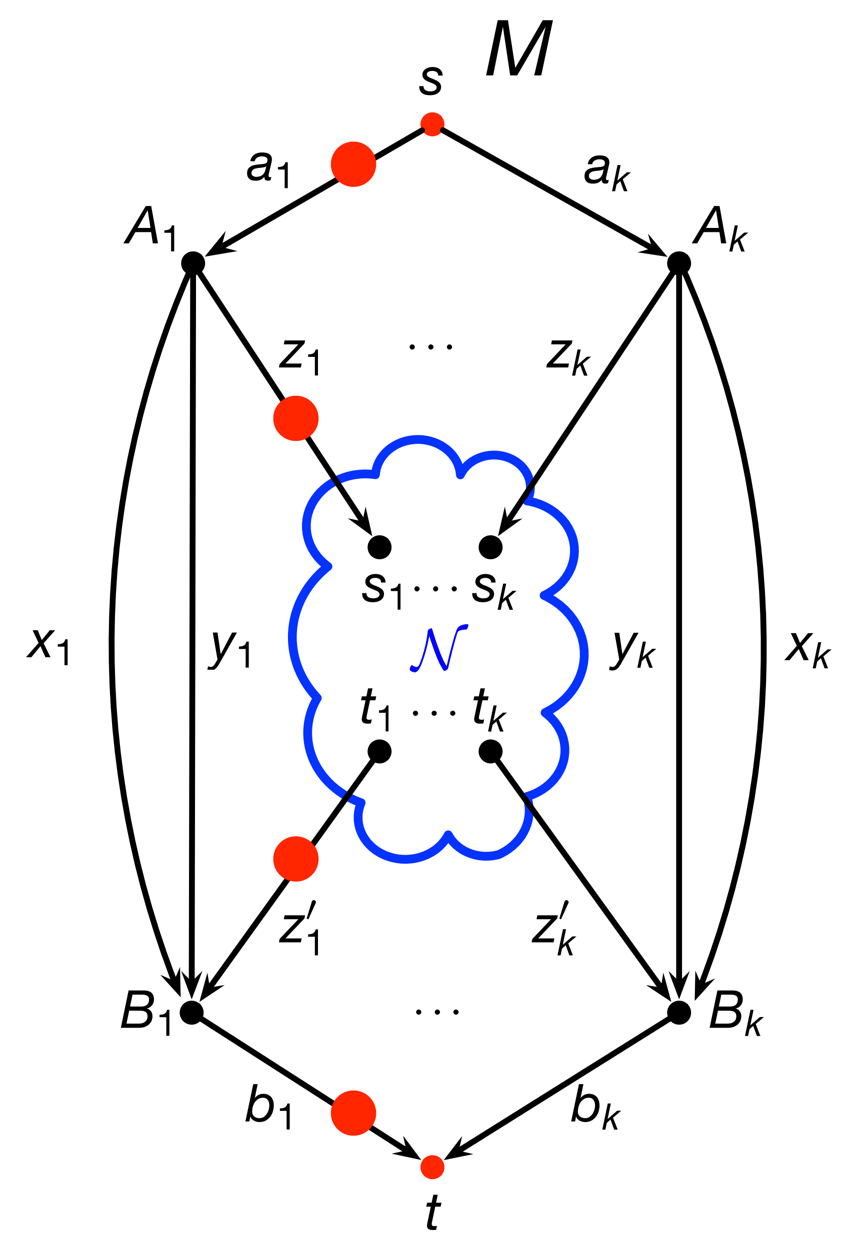

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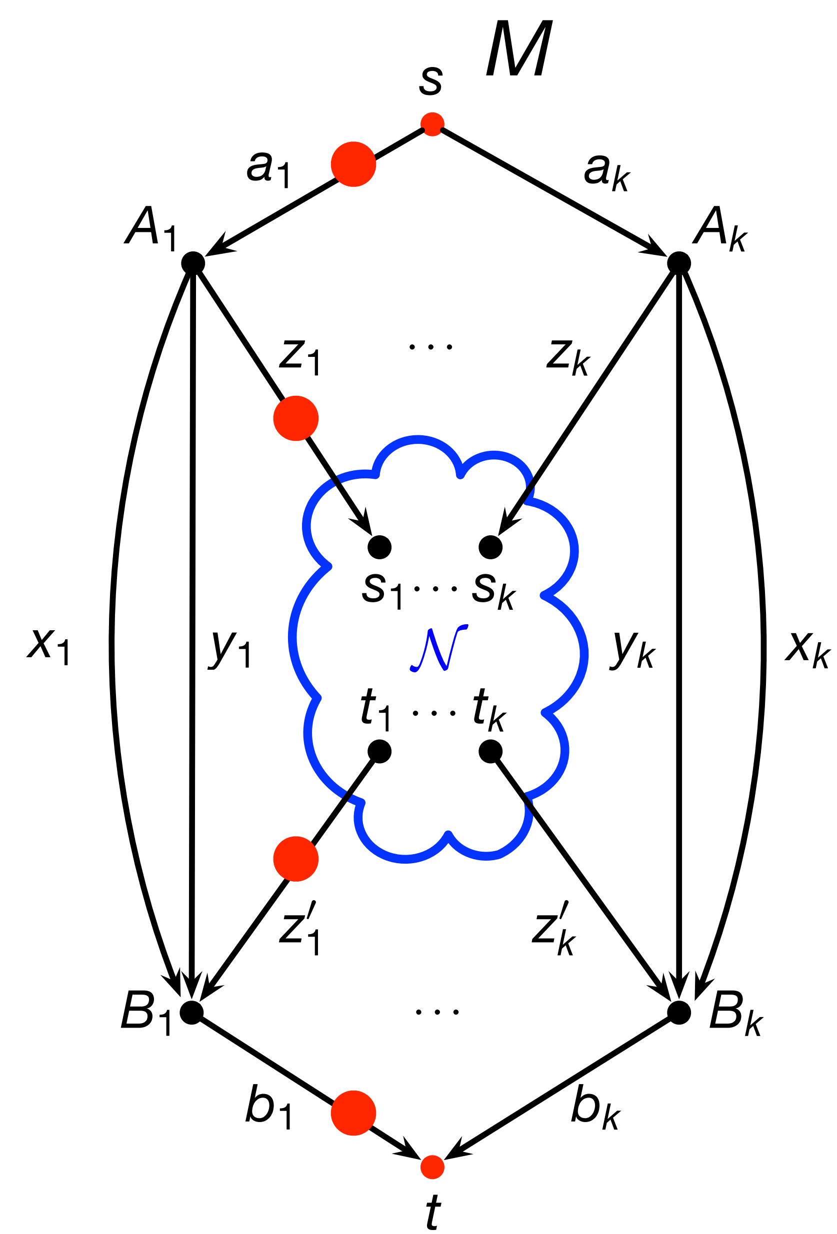
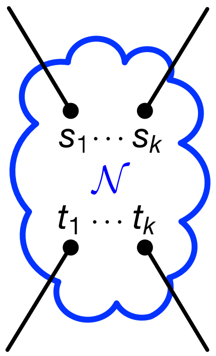
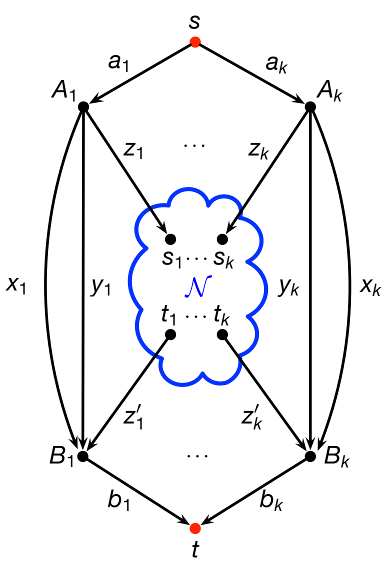
- In summary: $\lim_{n\to\infty,\epsilon\to 0} I(z_i;z_i')/n = 1$
- Block codes: MU with $\epsilon > 0$ possible



Current Understanding





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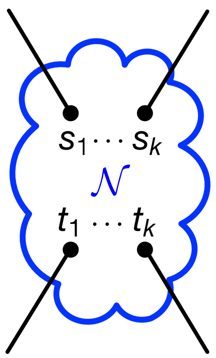

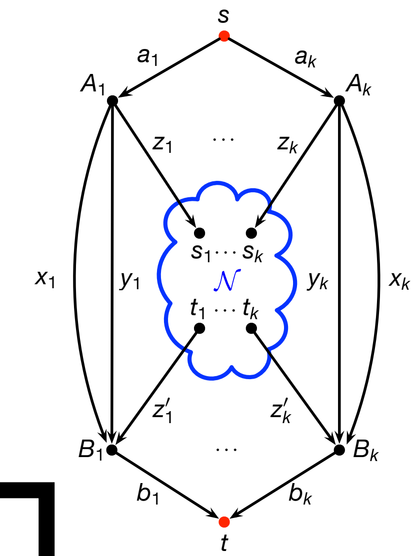

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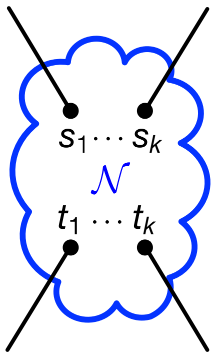

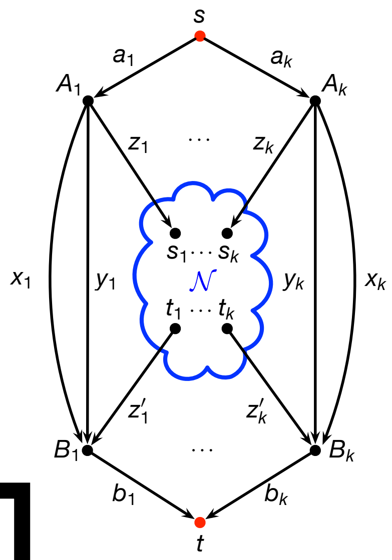

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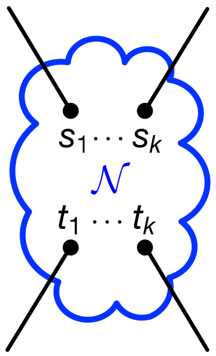

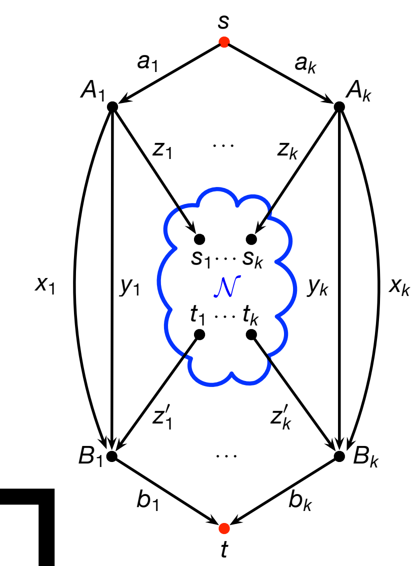

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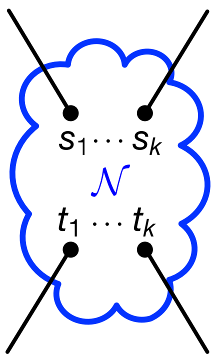

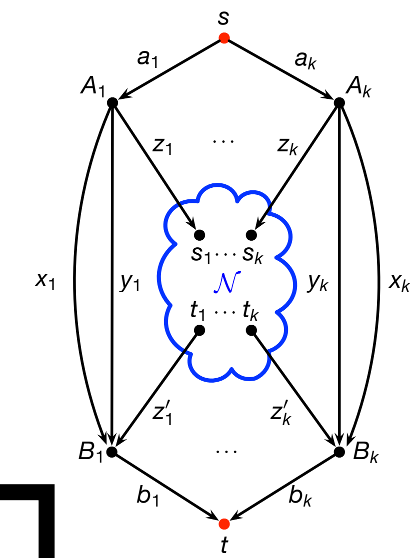

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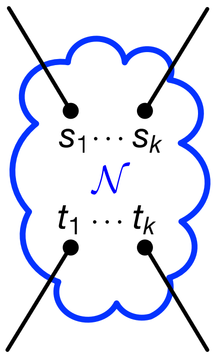

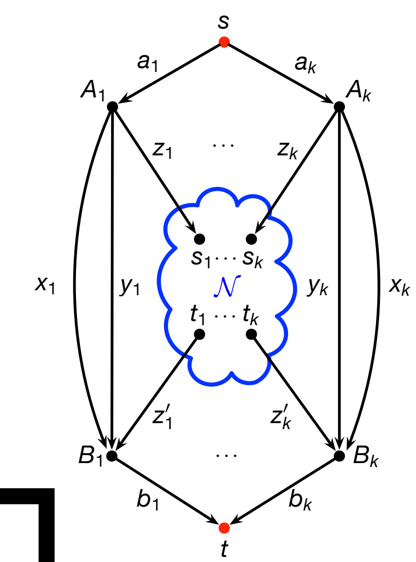

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Infeasibility of Multiple Unicast

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There exist networks $\mathcal{N}$ and $\mathcal{N}'$ such that rate *k* is asymptotically achievable on $\mathcal{N}'$ but $(1, 1, \ldots, 1)$ is not asymp. achievable on $\mathcal{N}$.

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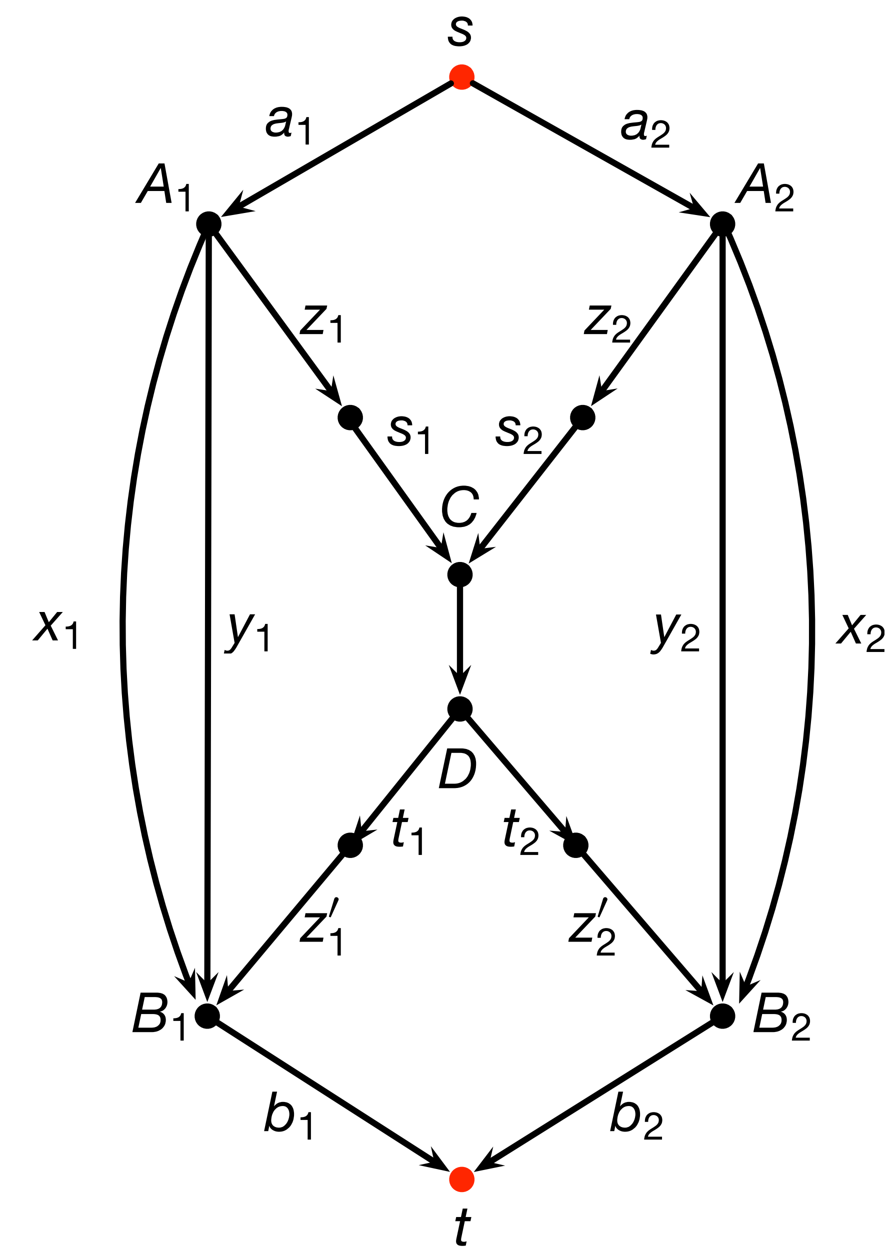

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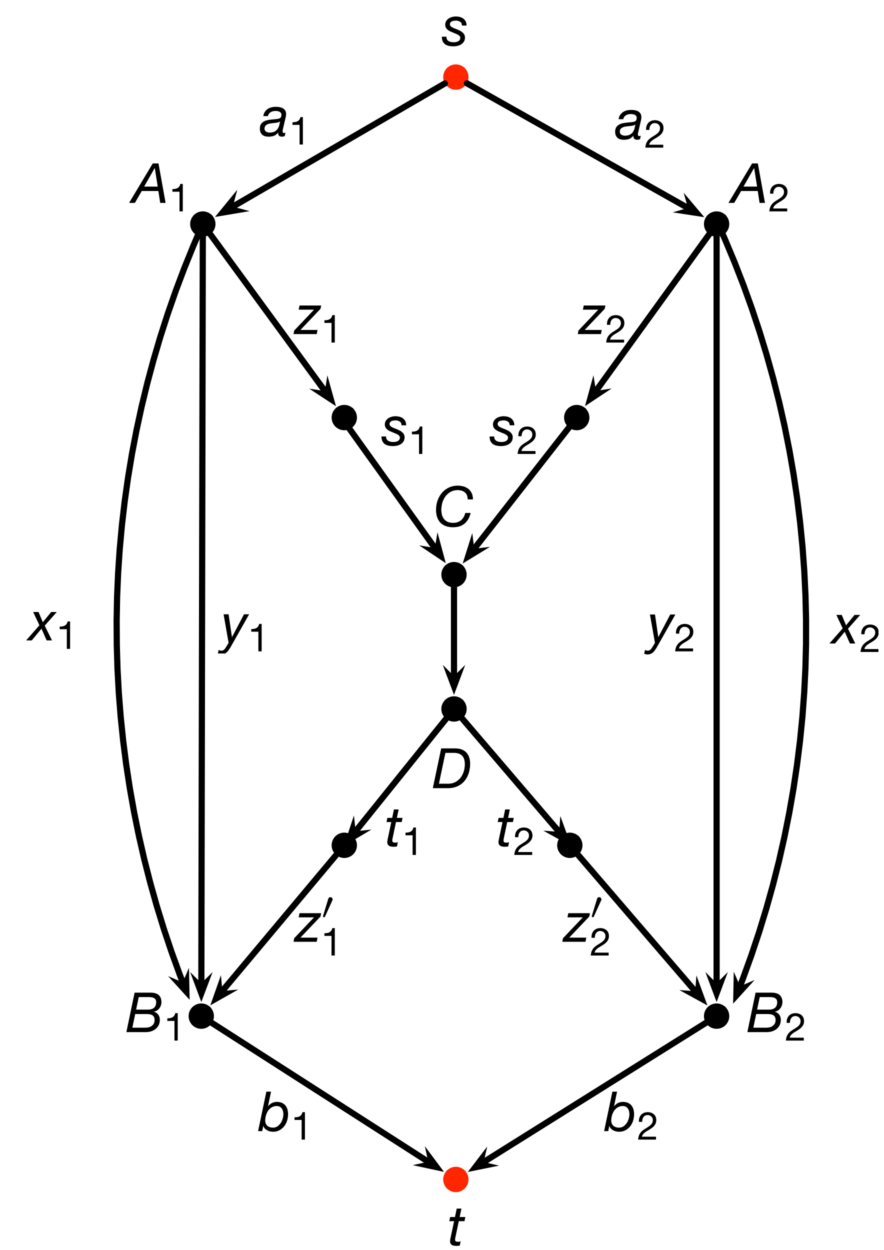

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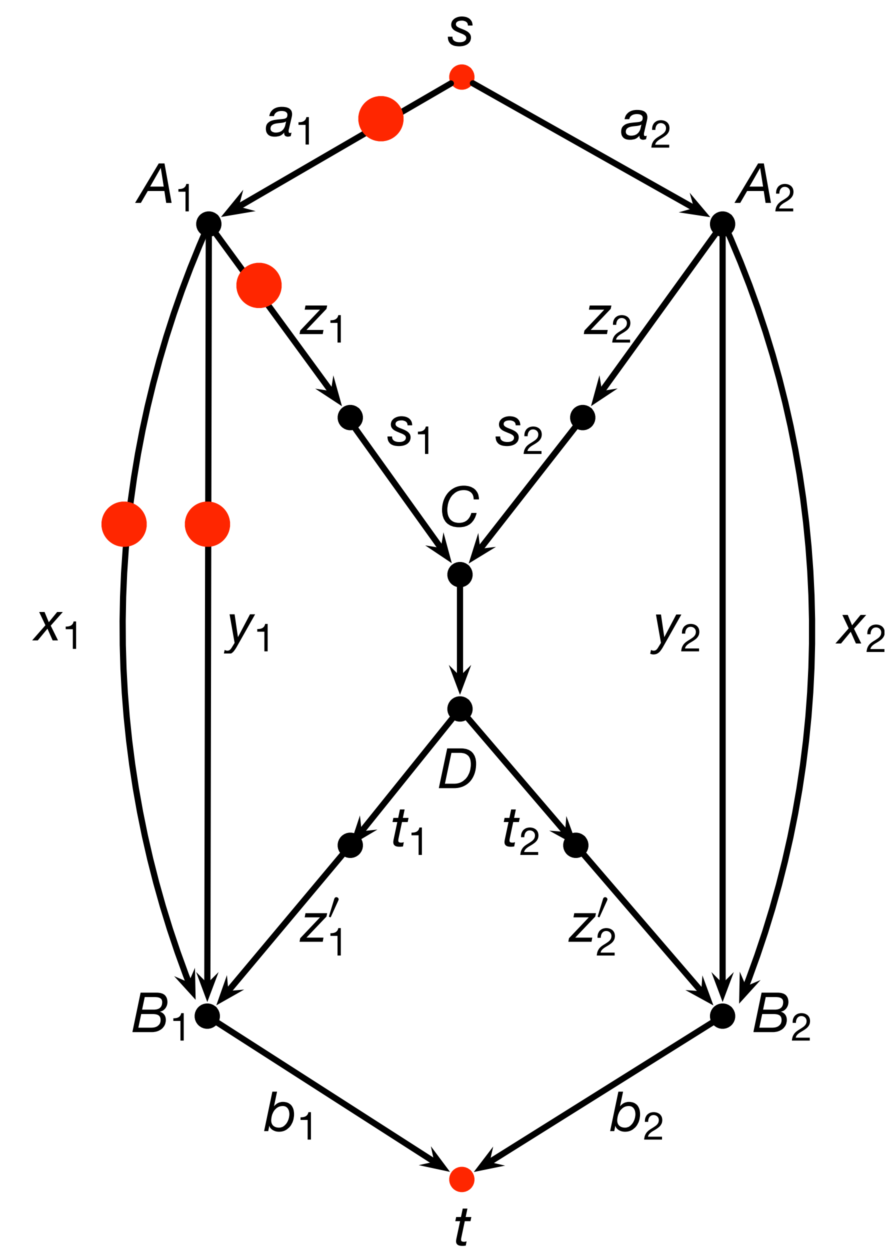

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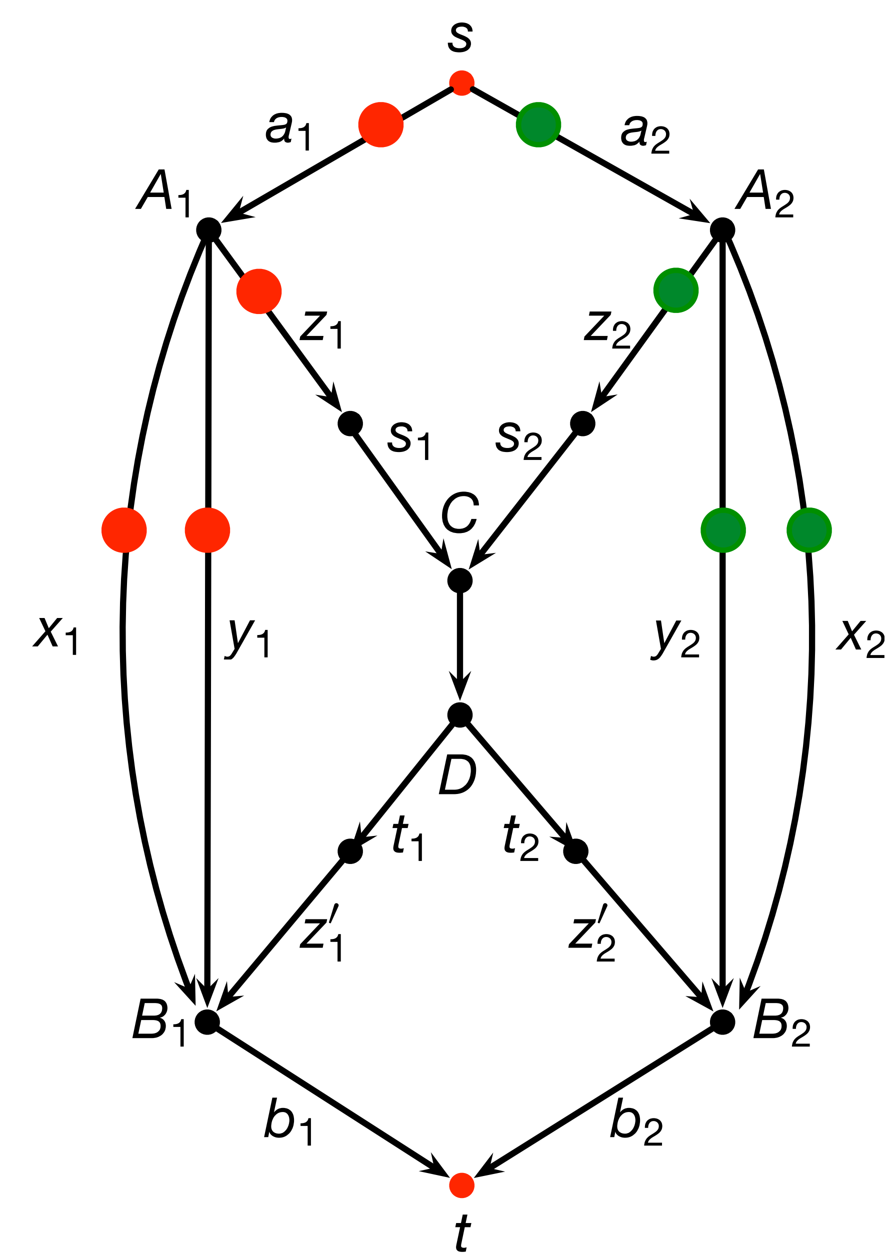

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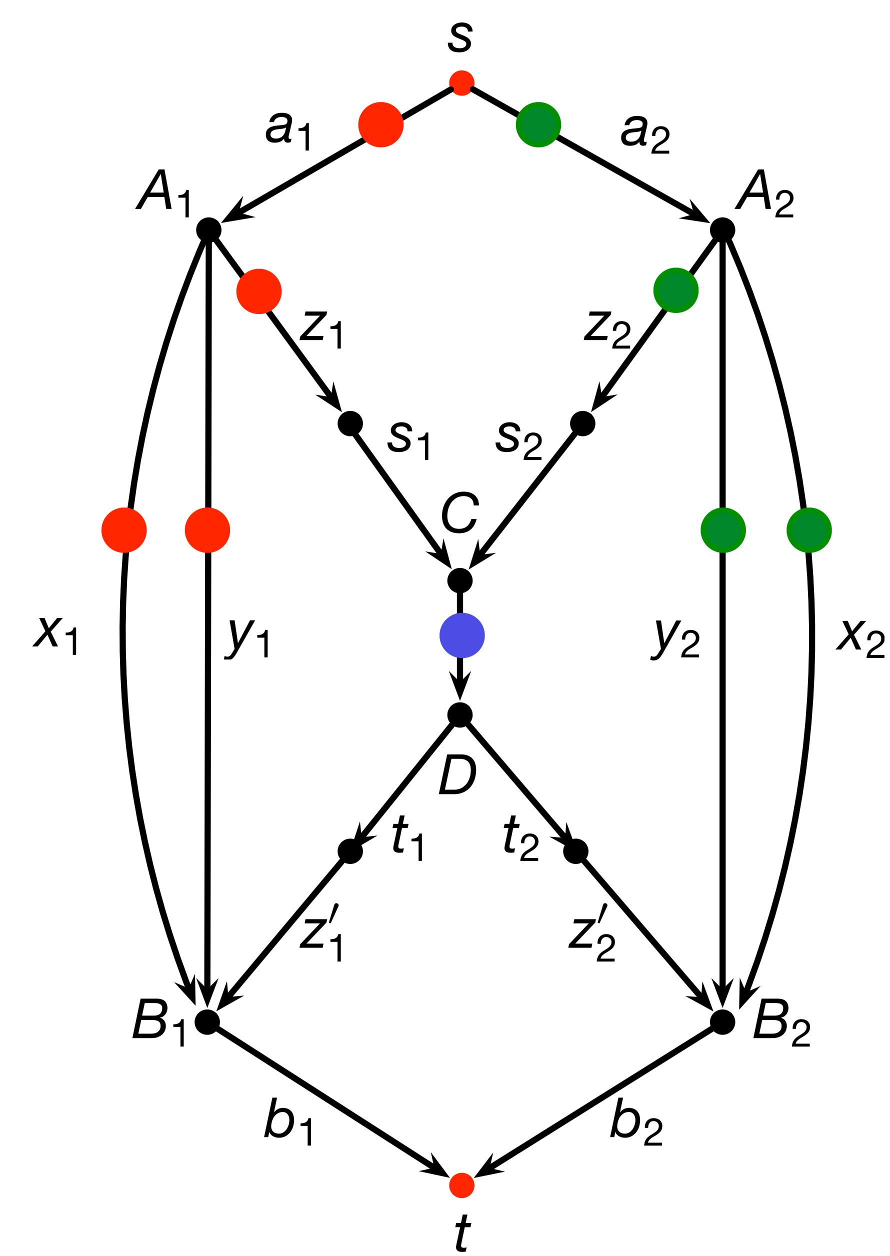
  $z_1'(M) = z_2'(M) = M_1 + M_2$



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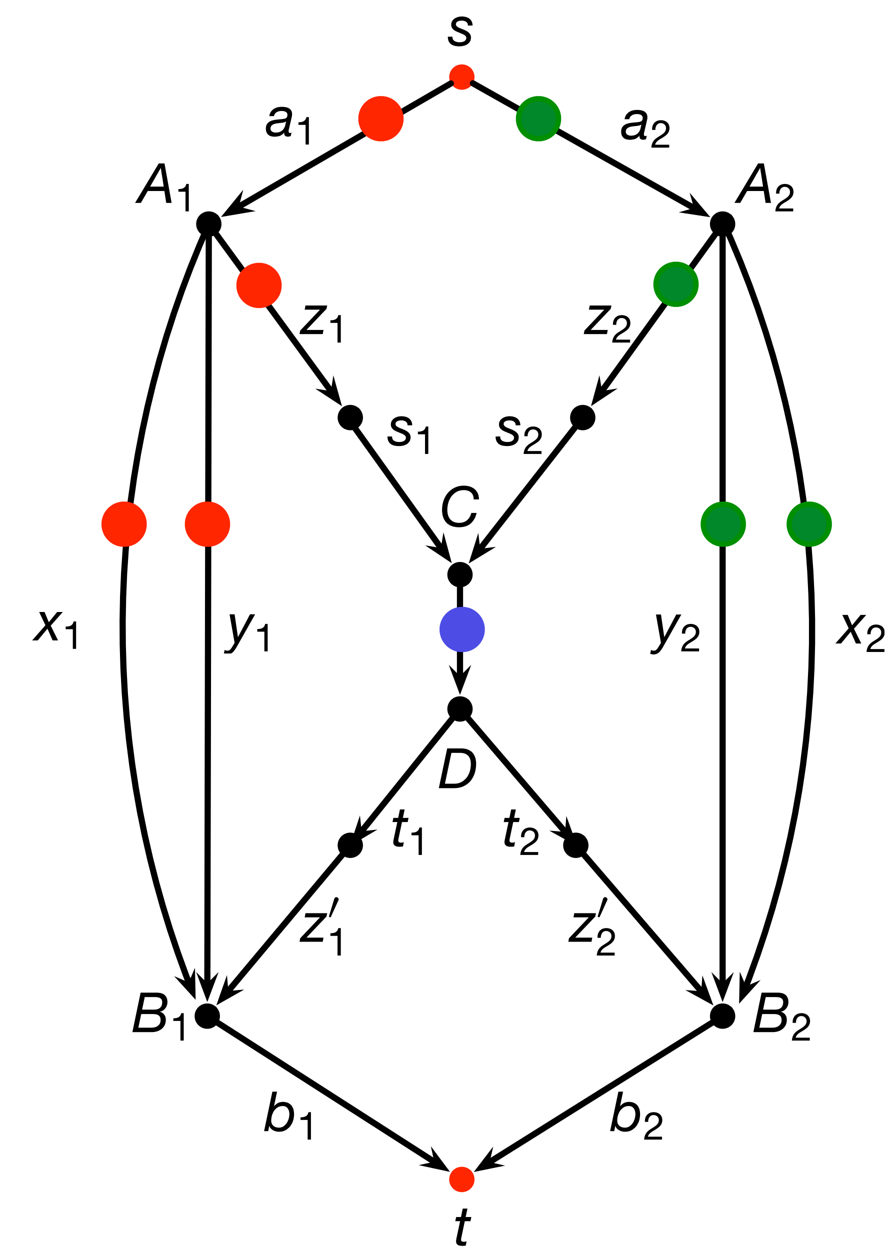

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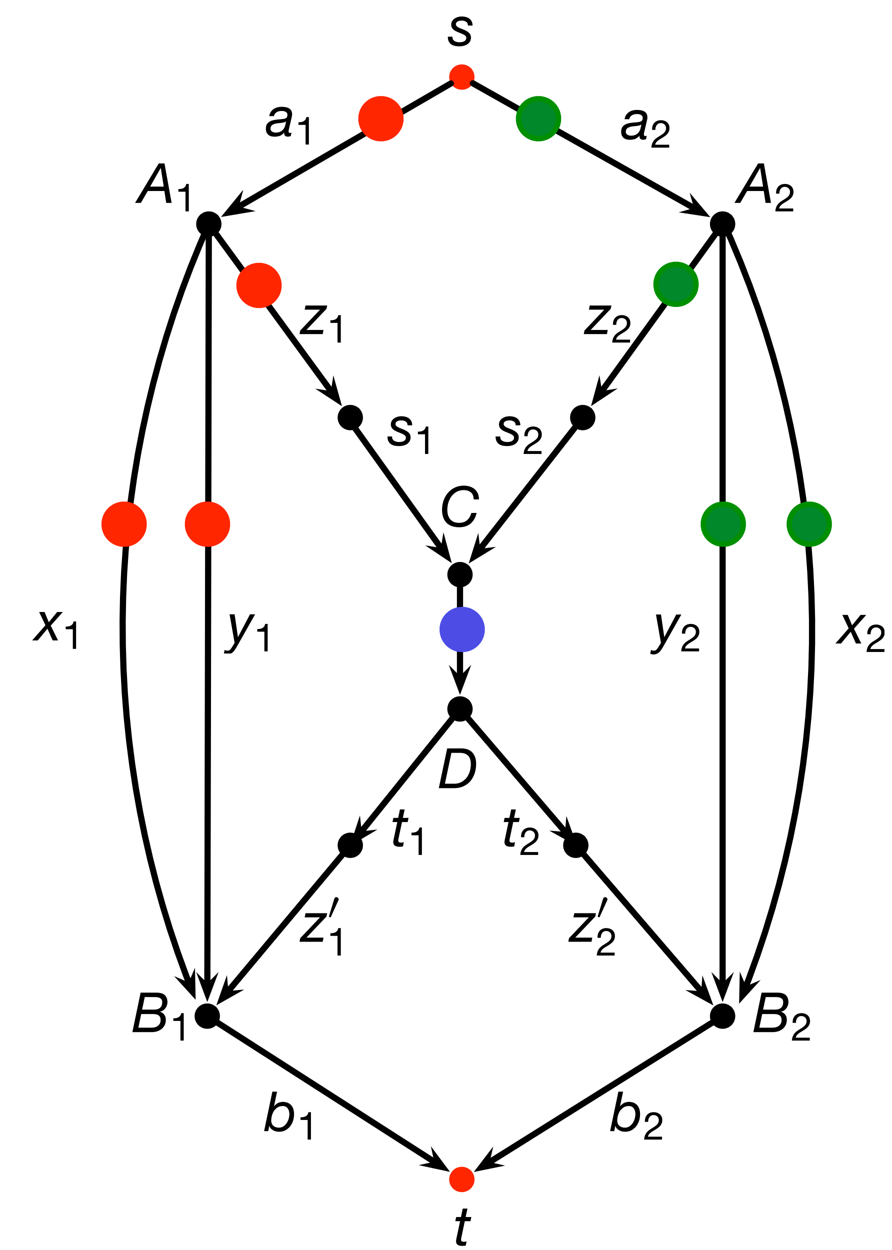

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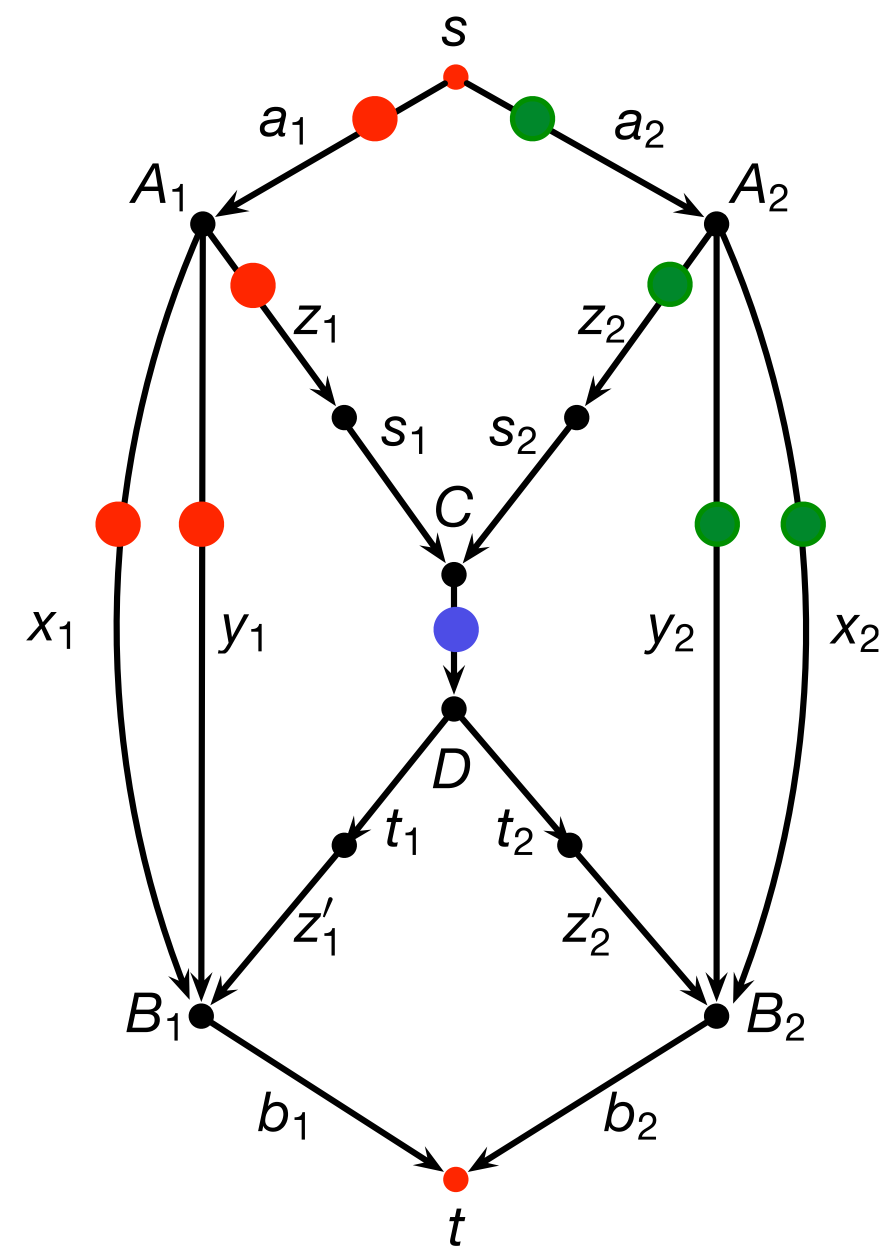
- t is able to decode (M_1, M_2) correctly at asymptotic rate 2
- Multiple unicast with rate (1, 1) is not feasible



Take Aways

- Error correction in a simple network setting
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- [Huang, Langberg, Kliewer, accepted for ISIT 2015]