Graphical Inequalities for the Linear Ordering Polytope

Jean-Paul Doignon

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Joint work with

Samuel Fiorini and *Gwenaël Joret*

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Binary Choice Probabilities

Take

Z some finite set of cardinality n,

Π the collection of the $n!$ rankings or linear orderings of Z.

To each probability distribution P on Π,

we associate the

binary choice probabilities p_{ij}, for $i, j \in Z$ and $i \neq j$,

defined by

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\begin{aligned}
p_{ij} &= P\{\, i \text{ is ranked before } j \,\} \\
&= \sum \{\, P(L) : L \in \Pi \text{ and } i\,L\,j \,\}.
\end{aligned}
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Binary Choice Probabilities on $\{a, b, c\}$

Example

For

$$Z = \{a, b, c\},$$
$$\Pi = \{\, abc,\ acb,\ bac,\ bca,\ cab,\ cba \,\},$$

we have by definition

$$
\begin{aligned}
p_{ab} &= P(abc) + P(acb) + P(cab),\\
p_{ba} &= P(bac) + P(bca) + P(cba),\\
p_{ac} &= P(abc) + P(acb) + P(bac),\\
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A Question

Can the following data be produced in this way?

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\begin{aligned}
p_{ab} &= 0.12, & p_{ba} &= 0.82,\\
p_{ac} &= 0.56, & p_{ca} &= 0.44,\\
p_{bc} &= 0.75, & p_{cb} &= 0.25.
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More precisely: is there some probability distribution P on Π that would give the following?

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Main Problem: Characterizing Binary Choice Prob.

Given real numbers p_{ij} for all $i, j \in Z$ with $i \neq j$,

can we find some probability distribution P on Π such that the p_{ij}'s are the binary choice probabilities defined by P?

More precisely:

find a necessary and sufficient condition on the p_{ij}'s for the existence of P.

The usual comment:

characterizing binary choice probabilities is . . .

. . . a hopeless problem!

An algorithmically tractable answer would lead to P = NP.

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Some Obvious Necessary Conditions

Binary choice probabilities always satisfy

$$\begin{aligned} p_{ij} &\geq 0, \\ p_{ij} + p_{ji} &= 1, \\ p_{ij} + p_{jk} + p_{ki} &\leq 2. \end{aligned}$$

These necessary conditions are also sufficient

exactly when $n \leq 5$:

Motzkin ($\leq$ 1960); … (19..); Dridi (1980); … (19..)

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A Geometric Point of View

Vectors of binary choice probabilities p belong to $\mathbb{R}^{Z \ltimes Z}$

(a space with one real coordinate for each pair (i, j) of distinct objects).

Example

For $Z = \{a, b, c\}$, we have 6-dimensional vectors

$$(\, p_{ab},\ p_{ba},\ p_{bc},\ p_{cb},\ p_{ac},\ p_{ca} \,).$$

As we know $\quad p_{ab} + p_{ba} = 1, \quad p_{ac} + p_{ca} = 1, \quad p_{bc} + p_{cb} = 1,$
we may work with only

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The collection of all (projected) vectors form a polyhedron in $\mathbb{R}^3$:

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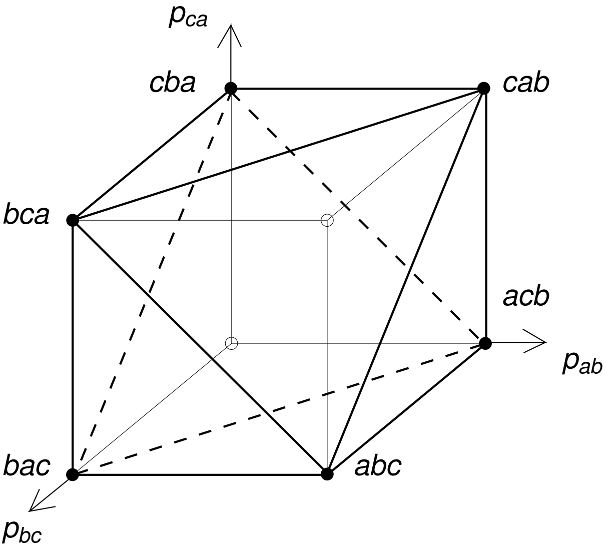
The Projected Polyhedron for $Z = \{a, b, c\}$



The Linear Ordering Polytope

Let $n = |Z|$.

The binary choice probabilities form a convex polytope in $\mathbb{R}^{Z \ltimes Z}$

of dimension $\dfrac{n \cdot (n-1)}{2}$,

with one vertex x^L per ranking L of Z:

$$x^L_{ij} = \begin{cases} 1 & \text{if } i\,L\,j, \\ 0 & \text{if } j\,L\,i. \end{cases}$$

This polytope is the binary choice polytope

or linear ordering polytope P^Z_{LO}.

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Rephrasing the Main Problem

The linear ordering polytope P_{LO}^{Z} has the vertices x^{L}, for $L \in \Pi$;

find the facets of the linear ordering polytope P_{LO}^{Z}.

And the usual comment: the problem is hopeless!

A manageable solution would give $\mathtt{P} = \mathtt{NP}$.

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Origins of the Problem

In mathematical psychology/economics:

Guilbaud (1953), Block and Marschak (1960).

In discrete mathematics:

Megiddo (1977).

In operations research:

Grötschel, Jünger and Reinelt (1985).

In voting theory:

Saari (1999).

Examples of Facet-defining Inequalities for P^n_{LO}

Remember our obvious necessary conditions.

Theorem

The following affine (linear) inequalities on $\mathbb{R}^{Z \ltimes Z}$ define facets:

$$p_{ij} \geq 0 \qquad \text{(trivial inequalities)},$$
$$p_{ij} + p_{jk} + p_{ki} \leq 2 \qquad \text{(triangular inequalities)}.$$

A first scheme of nonobvious facets is due independently to

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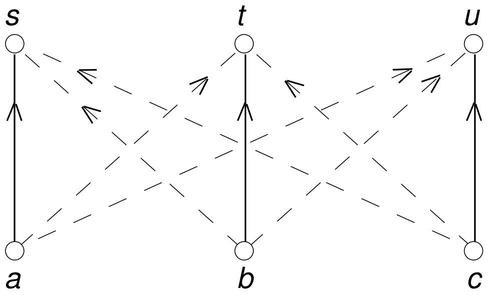
First Example of Fence Inequality

The following inequality is facet-defining:

$$x_{as} + x_{bt} + x_{cu} \quad - (x_{at} + x_{bs}) - (x_{au} + x_{cs}) - (x_{bu} + x_{ct}) \quad \leq \quad 1.$$



The Fence Inequality

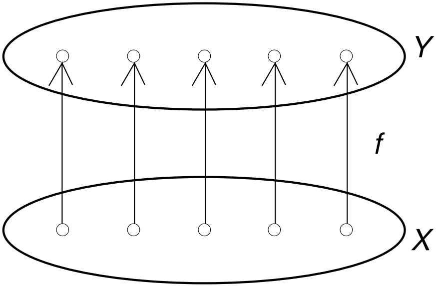
In general, let X, $Y \subset Z$ with

$$X \cap Y = \varnothing,$$

$$|X| = |Y|,$$

$f : X \to Y$ a bijective mapping

(we keep the notation throughout).



The Fence Inequality

Definition

The fence inequality is

$$\sum_{i \in X} x_{i\,f(i)} - \sum_{i,j \in X,\ i \neq j} \left(x_{i\,f(j)} + x_{j\,f(i)} \right) \quad \leq \quad 1.$$

Theorem (Cohen and Falmagne, 1978; Grötschel, Jünger and Reinelt, 1985)

For $|X| \geq 3$, the fence inequality defines a facet of the linear ordering polytope P^n_{LO}.

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A Structural Generalization of the Fence Inequality

Several steps:

McLennan (1990), Fishburn (1990), Koppen (1991), etc.

leading to a marvelous result by Koppen (1995).

Let $G = (V, E)$ be a (simple) graph.

The stability number $\alpha(G)$ of G is the largest number of vertices no two of which are adjacent.

Assume $f : X \to Y$ as before, and moreover $V = X$.

Definition
The graphical inequality of G reads

$$\sum_{i \in V} x_{i,f(i)} - \sum_{\{i,j\} \in E} (x_{i,f(j)} + x_{j,f(i)}) \leq \alpha(G).$$

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An Example of Graphical Inequality

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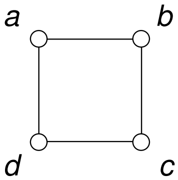
For the graph



with the bijection

$$f: \quad a \mapsto s, \quad b \mapsto t, \quad c \mapsto u, \quad d \mapsto v,$$

we get the inequality

$$x_{as} + x_{bt} + x_{cu} + x_{dv} - (x_{at} + x_{bs}) - (x_{bu} + x_{ct}) - (x_{cv} + x_{du}) - (x_{ds} + x_{av}) \leq 2.$$

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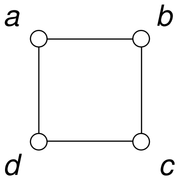

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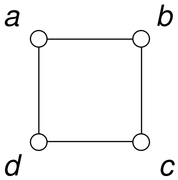

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Main Result in Koppen (1995)

Theorem (Koppen, 1995)

The graphical inequality of G is valid for the linear ordering polytope.

It defines a facet if and only if G is
different from K_2,
connected,
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A graph is stability critical when its stability number increases whenever any of its edges is deleted.

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An Example of Stability-Critical Graph

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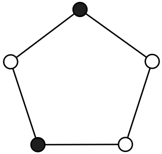
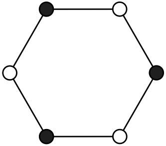
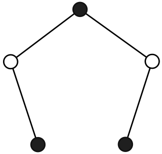
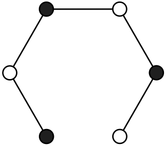




Delete any edge:





Thus: the 5-cycle is stability critical but the 6-cycle is not.

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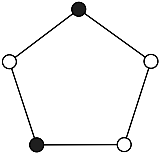

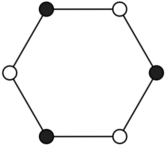

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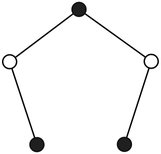

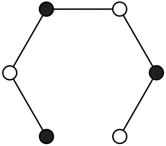

Thus: the 5-cycle is stability critical but the 6-cycle is not.

A Weighted Generalization of the Fence Inequality

Independently: Leung and Lee (1994), Suck (1992).

Theorem

For $|X| \geq 3$, the reinforced fence inequality

$$\sum_{i \in X} t\, x_{i,f(i)} \quad - \sum_{i,j \in X,\ i \neq j} \left(x_{i,f(j)} + x_{j,f(i)}\right) \quad \leq \quad \frac{t(t+1)}{2}$$

defines a facet of P^n_{LO} if and only if the constant value t satisfies

$$1 \ \leq\ t \ \leq\ |X| - 2.$$

Our Contribution (D., F. and J.)

Schematically:

fence inequality

graphical inequality of
a stability critical graph

reinforced fence inequality
(of a complete graph)

A common generalization?

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Preparing a General Graphical Inequality

Let (G, μ) be a weighted graph, with $G = (V, E)$ and $\mu : V \to \mathbb{Z}$.

Definition

For $S \subseteq V$, the worth (or net weight) $w(S)$ equals the total weight $\mu(S)$ minus the number of edges in S.

A subset of S is tight if it maximizes the worth.

Notation

$$\alpha(G, \mu) = \max_{S \subseteq V} w(S).$$

Remark

If $\mu = \mathbf{1}$ (constant weight 1), then $\alpha(G, \mathbf{1}) = \alpha(G)$.

Thus $\alpha(G, \mu)$ is a true generalization of $\alpha(G)$.

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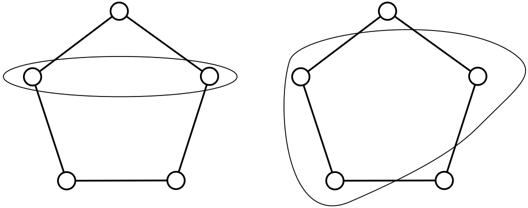
Examples of Tight Sets

Example

For the pentagon with $\mu = \mathbf{1}$, here are tight sets:



Remember that tight sets S maximize

$$w(S) = \mu(S) - ||S||.$$

Graphical Inequalities

Let (G, μ) be a weighted graph, with $G = (V, E)$ and $\mu : V \to \mathbb{Z}$.

Definition

Let $f : X \to Y$ be bijective with $X, Y \subset Z$, $X \cap Y = \varnothing$, and assume $V = X$.

The graphical inequality of (G, μ) reads

$$\sum_{i \in V} \mu(i)\, x_{i,f(i)} \quad - \sum_{\{i,j\} \in E} (x_{i,f(j)} + x_{j,f(i)}) \quad \leq \quad \alpha(G, \mu).$$

Proposition

The graphical inequality is always valid for the linear ordering polytope P_{LO}^{Z}.

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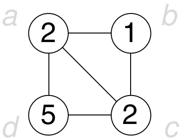
Example

Consider $X = \{a, b, c, d\}$, $Y = \{s, t, u, v\}$, and the bijection

$$f : \qquad a \mapsto s, \quad b \mapsto t, \quad c \mapsto u, \quad d \mapsto v.$$

Take the graph



Its graphical inequality is

$$2\,x_{as} + x_{bt} + 2\,x_{cu} + 5\,x_{dv} - (x_{at} + x_{bs}) - (x_{au} + x_{cs}) - (x_{av} + x_{ds}) - (x_{bu} + x_{ct}) - (x_{cv} + x_{du}) \leq 6.$$

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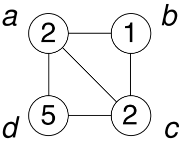

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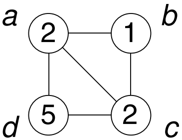

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A weighted graph is facet defining or a FDG if its graphical inequality defines a facet of P^n_{LO}.

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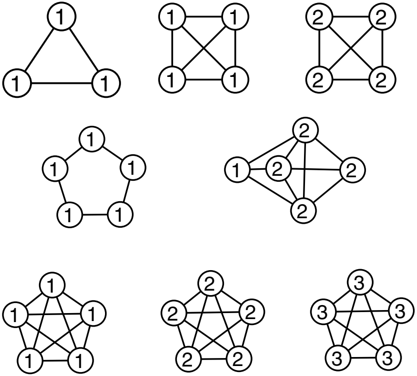

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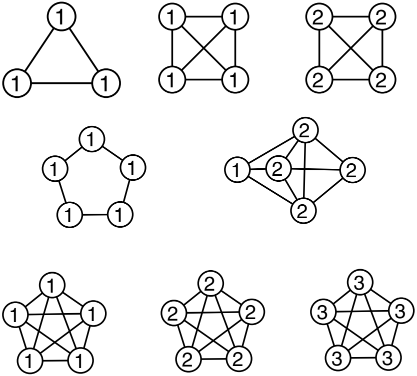

A Subsidiary Problem

Problem

To understand FDGs, e.g. to classify them.

Remark

FDGs include connected, stability critical graphs with more than 2 vertices.

Hard (although only partial) results were obtained in classifying the latter graphs, see e.g. Lovász (1993).

(**Remark**
Another weighted, generalization of stability critical graphs is investigated by Lipták and Lovász (2000, 2001).)

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Let (G, μ) be a weighted graph with more than two vertices.

Then (G, μ) is a FDG

if and only if

for each nonzero valuation $\lambda : V(G) \cup E(G) \to \mathbb{Z}$ there is a tight set T of (G, μ) with

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Sketch of the proof

Take $f: X \to Y$ as in the definition of the graphical inequality.

Take the restrictions to $X \times Y$ of all linear orderings L of Z.

The resulting relations from X to Y coincide with the "biorders" from X to Y (Doignon, Ducamp and Falmagne, 1984).

The biorder polytope $P_{\mathrm{Bio}}^{X \times Y}$ is defined in $\mathbb{R}^{X \times Y}$ (Christophe, Doignon and Fiorini, 2004).

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First Results on Facet Defining Graphs

Theorem
For any FDG (G, μ), the graph G is 2-connected.

Theorem
If (G, μ) is a FDG, so is $(G, deg - \mu)$.

[Here $(\deg - \mu)(v) = \deg(v) - \mu(v)$.]

Thus most stability critical graphs produce two FDGs:

one with $\mu = \mathbf{1}$, another one with $\mu = \deg - \mathbf{1}$.

Let's go back to stability critical graphs (FDGs when $\mu = \mathbf{1}$).

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The Defect of Stability Critical Graphs

For any graph $G = (V, E)$ (no weight here), define its defect

$$\delta(G) \quad = \quad |V| - 2\,\alpha(G).$$

Consider here a connected, stability critical graph G.

Theorem (Erdös and Gallai, 1961)

$$\delta(G) \ \geq \ 0.$$

Theorem (Hajnal, 1965)

Any vertex v of G satisfies $\quad deg(v) \ \leq \ \delta(G) + 1.$

Corollary (Hajnal, 1965)

$$\delta(G) \ = \ 0 \qquad \Longleftrightarrow \qquad G \ = \ K_2;$$
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Odd-cycles coincide with odd subdivisions of K_3.

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Odd Subdivisions of a Graph

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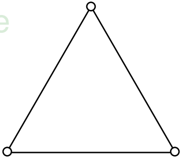
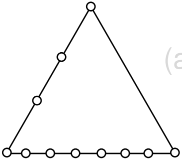
An odd subdivision of a graph G is obtained by replacing a number of edges of G with odd paths (of various lengthes).

Example

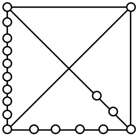




(an 11-cycle)

Theorem (Andrásfai, 1967)

The connected stability critical graphs with defect 2 are the odd-subdivision of K_4.



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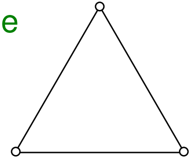

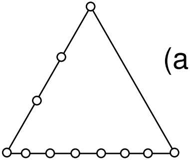

(an 11-cycle)

Theorem (Andrásfai, 1967)

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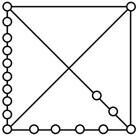

Odd Subdivisions of a Graph

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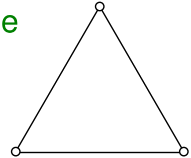

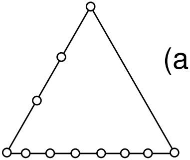

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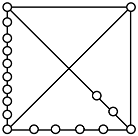


The Basis Theorem for Stability Critical Graphs

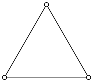
Theorem (Lovász, 1978)

For any natural number $\delta > 0$, there is a finite collection $\mathcal{S}_\delta$ of graphs such that

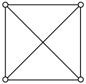
$$G \text{ is a connected stability critical graph with } \delta(G) = \delta$$
$$\Longleftrightarrow$$
$$G \text{ is an odd-subdivision of some graph in } \mathcal{S}_\delta.$$

Examples

$\mathcal{S}_1$:



$\mathcal{S}_2$:



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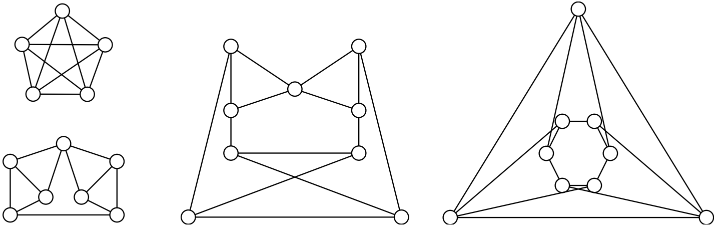
Among the graphs in $\mathcal{S}_3$, we show only those with minimum degree 3:



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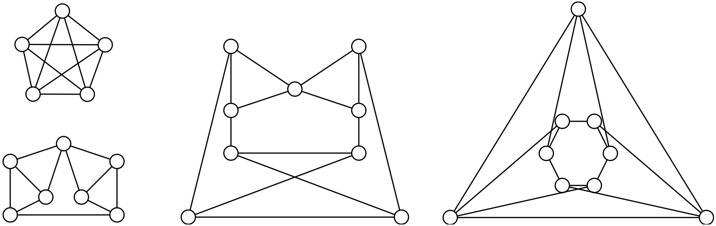

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The Defect of Facet-Defining Graphs (FDGs)

How to define the defect of a weighted graph (G, μ) ?

In our case, with $G = (V, E)$, we use

$$\delta(G, \mu) \quad = \quad \mu(V) \; - \; 2\,\alpha(G, \mu).$$

Notice $\quad \delta(G, \mathbf{1}) \; = \; \delta(G) \quad$ and $\quad \delta(G, \mu) \; = \; \delta(G, \deg - \mu).$

Let (G, μ) be any FDG.

Theorem

For each vertex v of G

$$1 \;\leq\; deg(v) - \mu(v) \;\leq\; \delta(G, \mu).$$

The proof is much more involved than in the case $\mu = \mathbf{1}$.

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$\mu(v) \leq \delta(G, \mu)$.

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If $\delta(G, \mu) = 1$, then $\mu = \mathbf{1}$ and G is an odd cycle.

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There is little hope that a computationally simple solution exists (otherwise, $\mathrm{P} = \mathrm{NP}$).

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