TU/e Technische Universiteit Eindhoven University of Technology

Sieving for shortest lattice vectors using fast search algorithms

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Rutgers University, New Brunswick, USA
(January 13, 2015)

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Lattices

What is a lattice?

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Lattices

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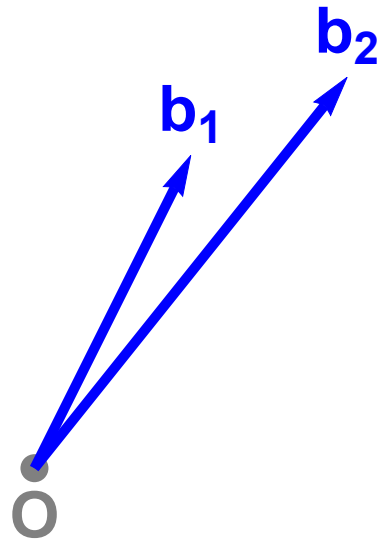

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Lattices

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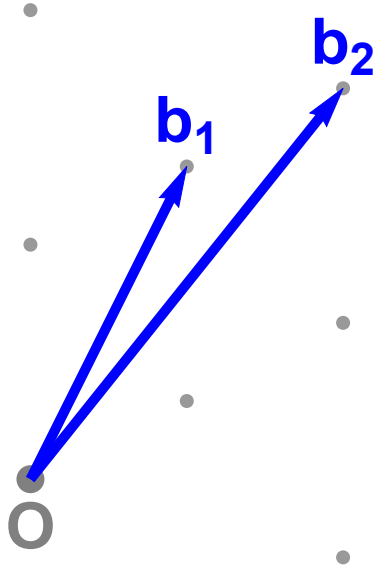

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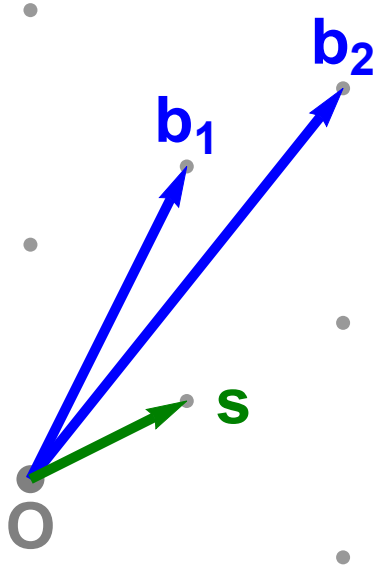
Lattices

Shortest Vector Problem (SVP)



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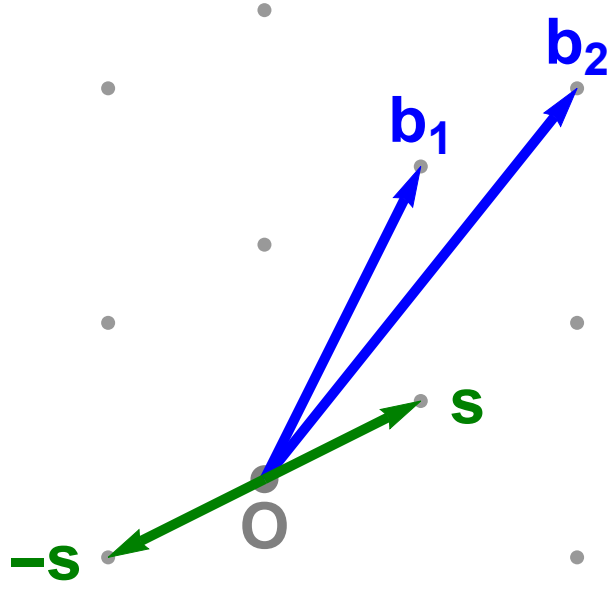

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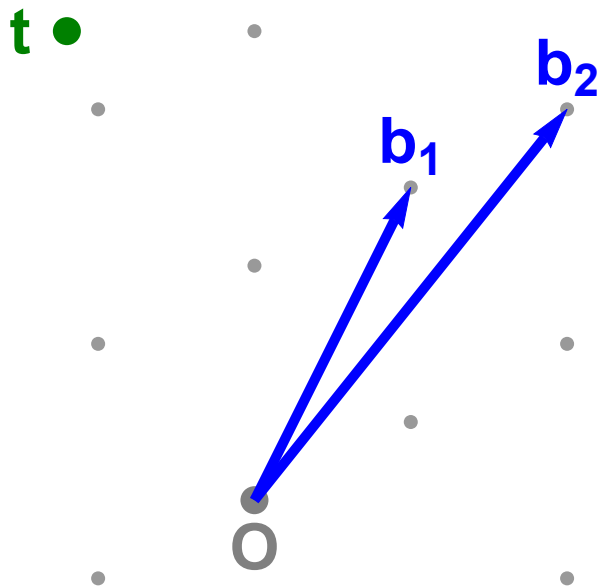

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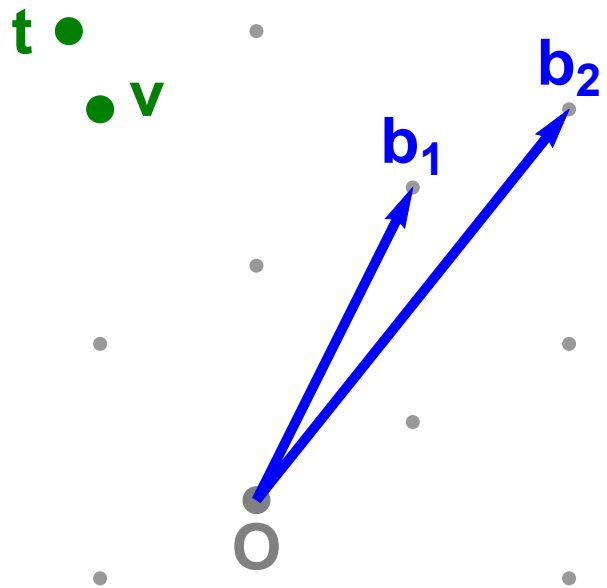

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Lattices

Applications

- "Constructive cryptography": Lattice-based cryptosystems
  - Based on hard lattice problems (SVP, CVP, LWE, SIS)
  - NTRU cryptosystem [HPS98, ...]
  - Fully Homomorphic Encryption [Gen09, ...]
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How hard are hard lattice problems such as SVP?

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Nguyen-Vidick sieve

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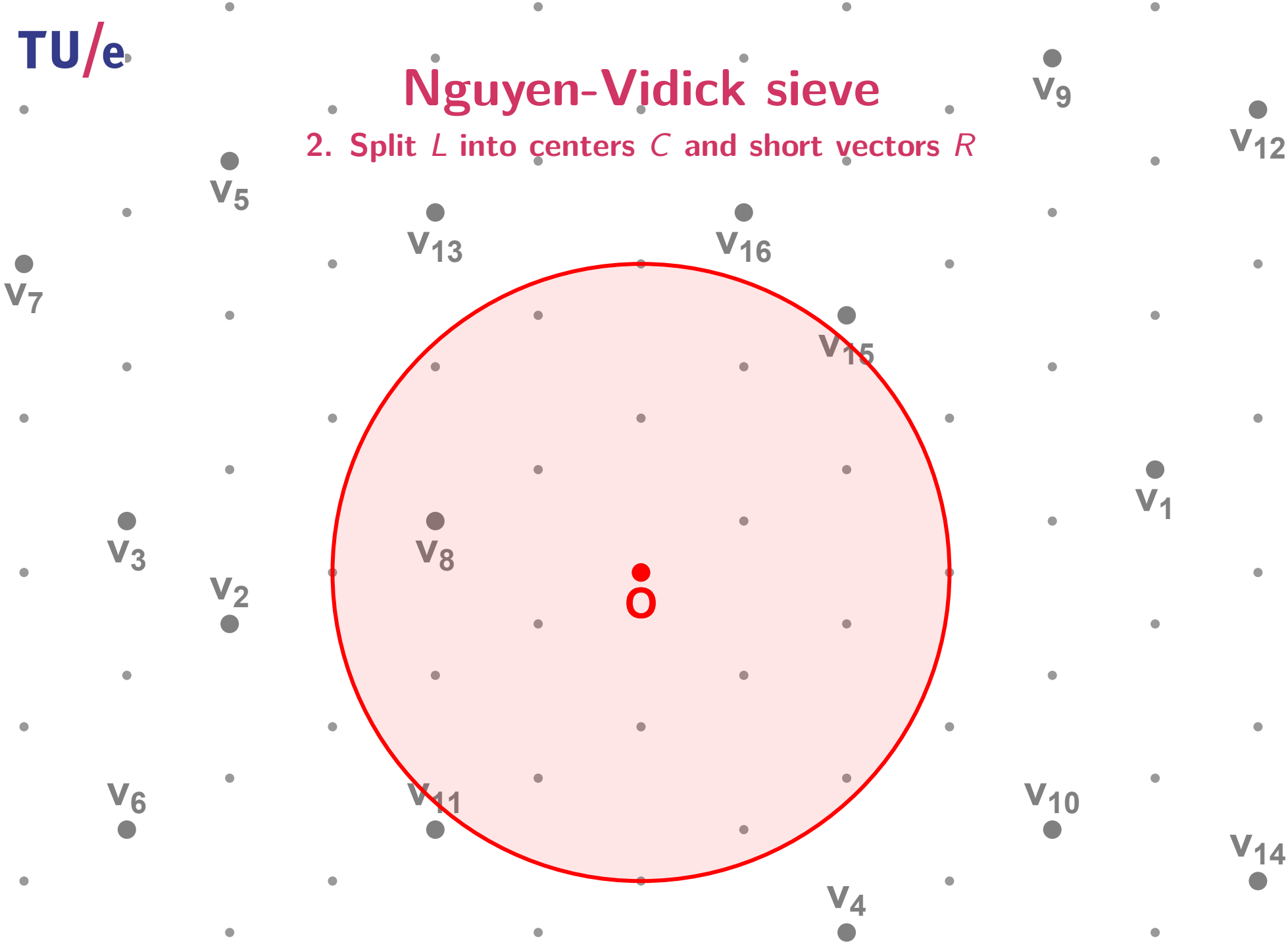

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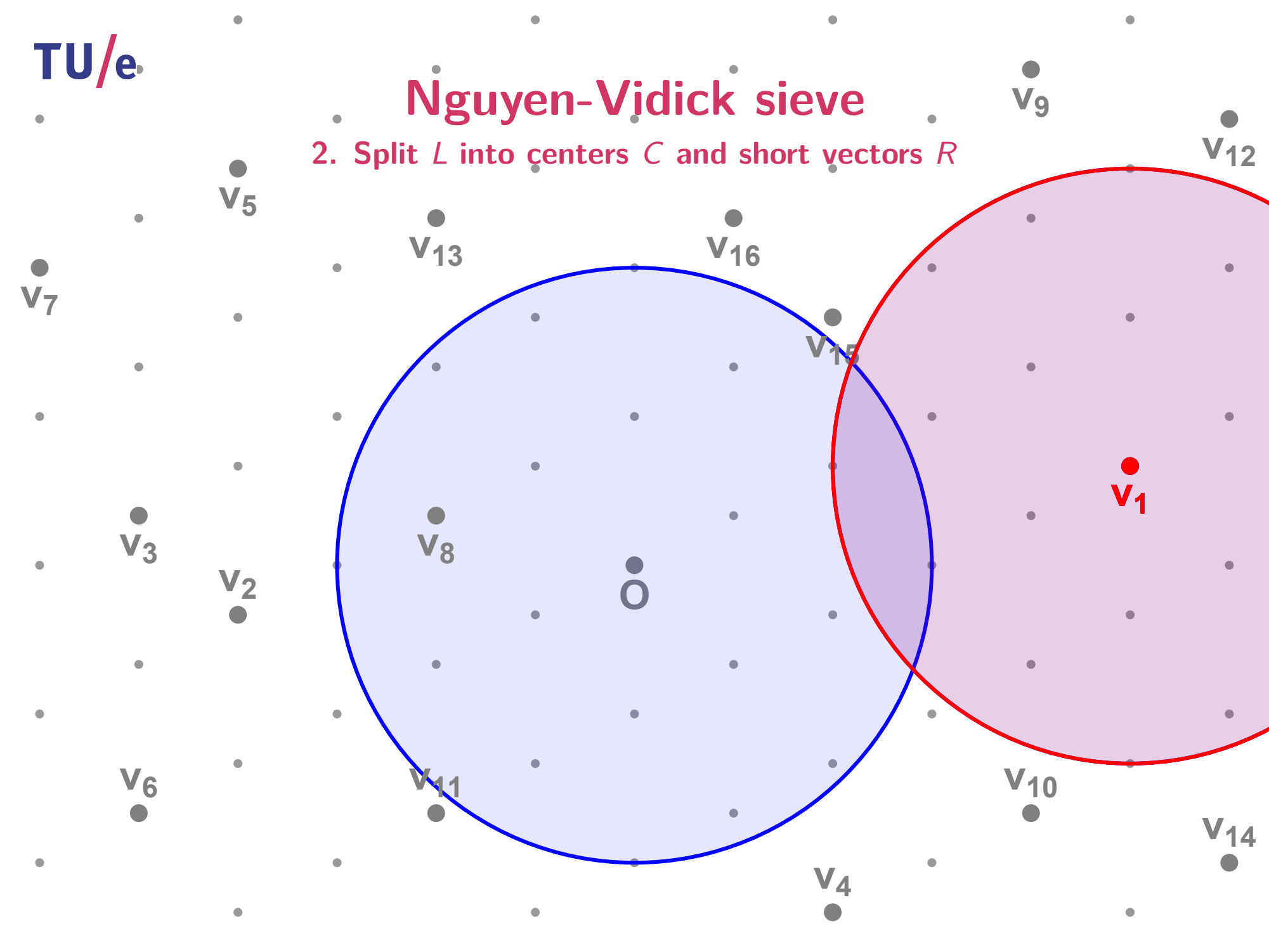

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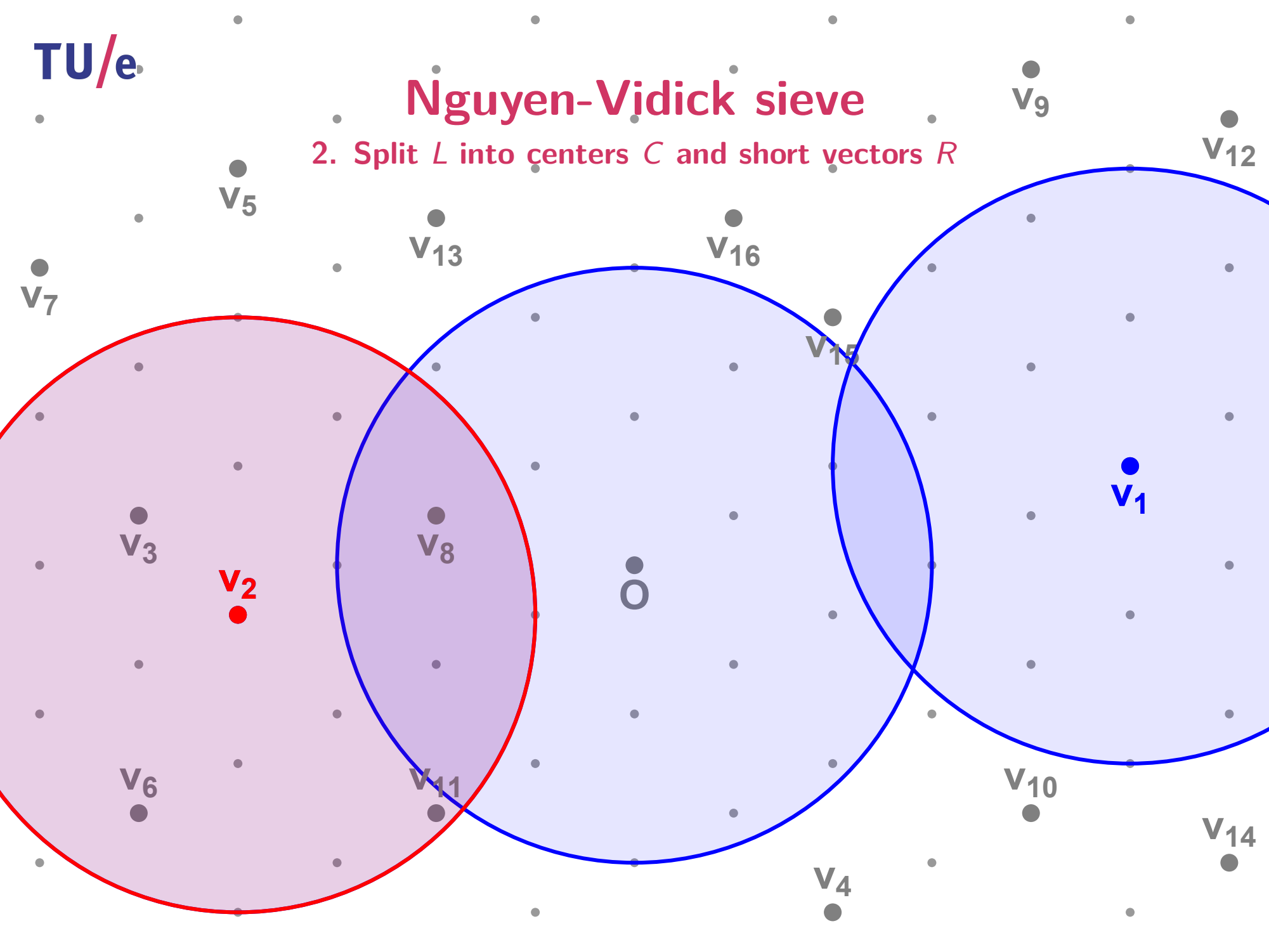

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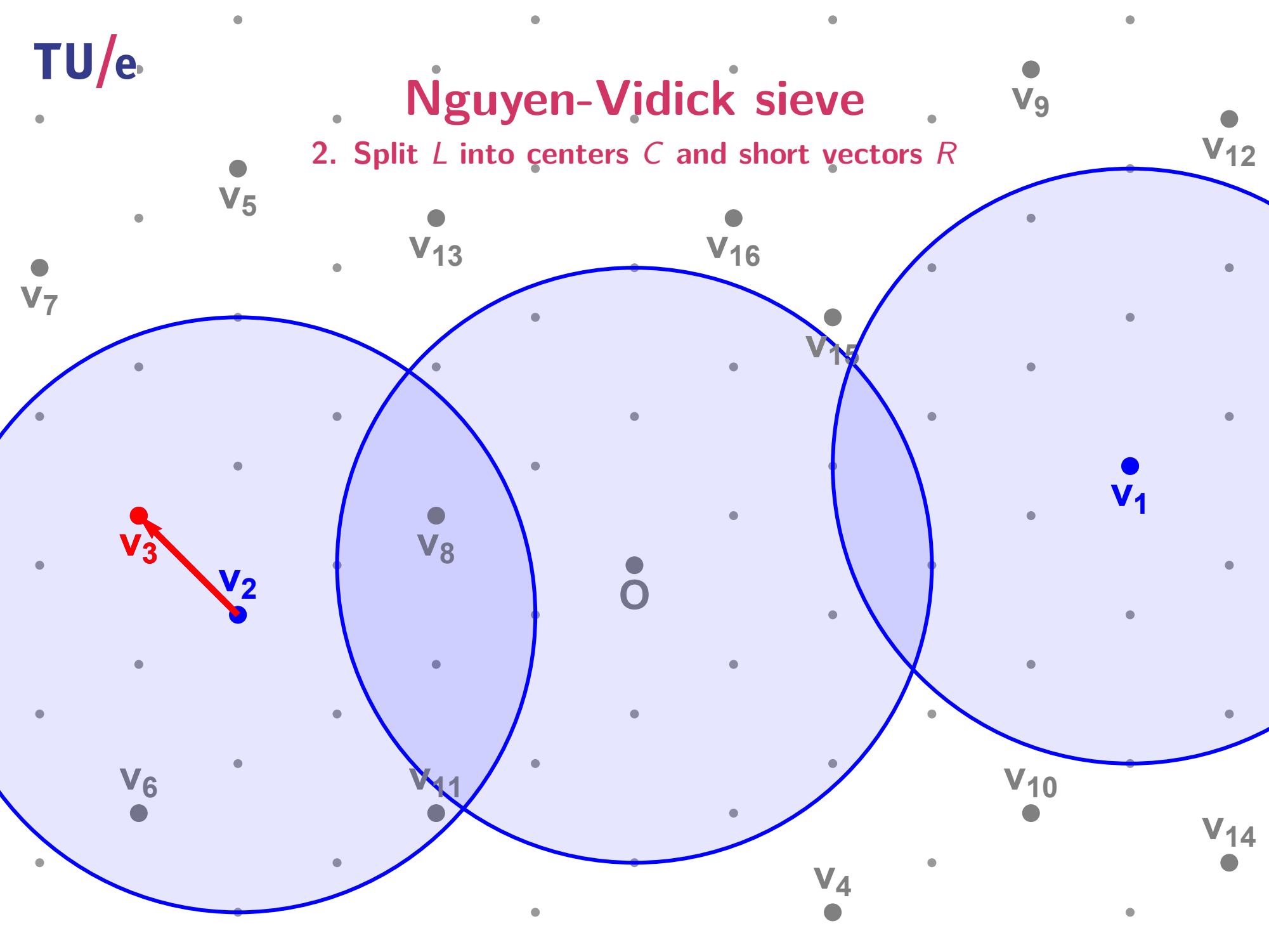

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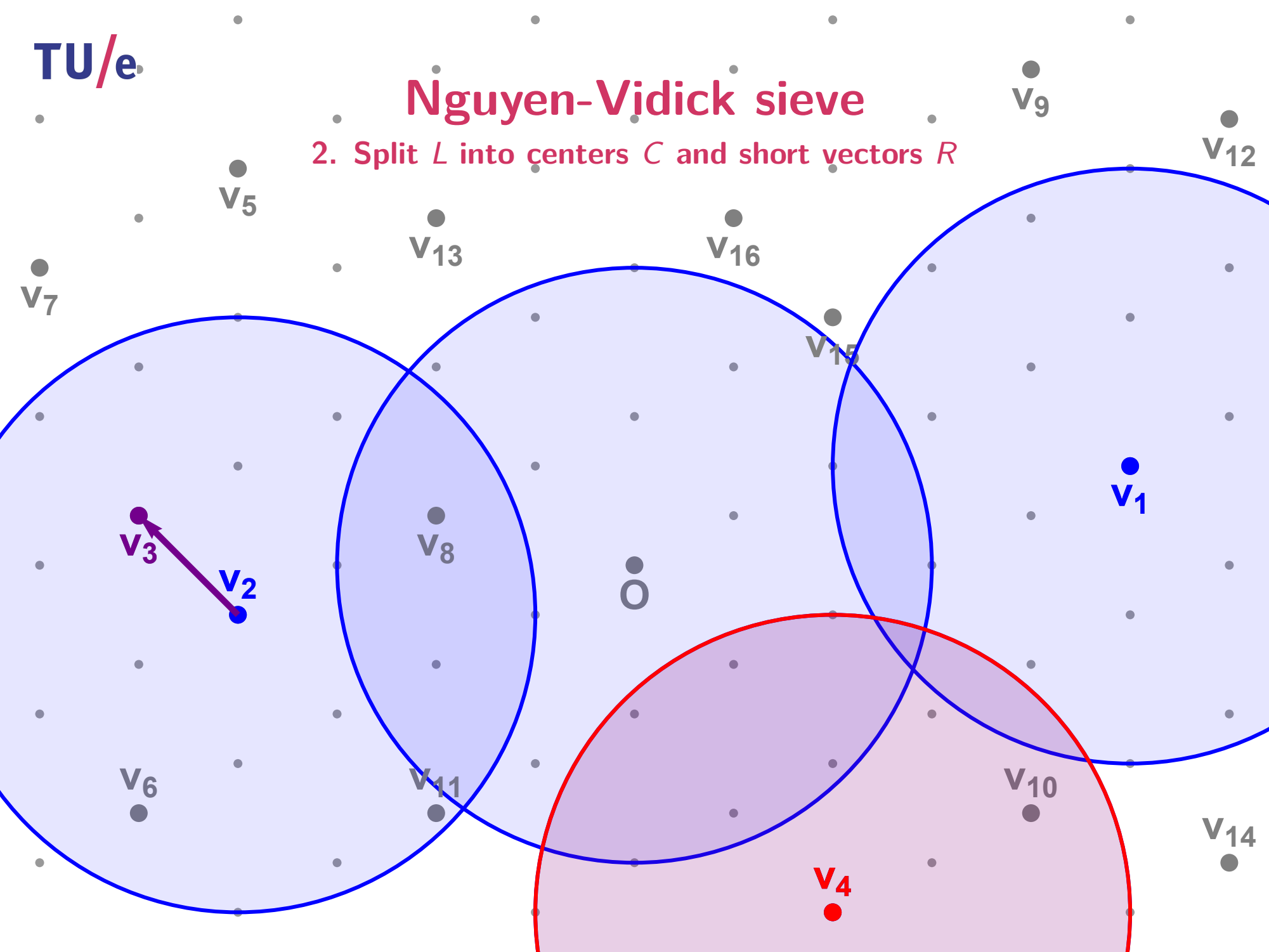

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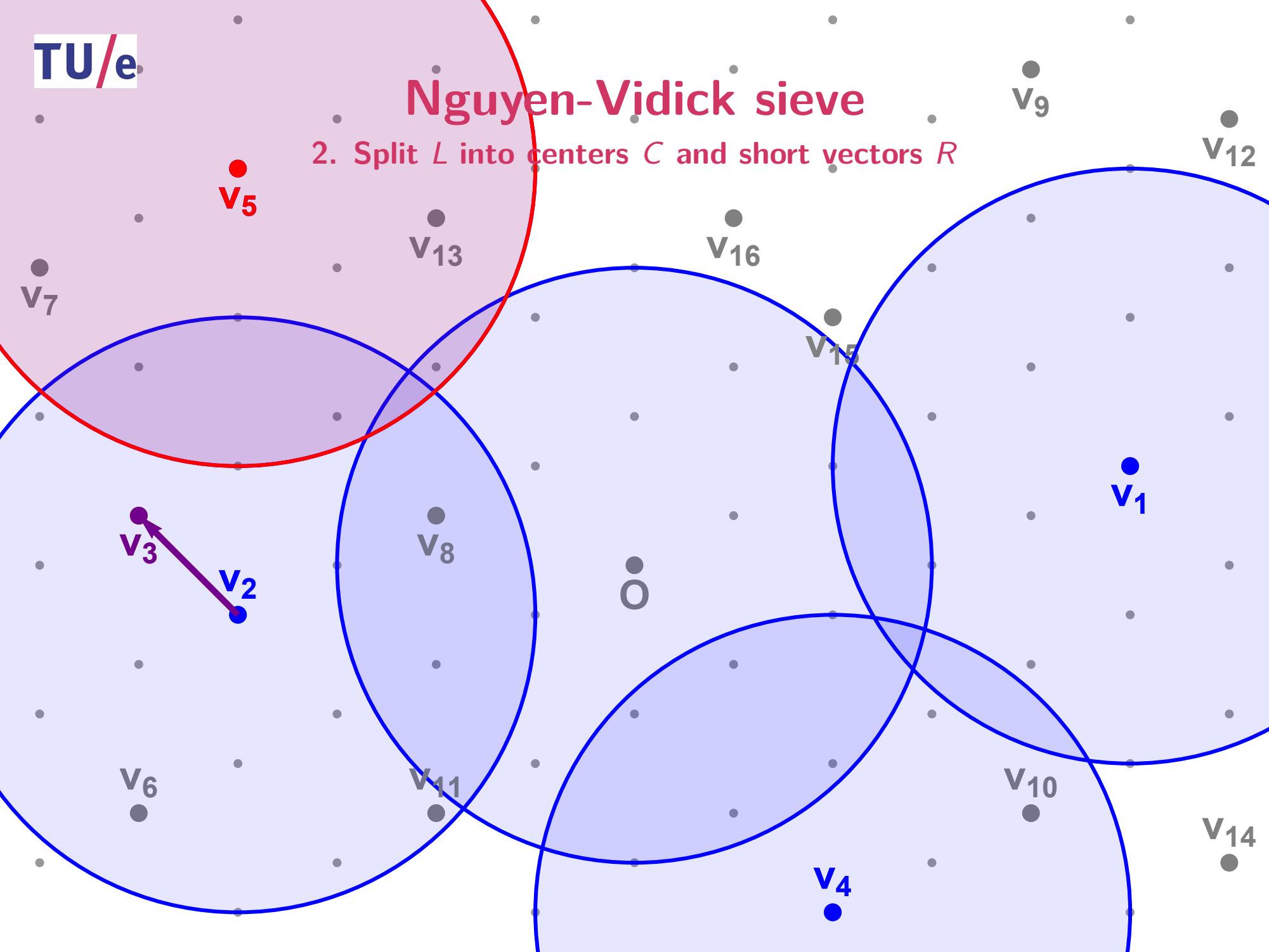

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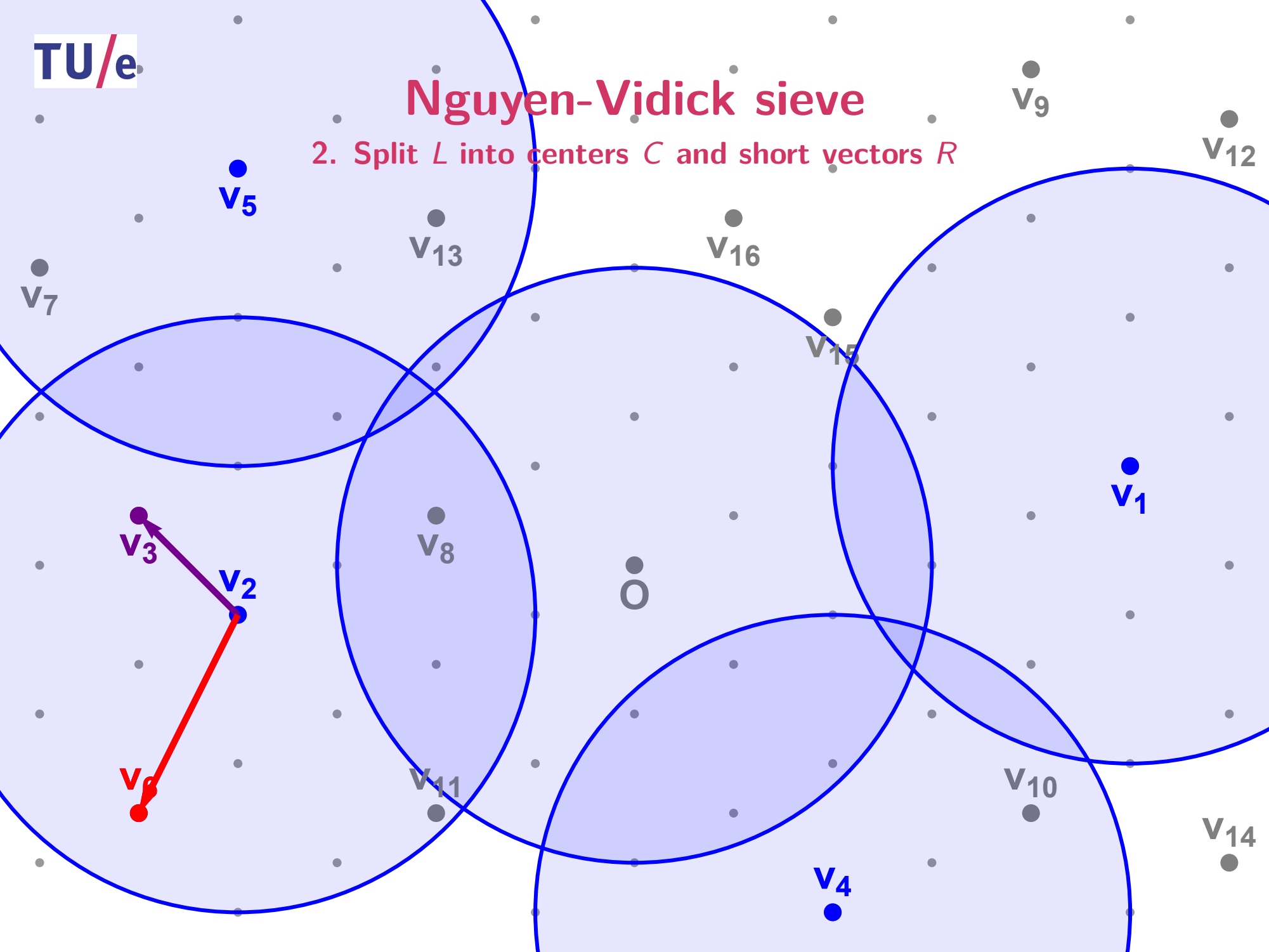

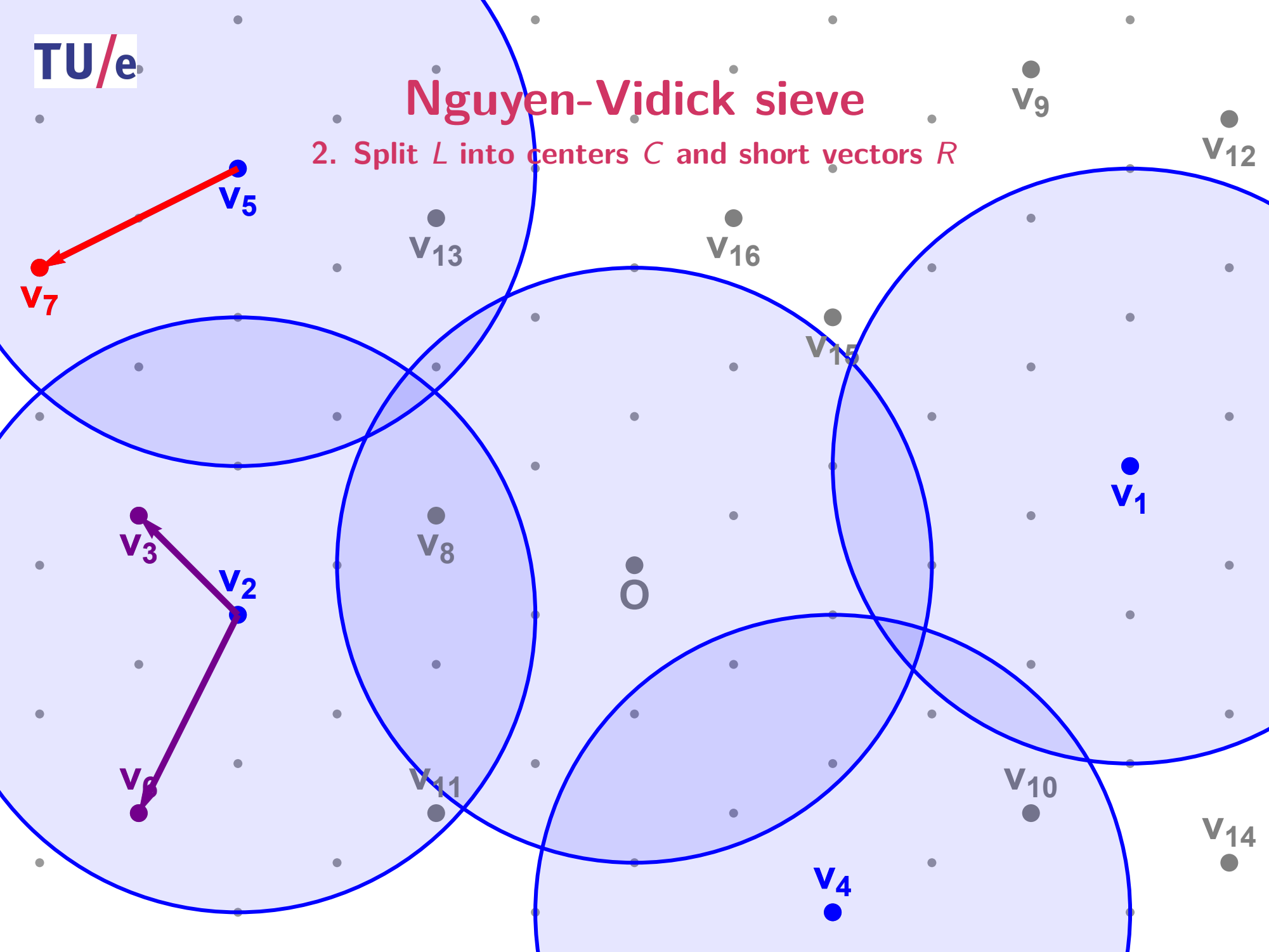
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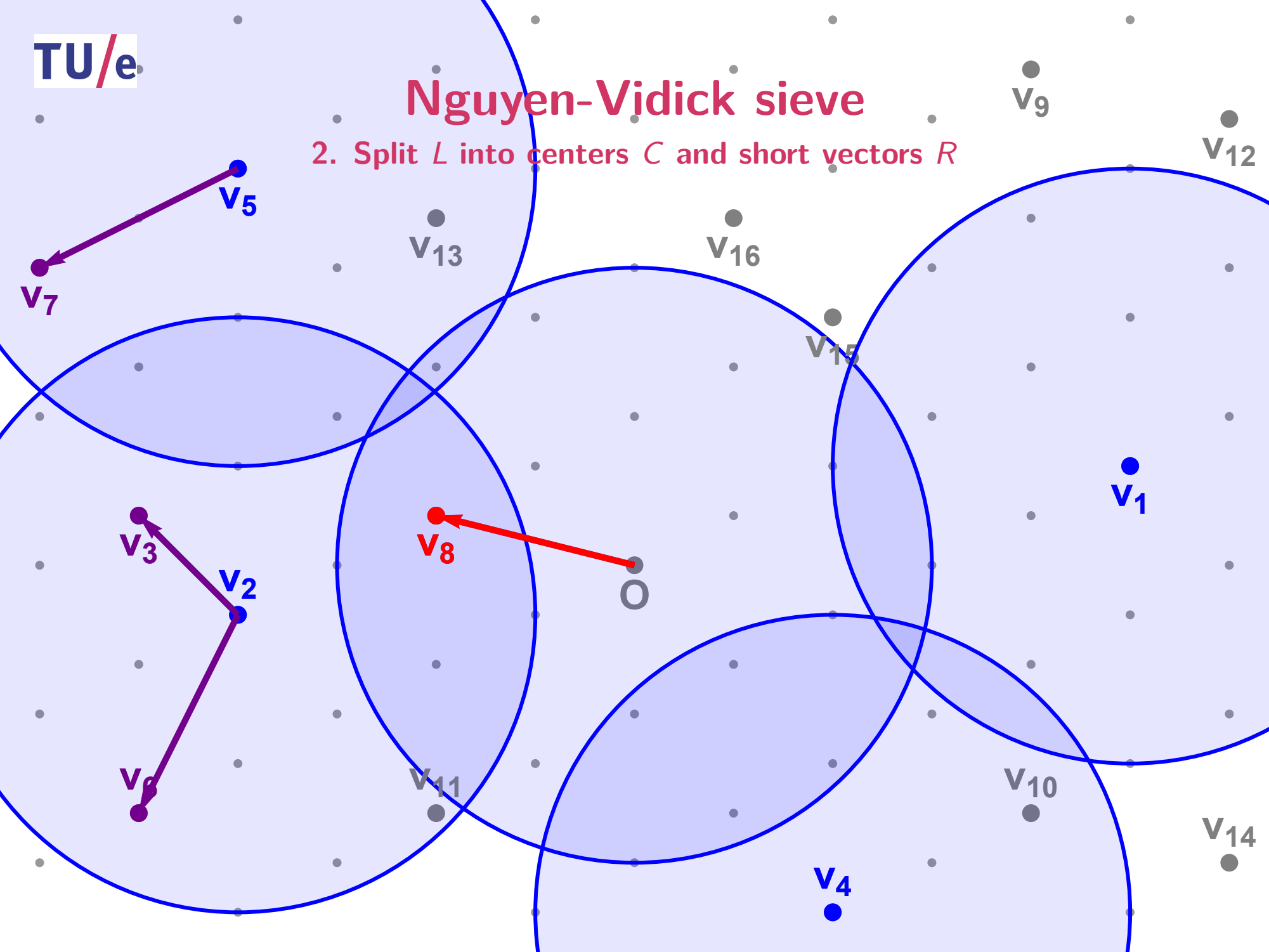

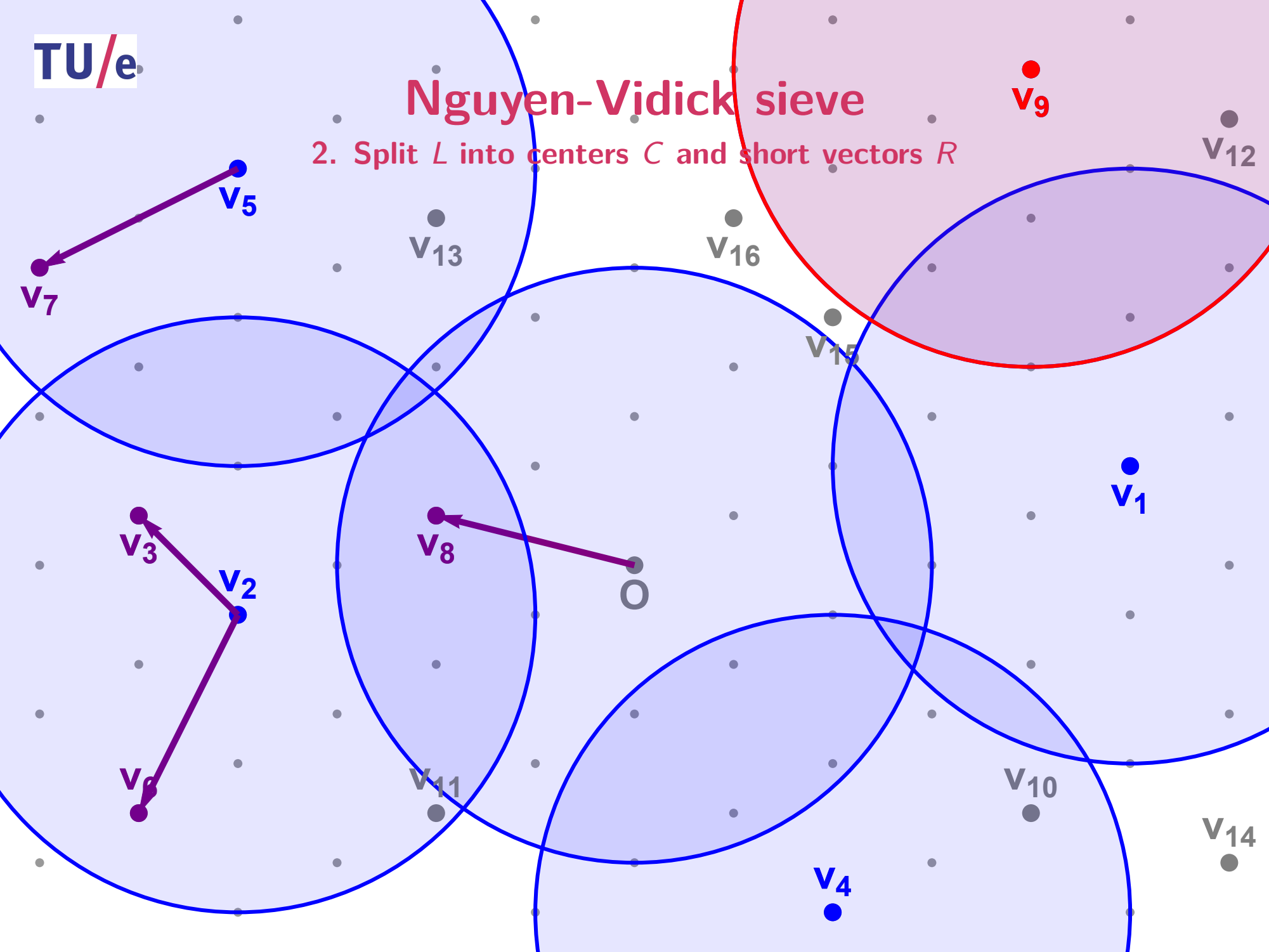
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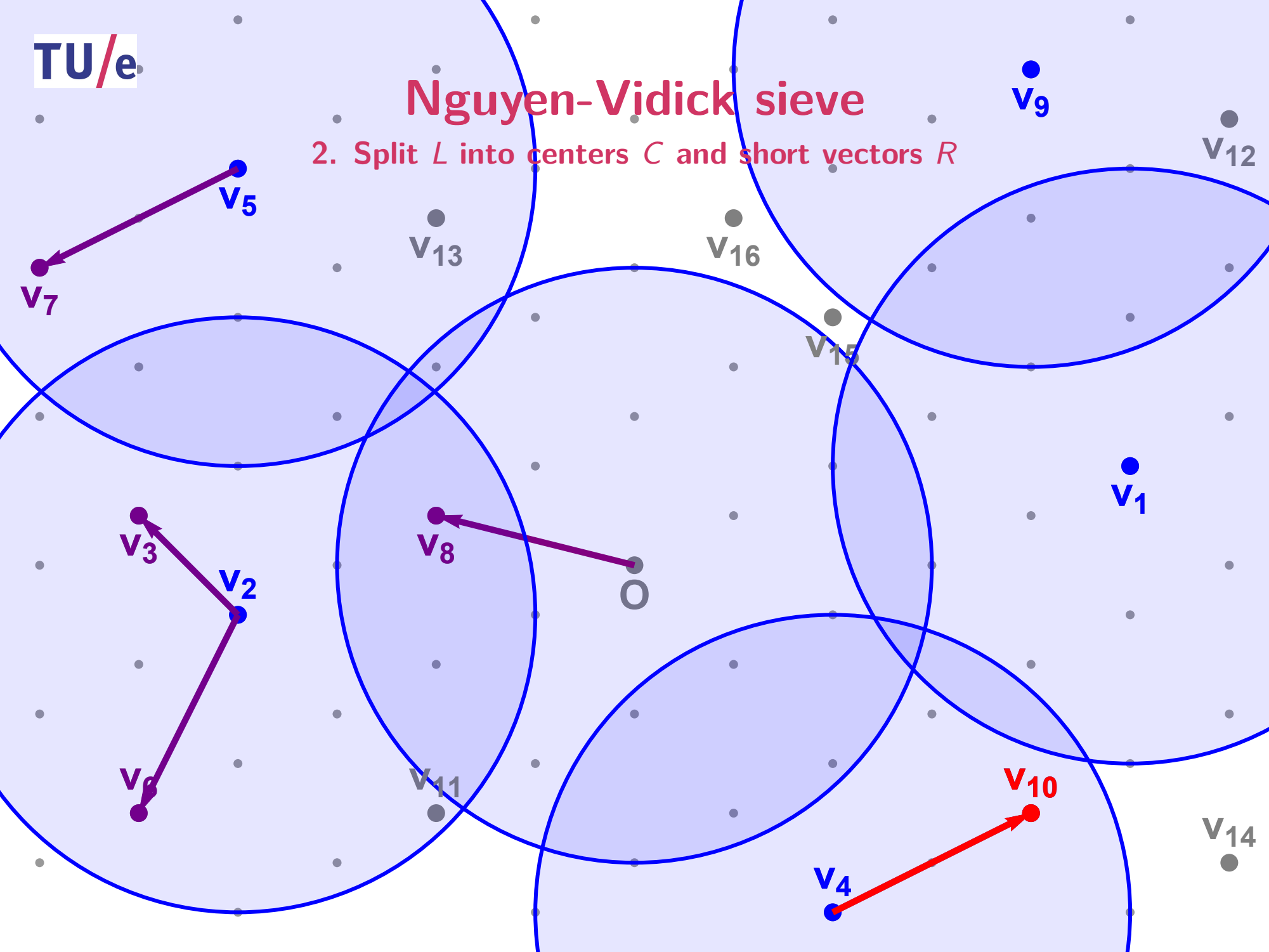
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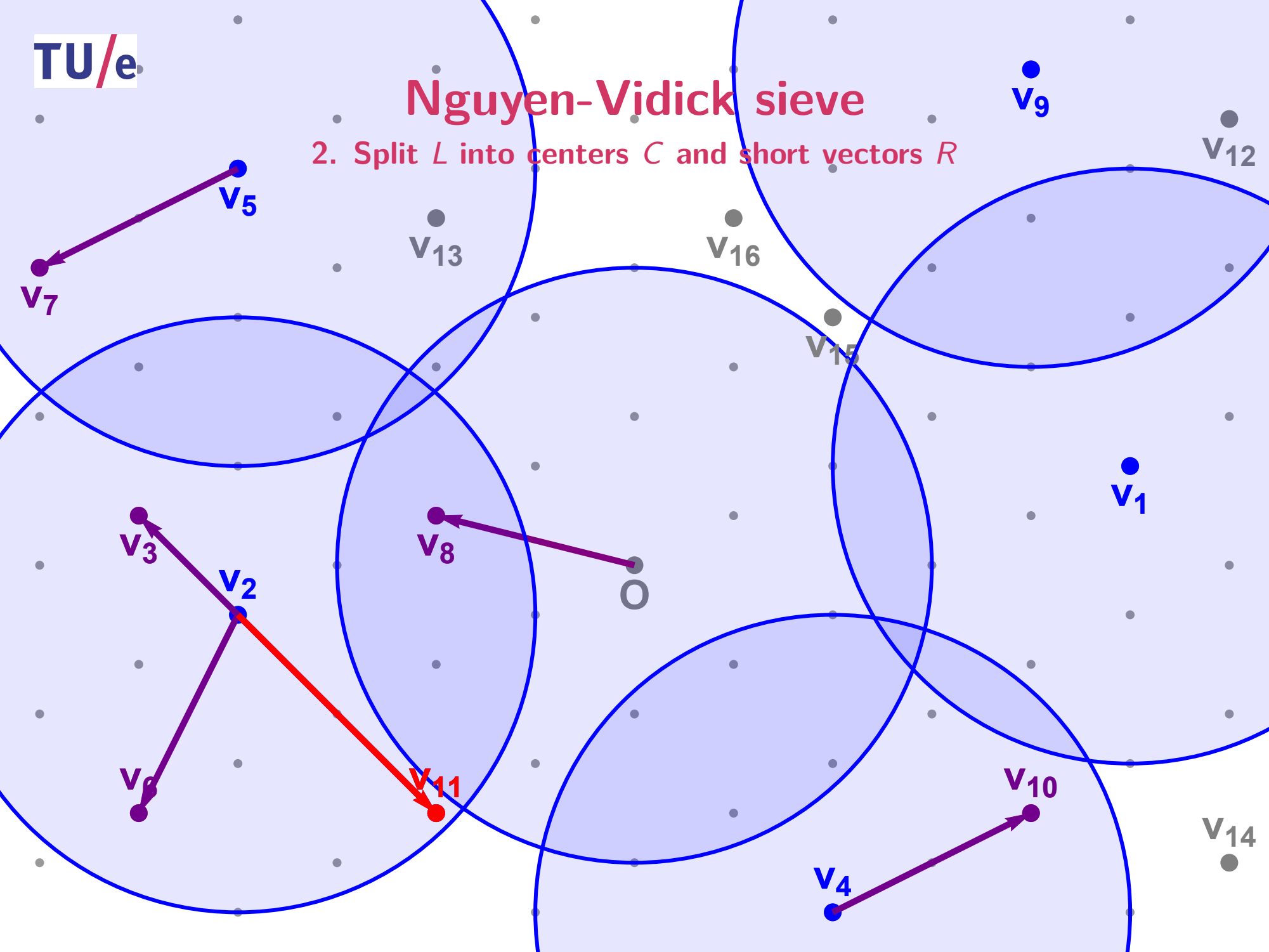
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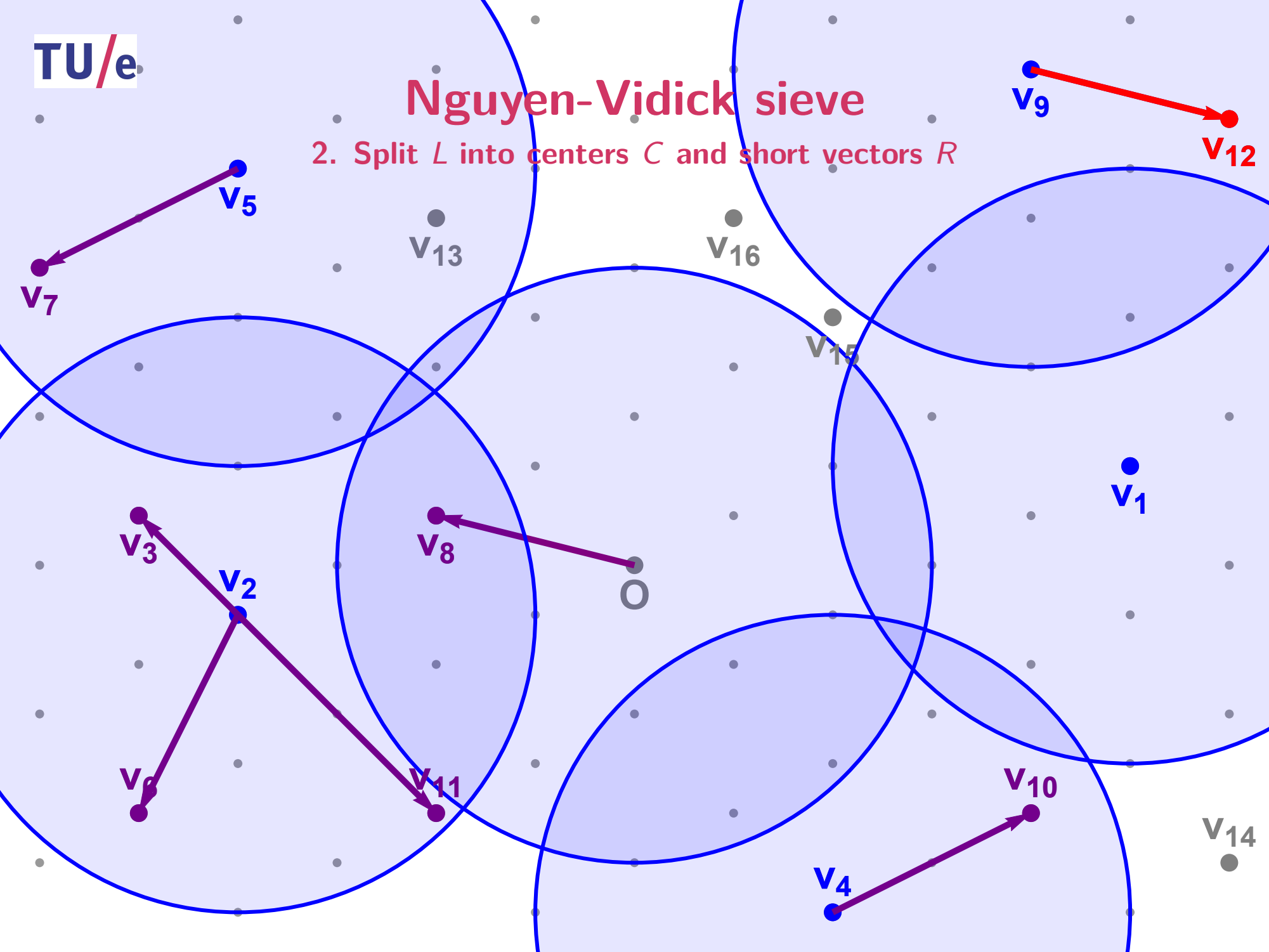
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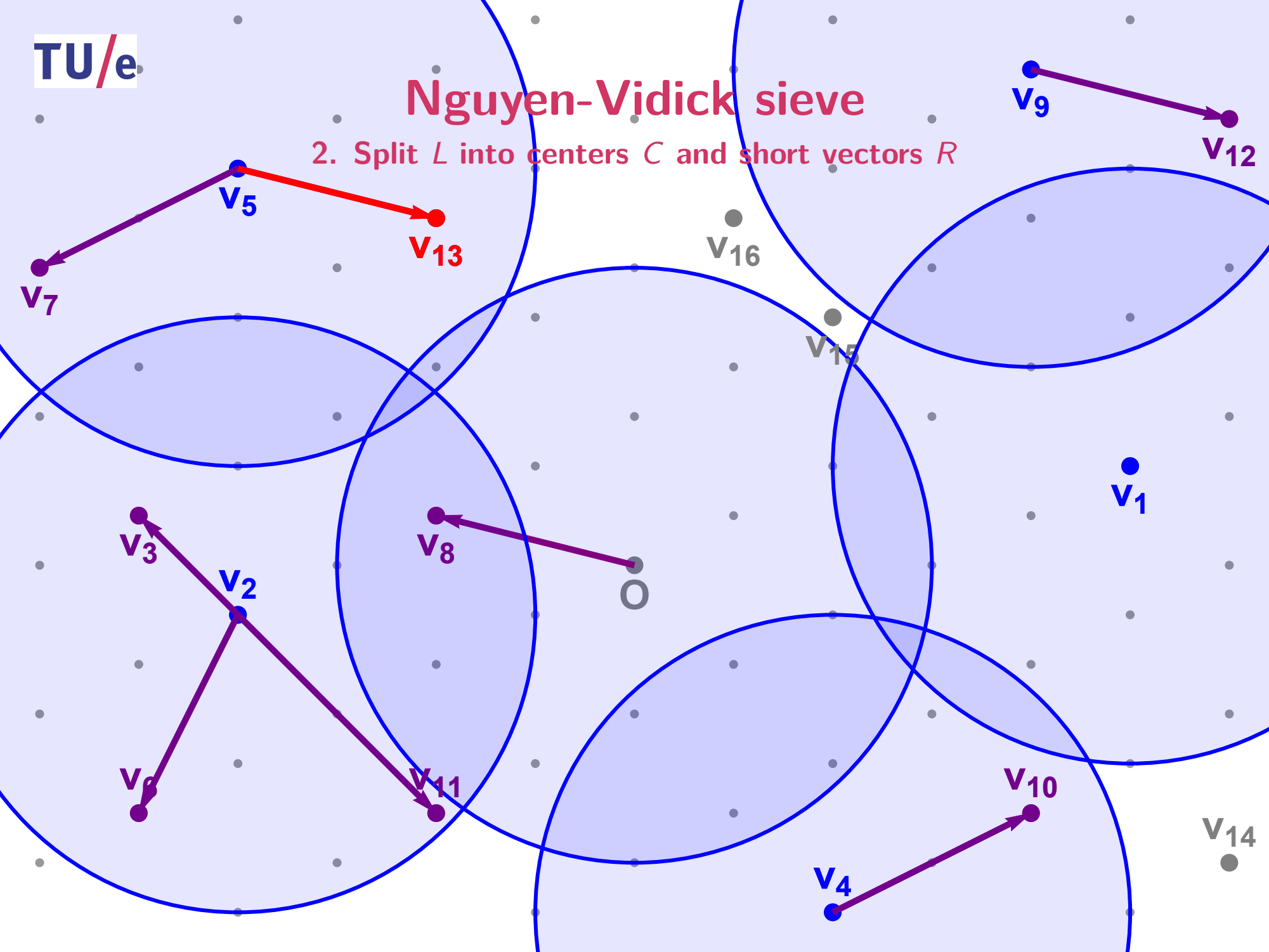
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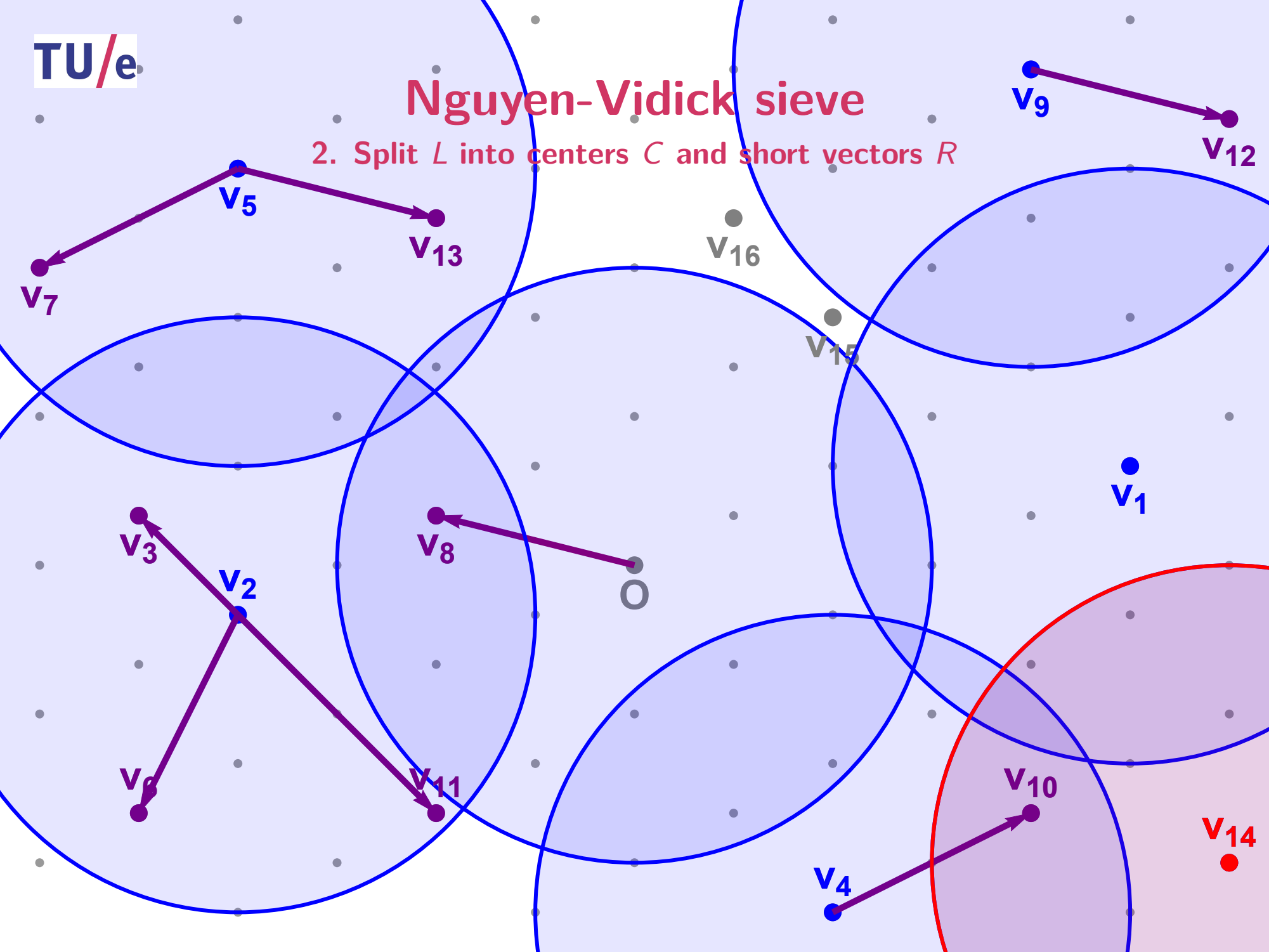

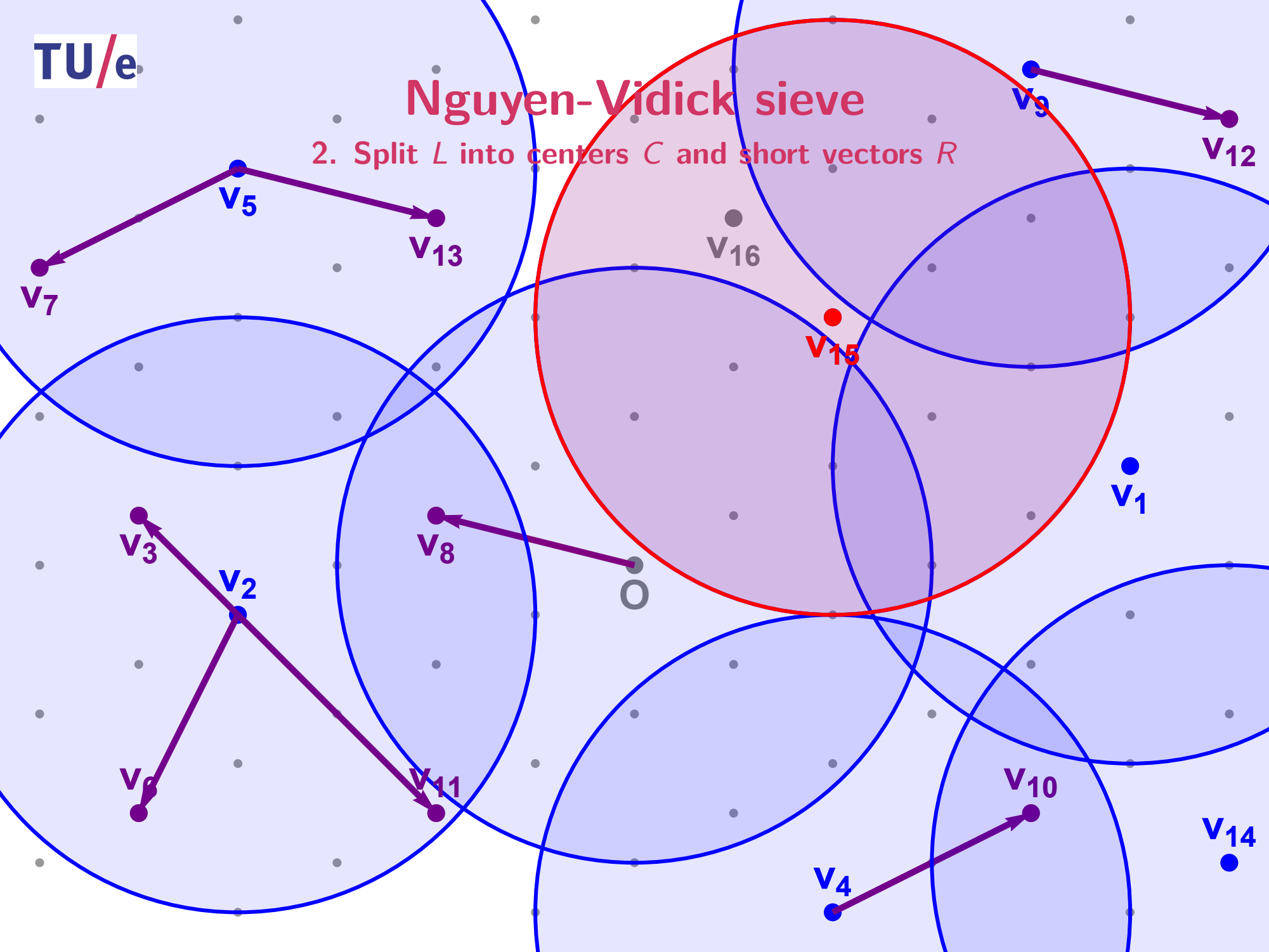
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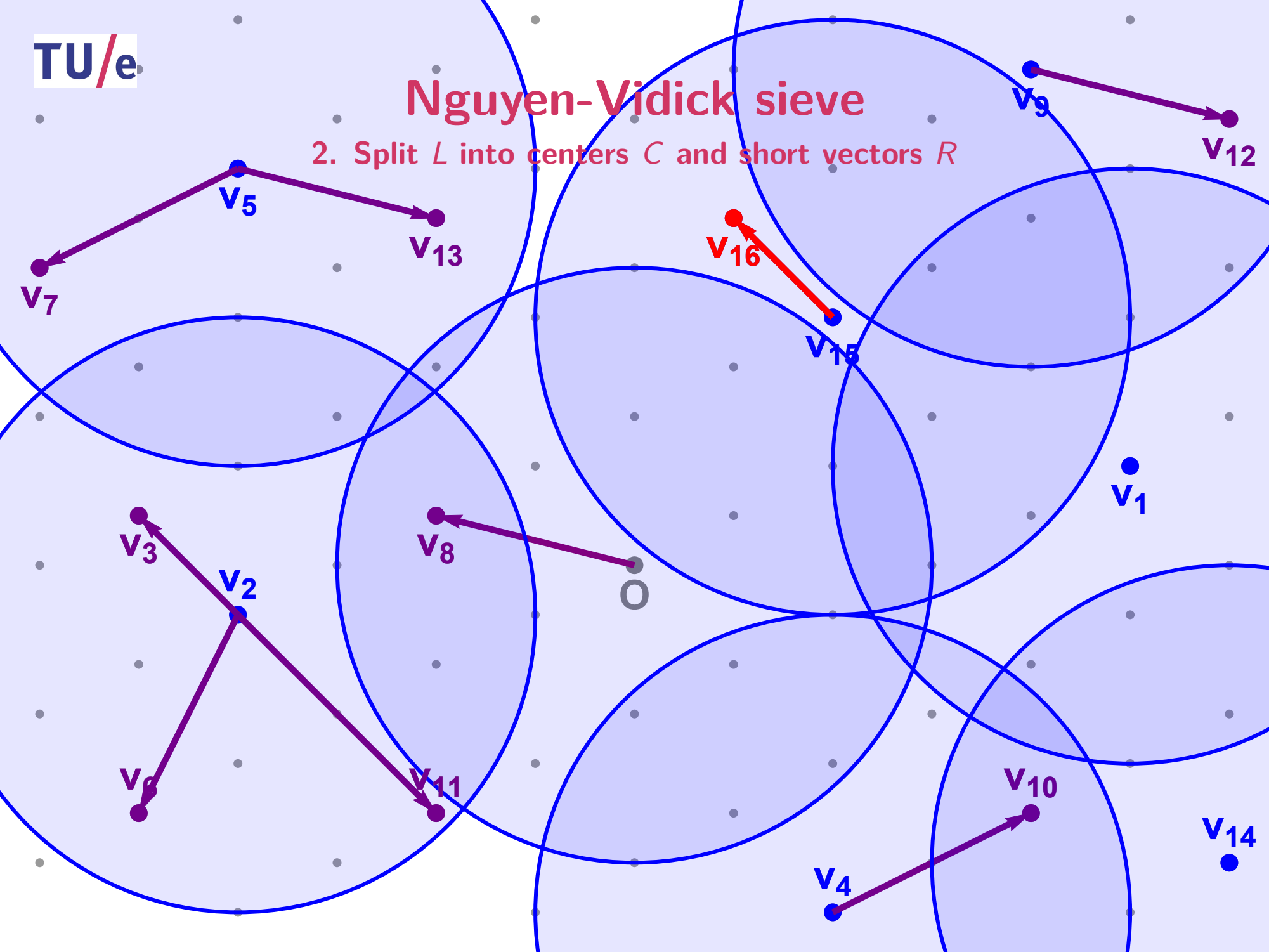

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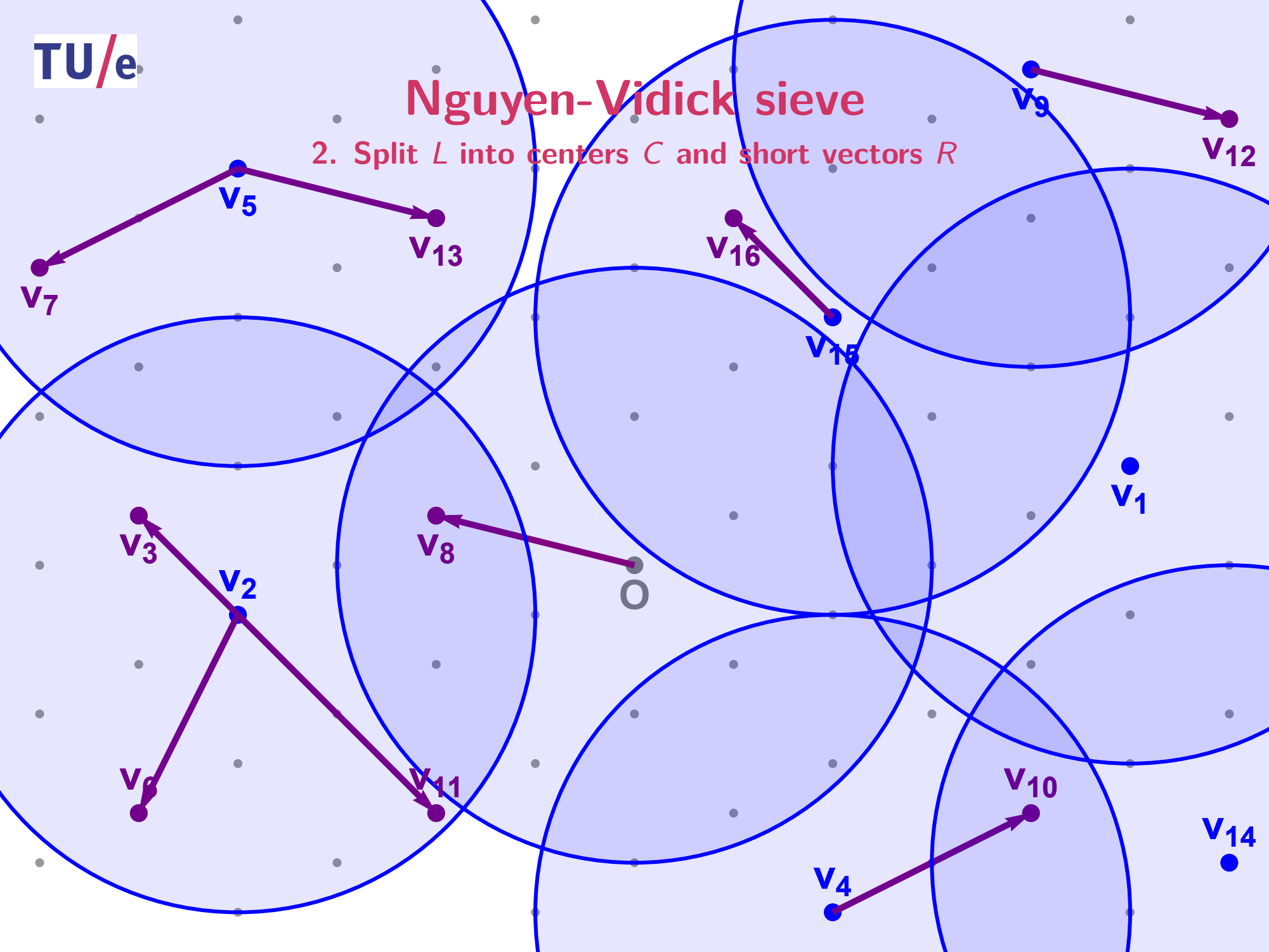
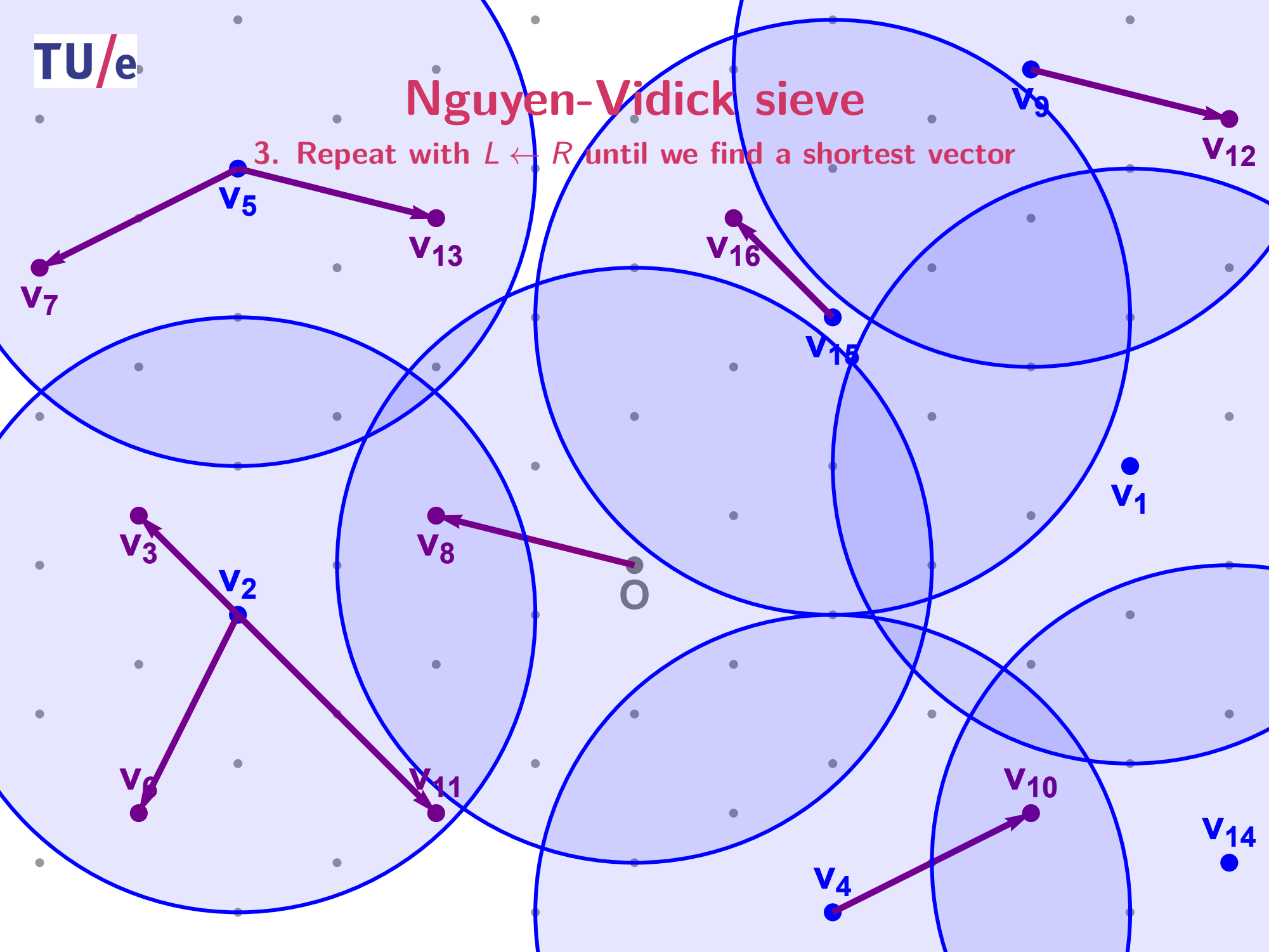


Nguyen-Vidick sieve

3. Repeat with $L \leftarrow R$ until we find a shortest vector



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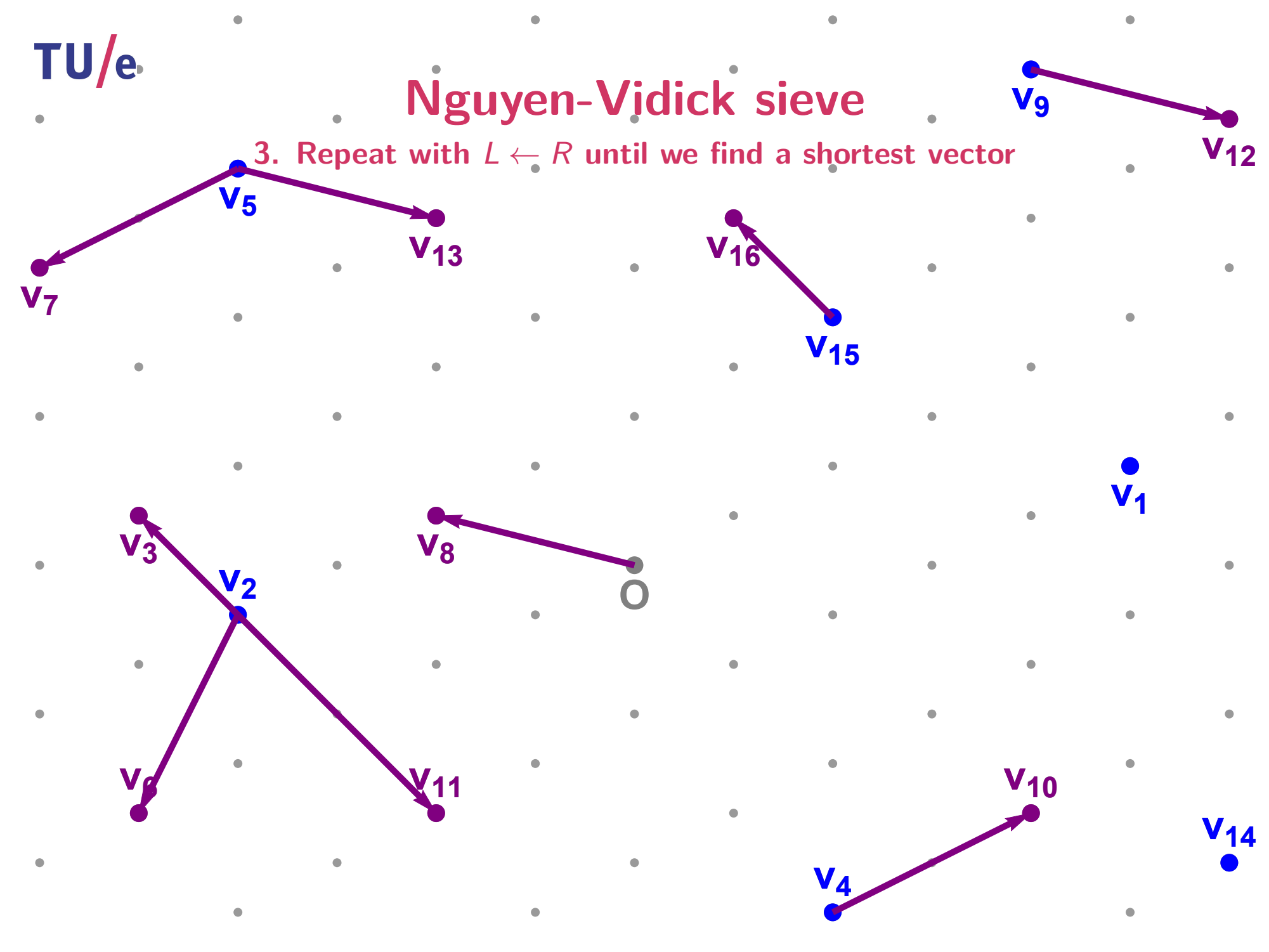

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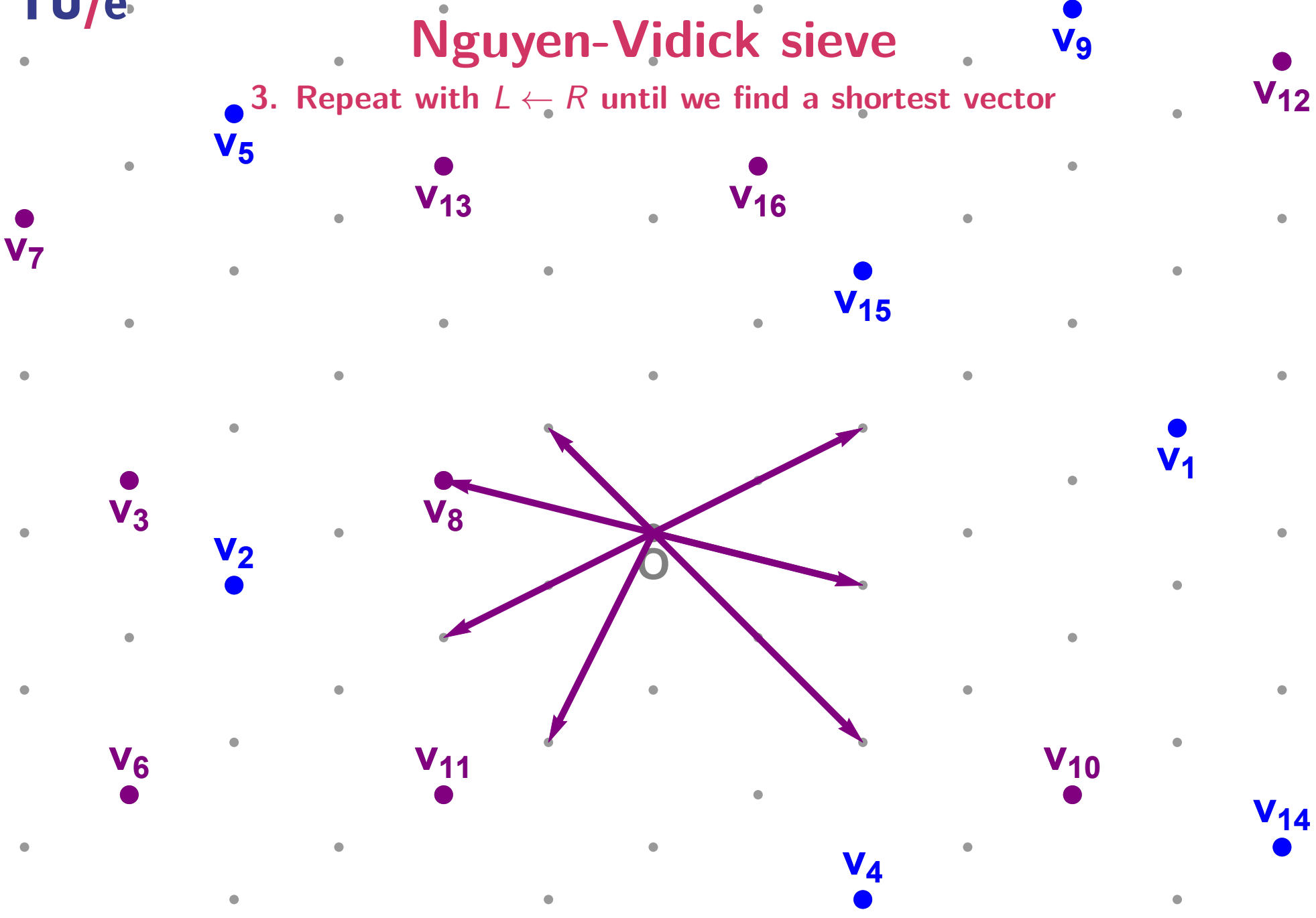

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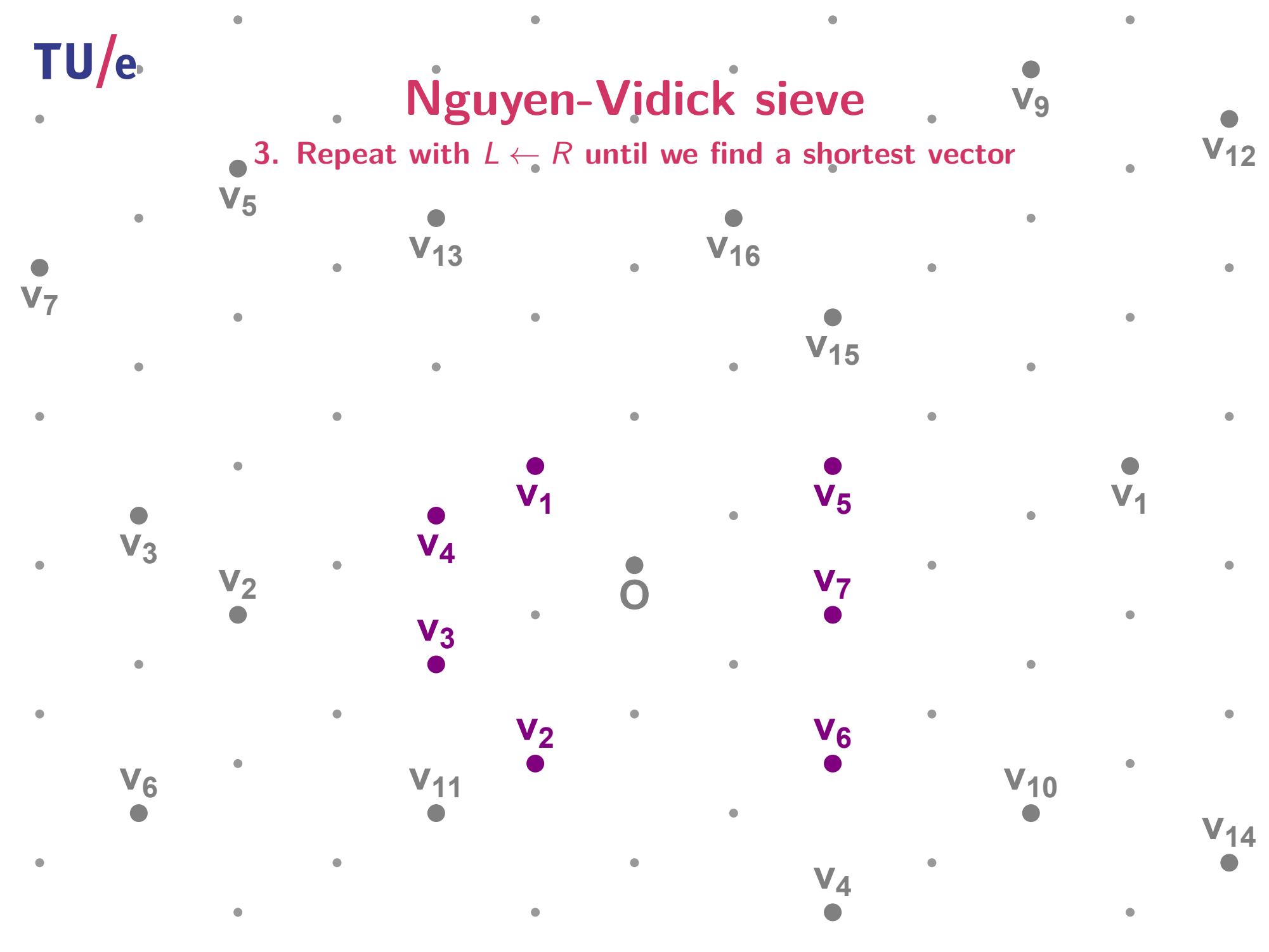

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Nguyen-Vidick sieve

Overview

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Nguyen-Vidick sieve

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Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The Nguyen-Vidick sieve runs in time $(4/3)^n$ and space $\sqrt{4/3}^n$.

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Nguyen-Vidick sieve

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The Nguyen-Vidick sieve runs in time $2^{0.415n}$ and space $2^{0.208n}$.

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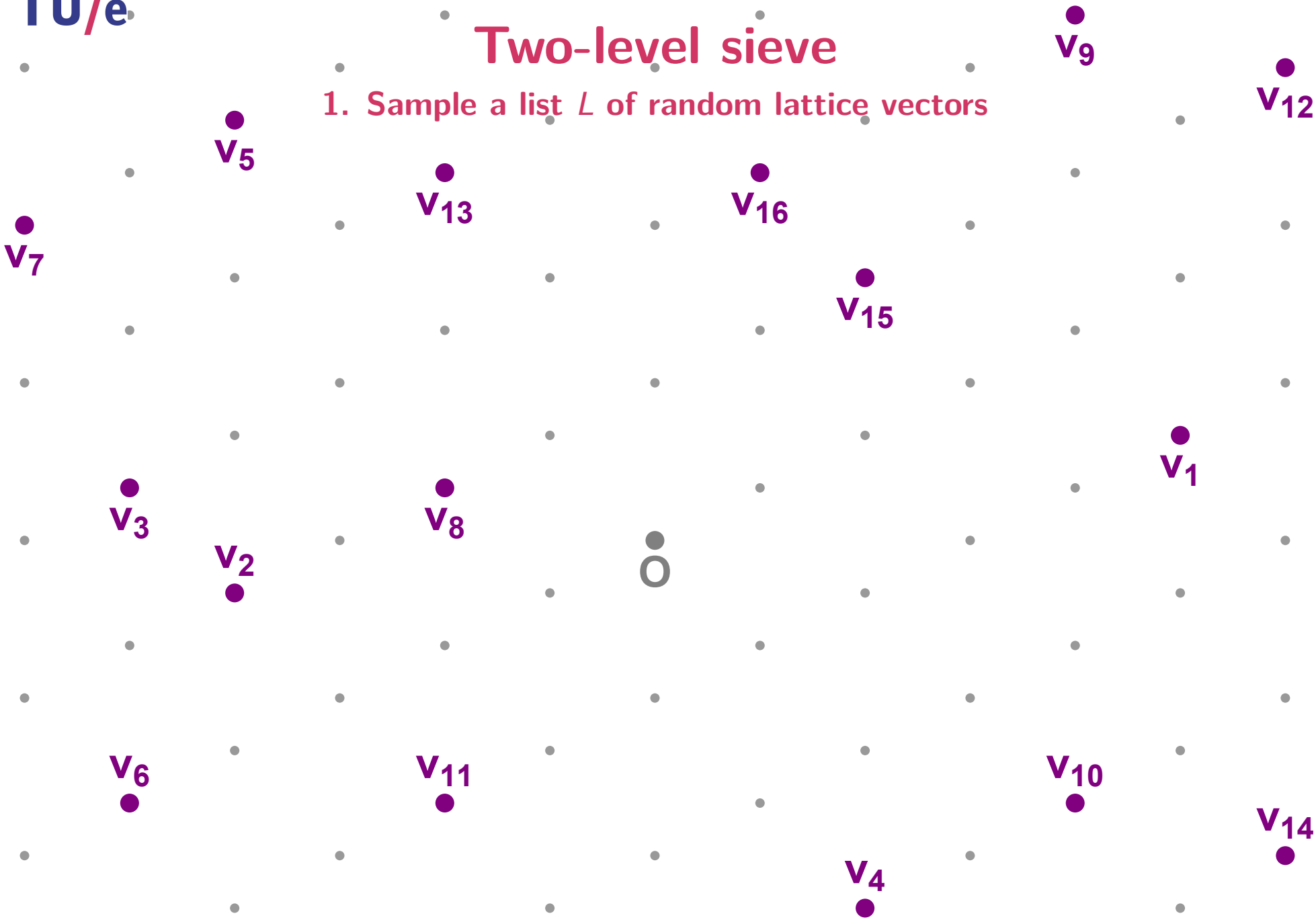

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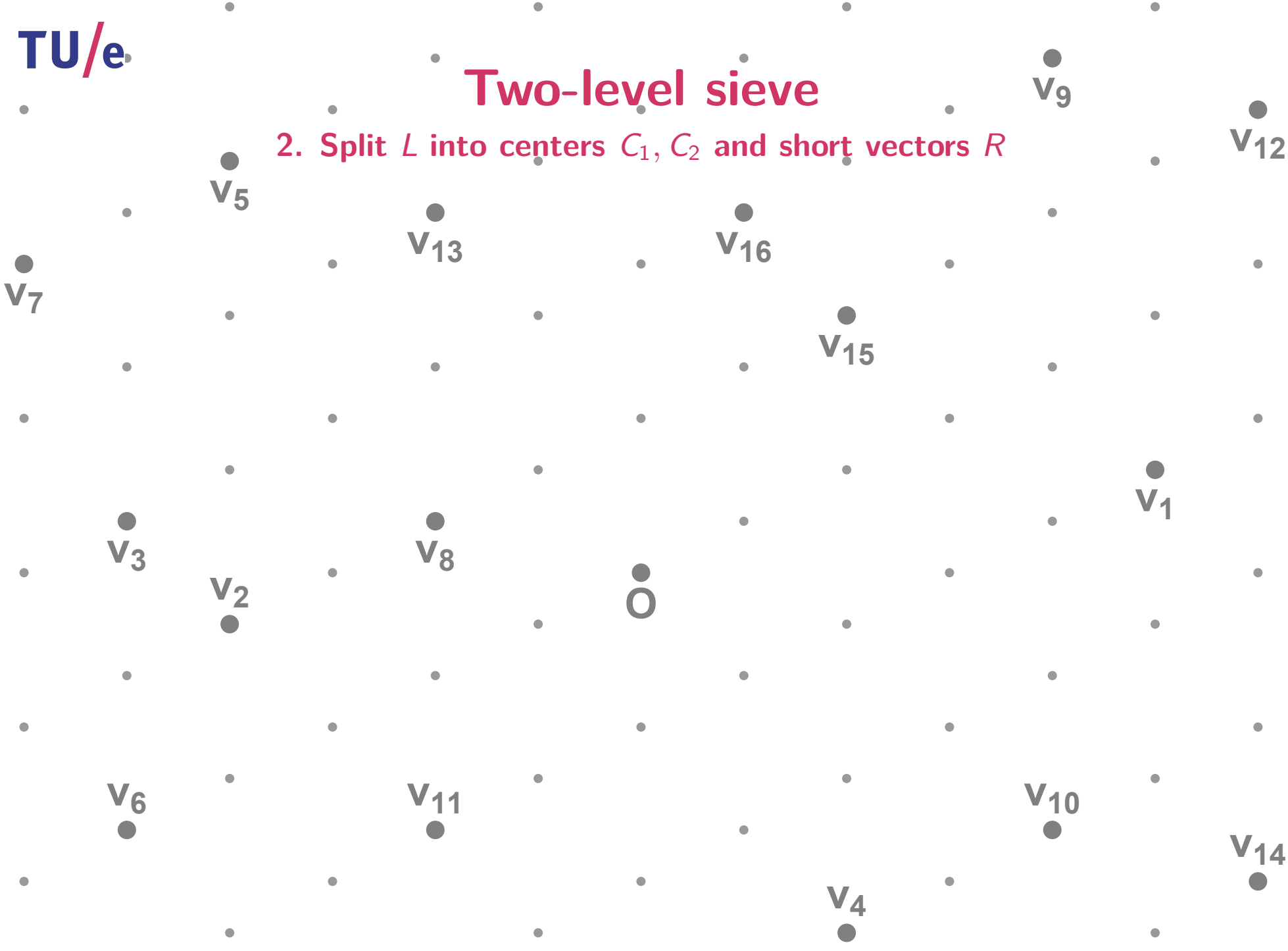

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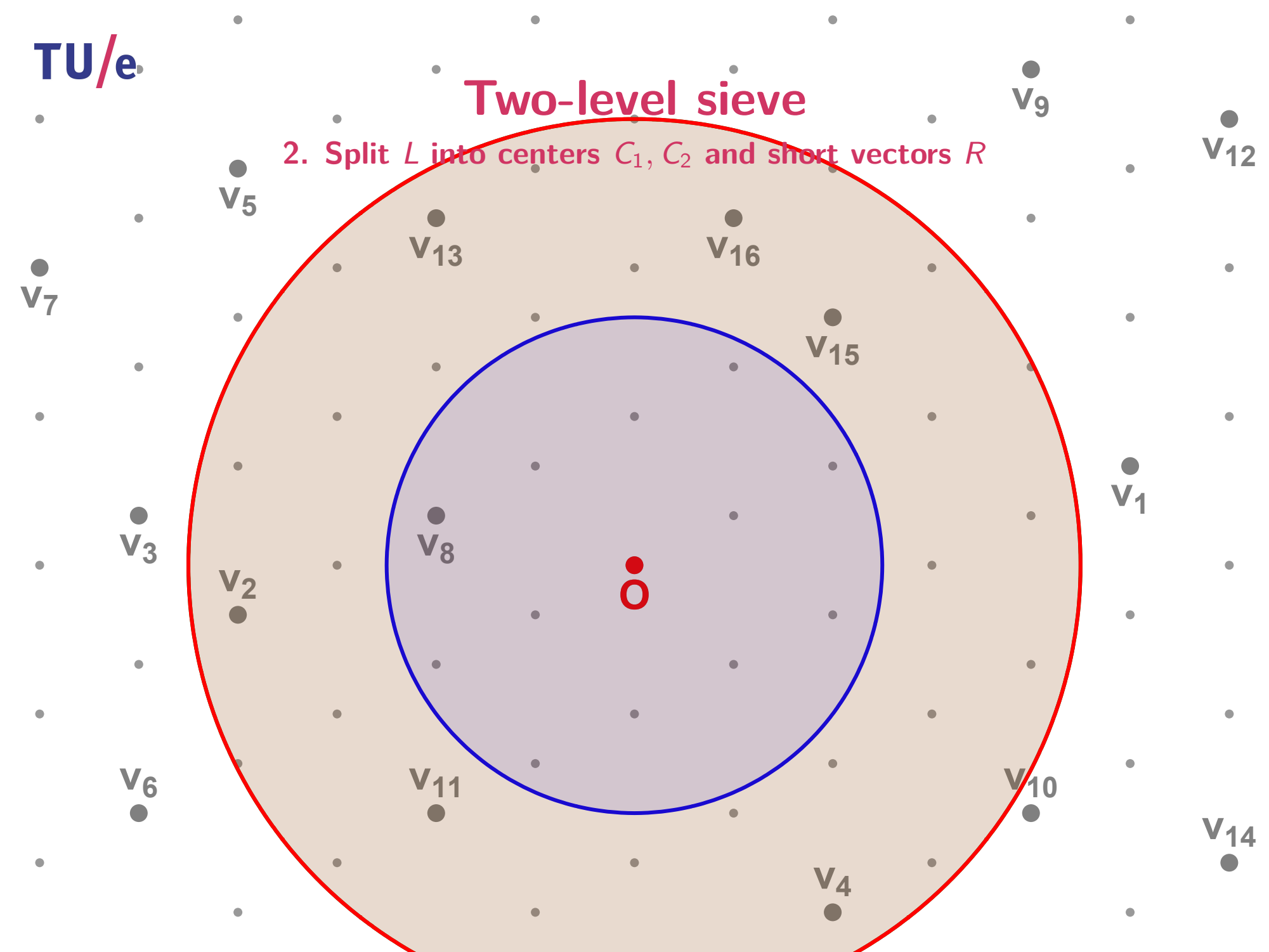

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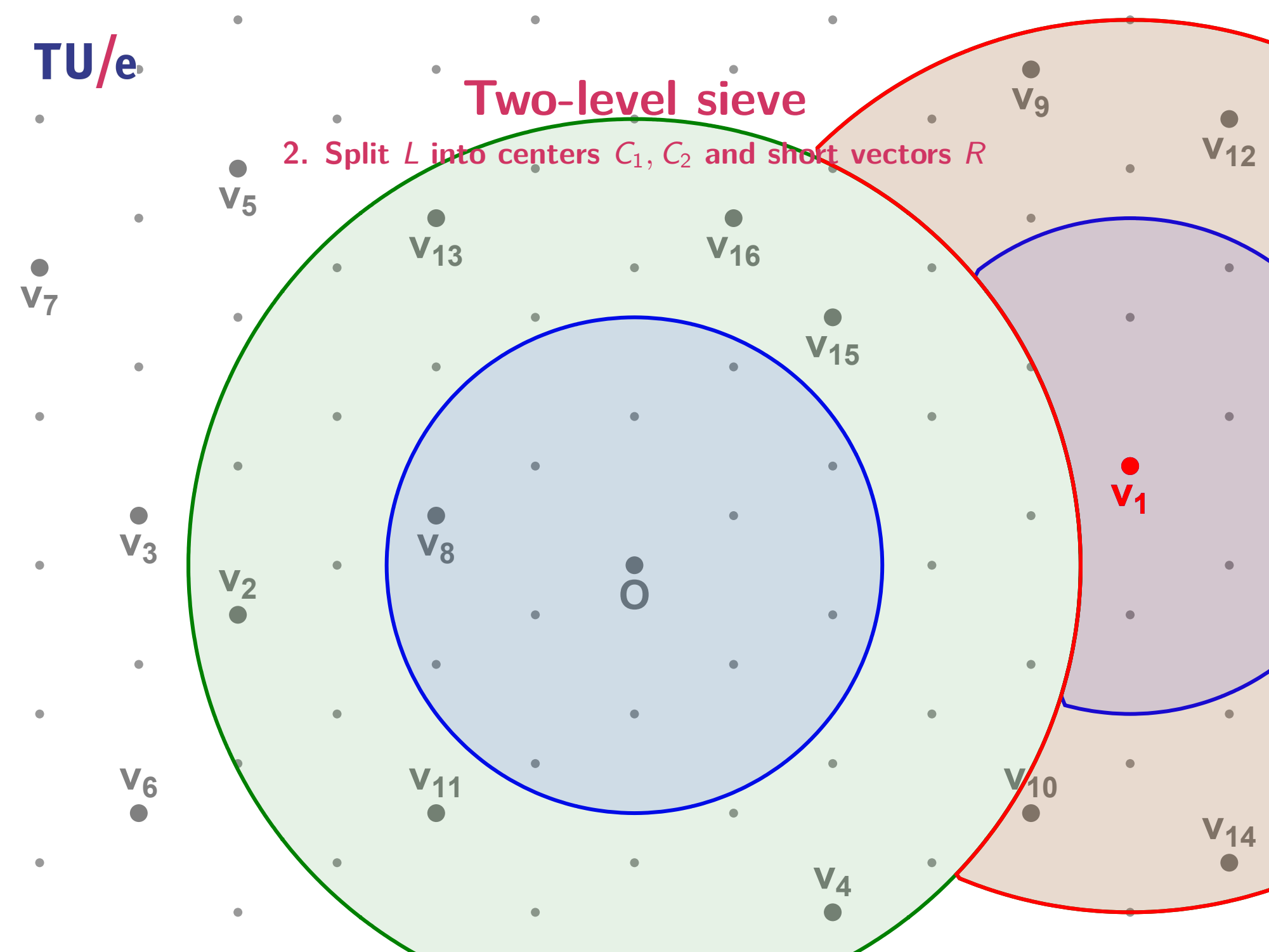

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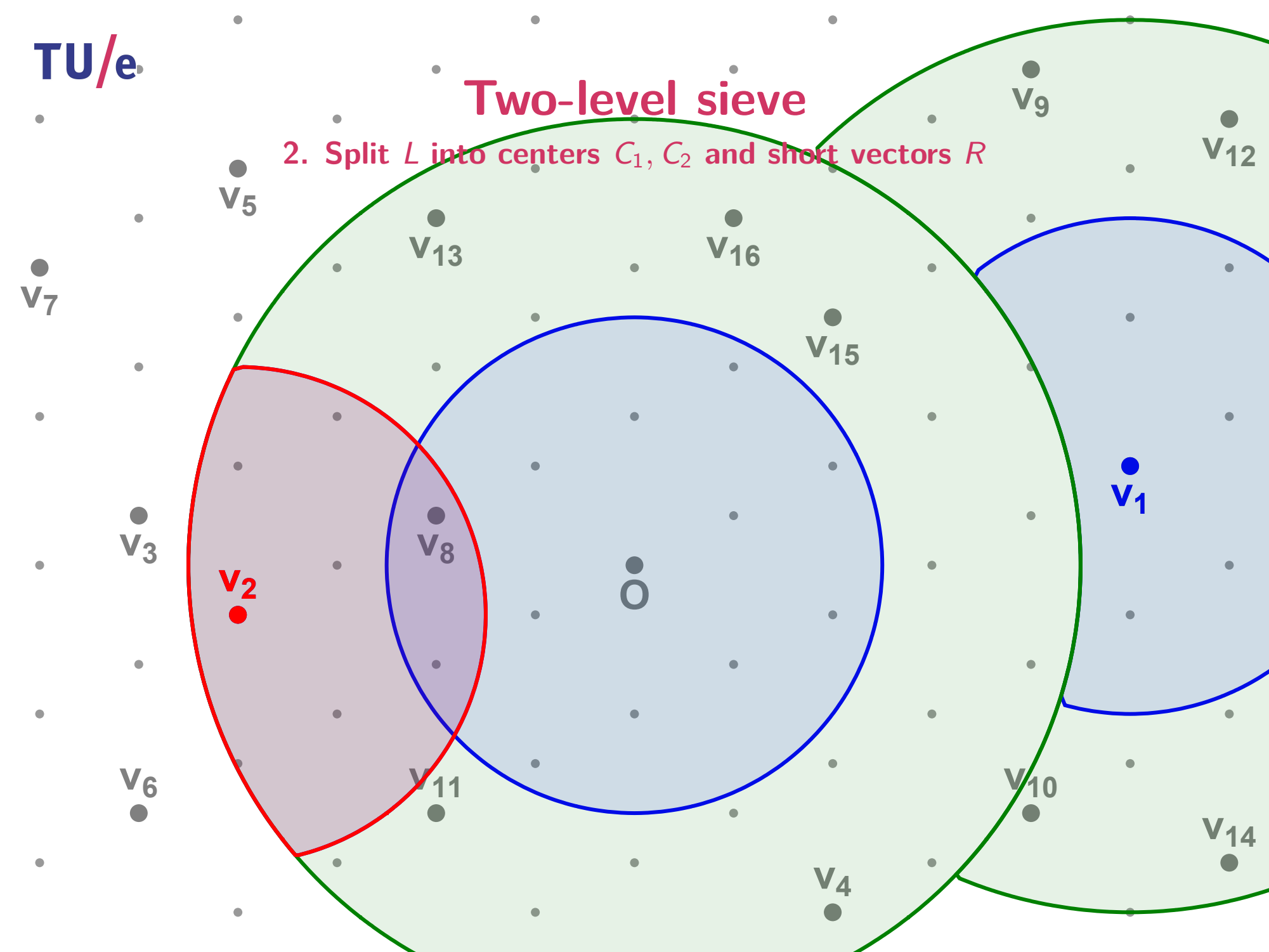

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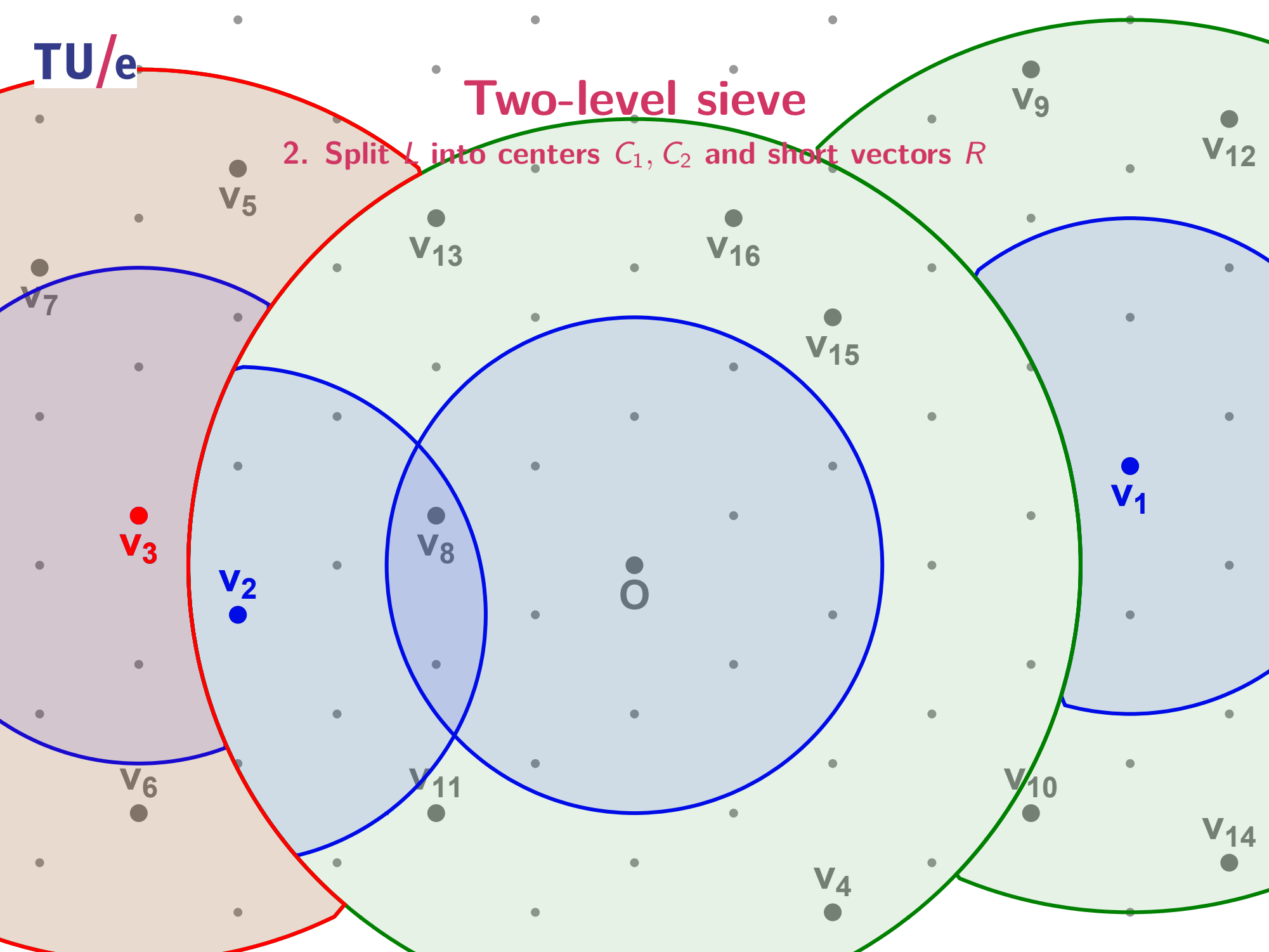

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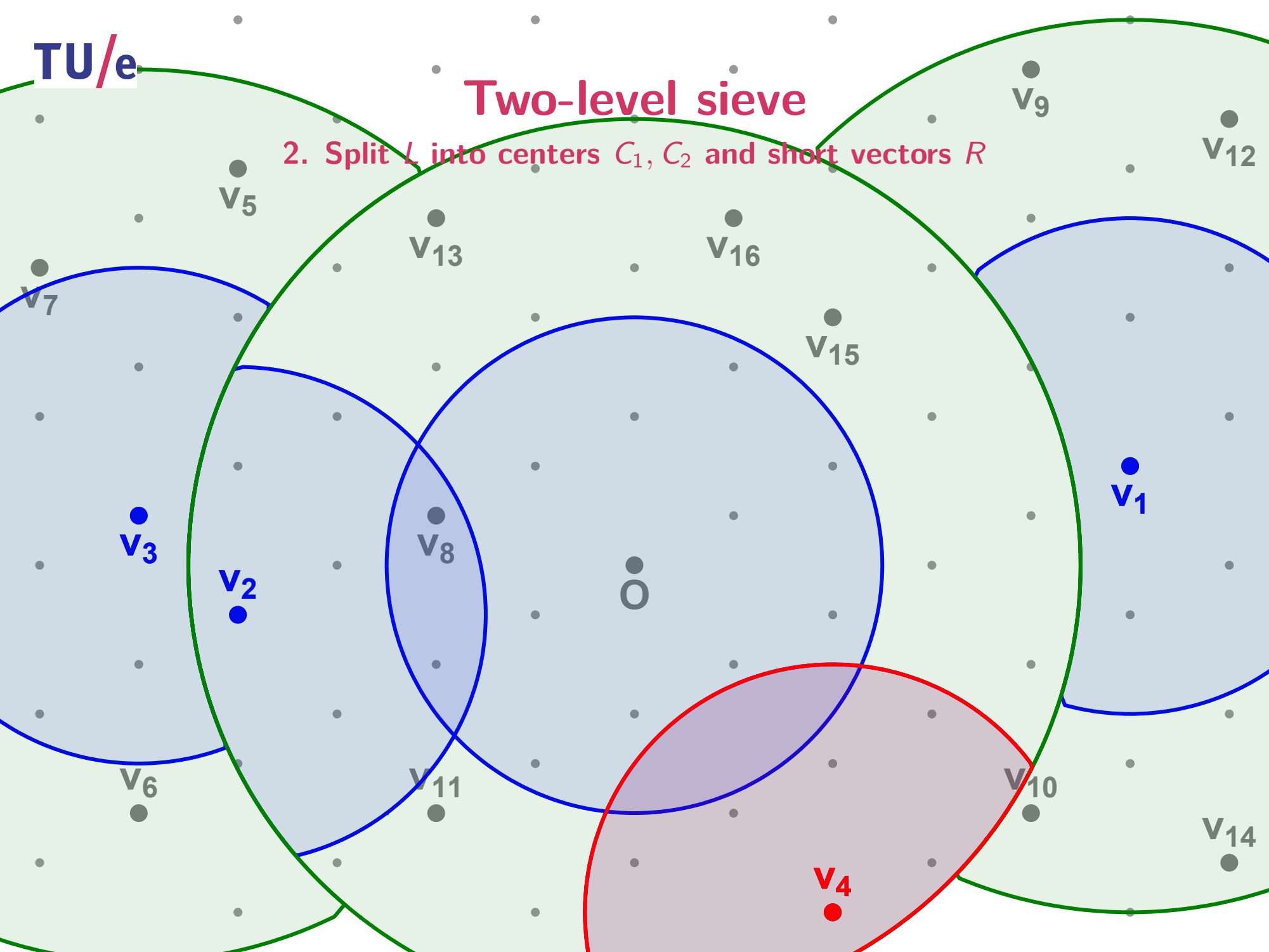

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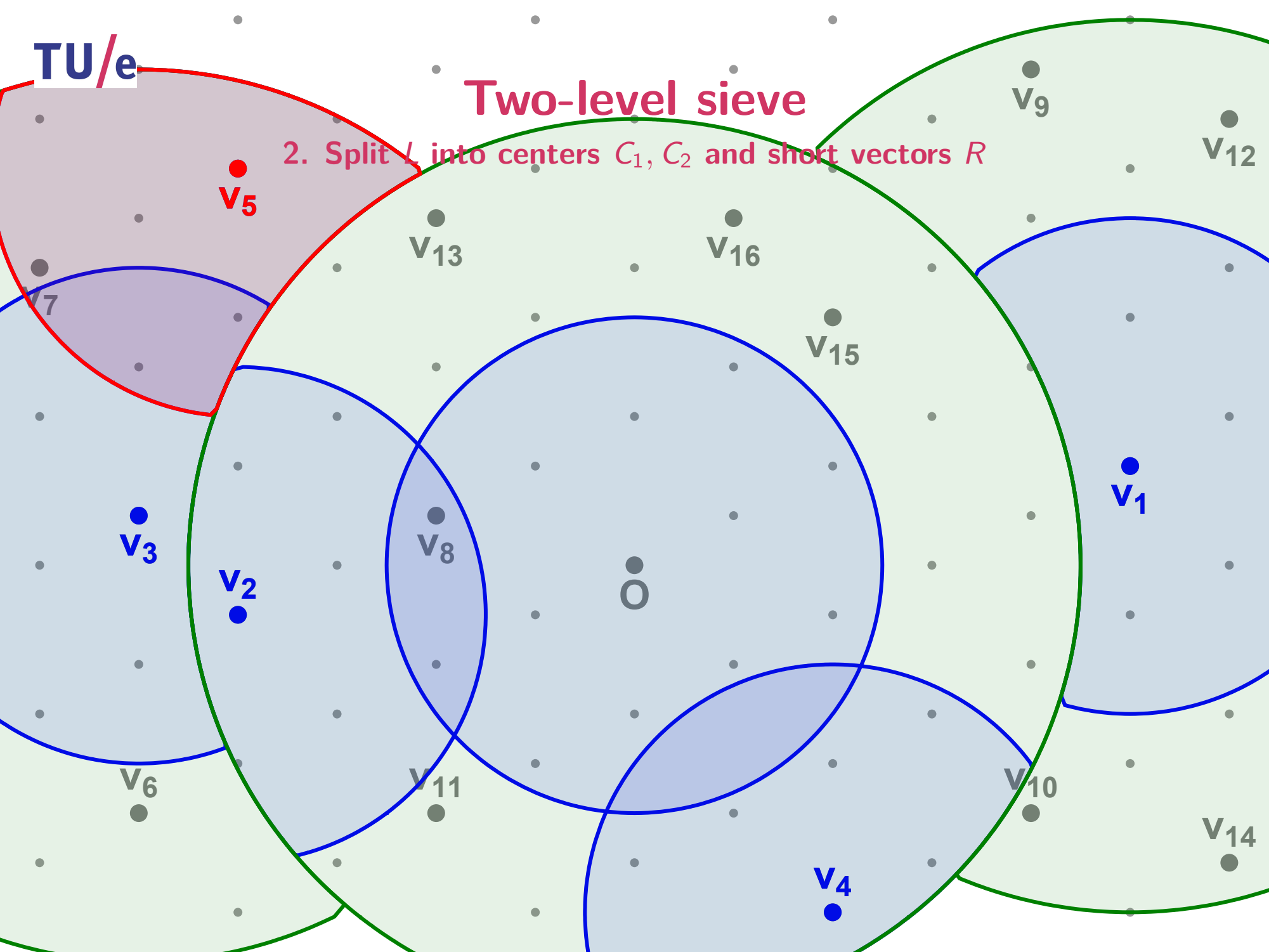

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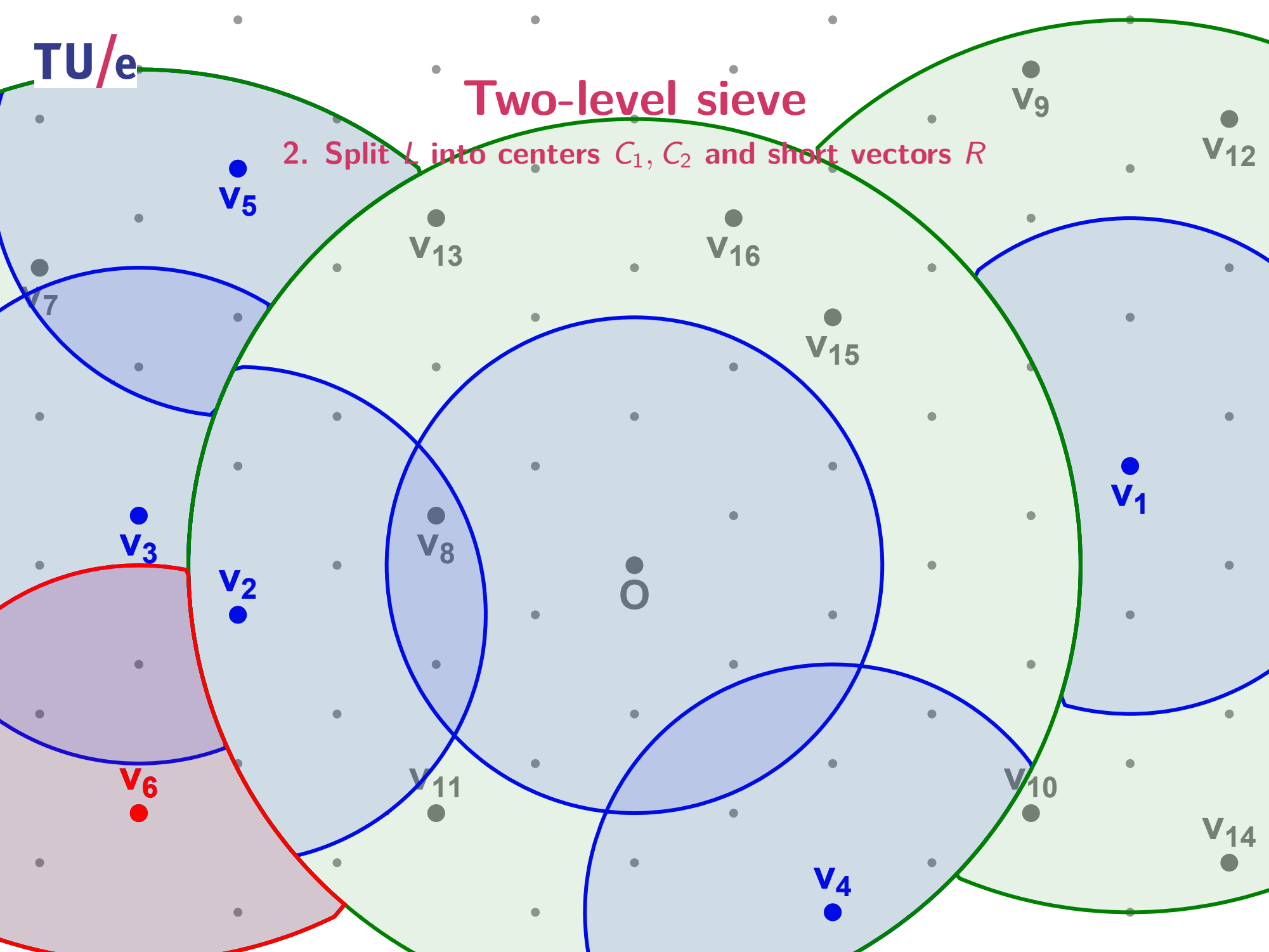

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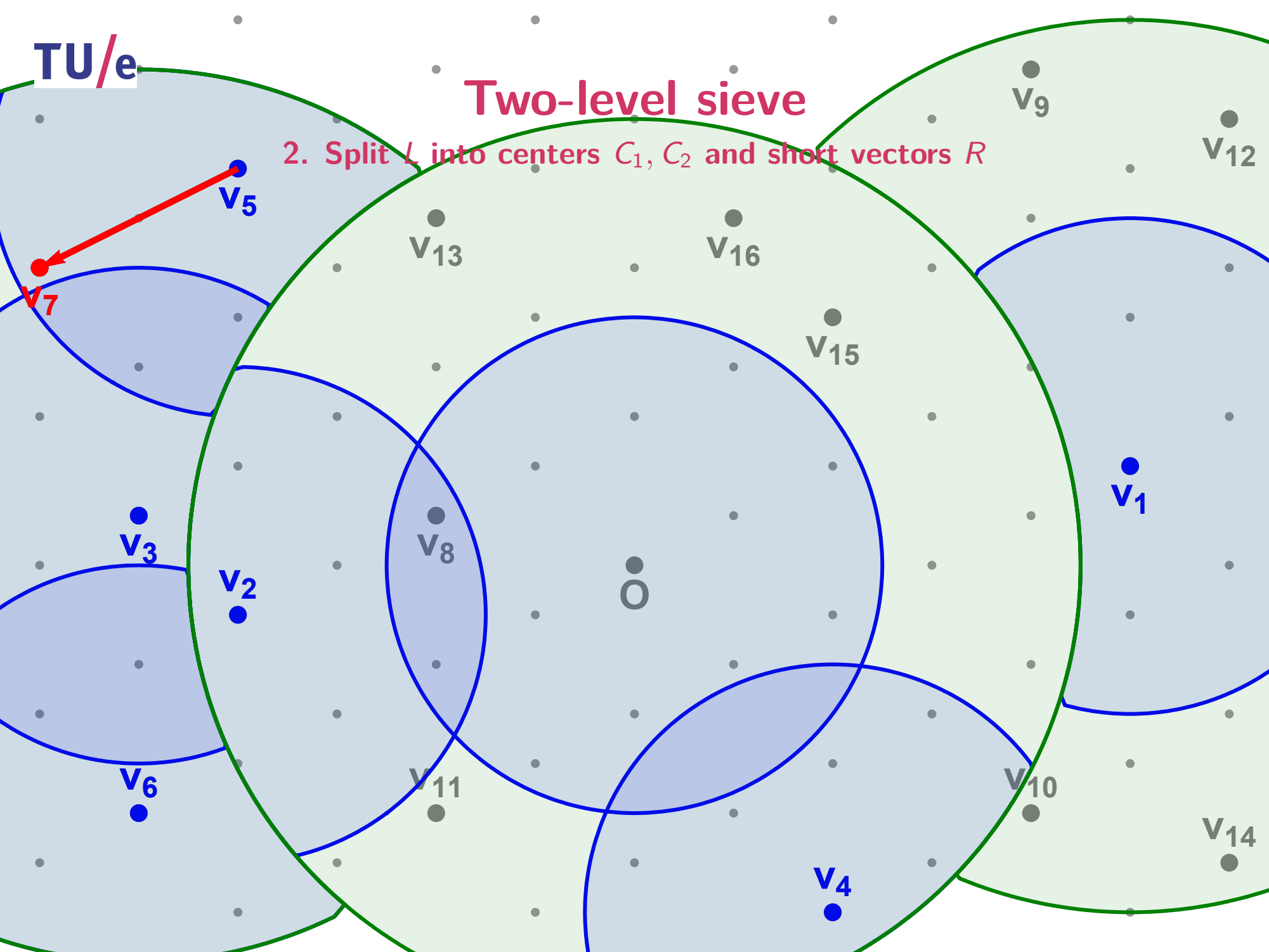

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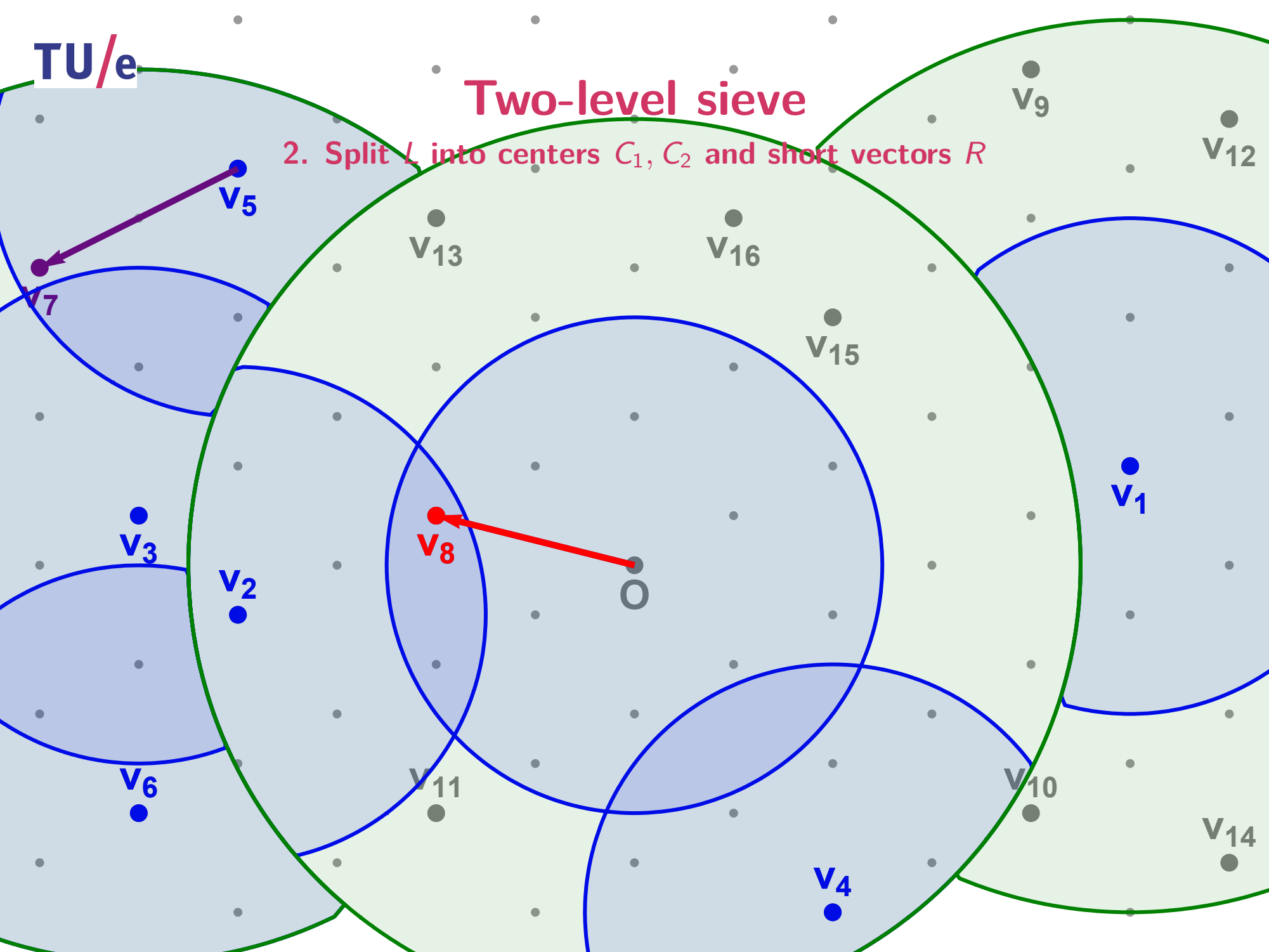

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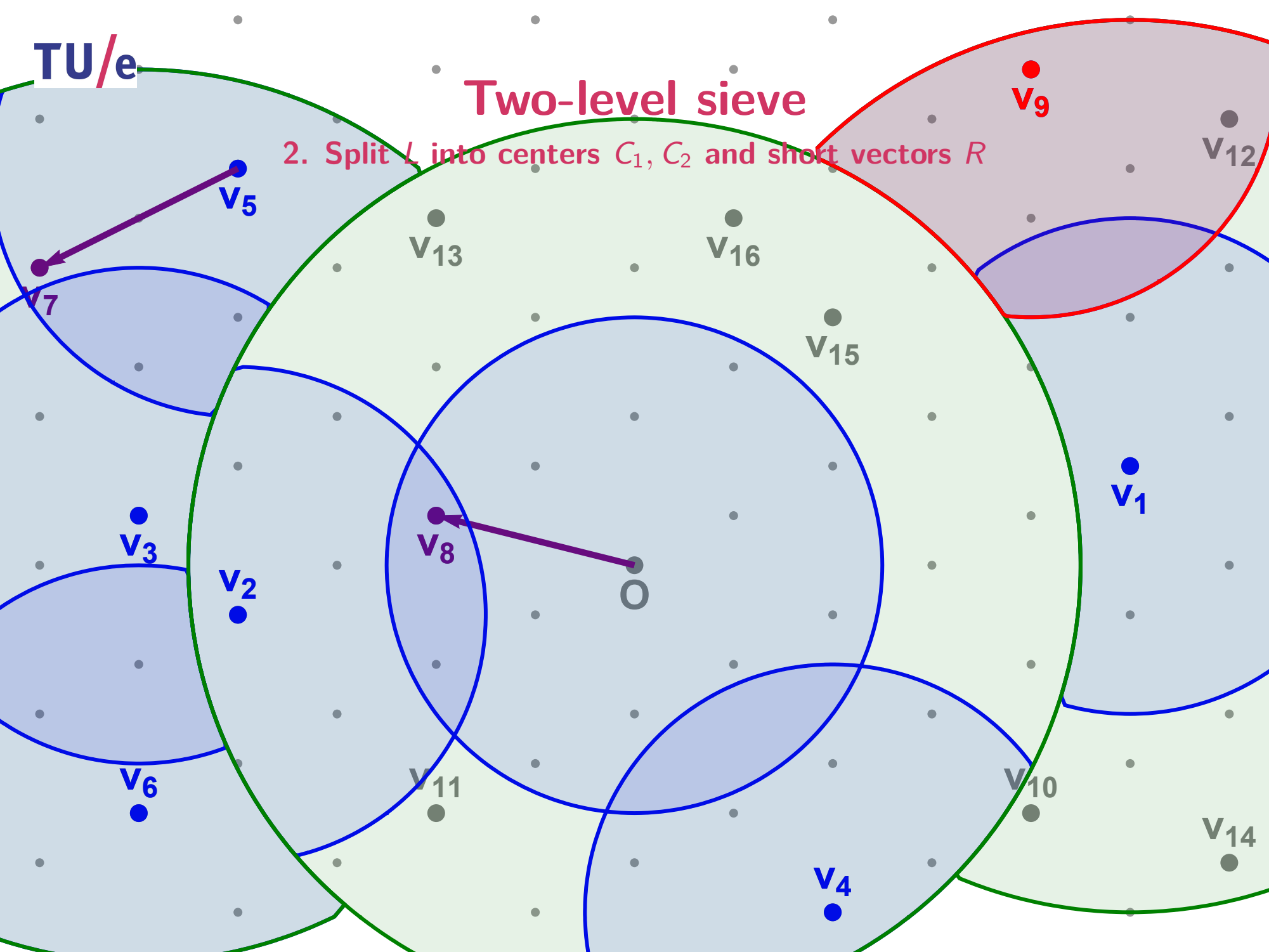

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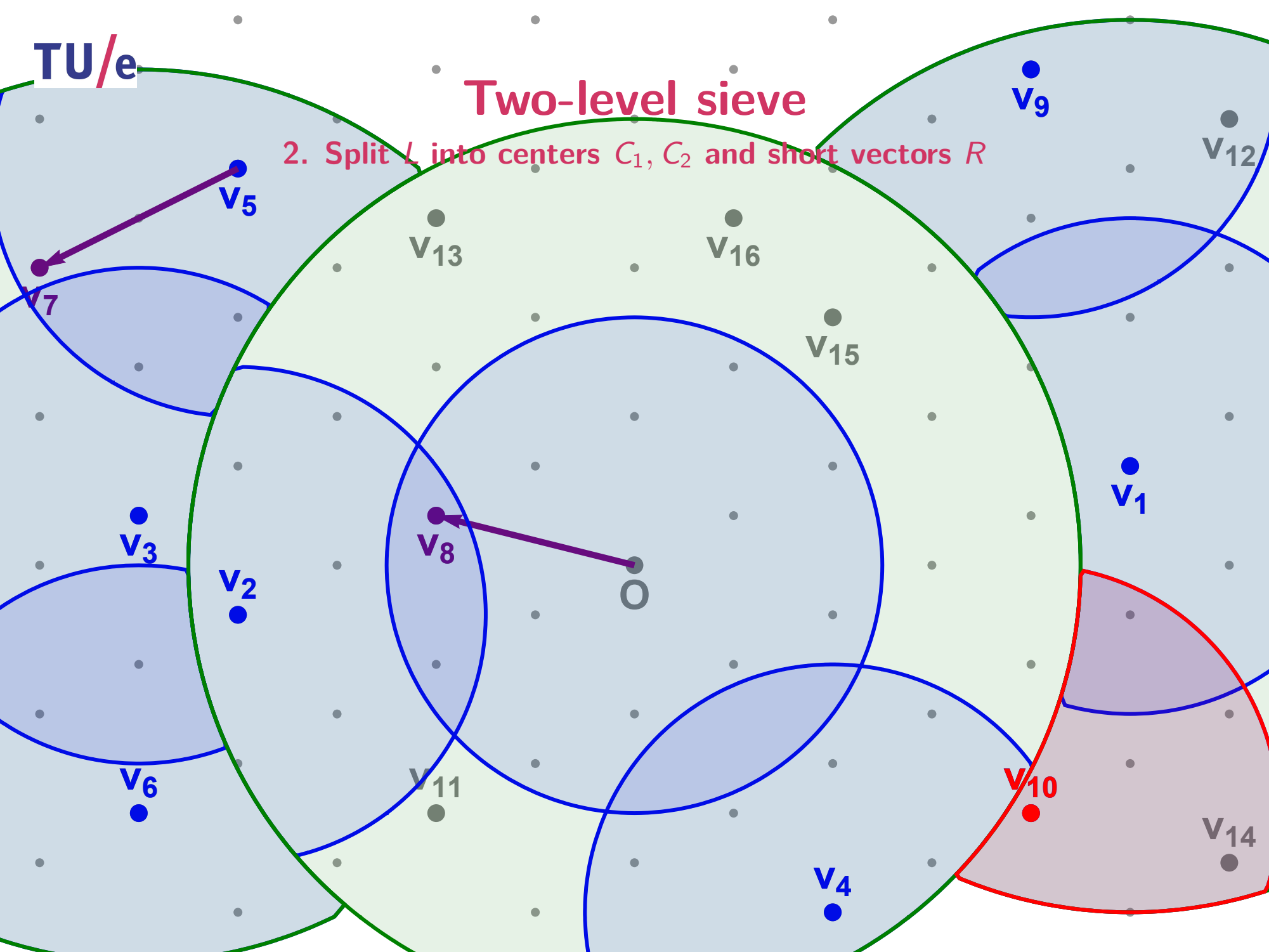

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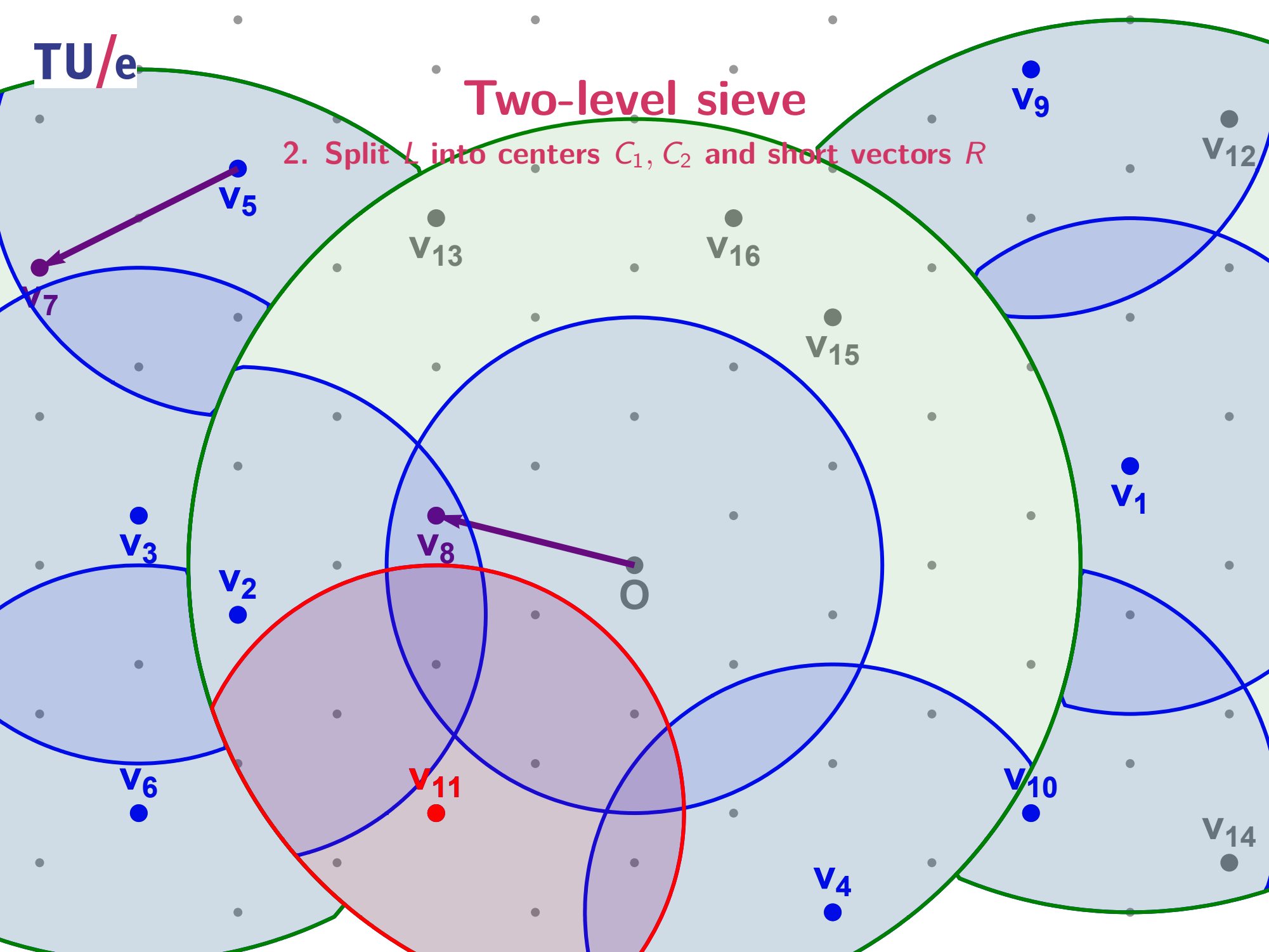

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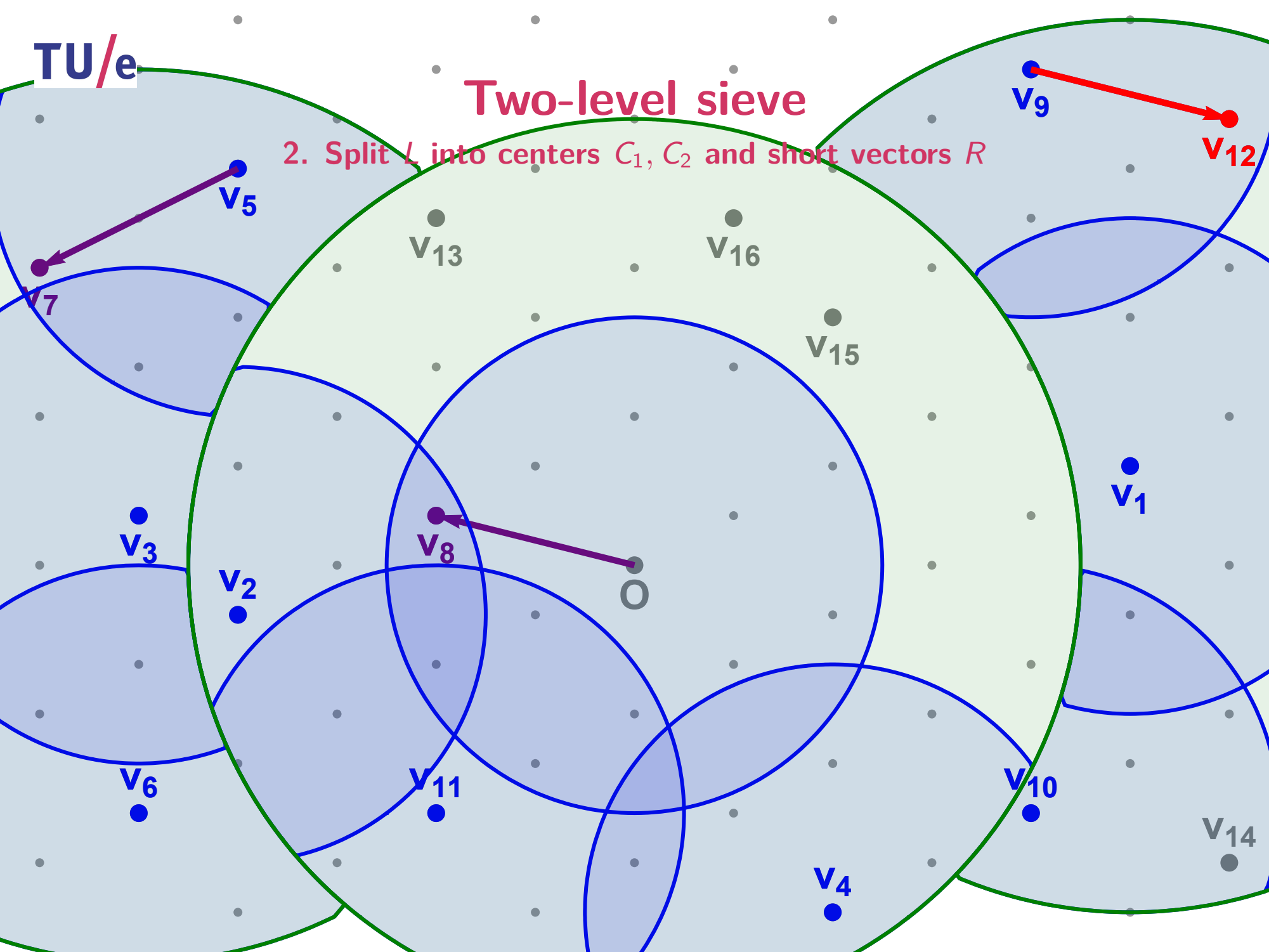

TU/e

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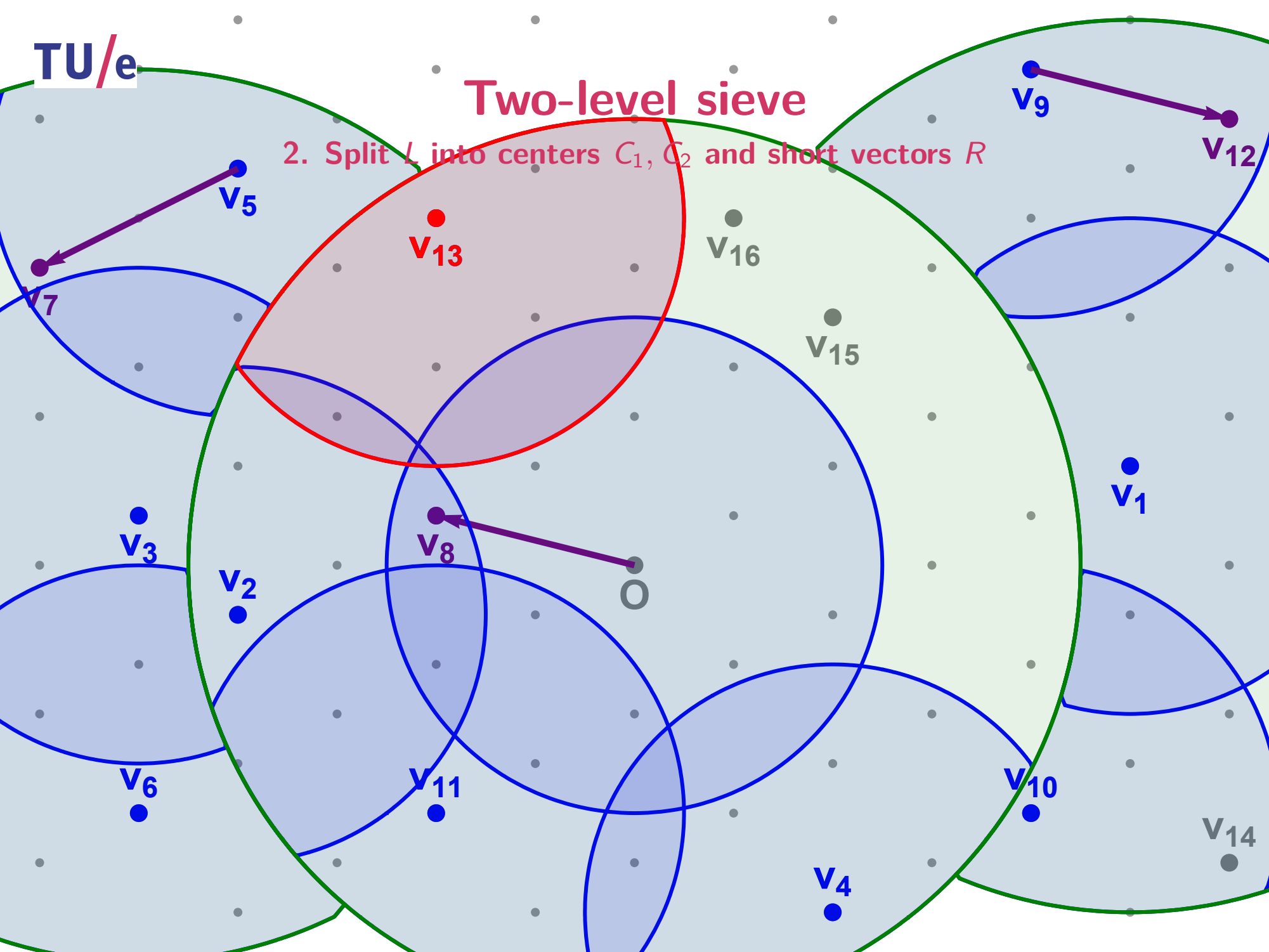

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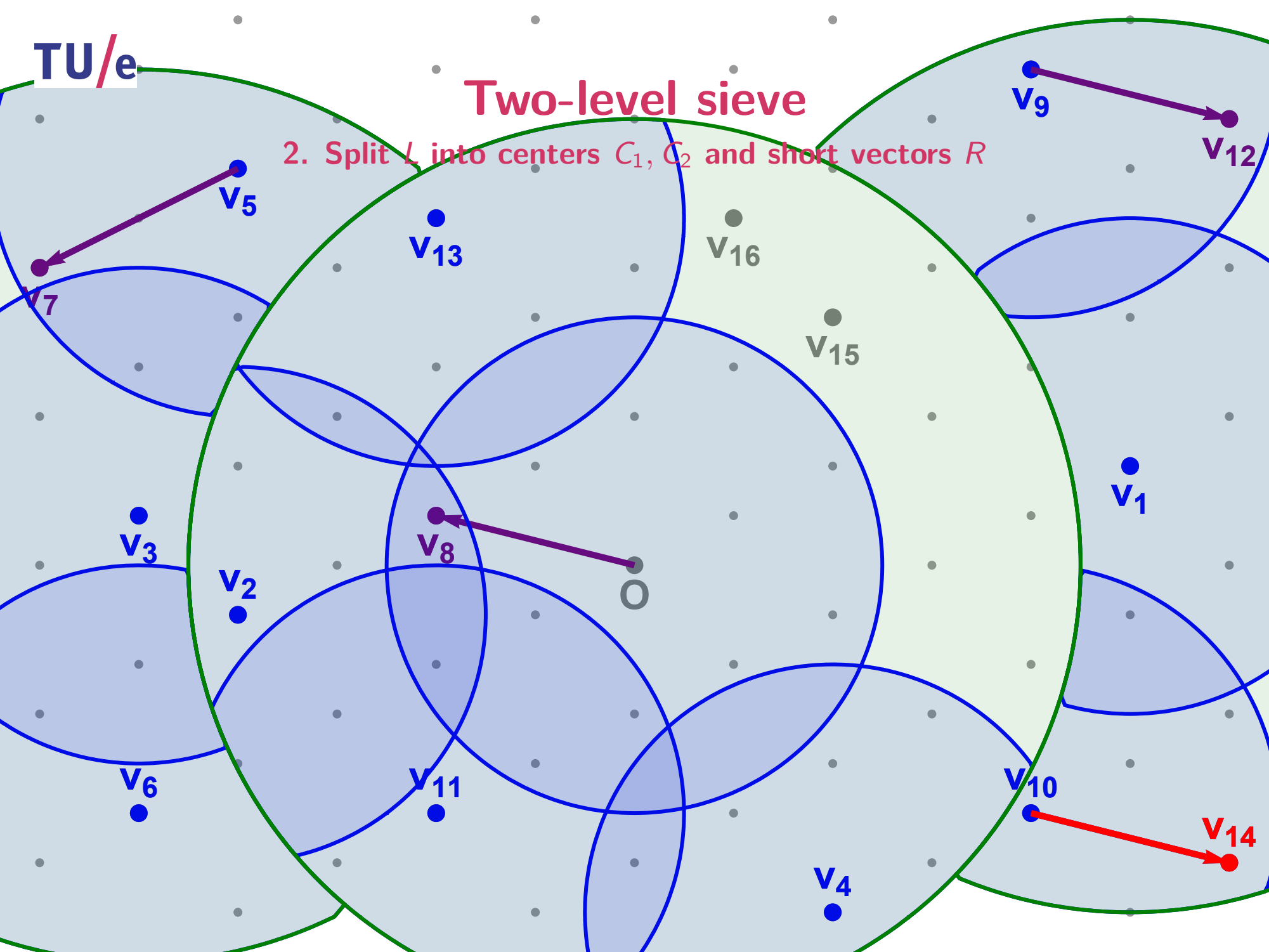

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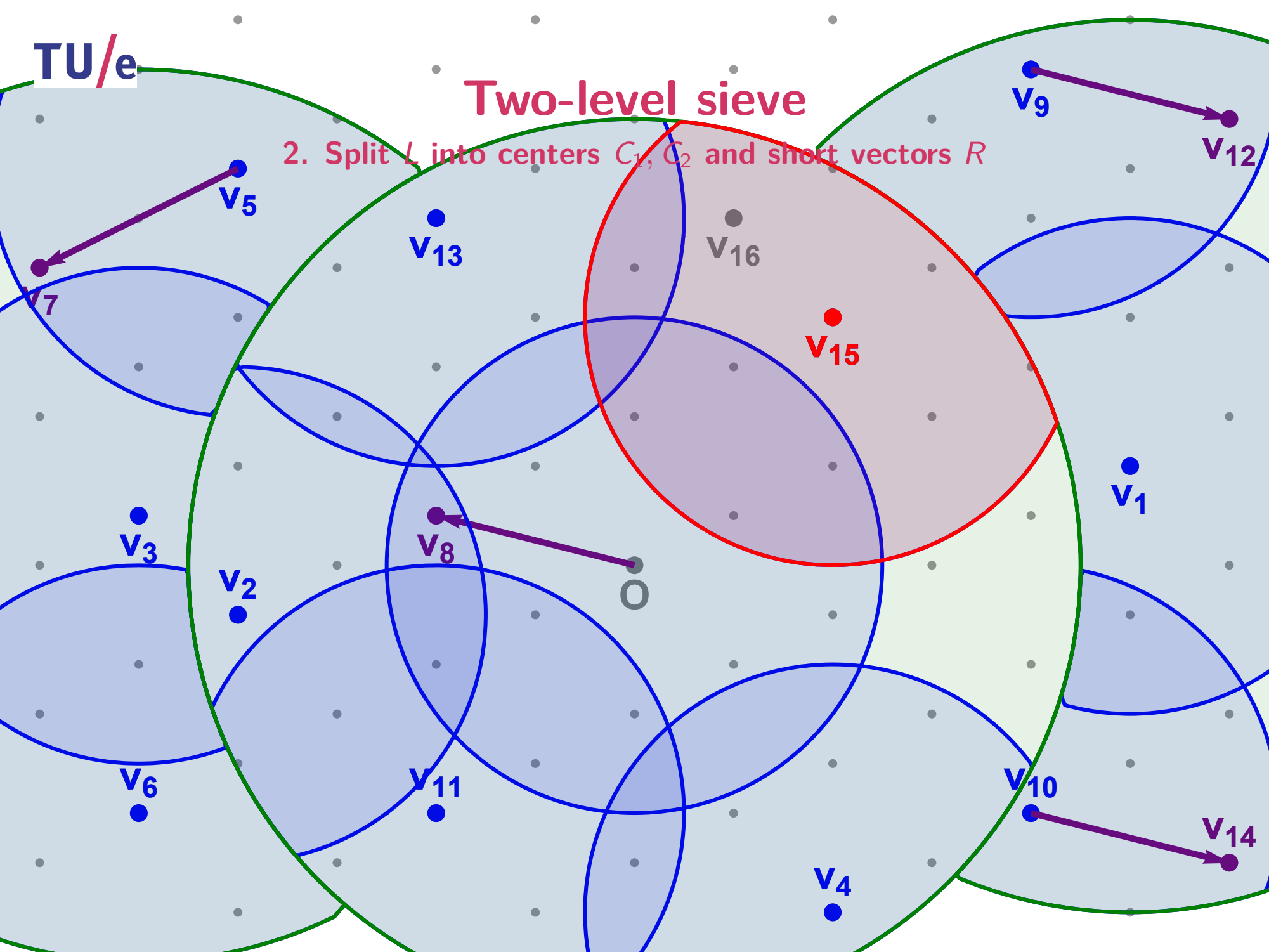

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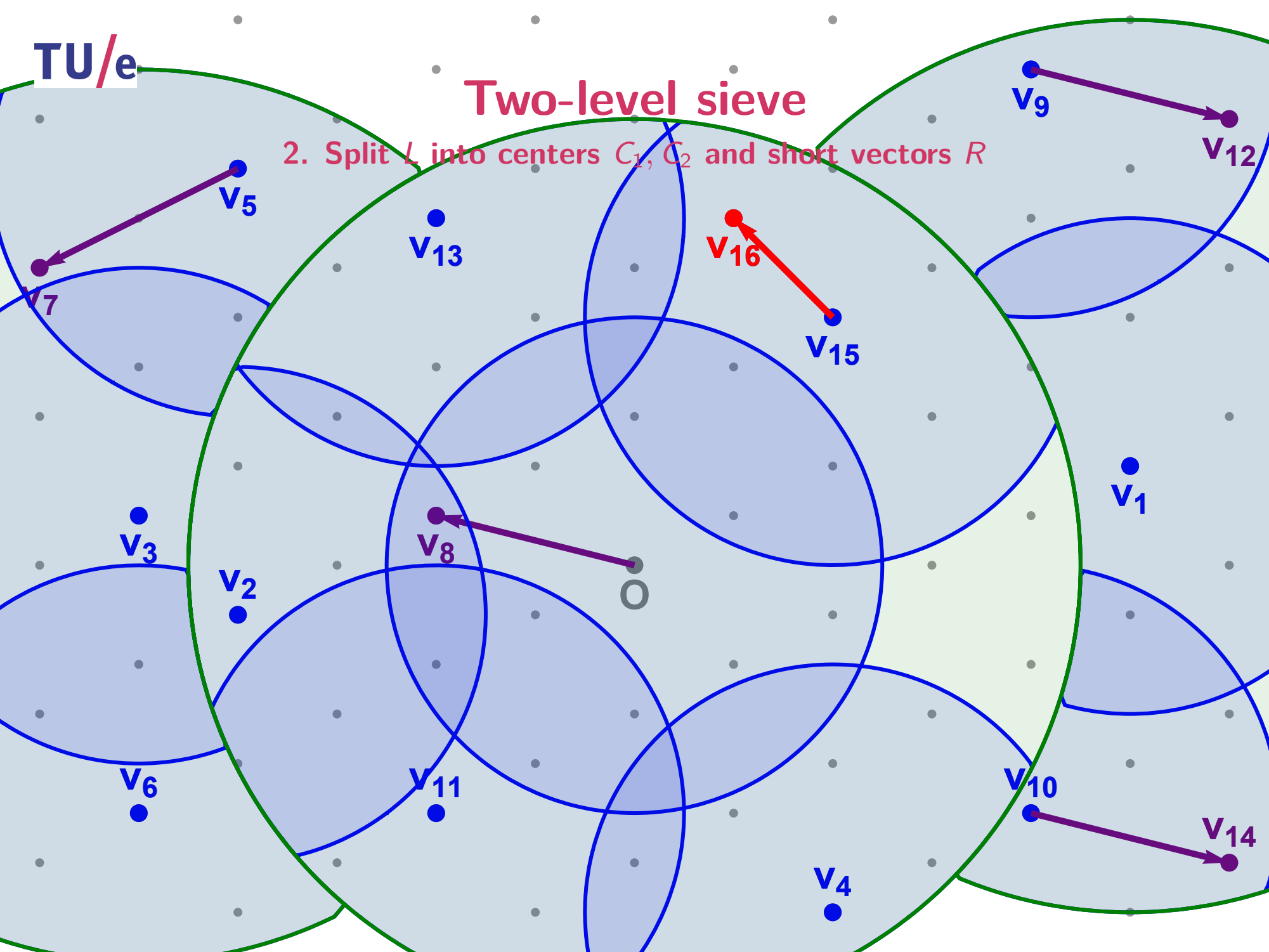

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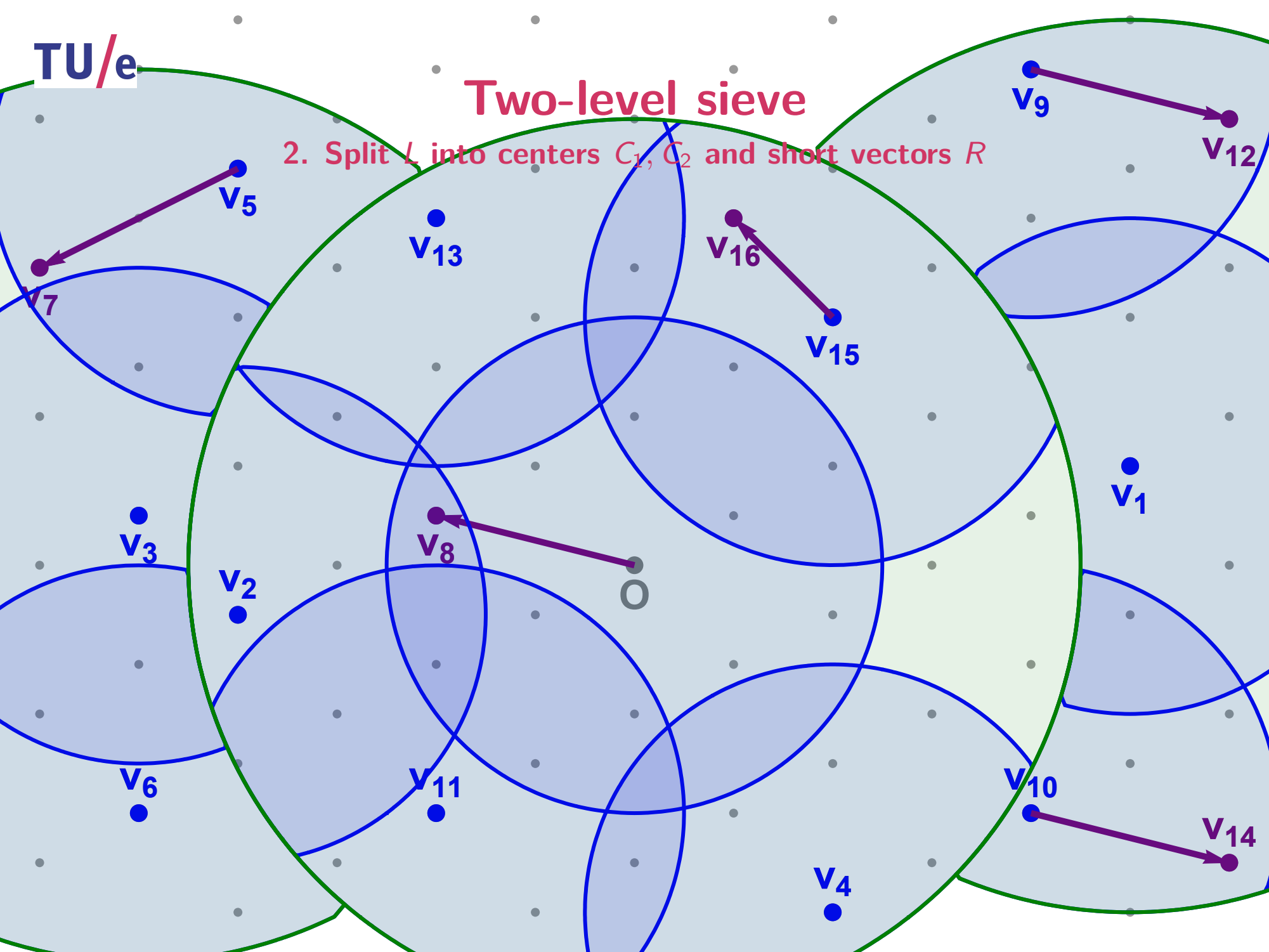

TU/e

Two-level sieve

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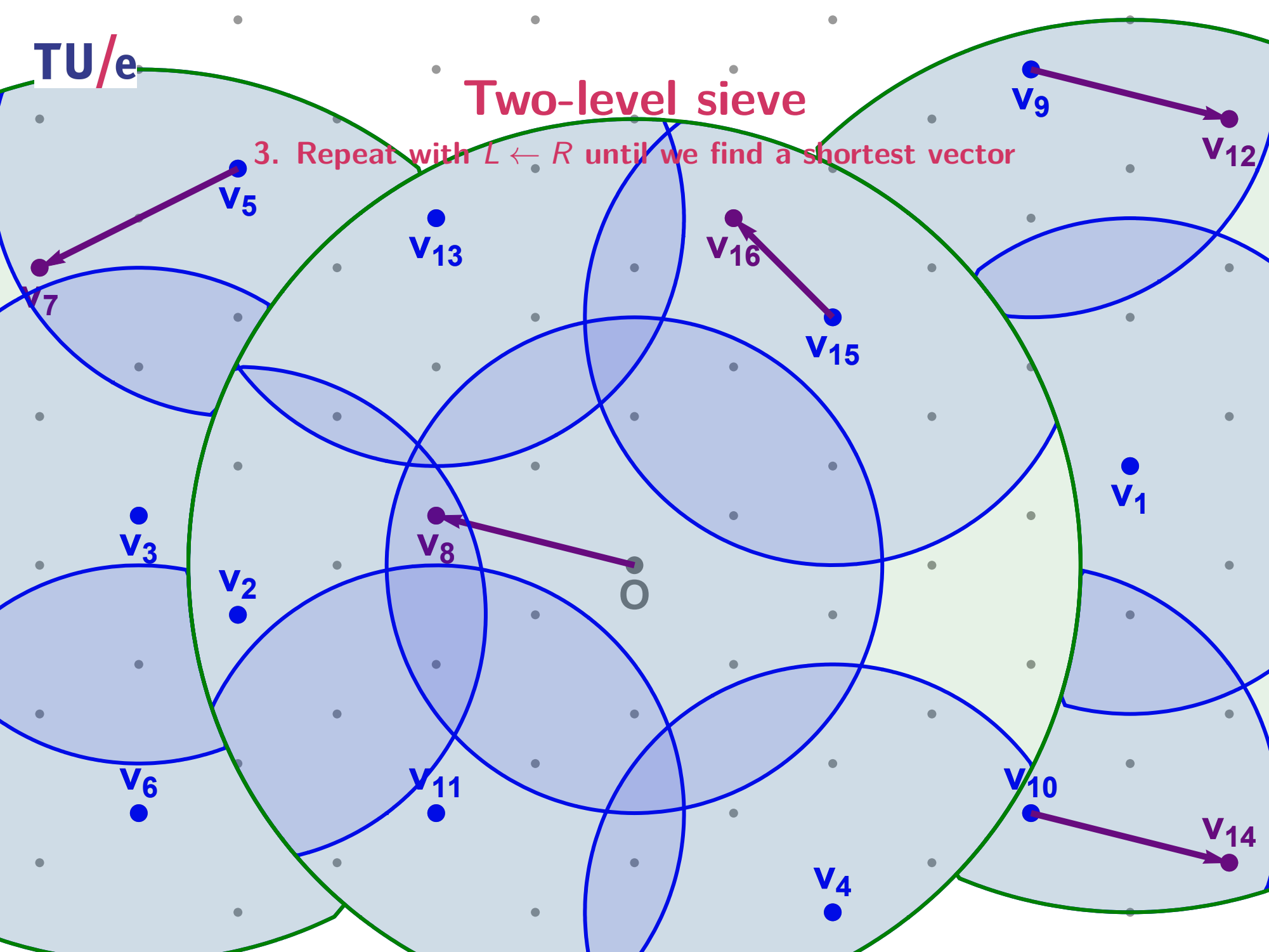

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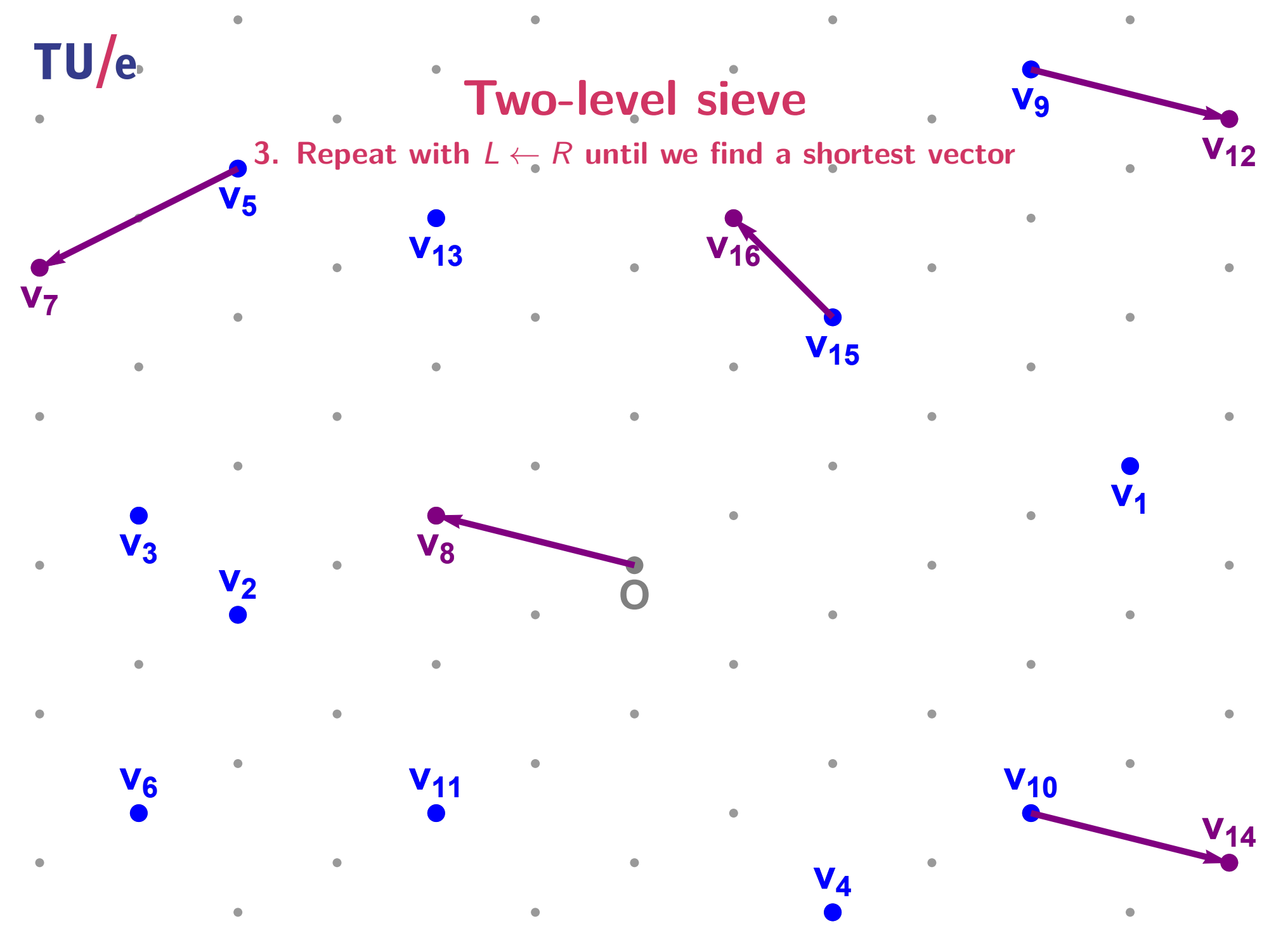

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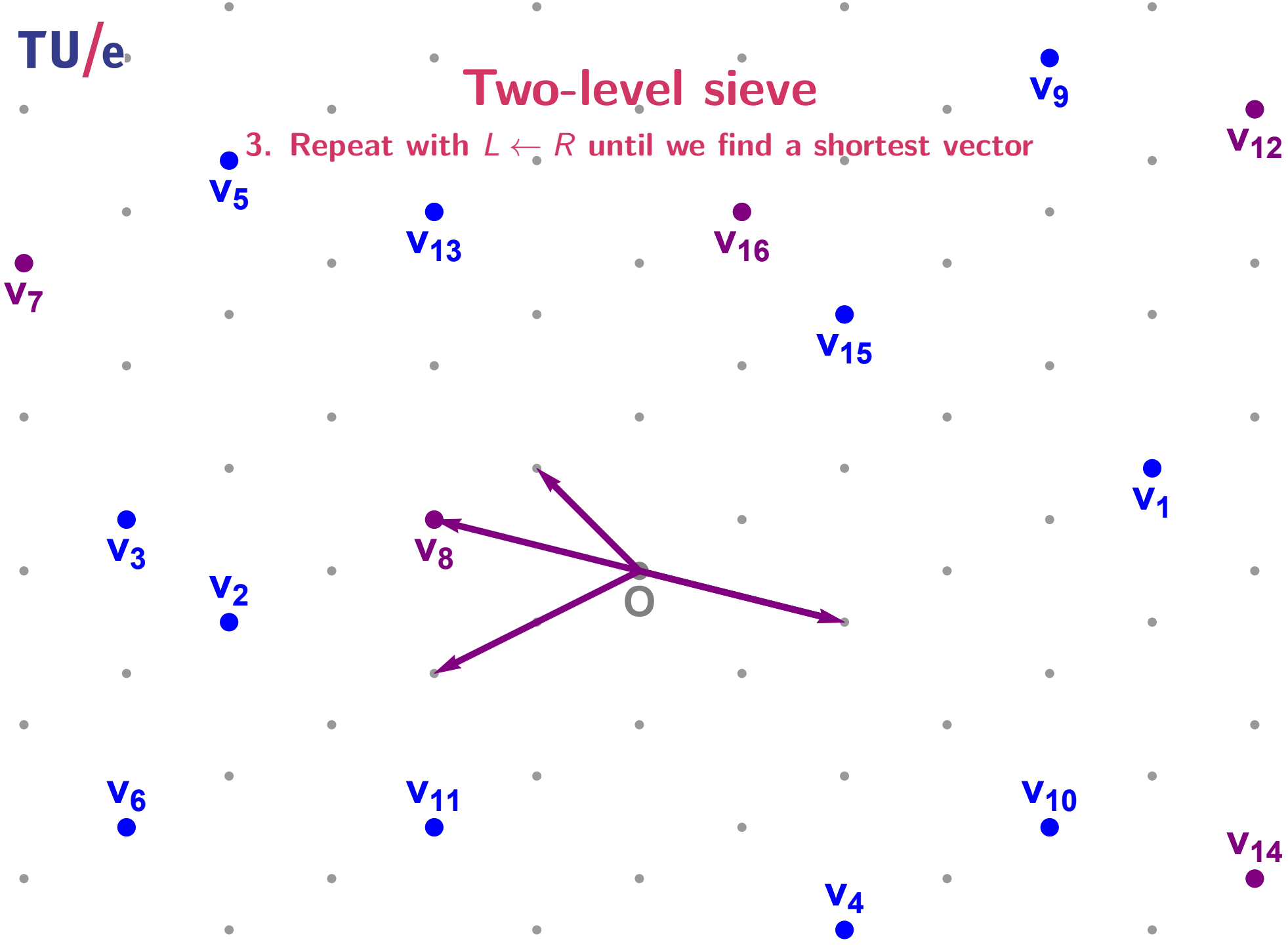

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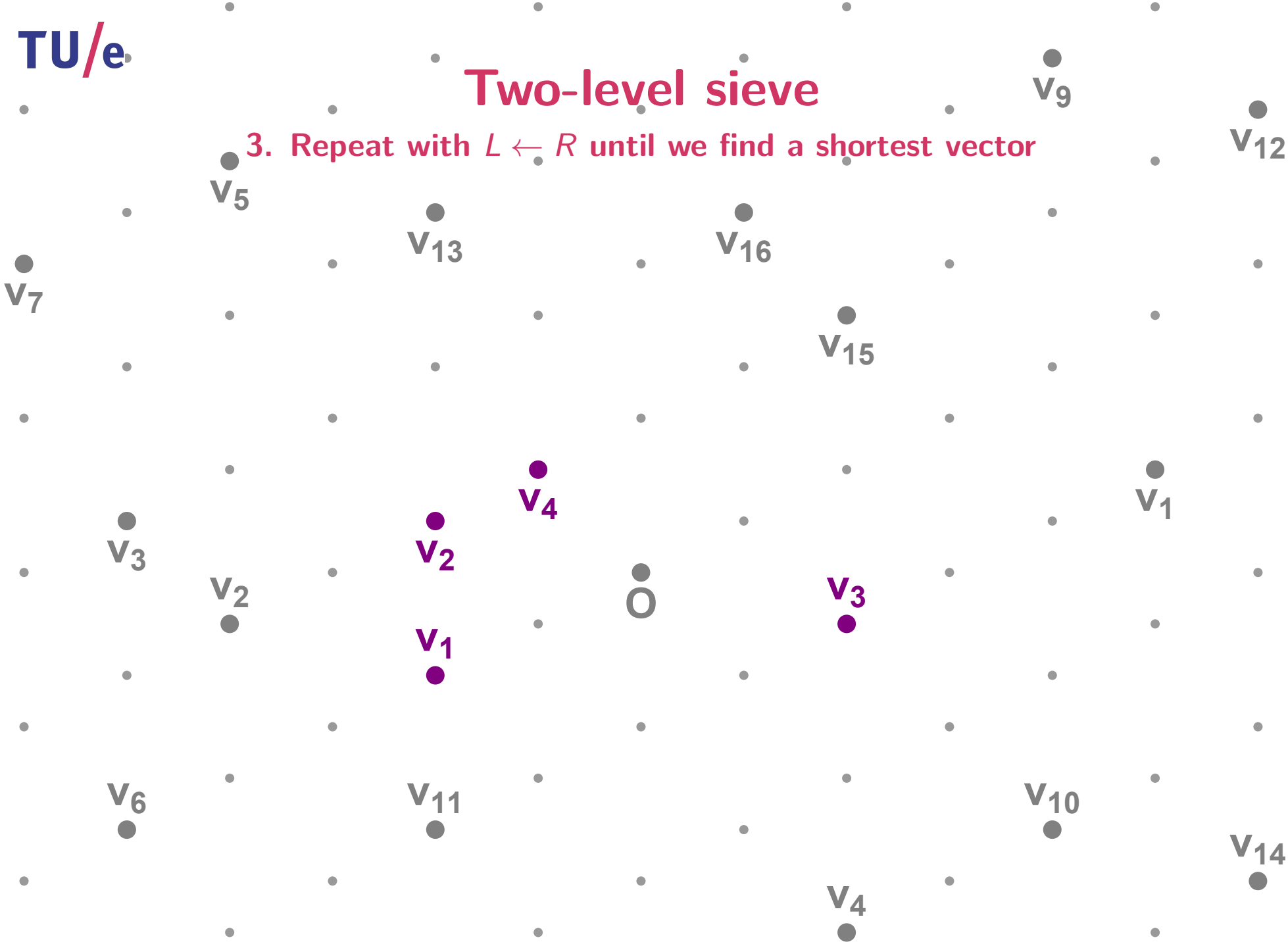
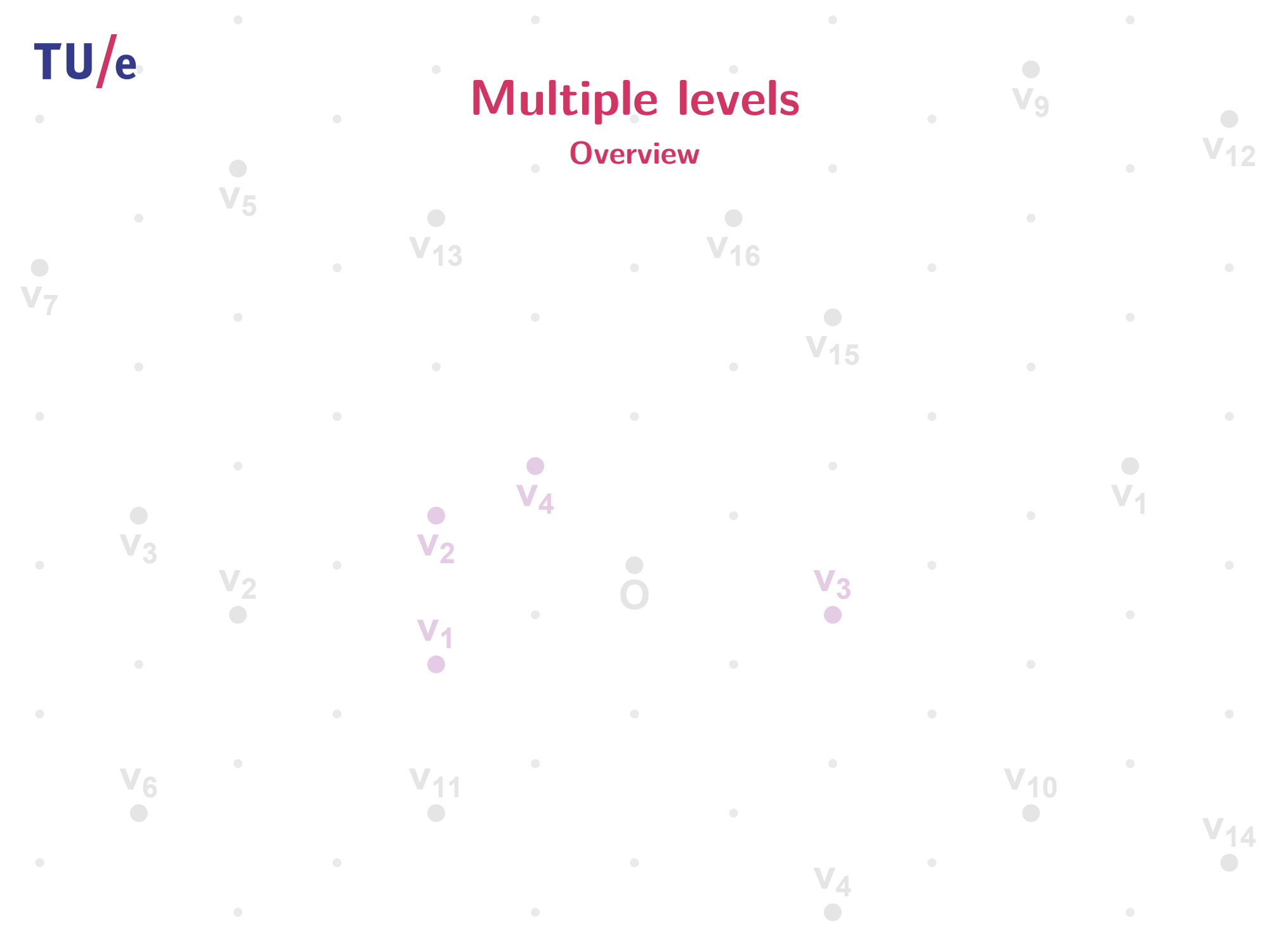


TU/e

Multiple levels

Overview



TU/e

Multiple levels

Overview

Heuristic (Nguyen and Vidick, J. Math. Crypt. '08)

The one-level sieve runs in time $2^{0.4150n}$ and space $2^{0.2075n}$.

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TU/e

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TU/e

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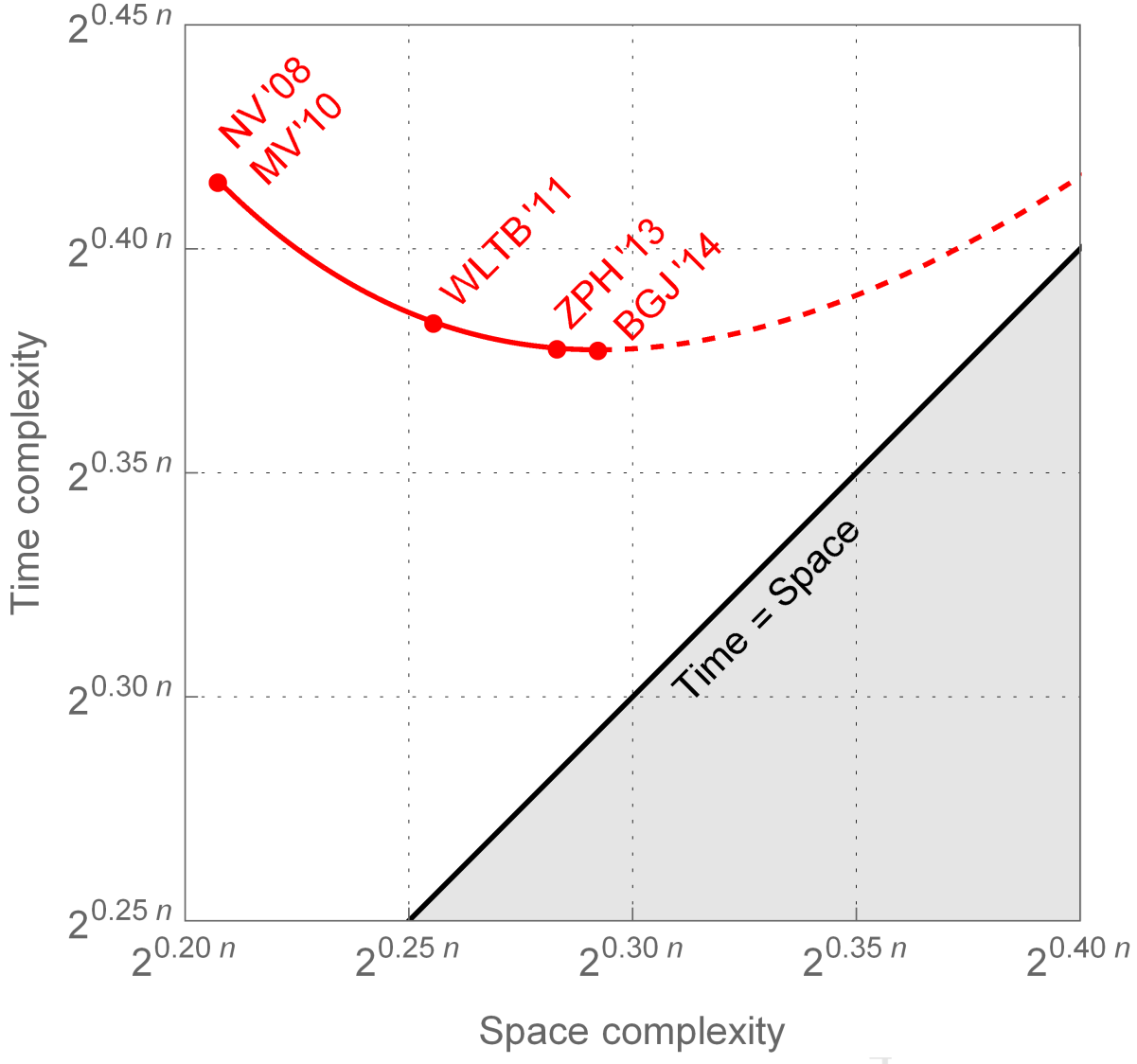
Conjecture

The four-level sieve runs in time $2^{0.3774n}$ and space $2^{0.2925n}$, and higher-level sieves are not faster than this.

Sieving

Space/time trade-off



TU/e

Quantum Search

Classical form

Problem: Given a list L of size N, and a function $f : L \to \{0, 1\}$ such that there is exactly one element $e \in L$ with $f(e) = 1$. Find this element e.

- Classical search: $\Theta(N)$ time
- Quantum search: $\Theta(\sqrt{N})$ time [Gro96]

TU/e

Quantum Search

General form

Problem: Given a list L of size N, and a function $f : L \to \{0, 1\}$ such that there are $c = O(1)$ elements $e \in L$ with $f(e) = 1$. Find one such element e.

- Classical search: $\Theta(N/c)$ time
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TU/e

Quantum Search

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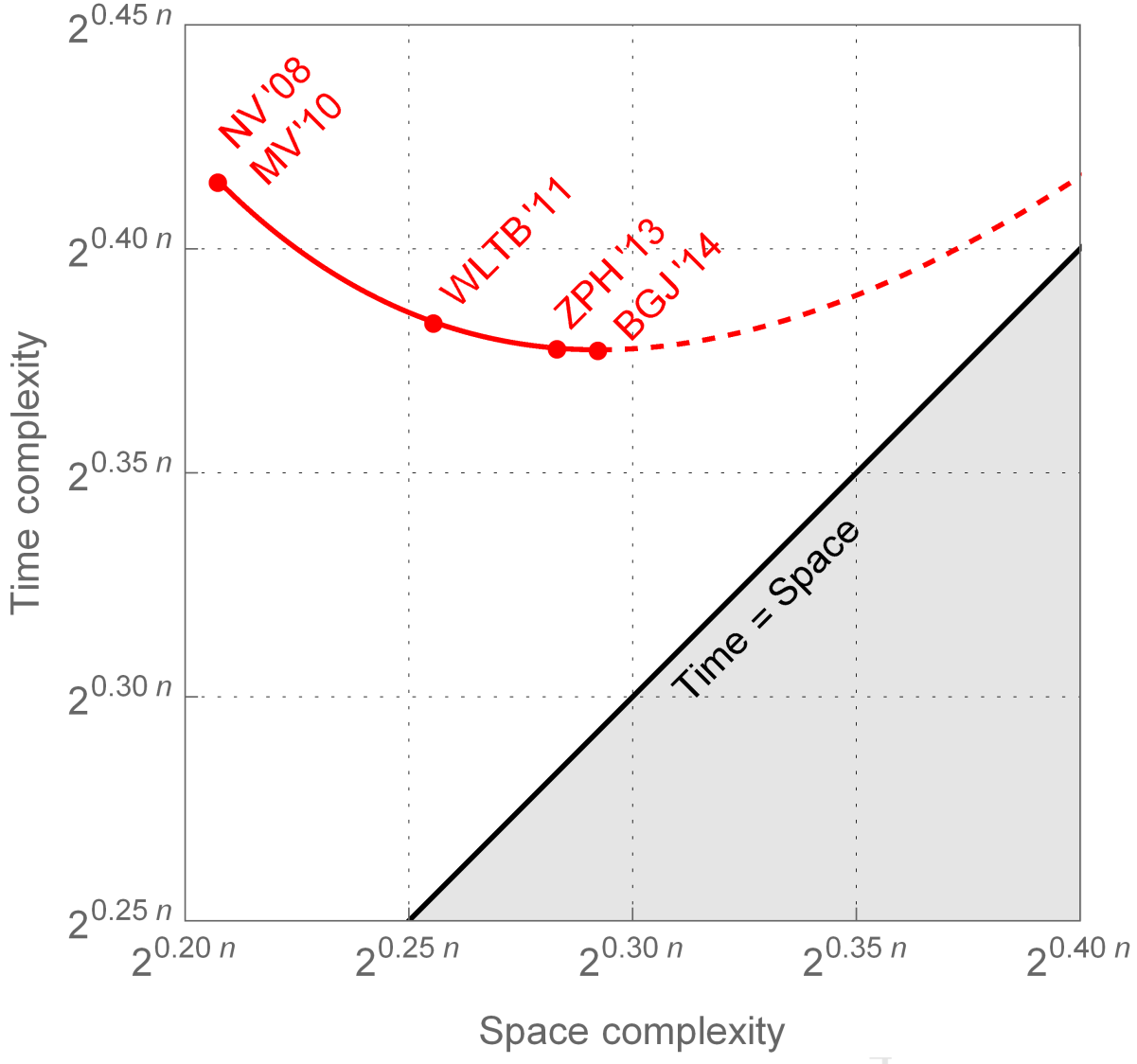
Potentially speed up search subroutines in sieving

Sieving

Space/time trade-off



TU/e

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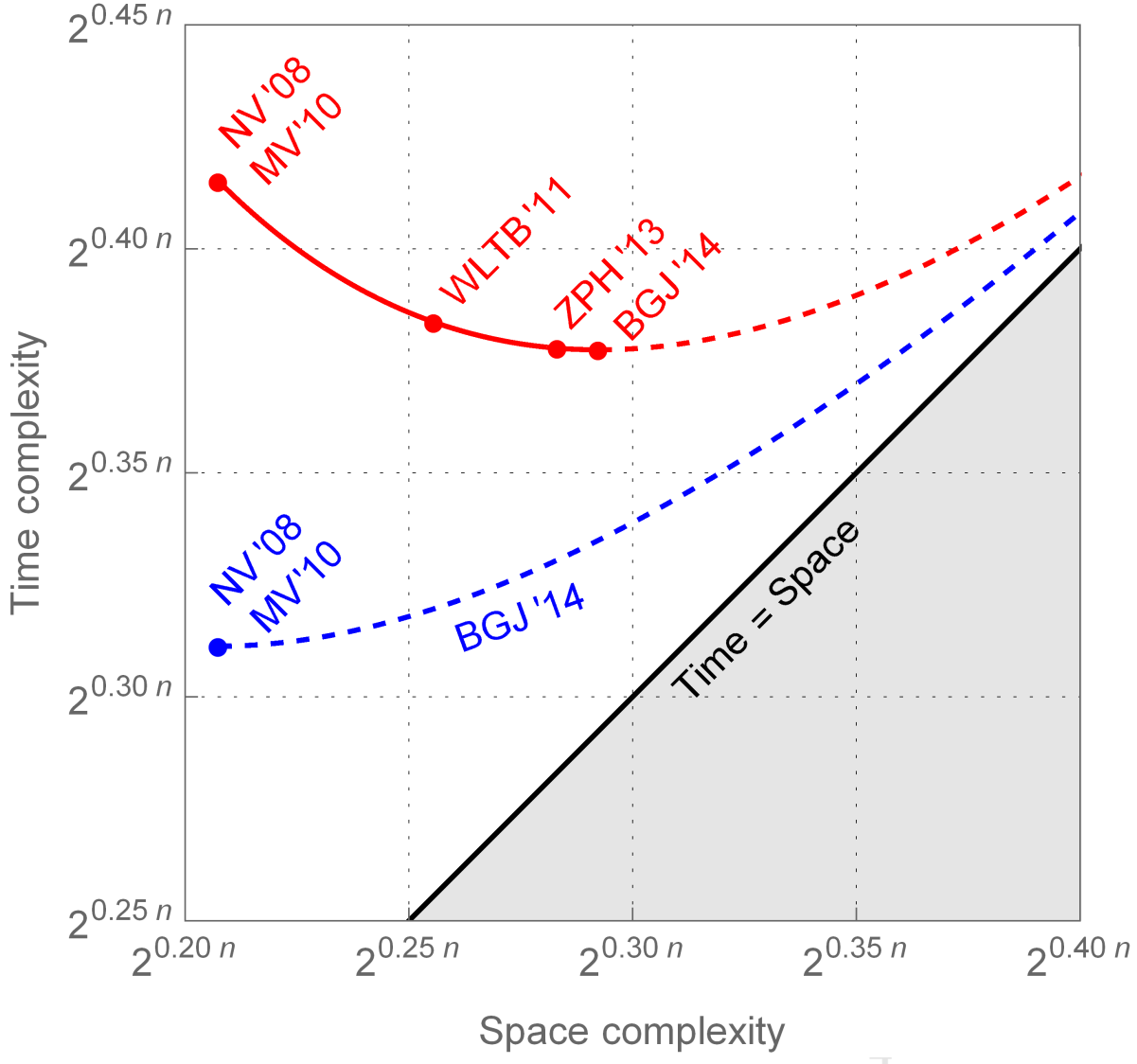

TU/e

Nguyen-Vidick sieve with LSH

1. Sample a list L of random lattice vectors

TU/e

Nguyen-Vidick sieve with LSH

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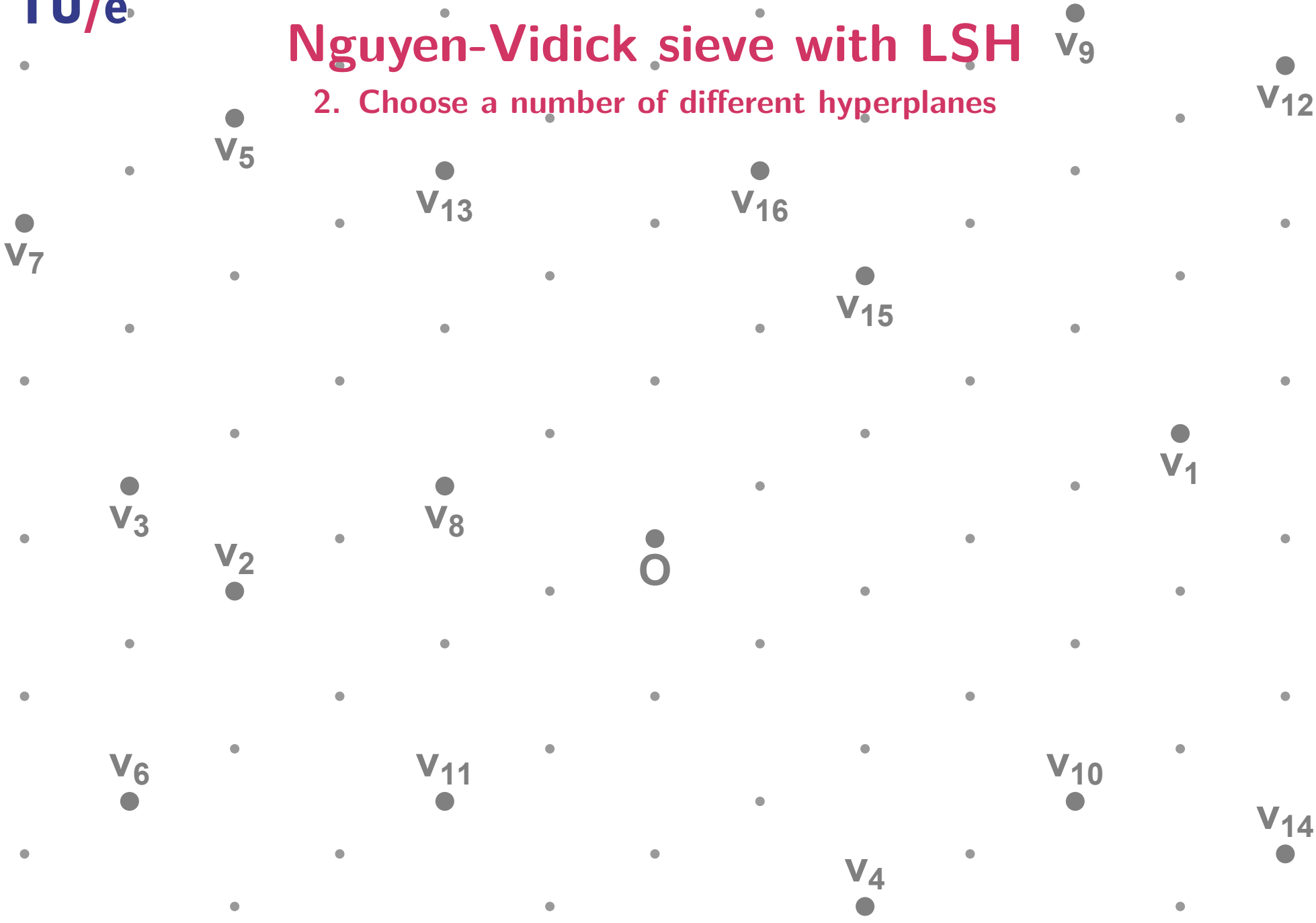
TU/e

Nguyen-Vidick sieve with LSH

2. Choose a number of different hyperplanes



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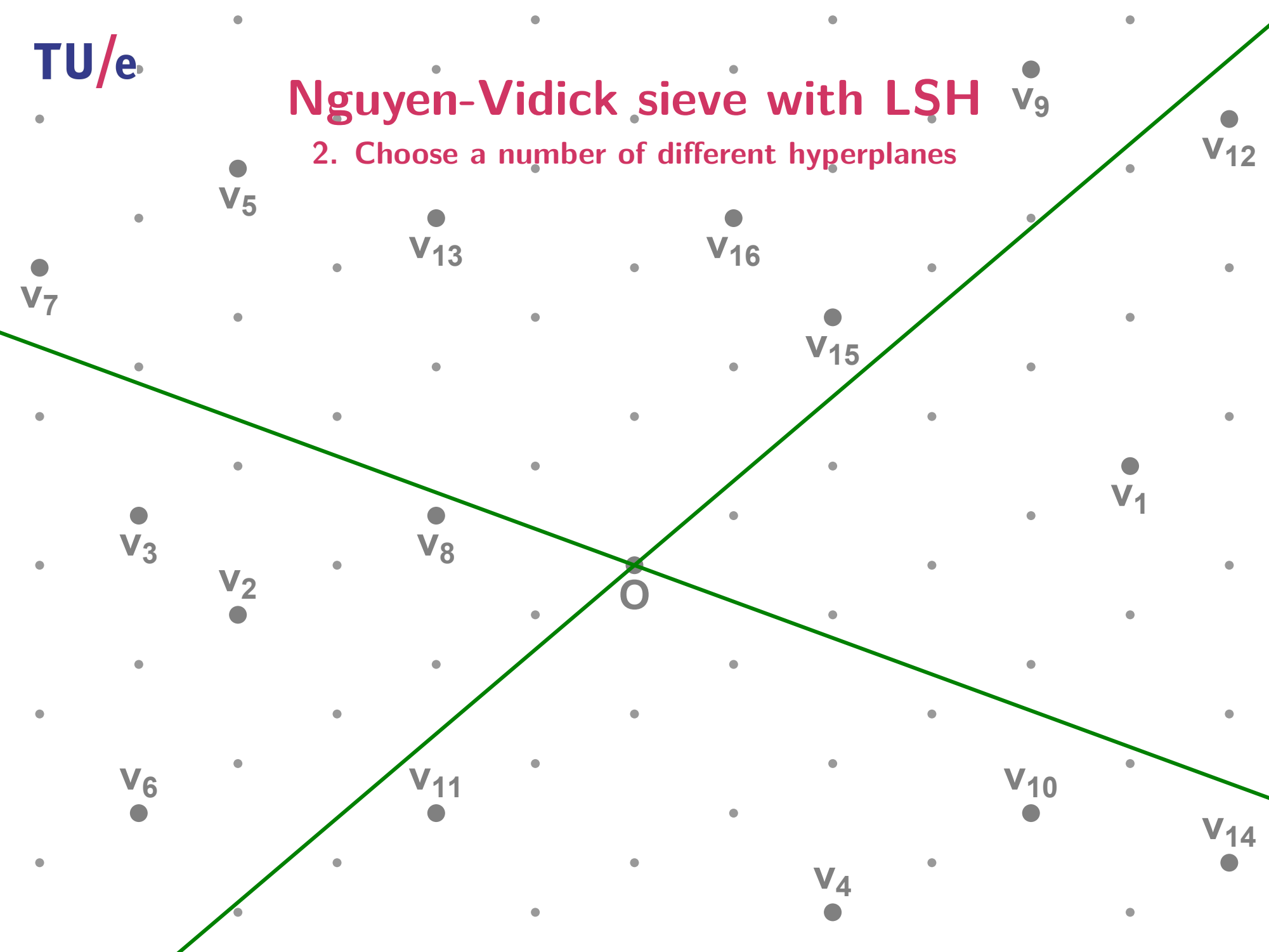
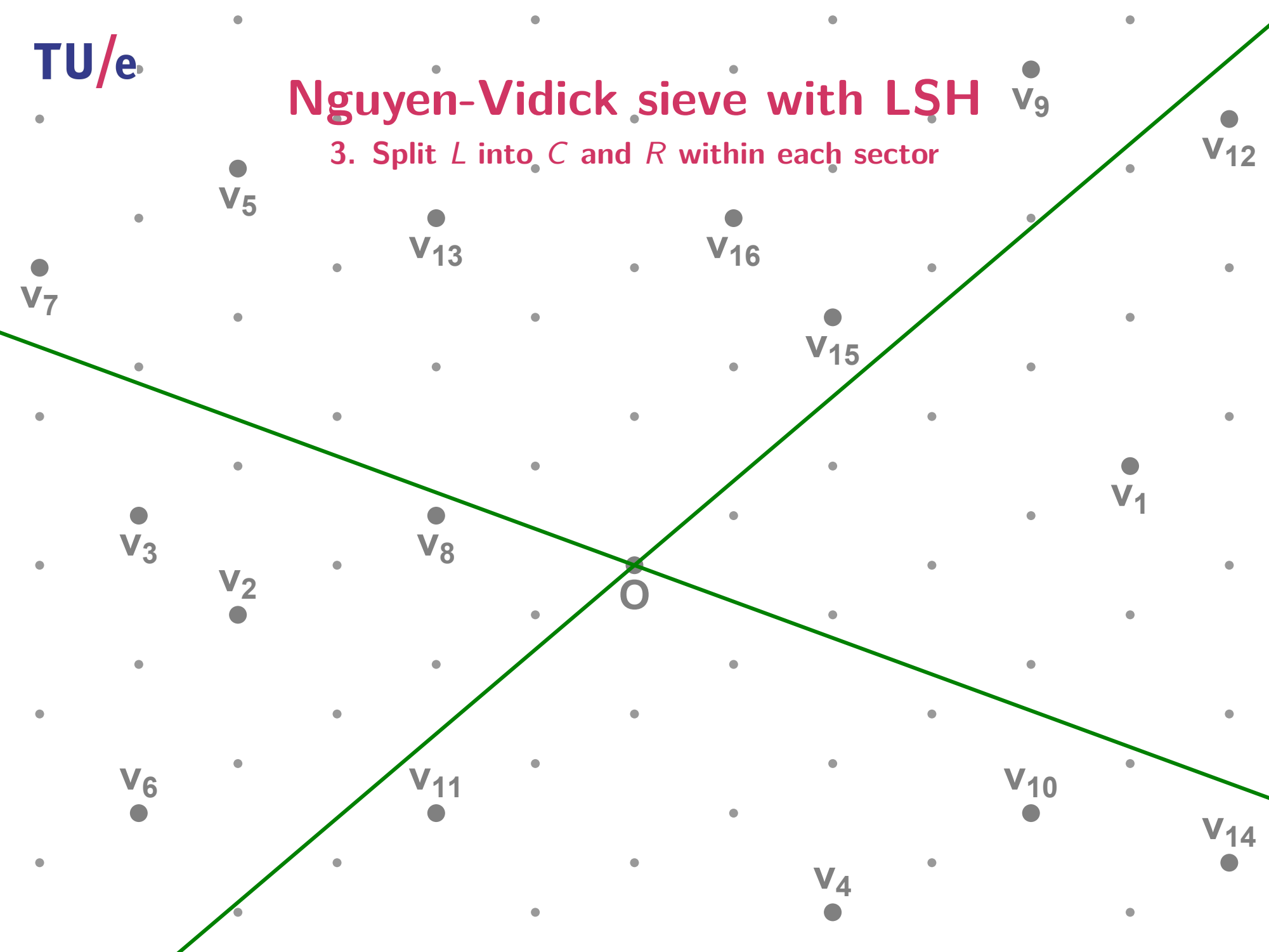


TU/e

Nguyen-Vidick sieve with LSH

3. Split L into C and R within each sector



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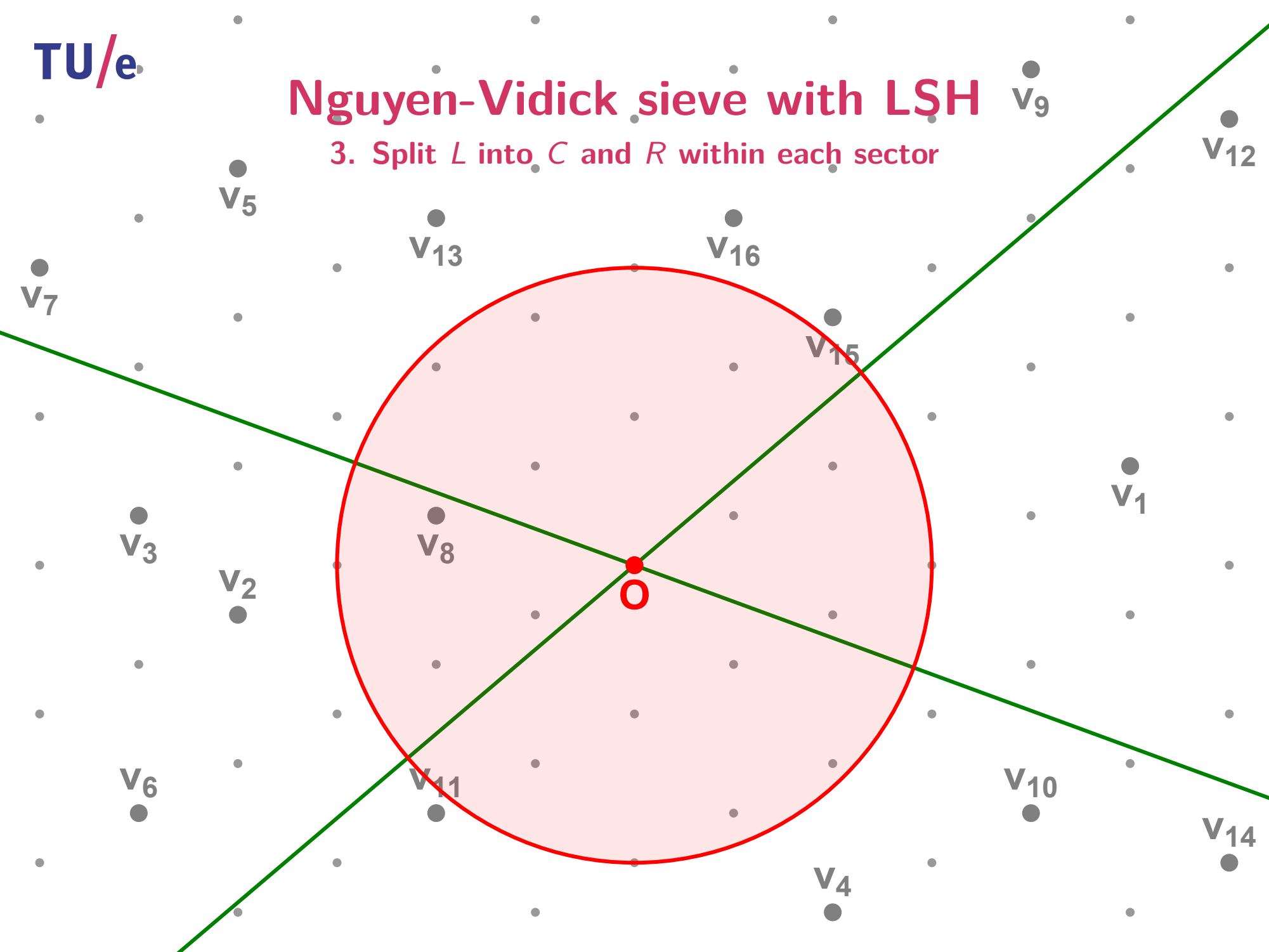

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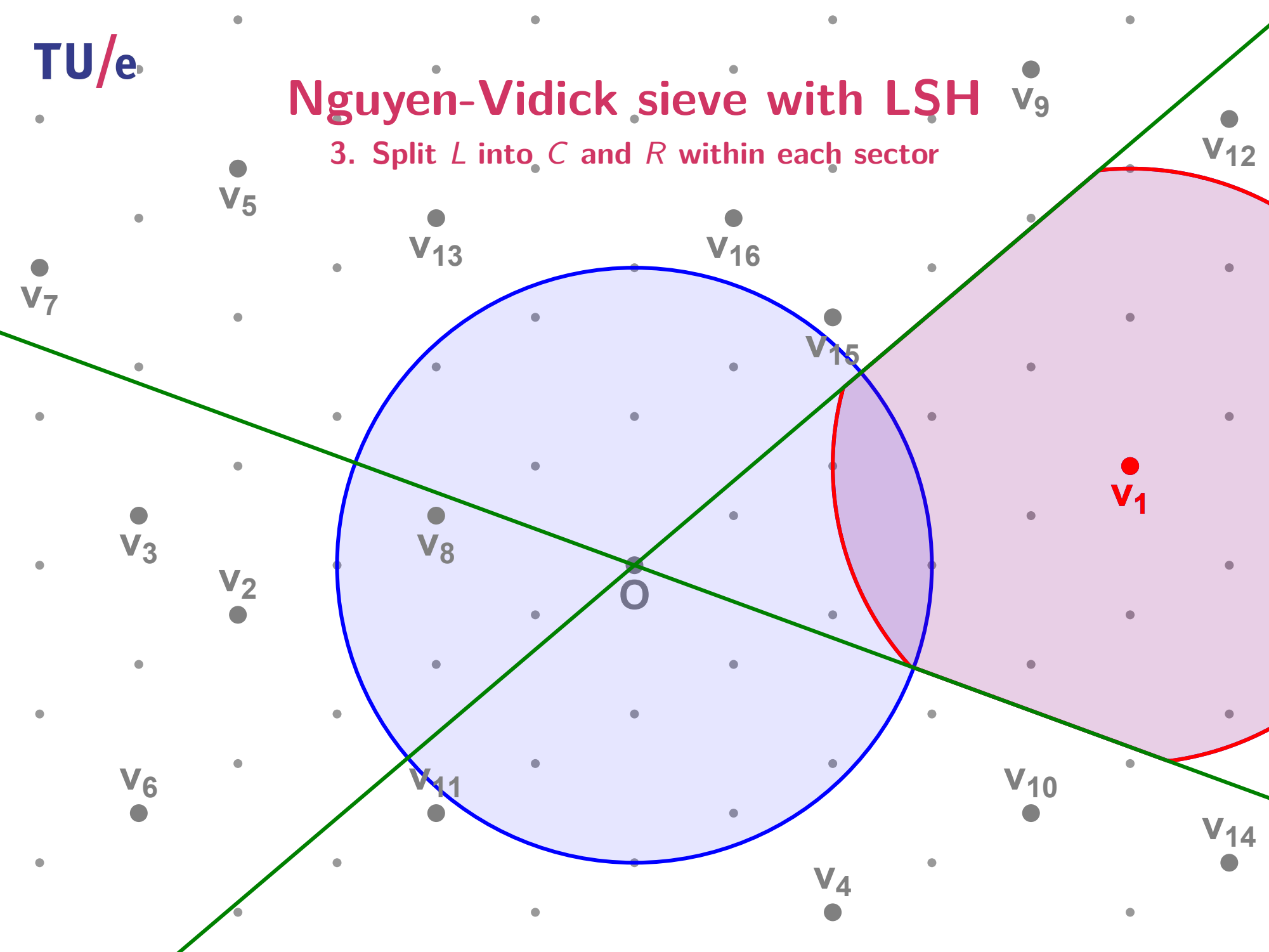

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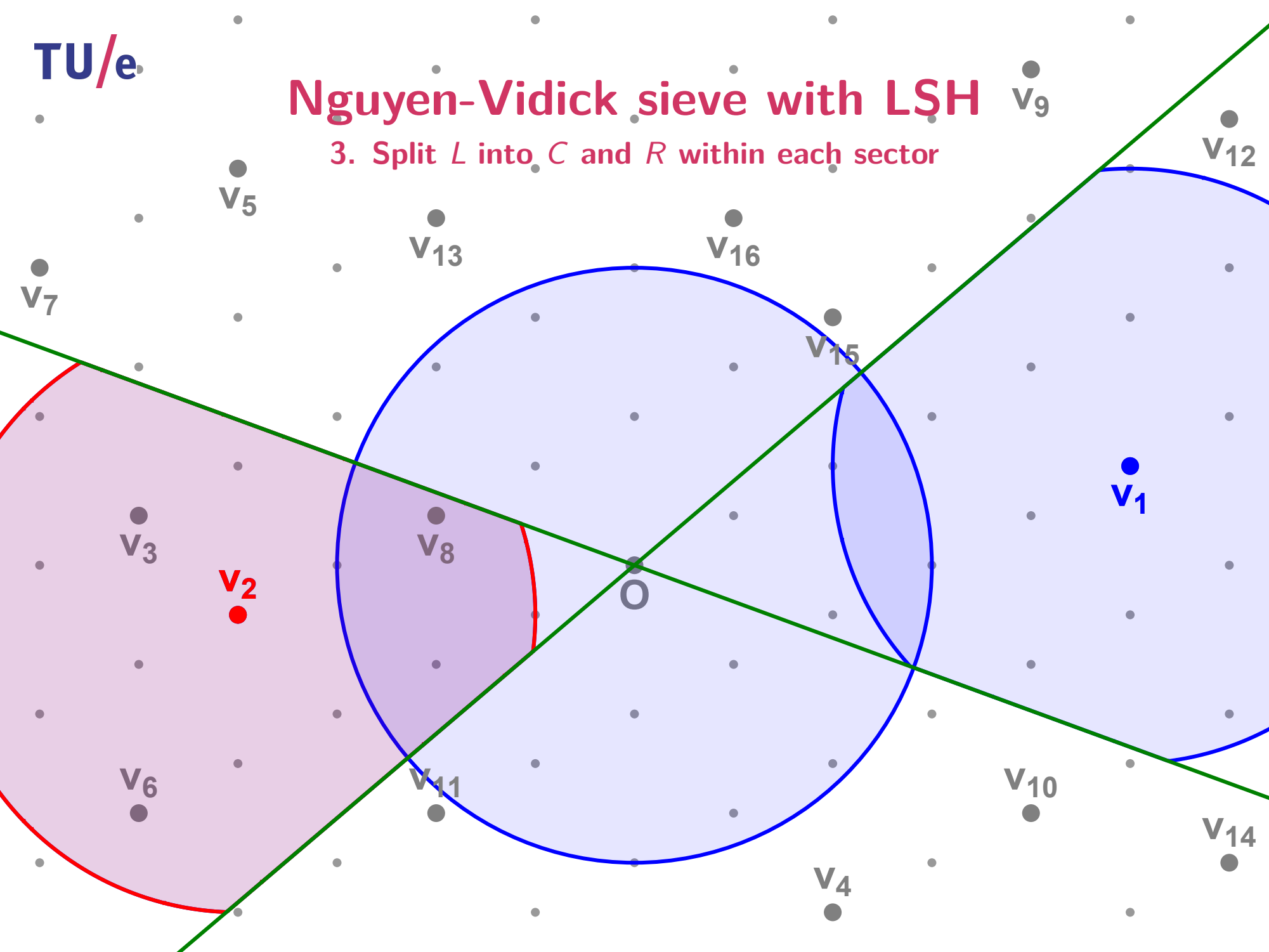

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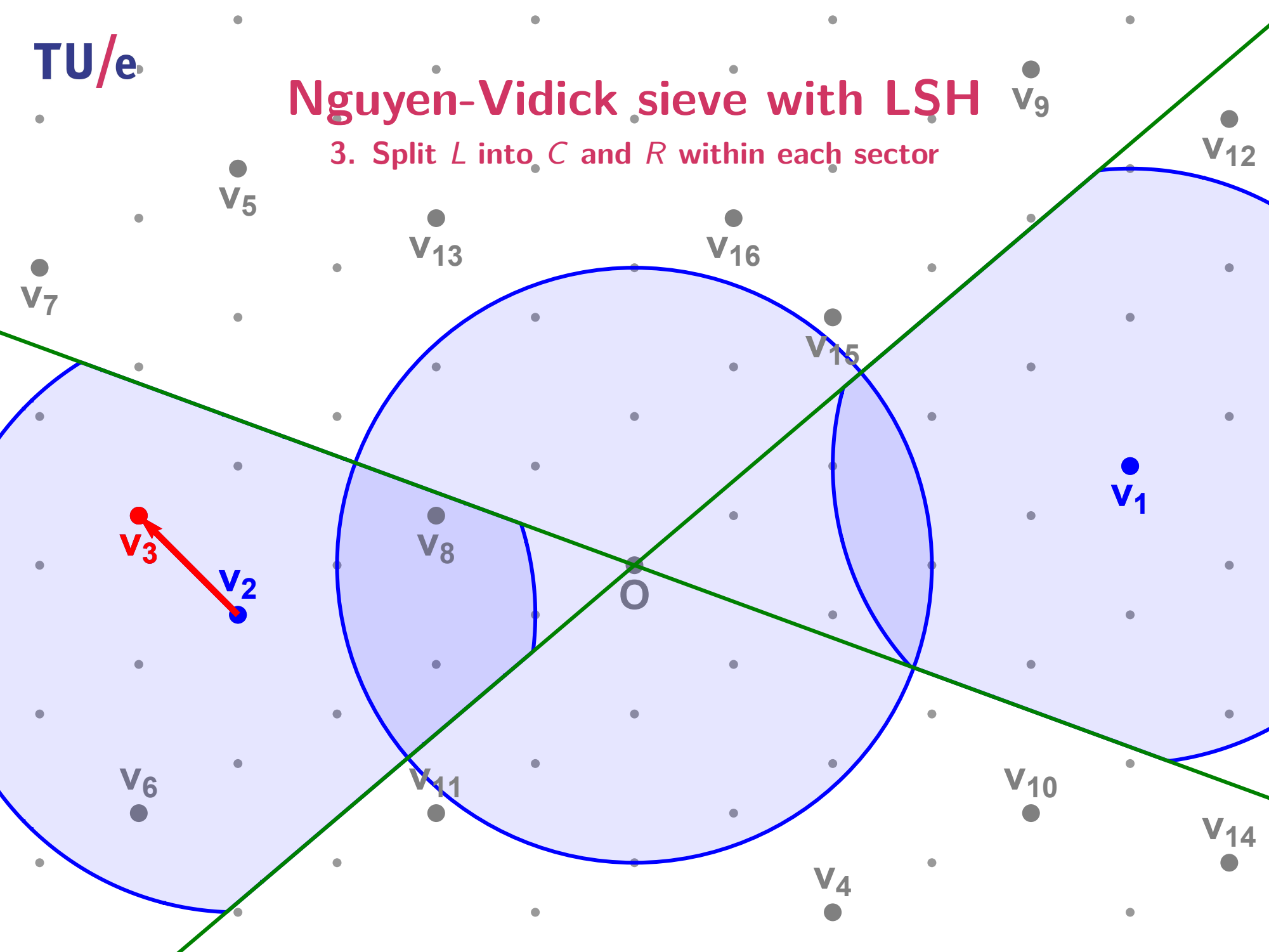

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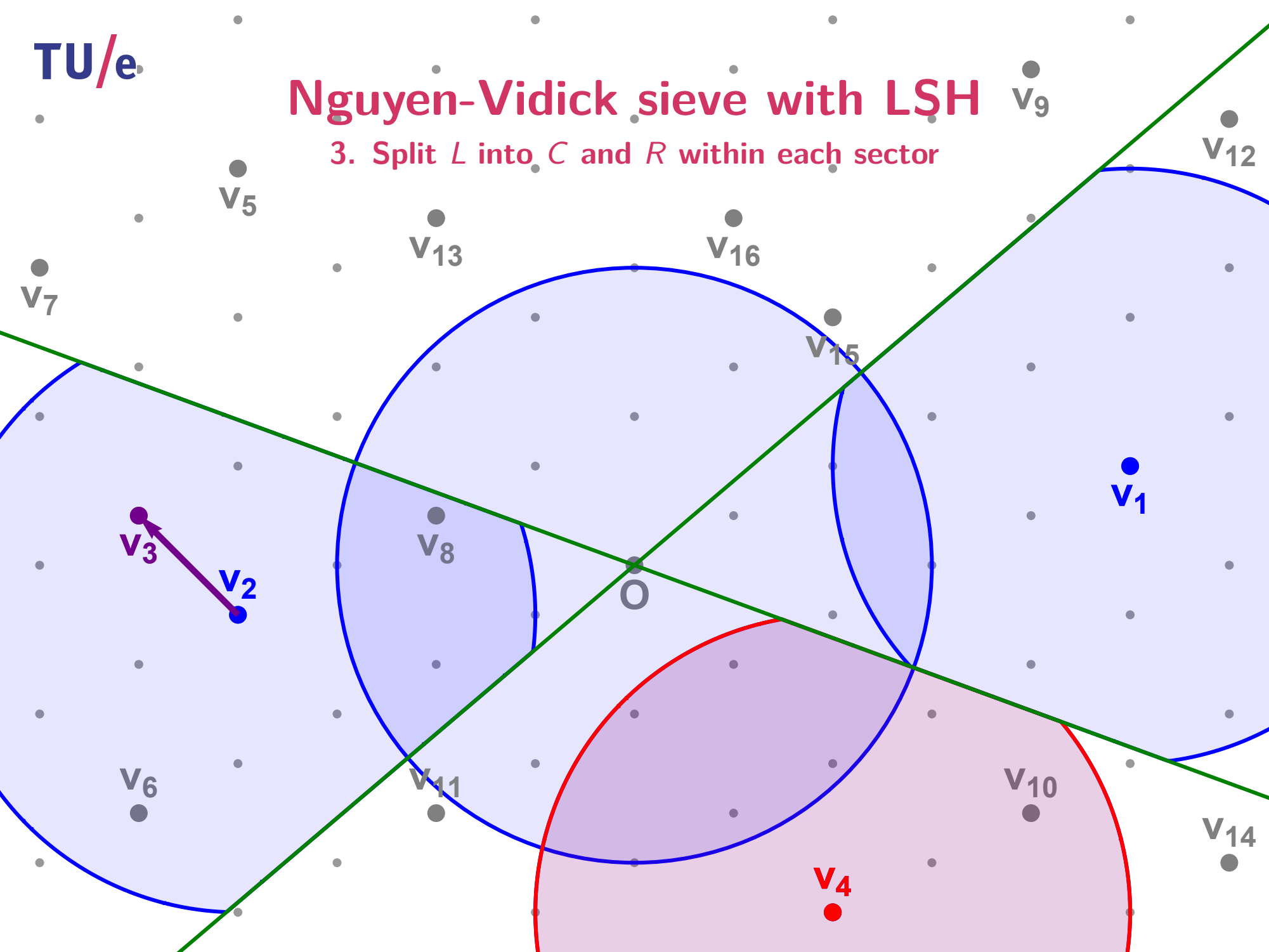

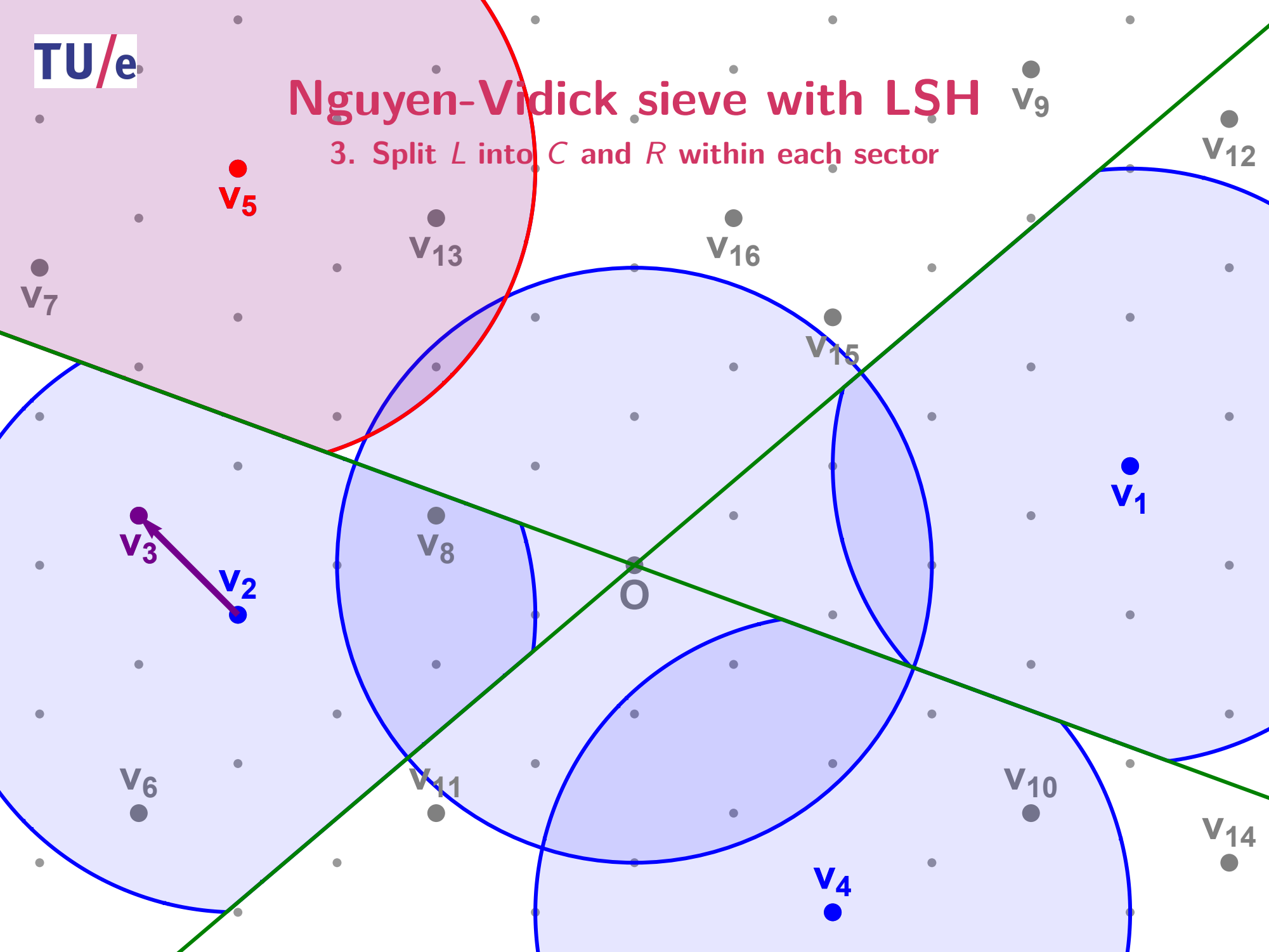
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TU/e

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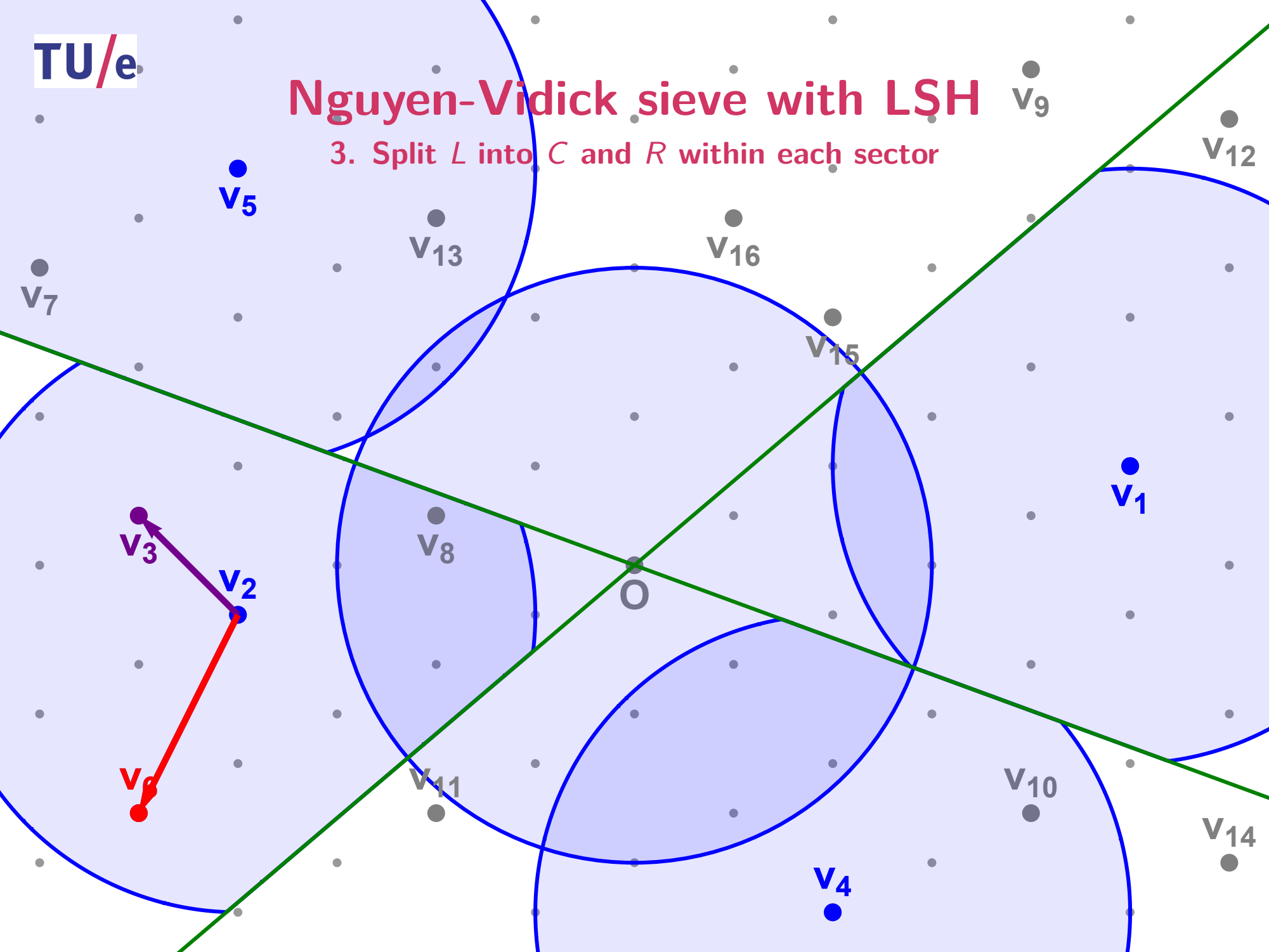

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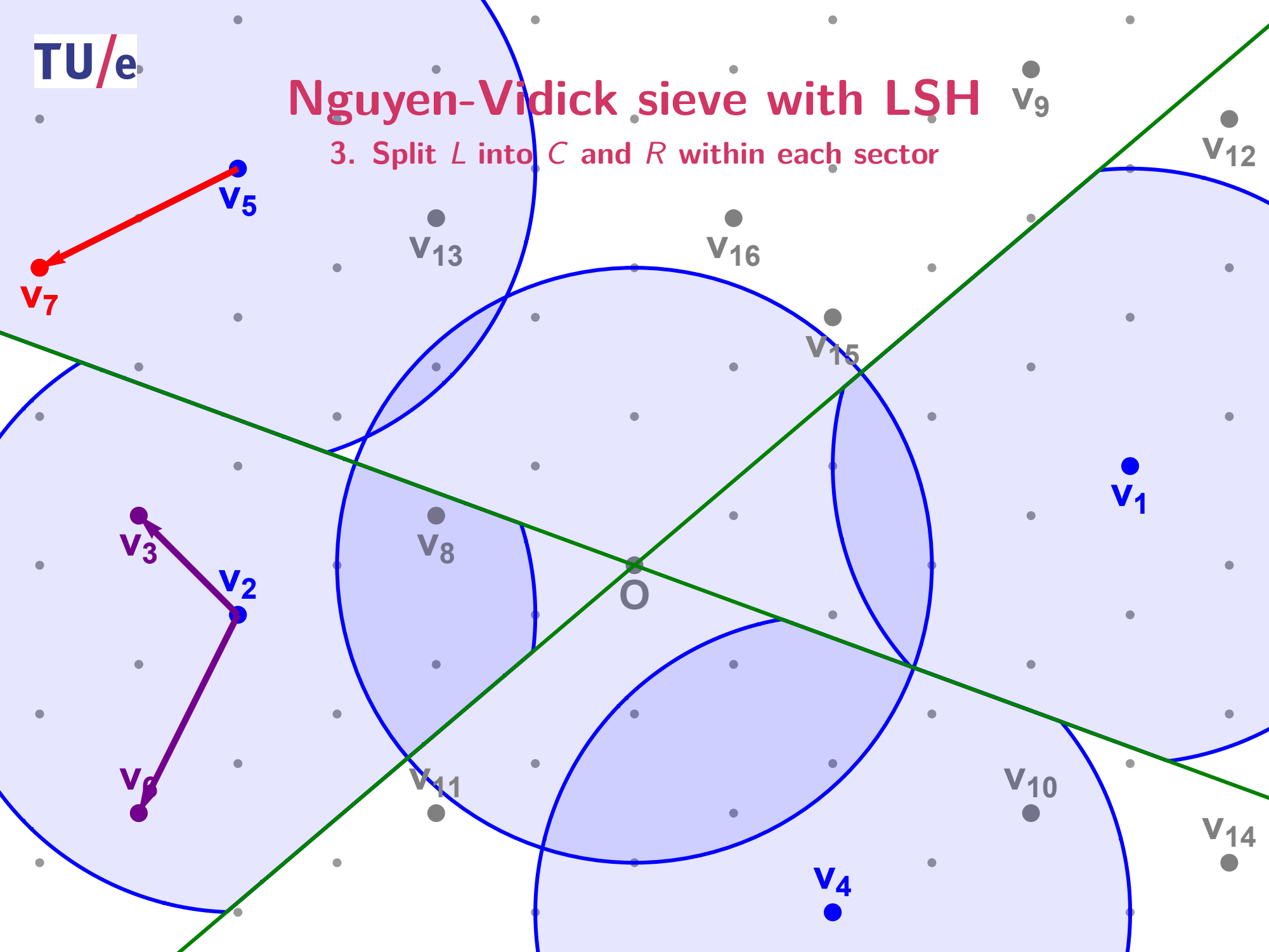

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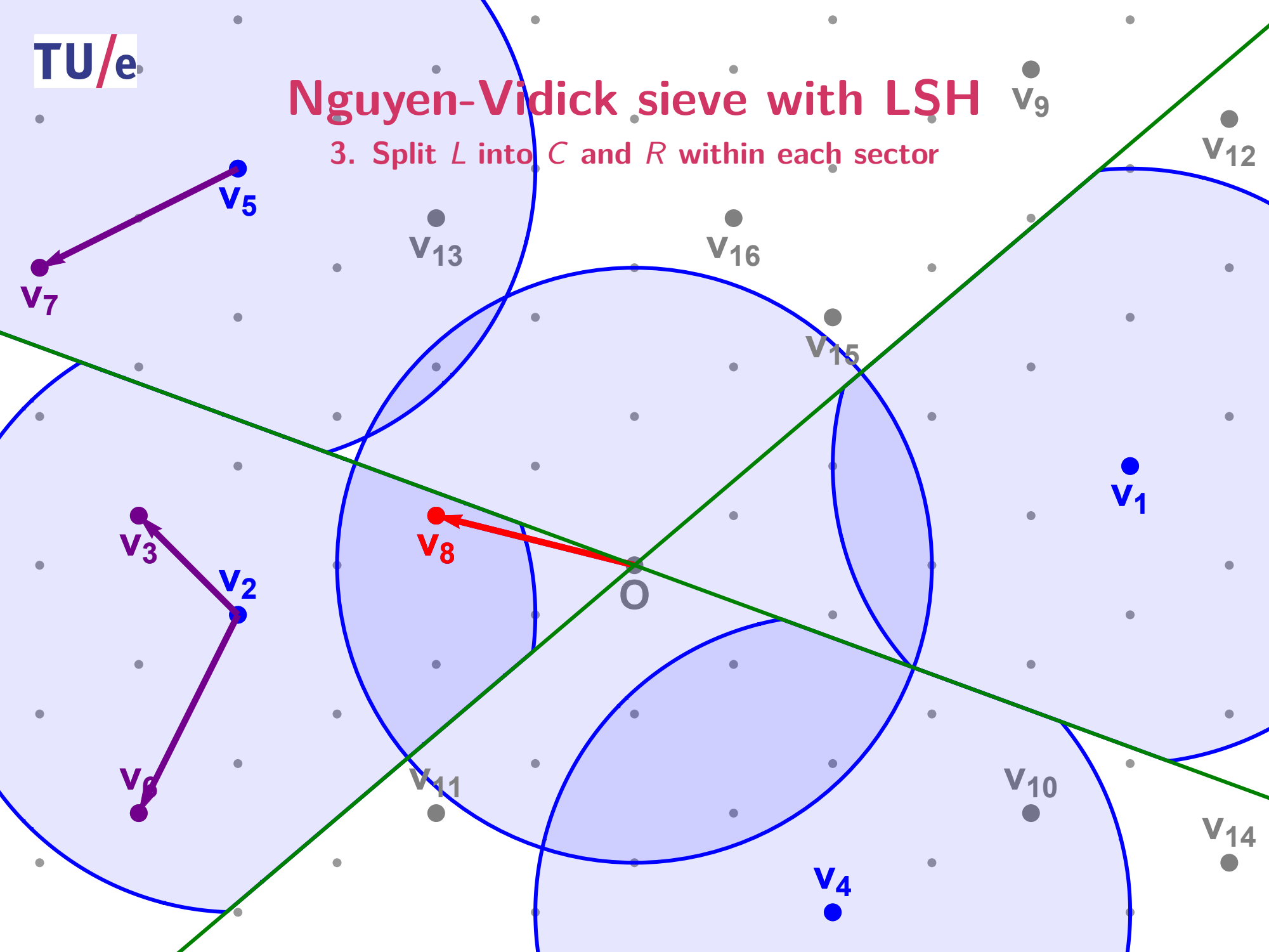

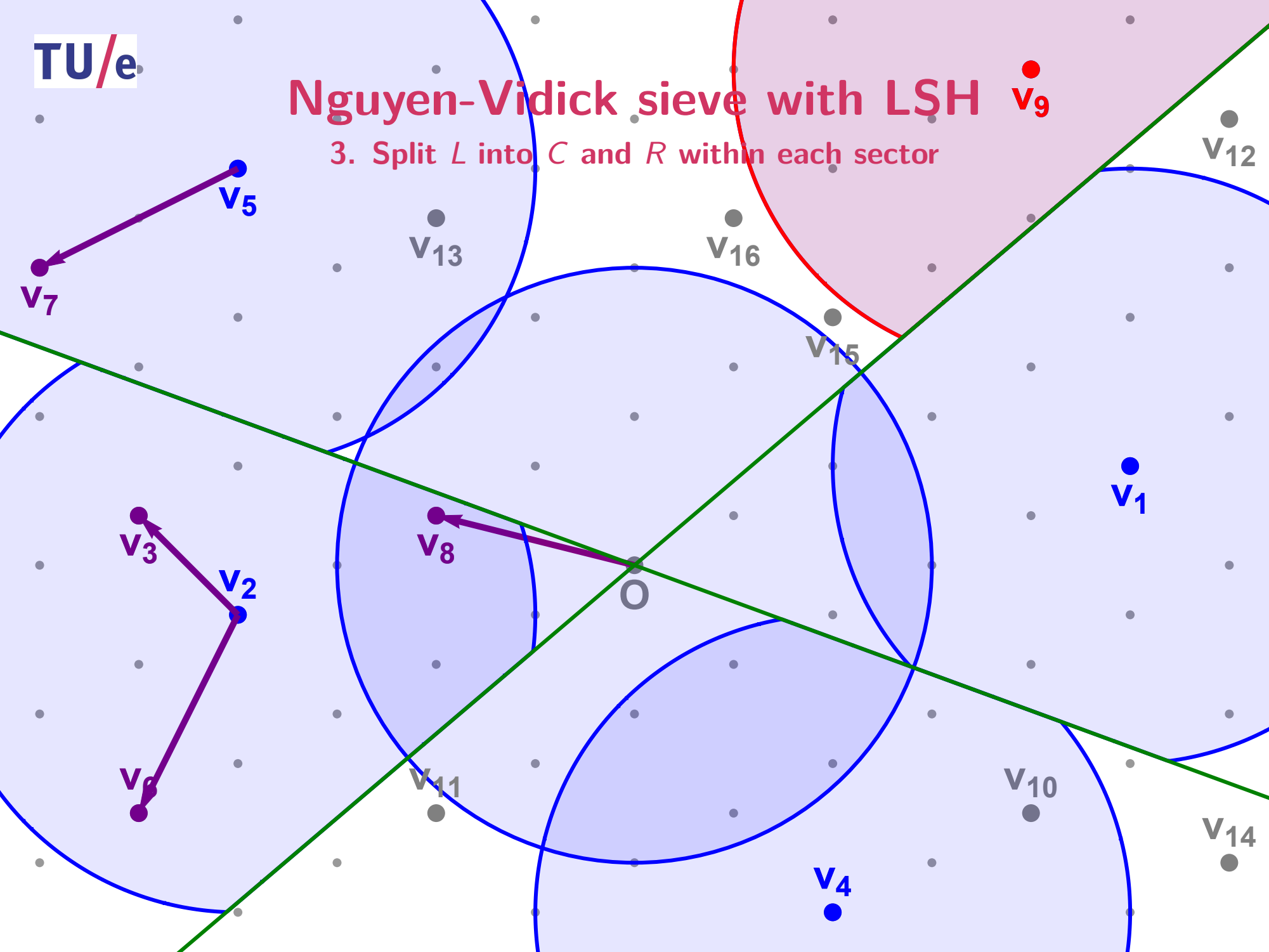
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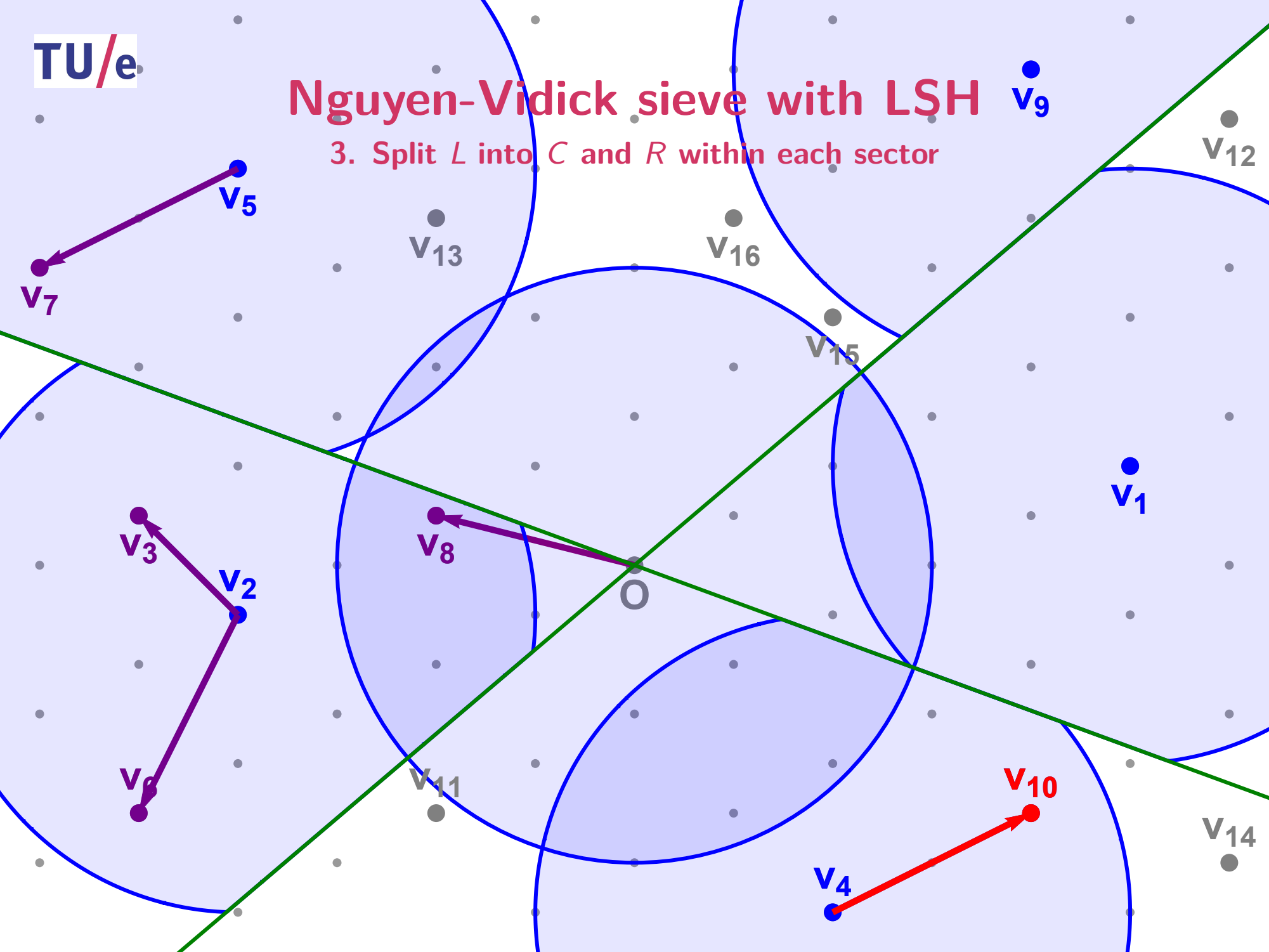
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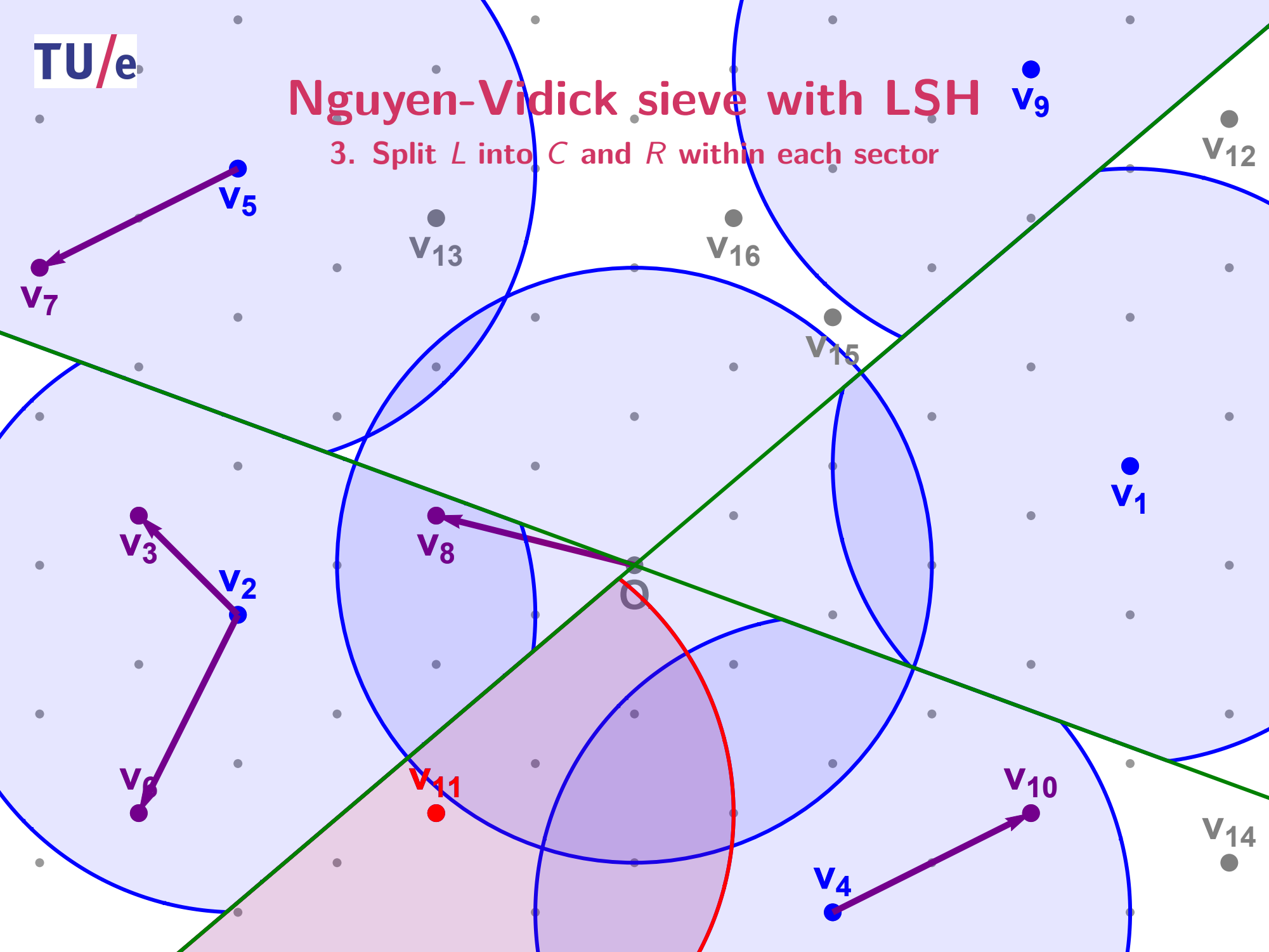
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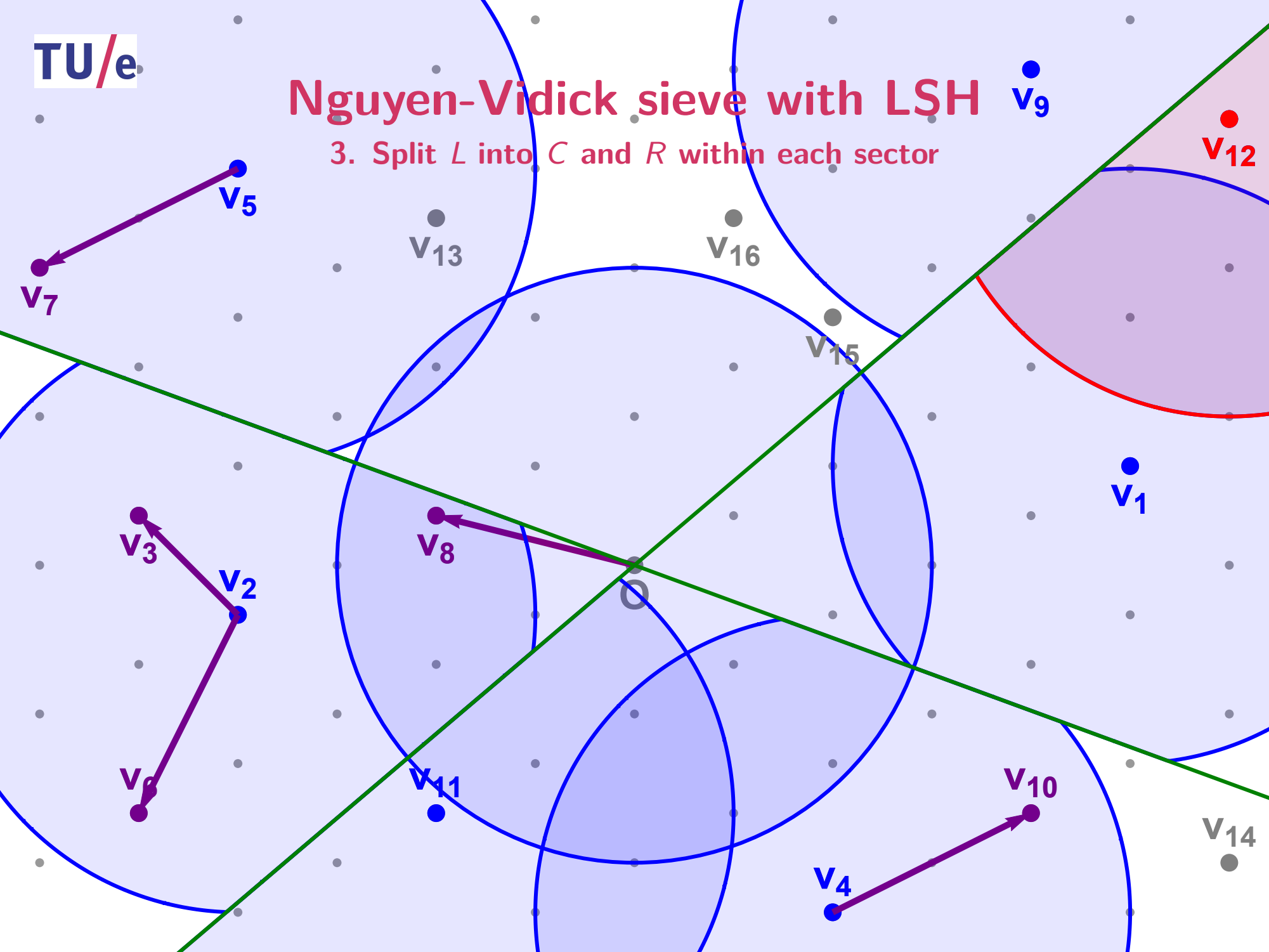
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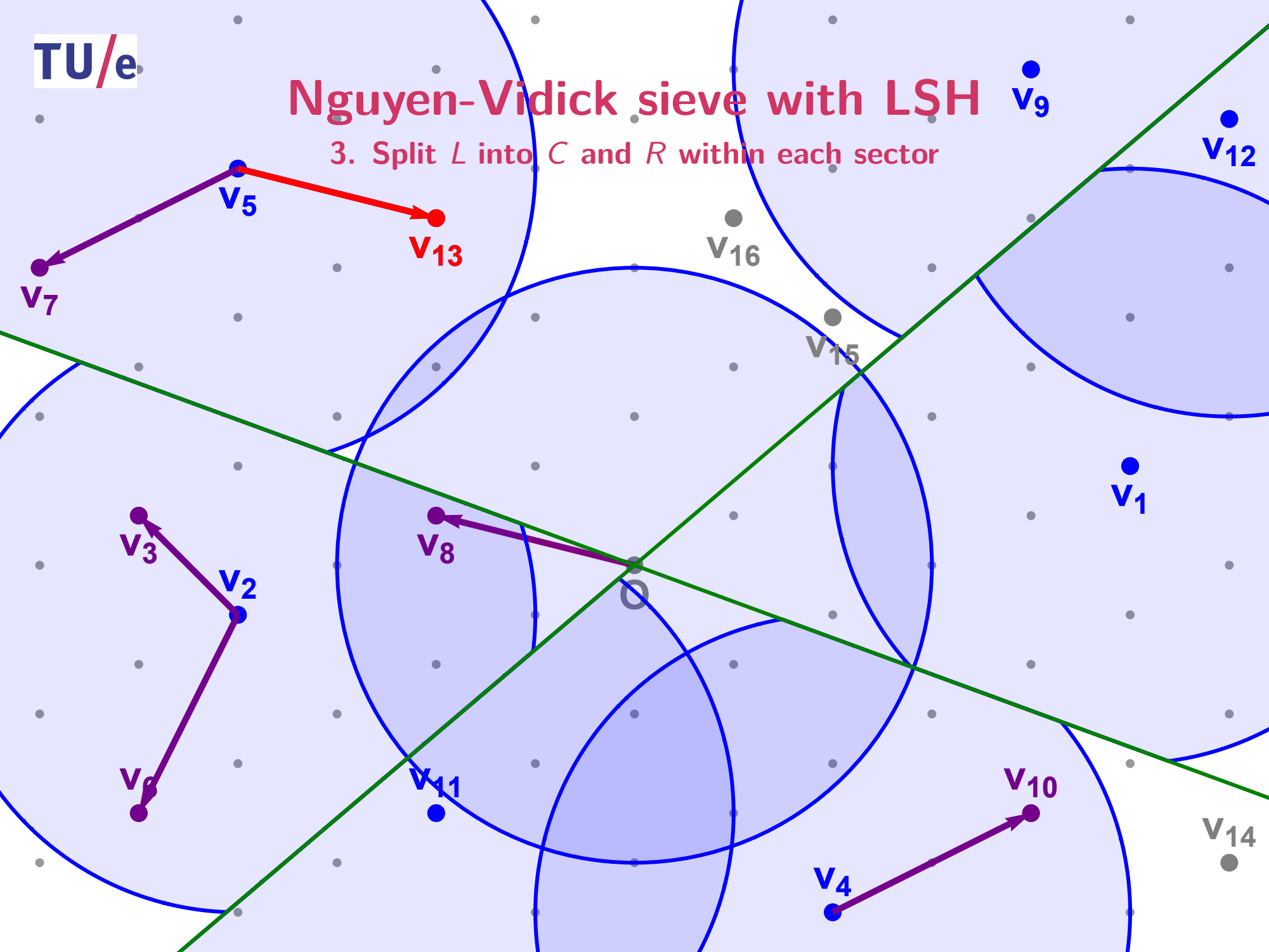
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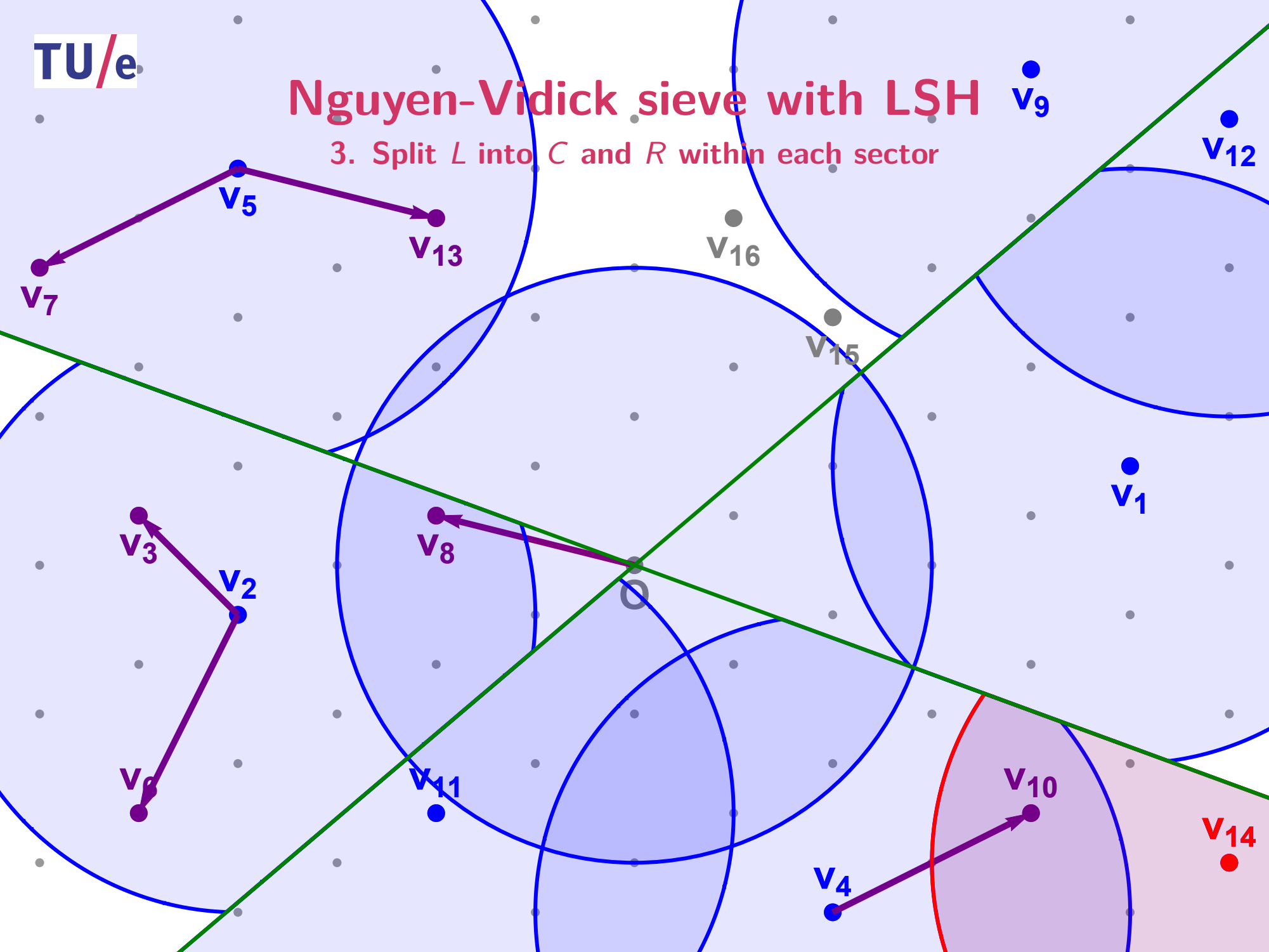
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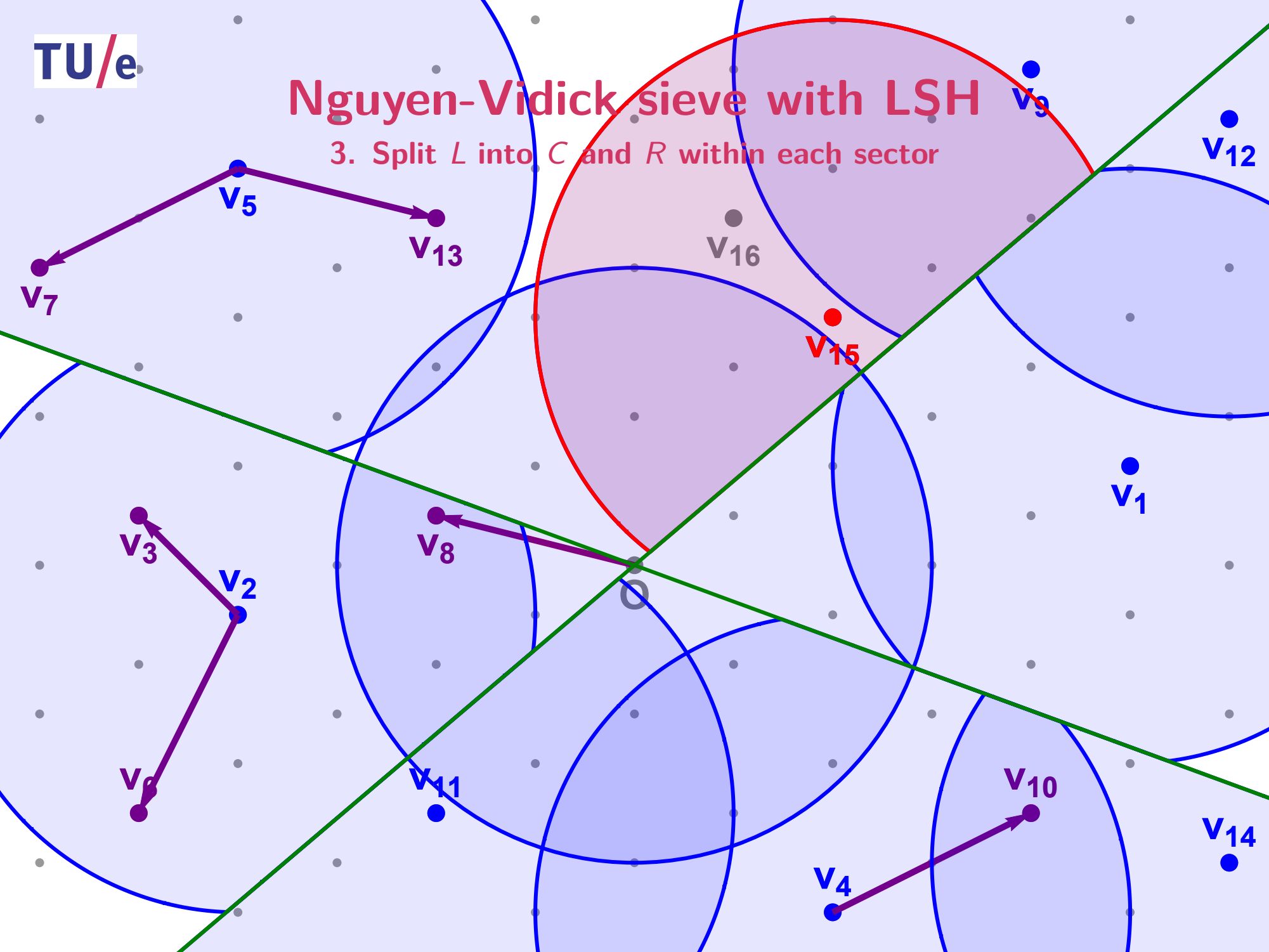
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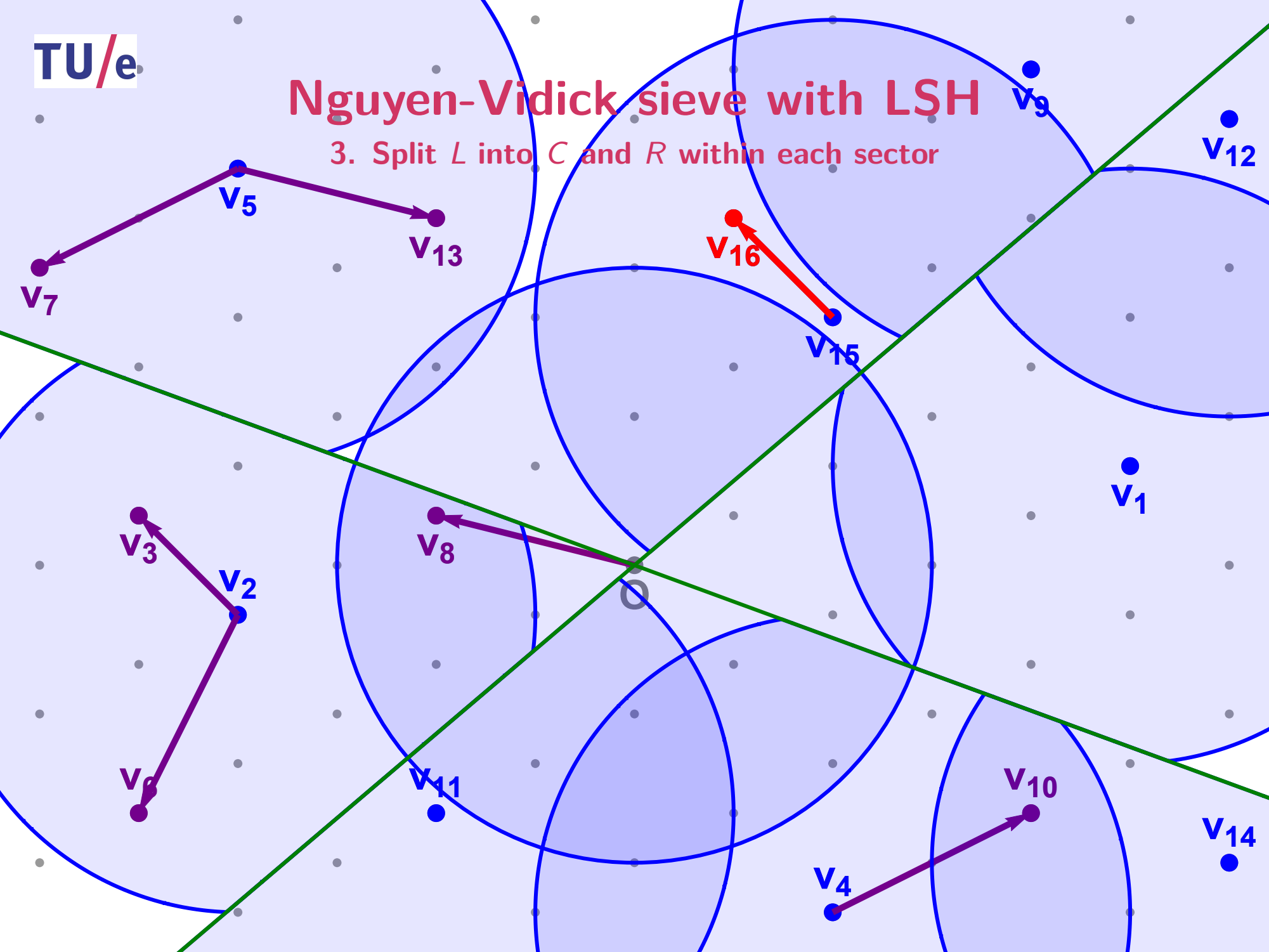
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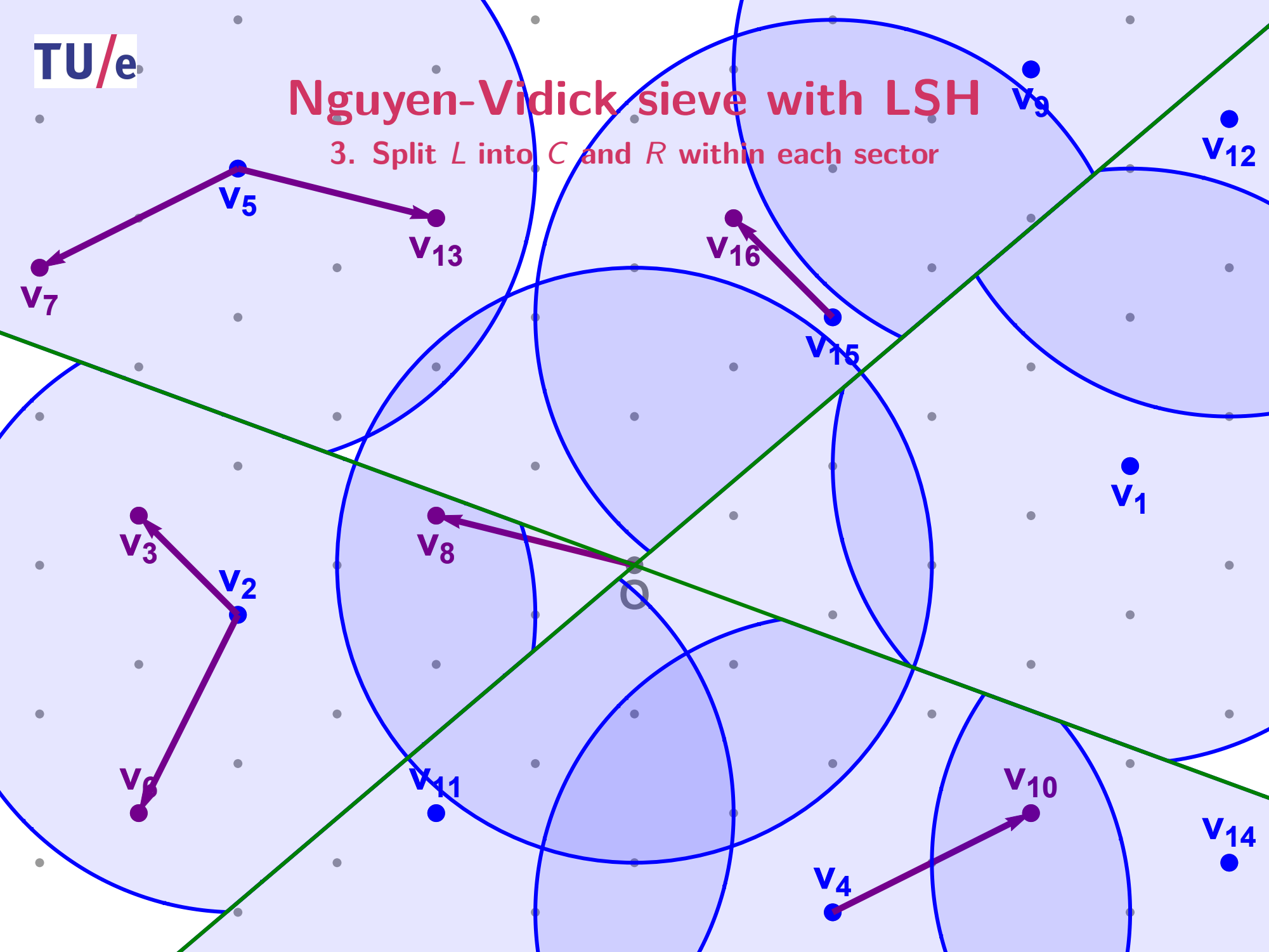

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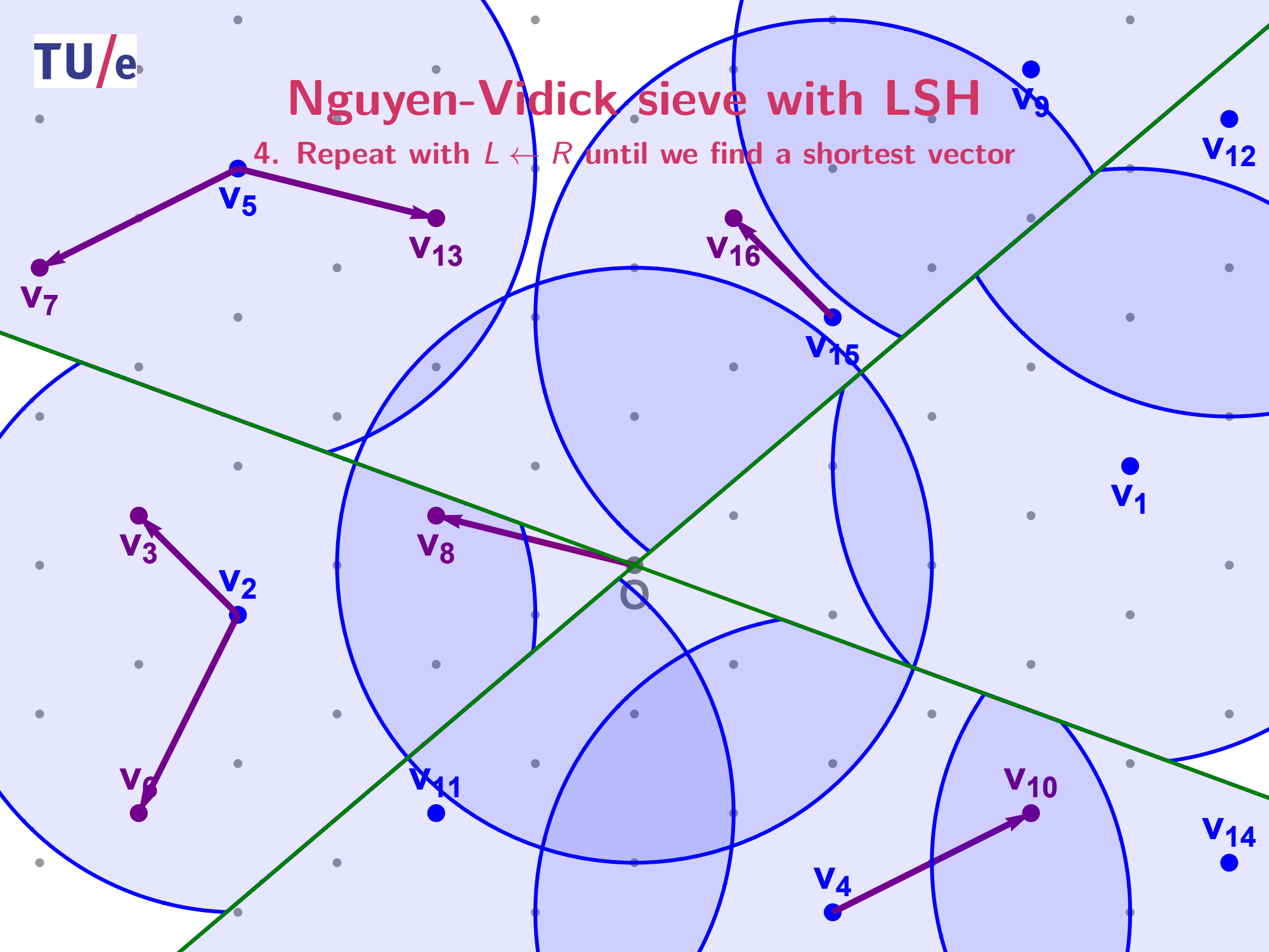

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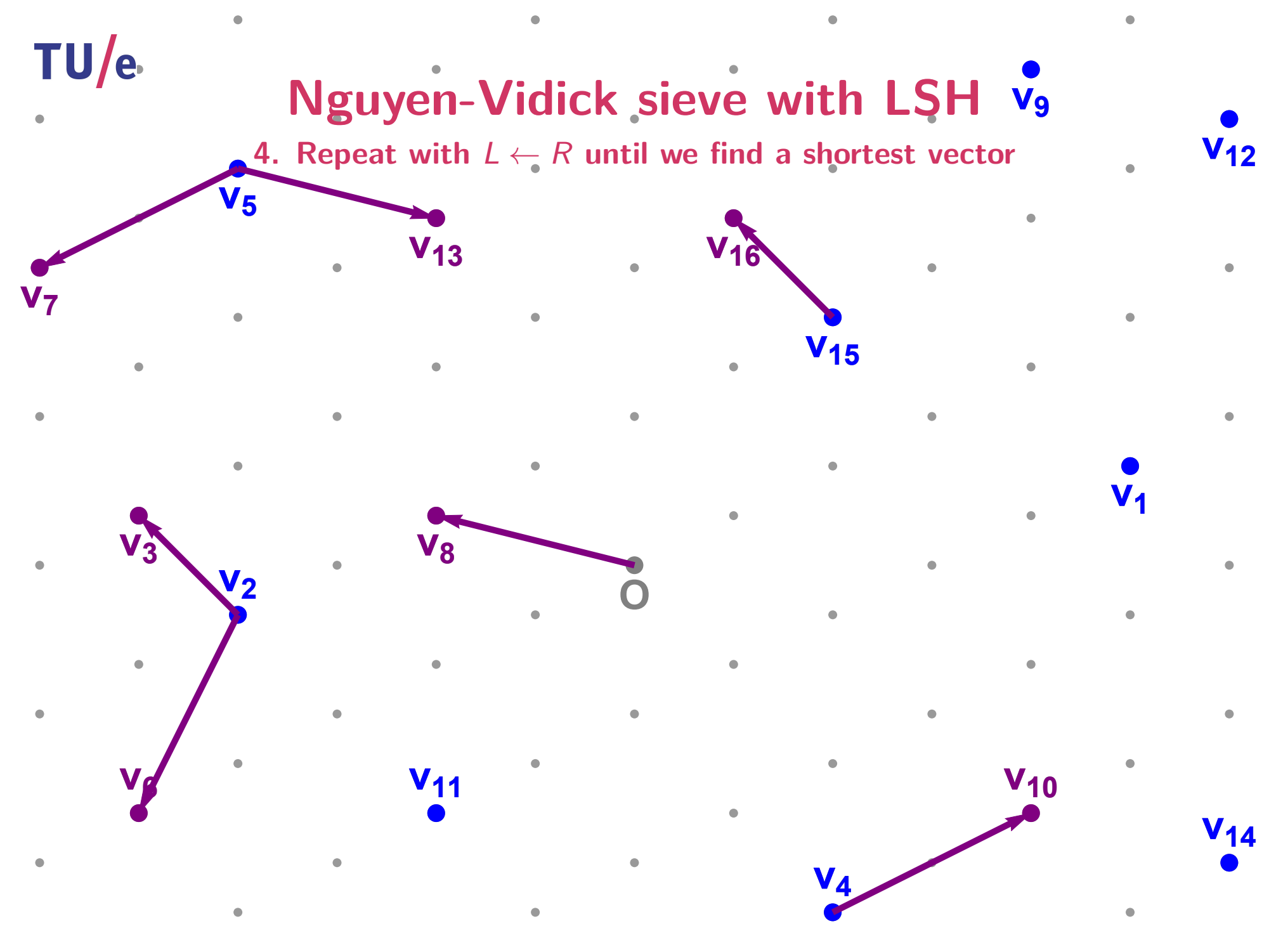

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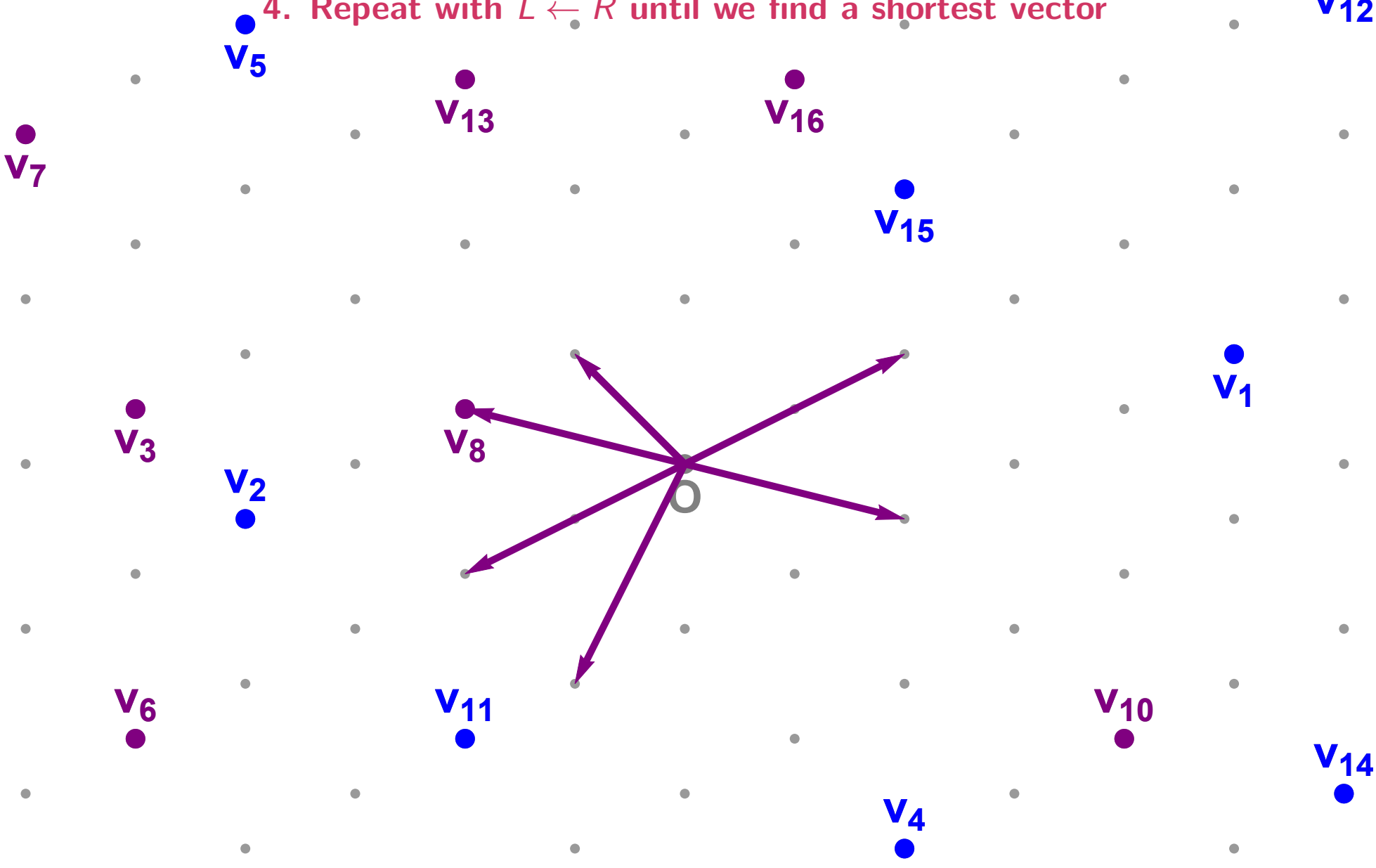

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Nguyen-Vidick sieve with LSH

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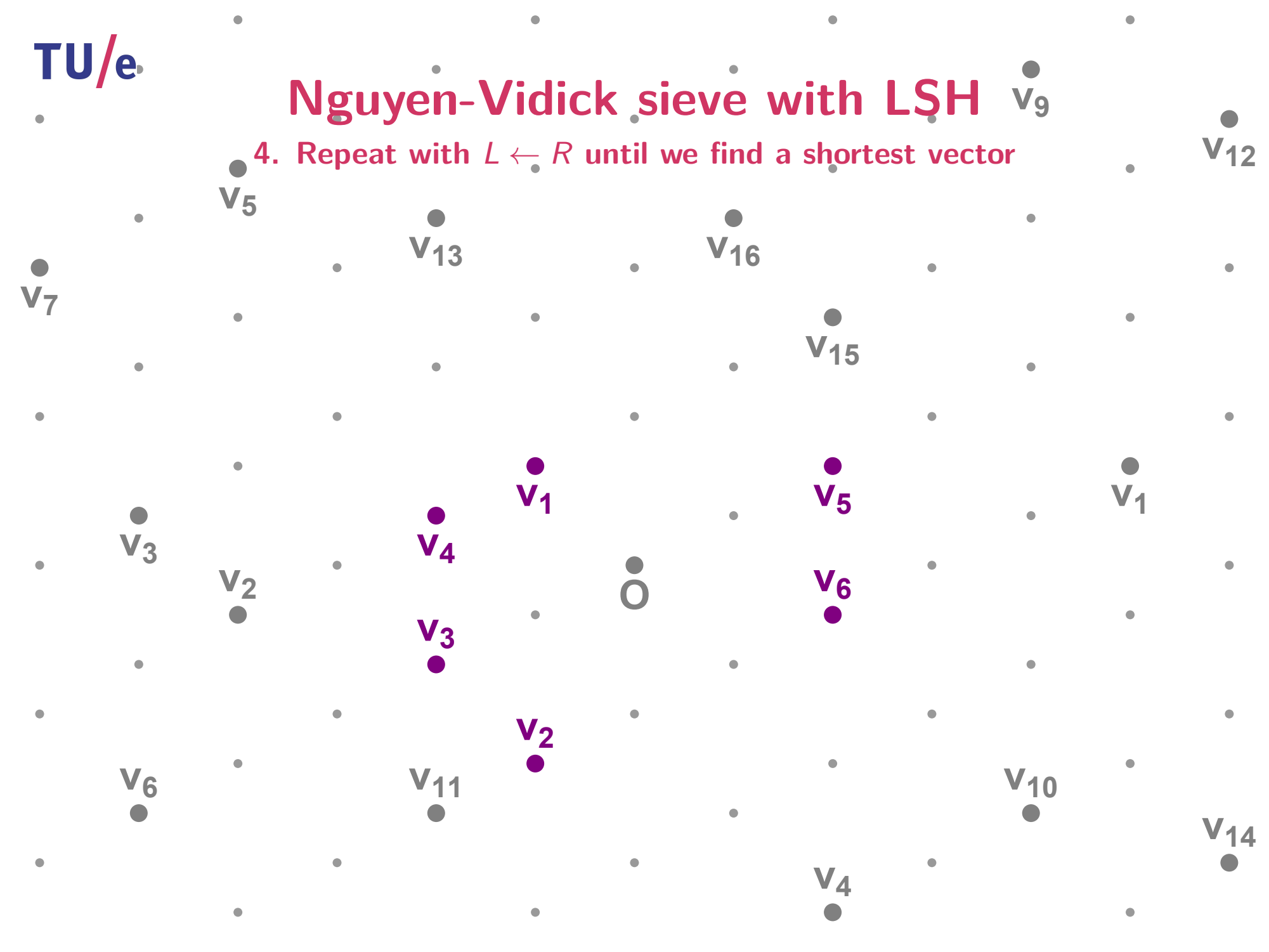

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Nguyen-Vidick sieve with LSH

Overview

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Nguyen-Vidick sieve with LSH

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- Two parameters to tune
  - $k = O(n)$: Number of hyperplanes, leading to 2^k regions
  - $t = 2^{O(n)}$: Number of different, independent "hash tables"

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Nguyen-Vidick sieve with LSH

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Heuristic

Sieving with LSH runs in time $2^{0.34n+o(n)}$ and space $2^{0.34n+o(n)}$.

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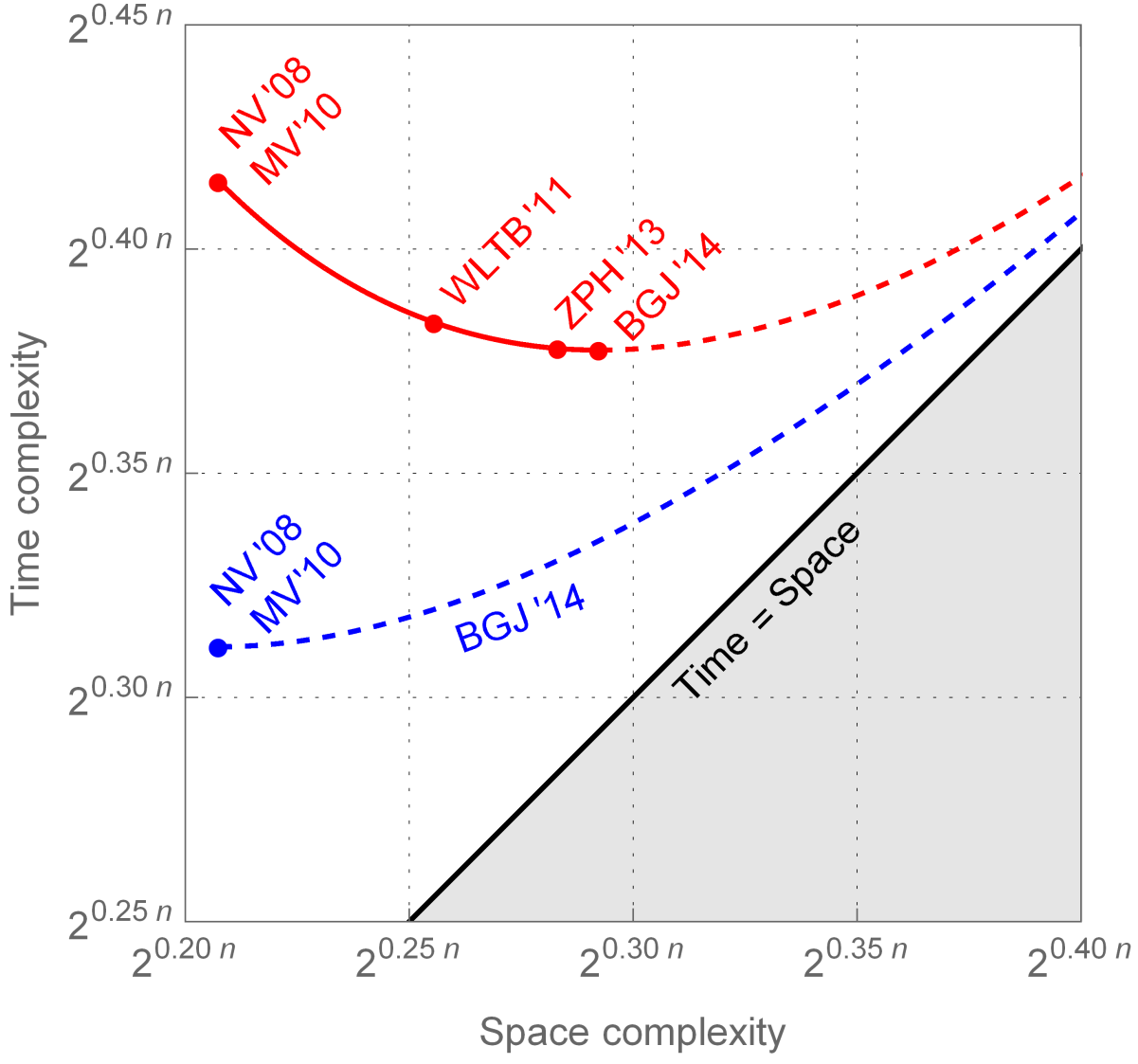
Sieving

Space/time trade-off



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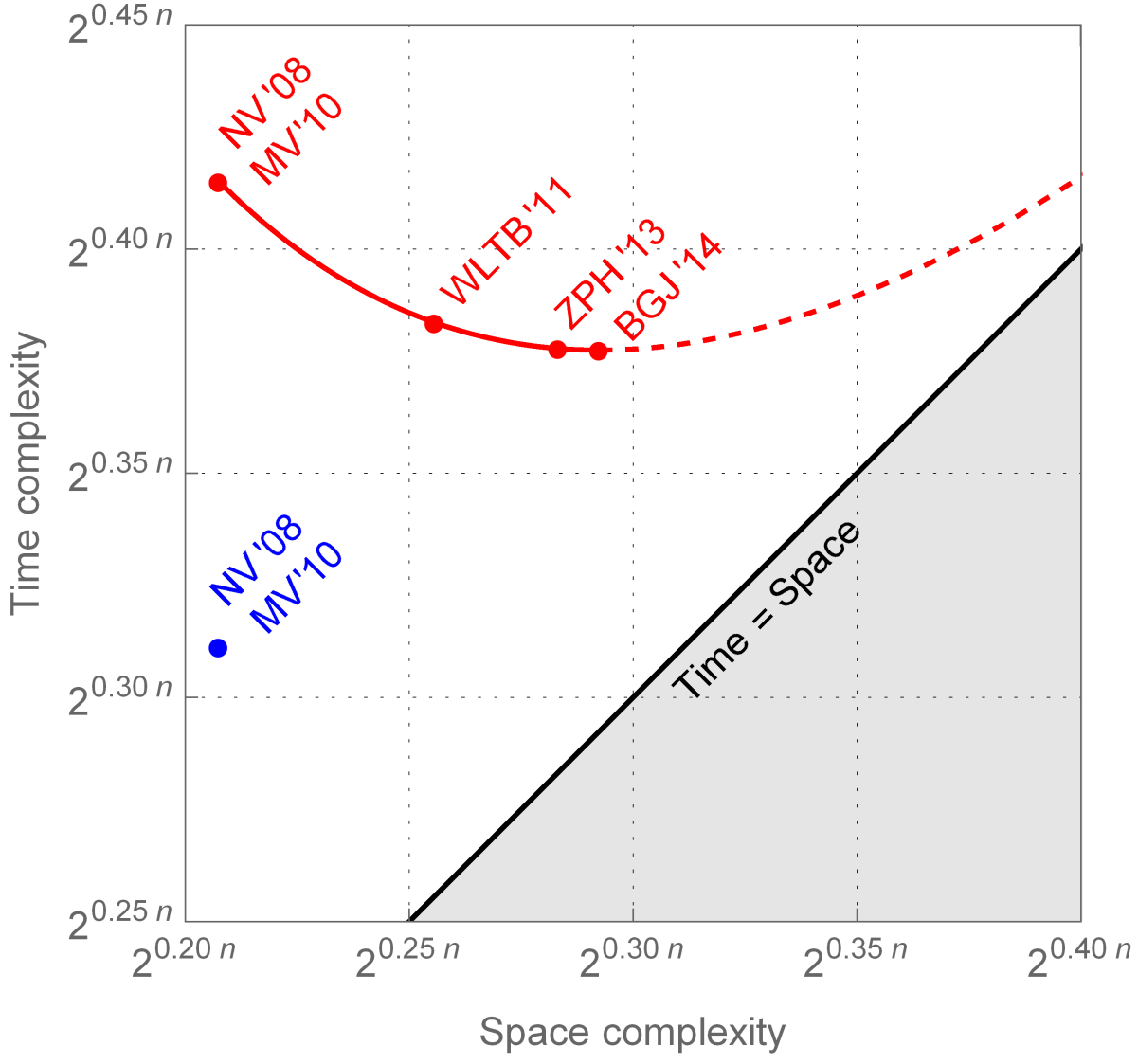
Sieving

Space/time trade-off



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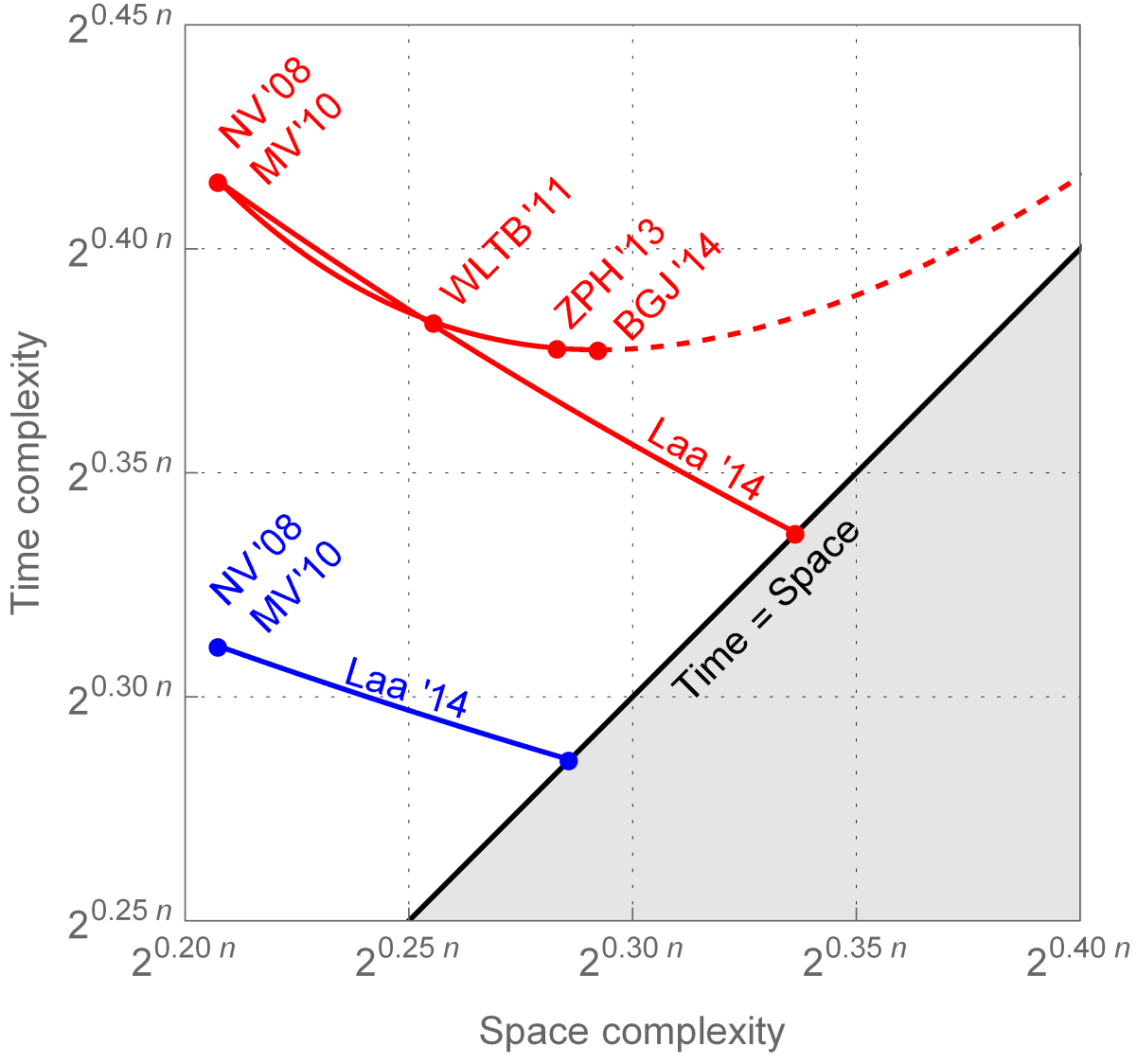
Sieving with LSH

Space/time trade-off



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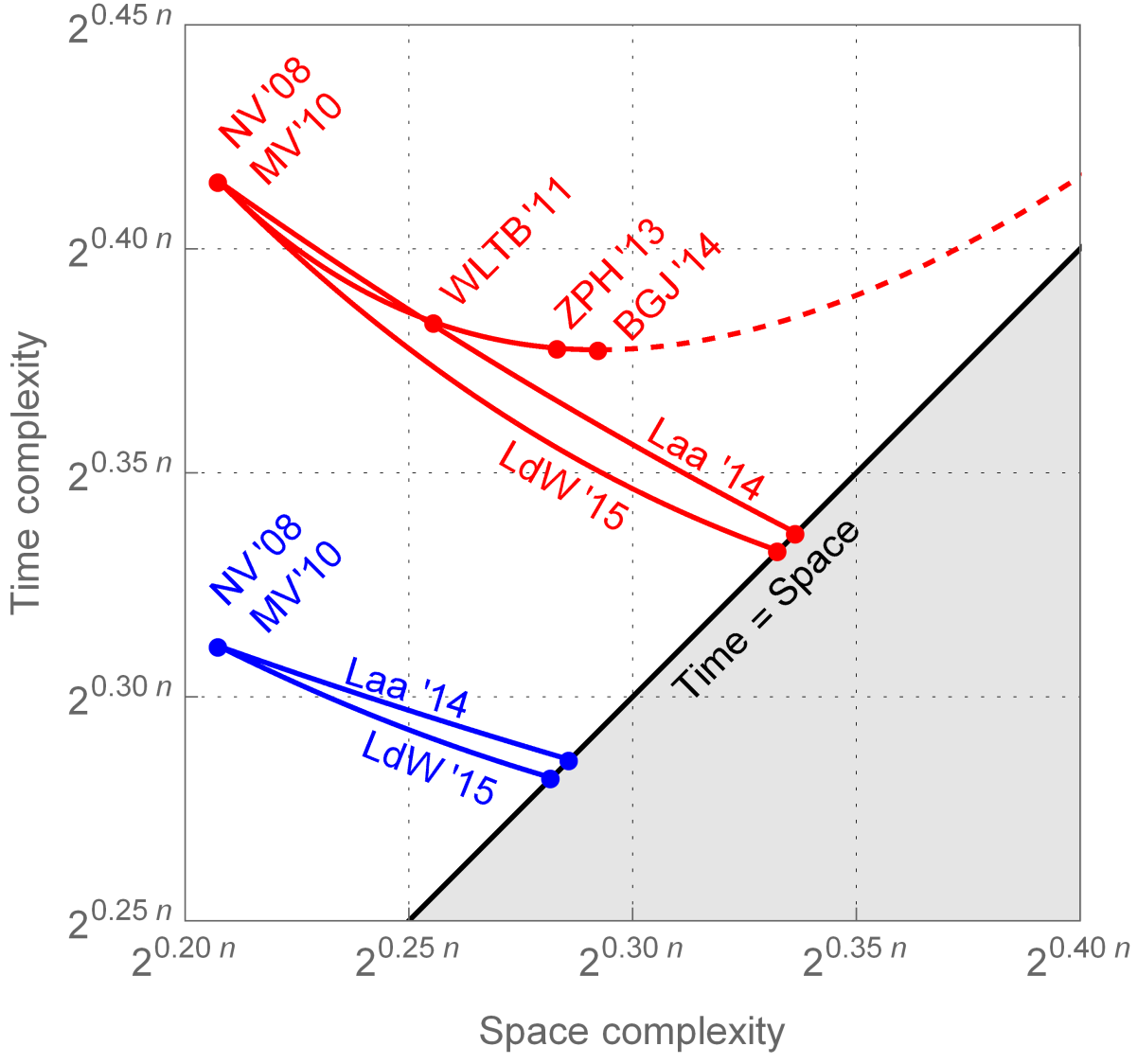
Sieving with LSH

Space/time trade-off



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Sieving with LSH

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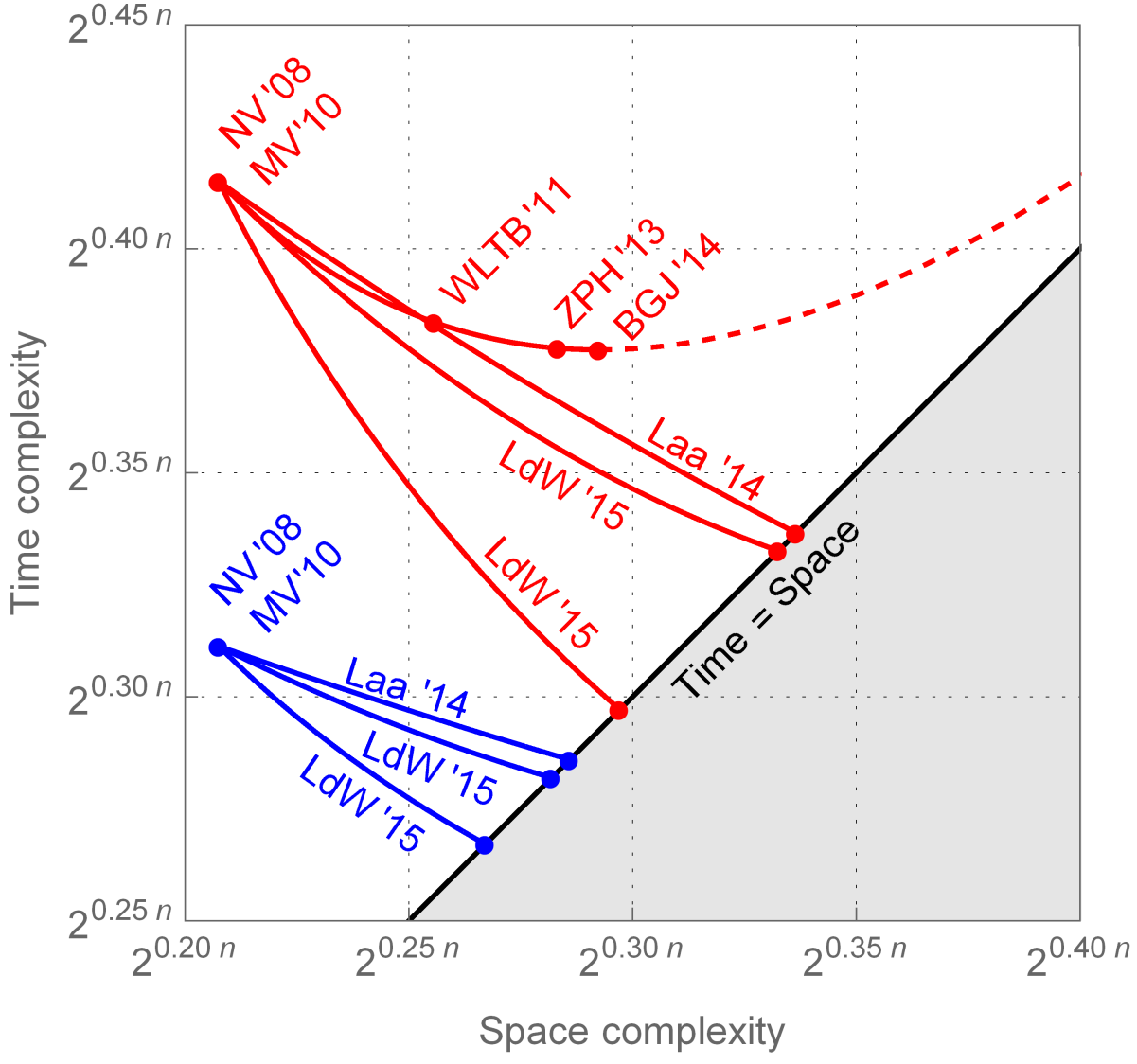

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Questions