Efficient **Parallel** approximation algorithms

lessons from facility location

Kanat Tangwongsan
Carnegie Mellon University

One thing I learned from greedy facility location....

which you can apply to set cover, max cover, min-sum set cover, etc.

(disclaimer: presented mostly without proof and might contain shameless plugs for upcoming talks)

Design Guidelines & Model

How to get the most out of our cores?

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lots of parallelism

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good locality

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depth - longest chain of dependencies

“small”

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work - total operation count

same as the sequential counterpart

Metric Facility Location

Free Food Project at Carnegie Mellon

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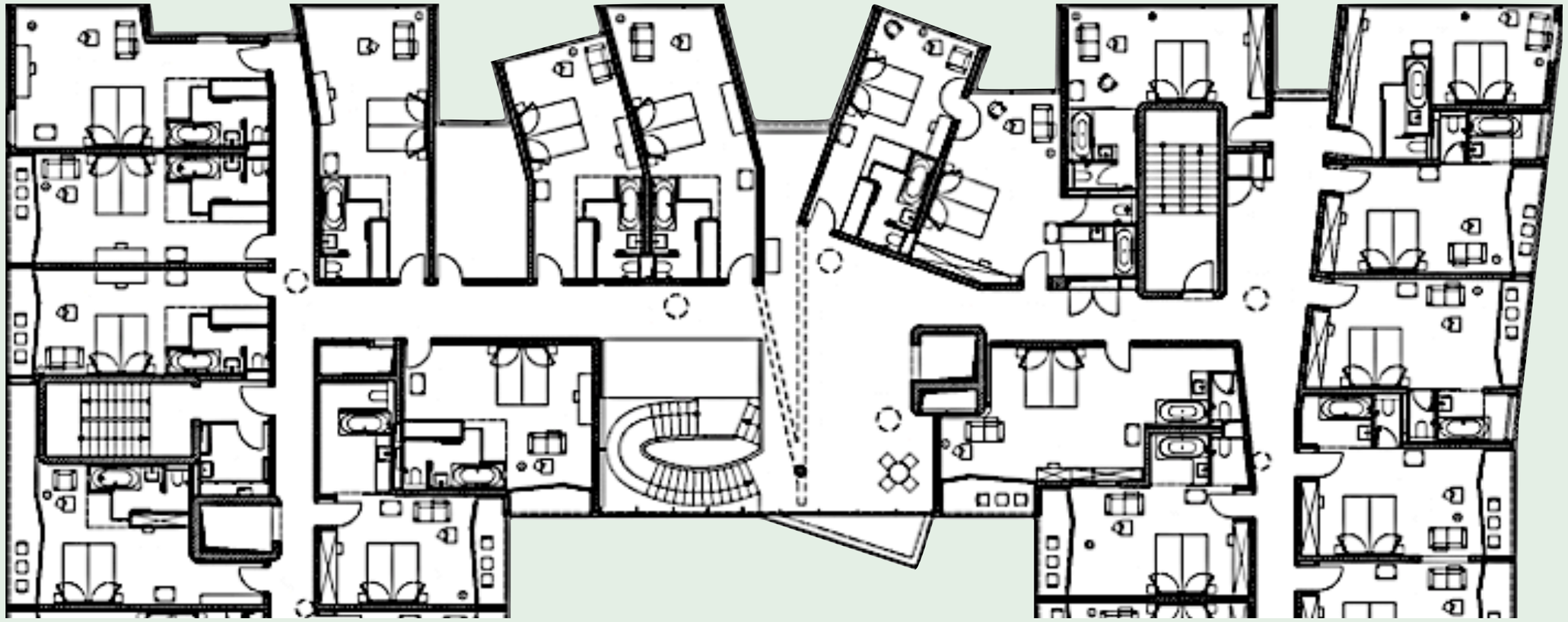

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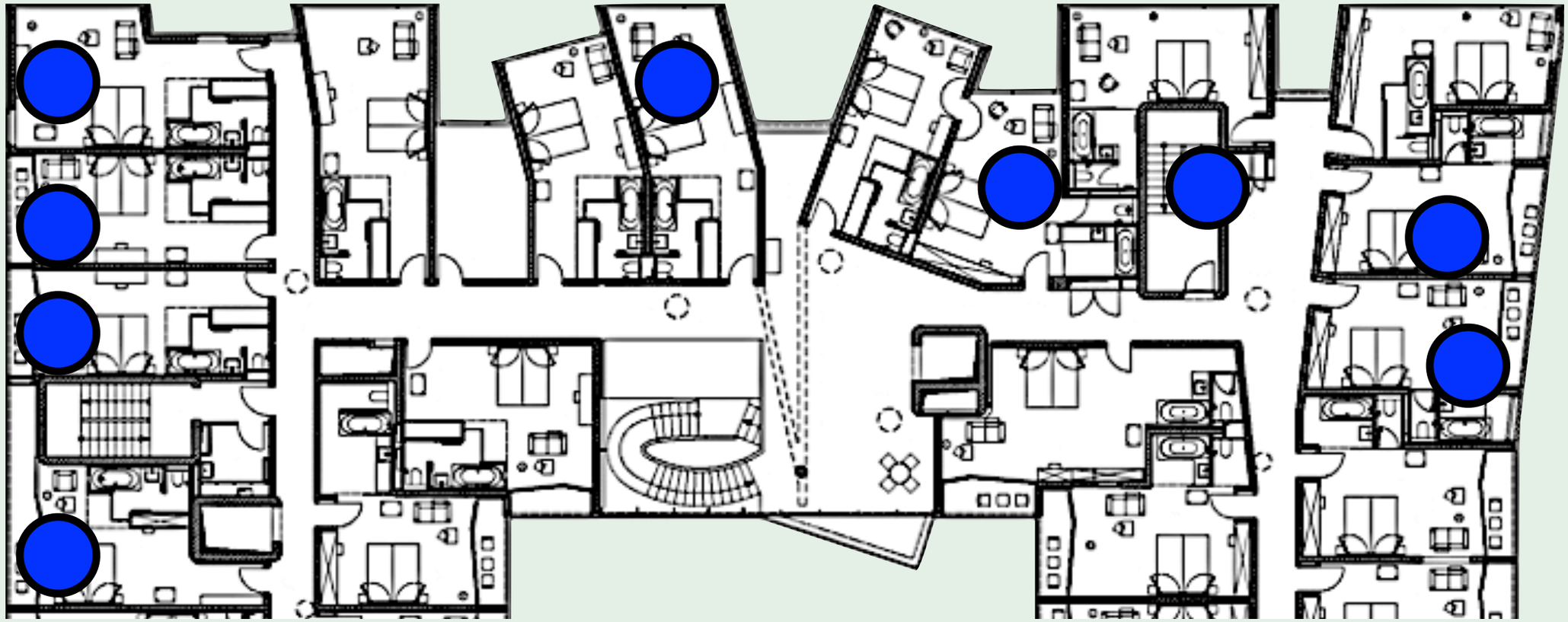

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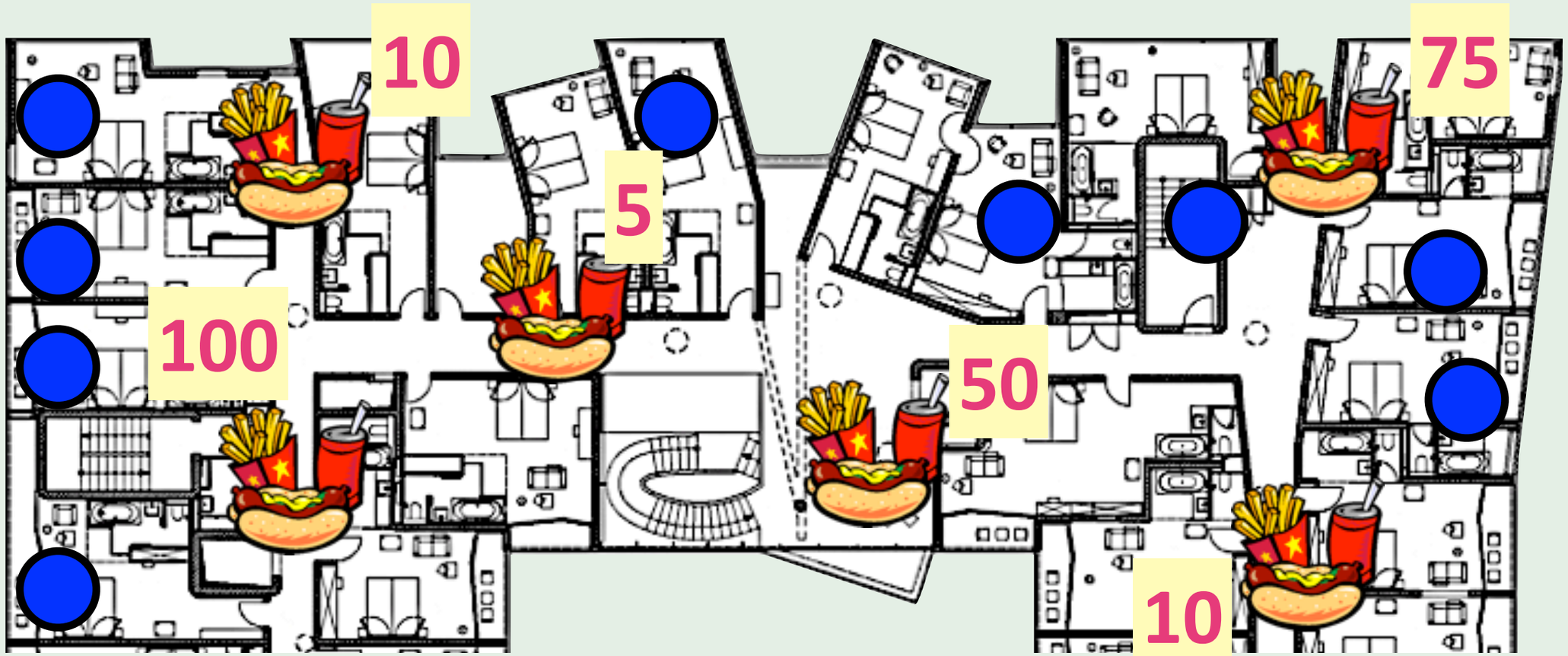

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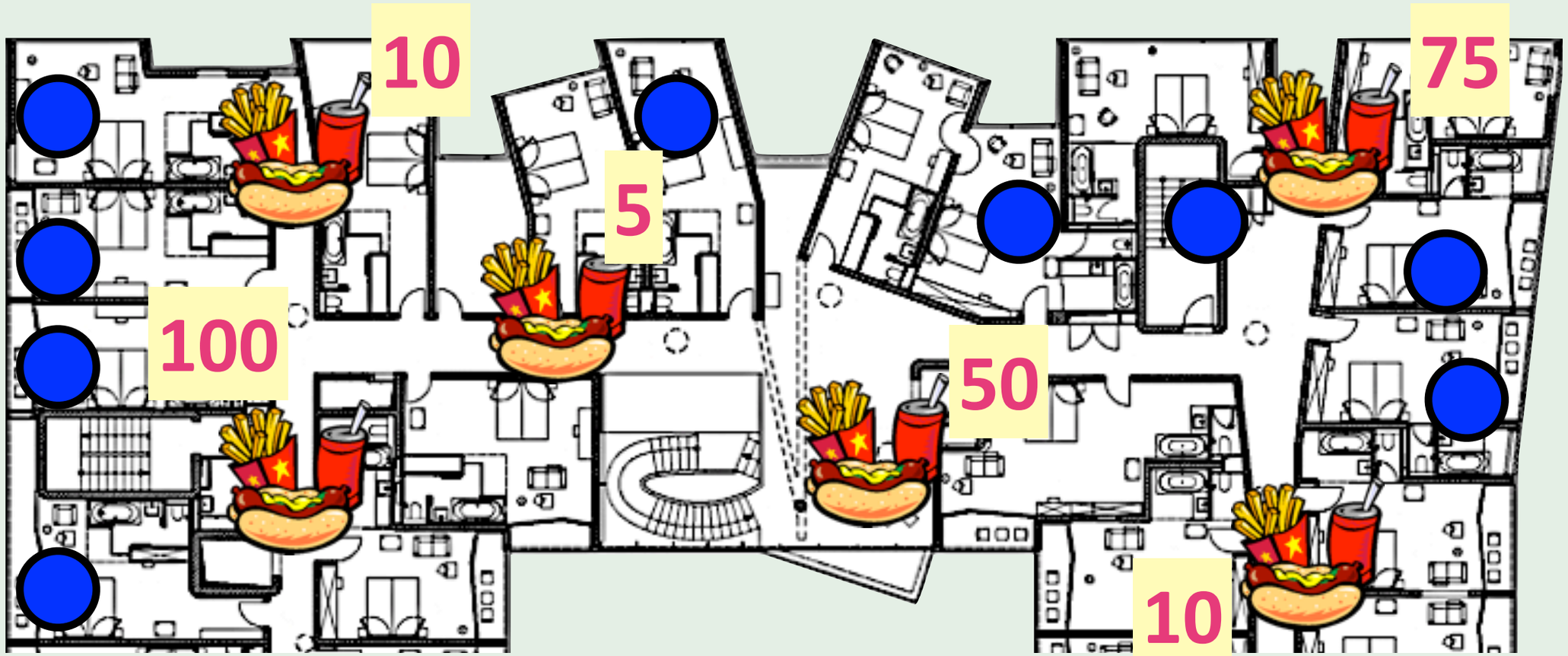

Q: Which food stations to open to minimize the cost?

Cost = Facility Cost + Connection Cost

$$\sum_{i:\text{opened}} f_i$$

$$\sum_{j \in C} d(j, \text{closest opened})$$

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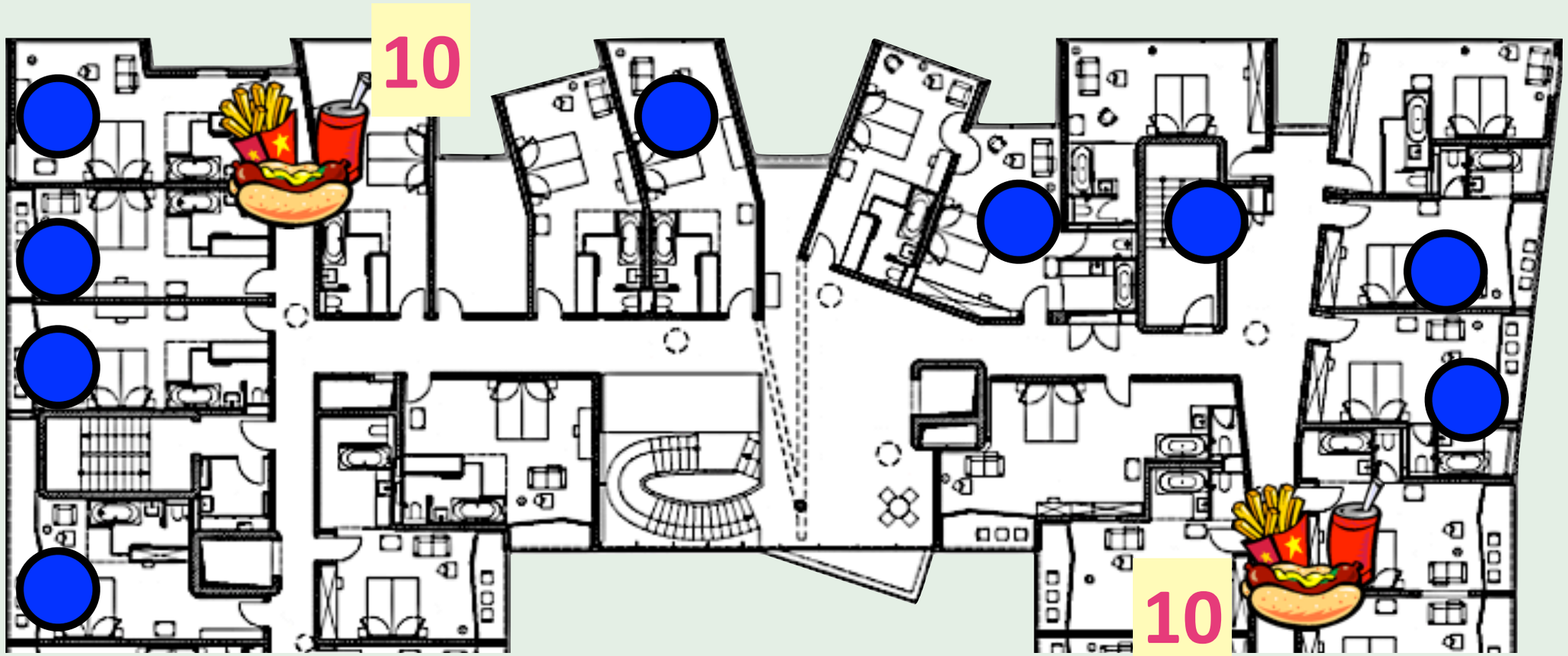

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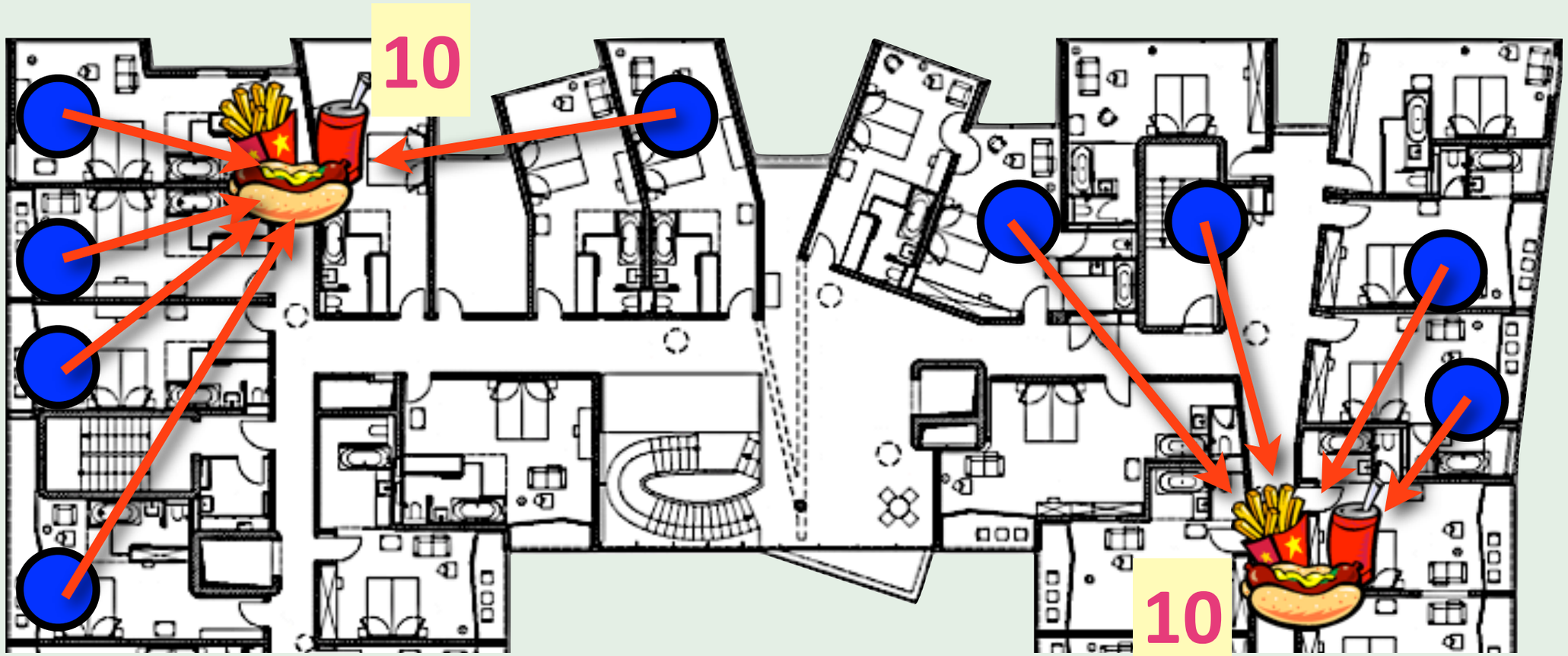

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Formal Definition

Input: facilities $\boldsymbol{F}$; cost f_i

clients $\boldsymbol{C}$

connection cost $d(i, j)$

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obeys triangle inequality

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Facility Cost + **Connection Cost**

Metric Facility Location

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Input: facilities ***F***; cost f_i

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connection cost $d(i, j)$

Applications:

clustering,
network design,
"testbed" for techniques, ...

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Metric Facility Location

What was known about this problem?

- **Hardness**

  NP-hard and factor-1.463 is hard [Guha and Khuller '99]

- **Several Constant Approximation Algorithms**

factor-4	linear program (LP) rounding	[Shmoys et al.'97]
factor-3	primal dual	[Jain and Vazirani'01]
factor-1.861	greedy	[Jain et al.'03]
	...	
factor-1.5	LP + scaling + ...	[Byrka'07]

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Theorem:

RNC O(*m* log *m*)-work, factor-(1.861+ε) *greedy-style* approximation algorithm.

  factor-1.5 LP + scaling + … [Byrka'07]

Greedy Facility Location

Jain et. al's Sequential Algorithm

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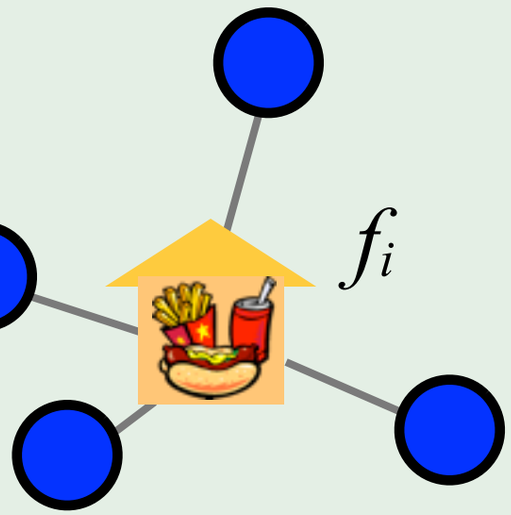

facility clients

$$\mathcal{S} = (i, T \subseteq C)$$

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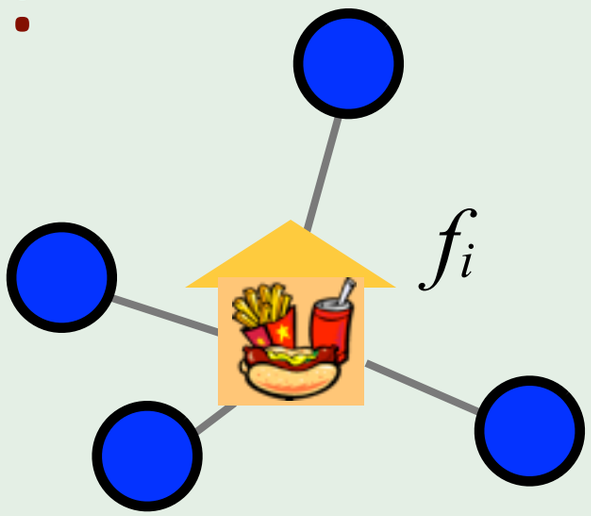

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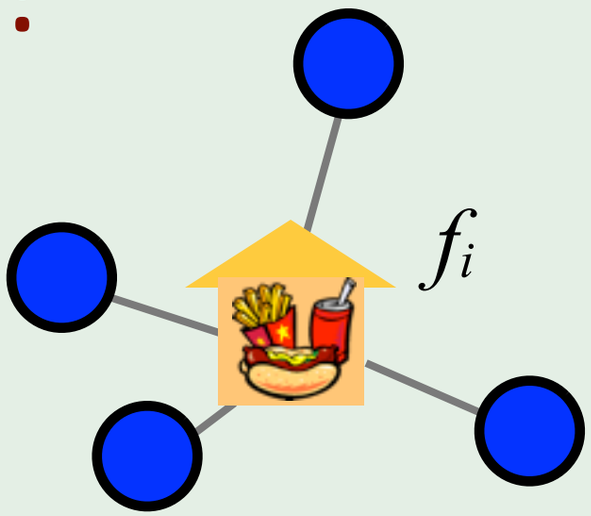

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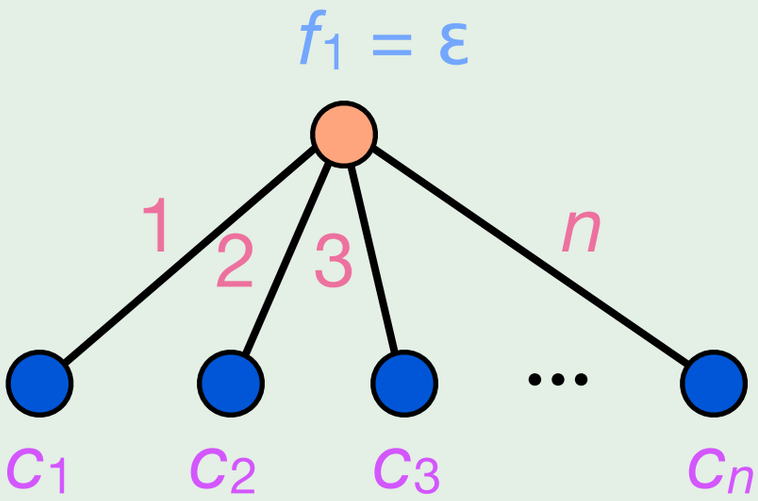

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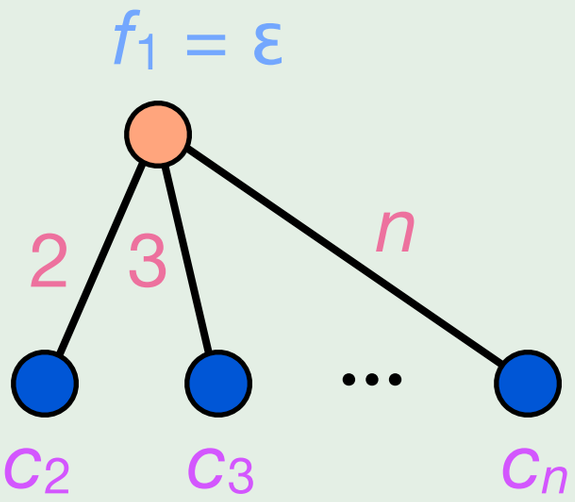

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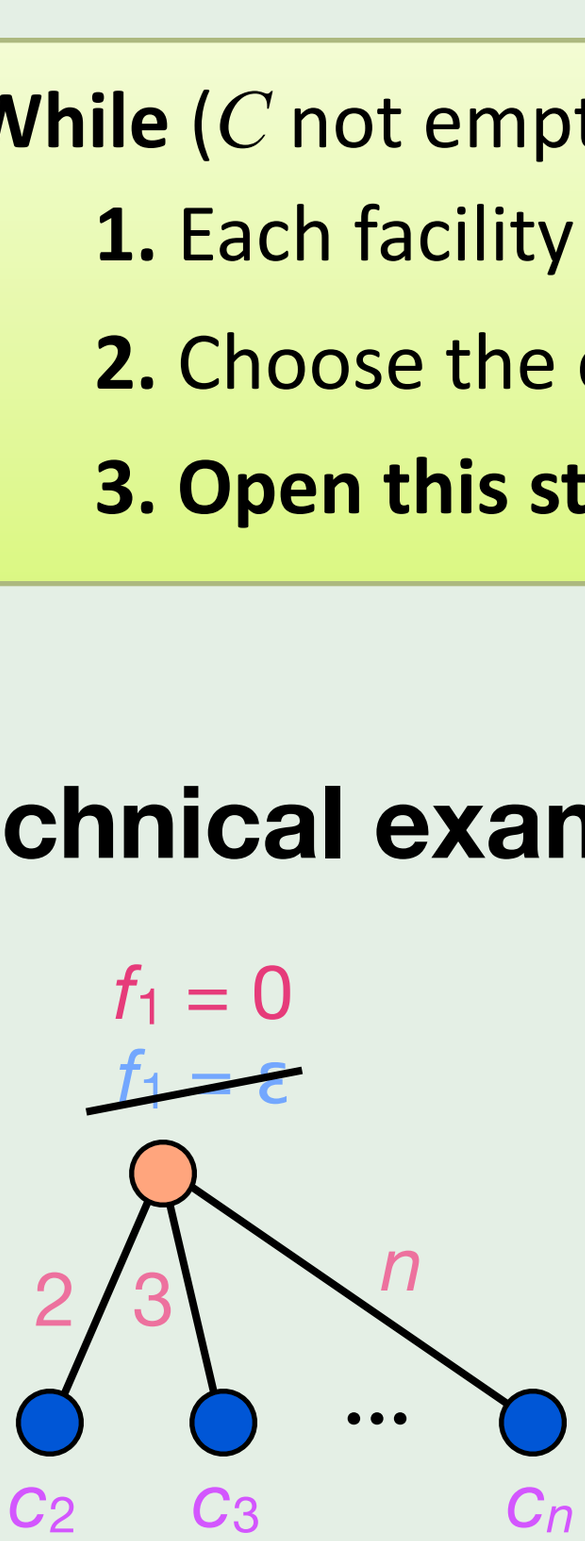

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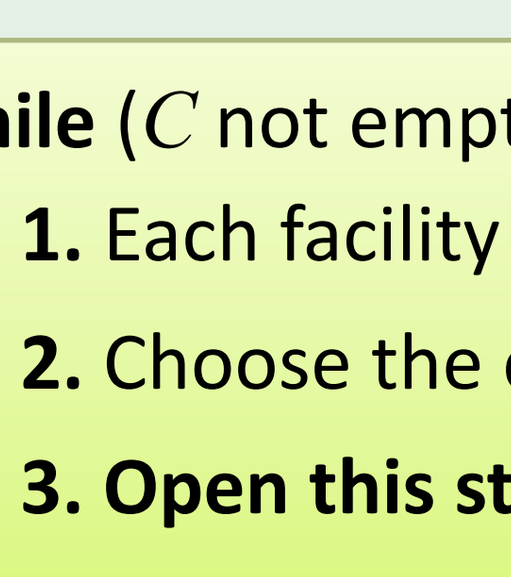

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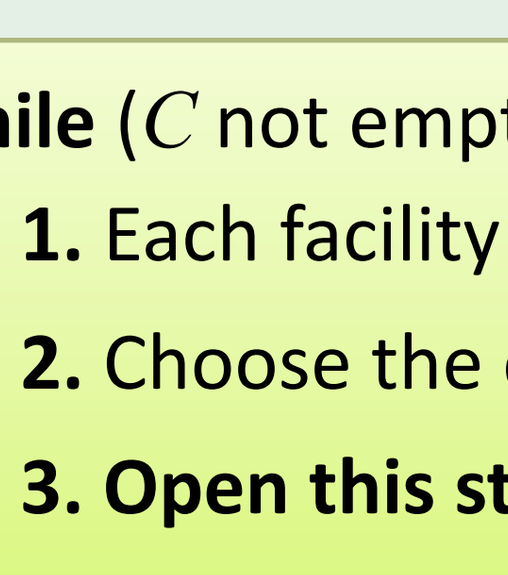

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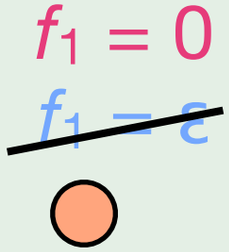

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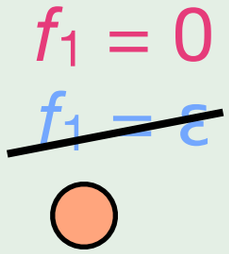

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Round 1:  price = 1.5

Round 2:  price = 2

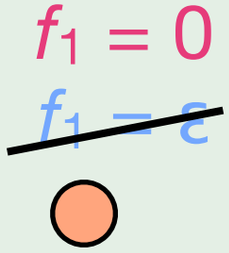
...

Round *n*:  price = n

Observations:

1. **Greedy process** can be ***inherently*** **sequential**
2. **Clients between stars don't overlap**

a technical example:



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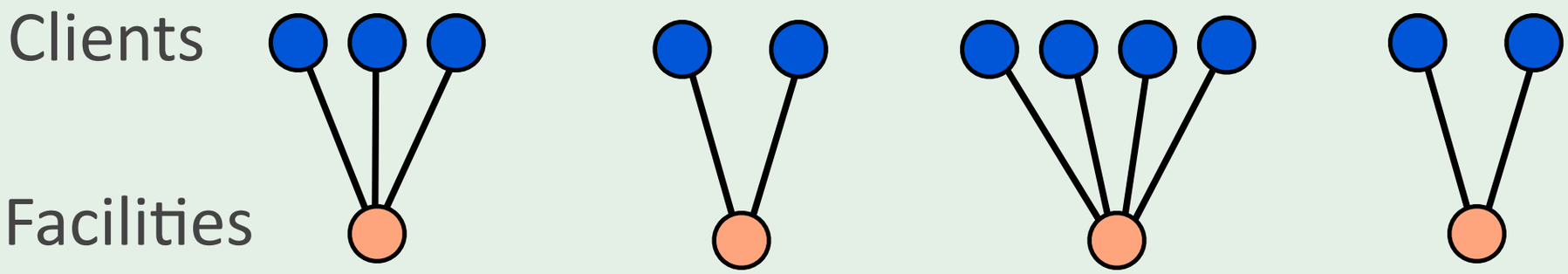

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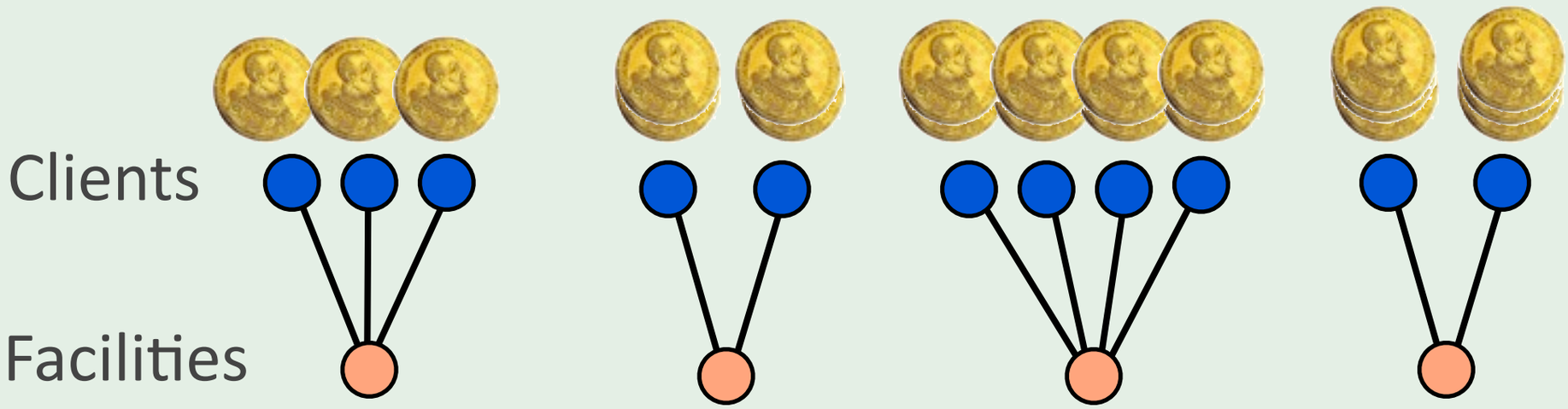

For each star *S*, put price(*S*) tokens on each client

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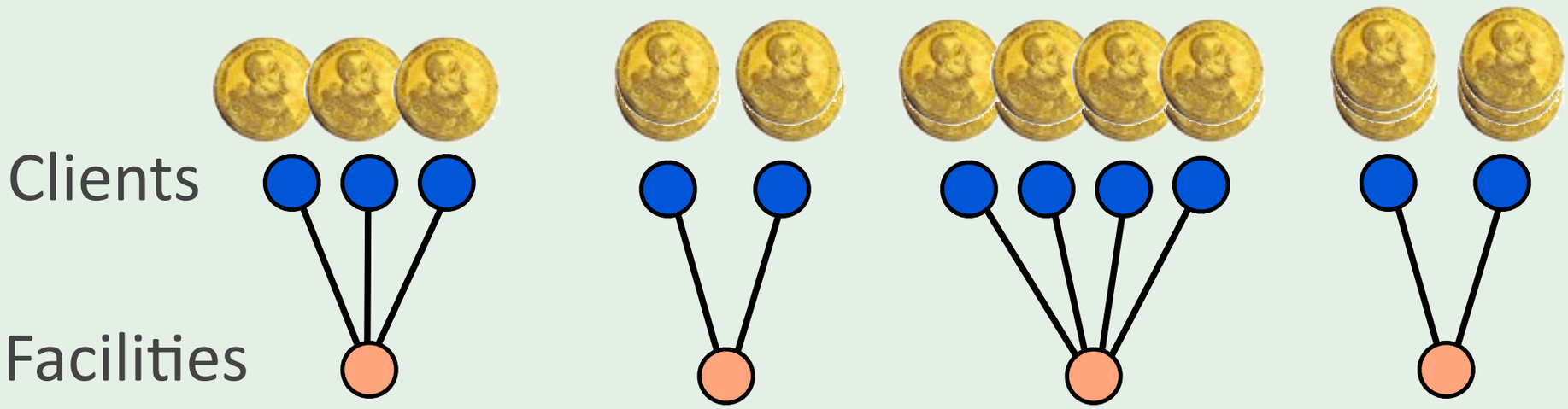

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The stars have no overlapping clients

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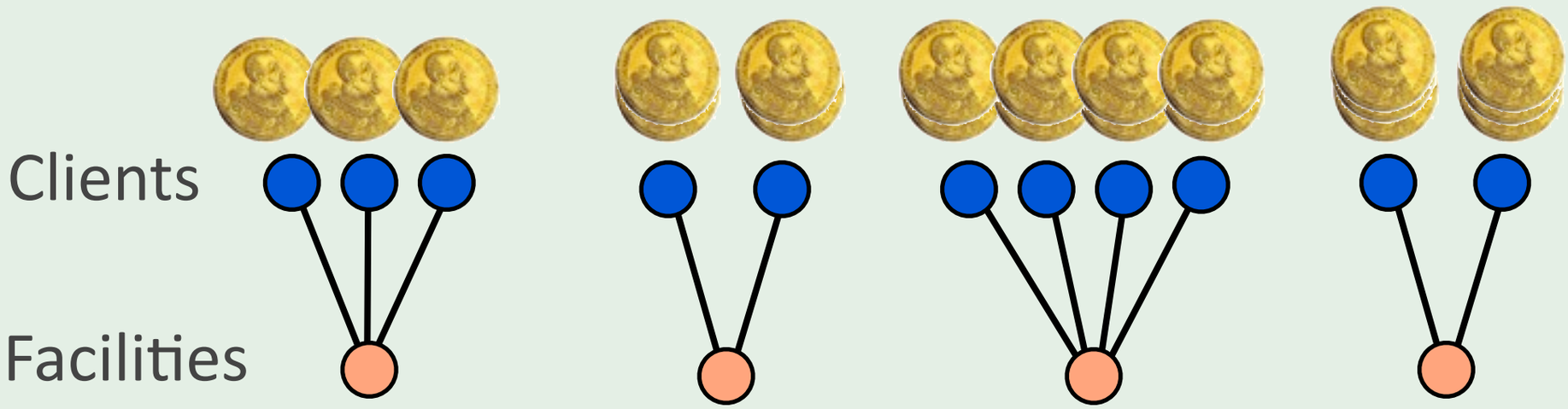

For each star *S*, put price(*S*) tokens on each client

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Lemma 2: Total #tokens ≤ 1.861**OPT**

factor-revealing LP + dual fitting

[Jain et al. ’03]

How to parallelize something that looks inherently sequential?

[Blelloch-T'10, Blelloch-Peng-T'11]

Idea #1: Geometric Scaling

Create opportunities for parallelism

Greedy Algorithm

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Greedy Algorithm → **Greedier Algorithm**

idea previously used in set cover, vertex cover, ...

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3. GOOD = { star centered at i *if price* ≤ $p(1+\varepsilon)$ }, open them:

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Good news: ≈ $\log_{1+\varepsilon} m$ rounds; price goes up by $(1 + \varepsilon)$

Problem: Stars Overlap

Opening all "good" stars is too aggressive

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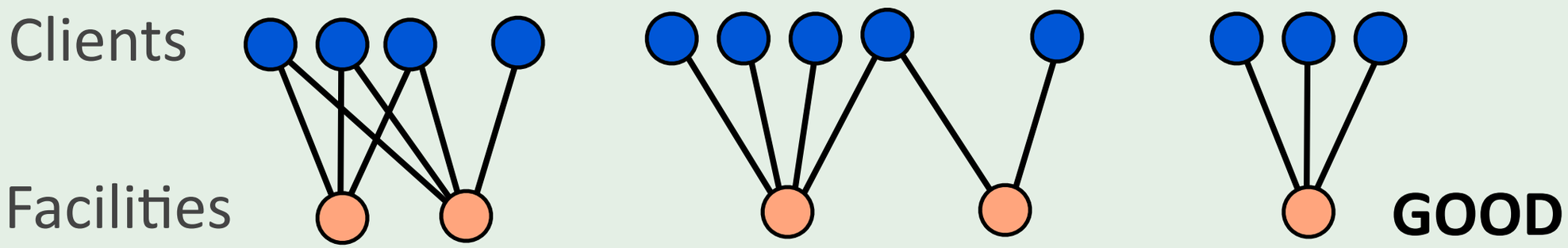
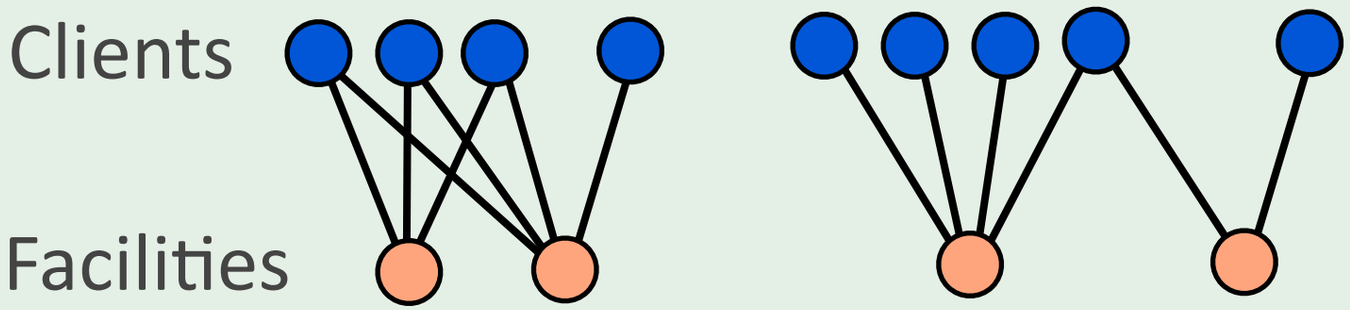


Idea #2: Subselect Facilities

Control how much overlap is allowed

In this round: cheapest star has price ***p***



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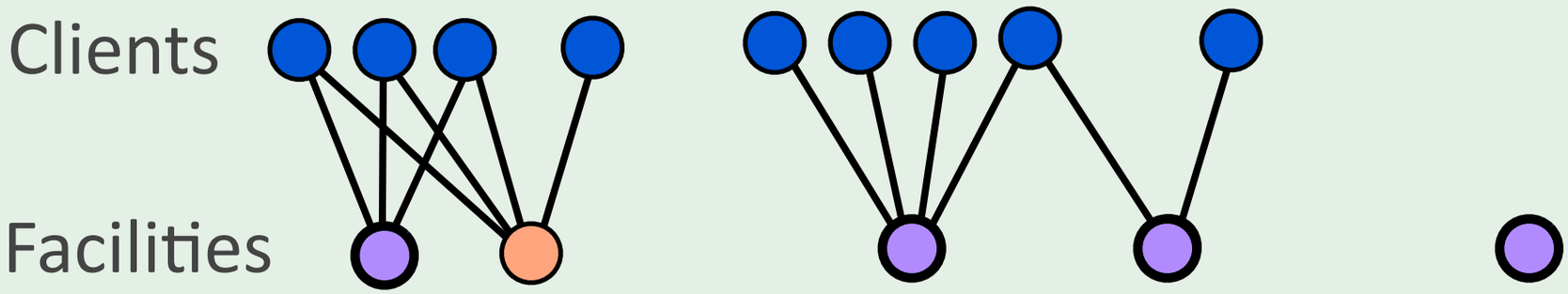

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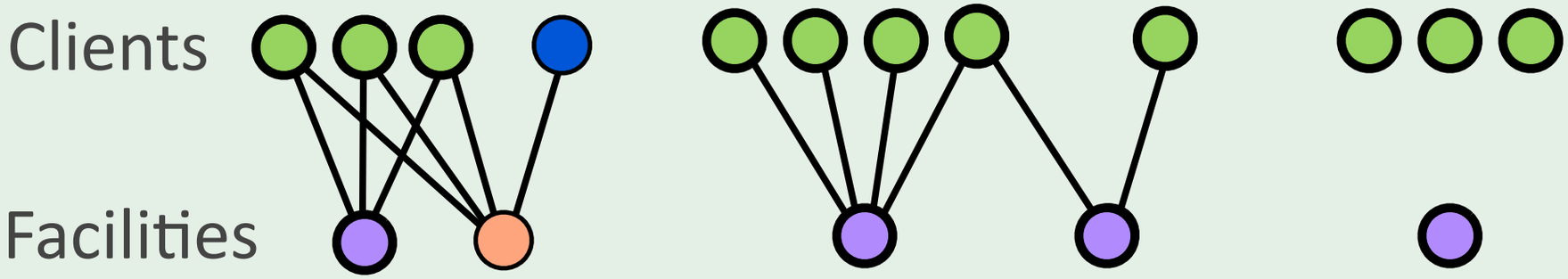

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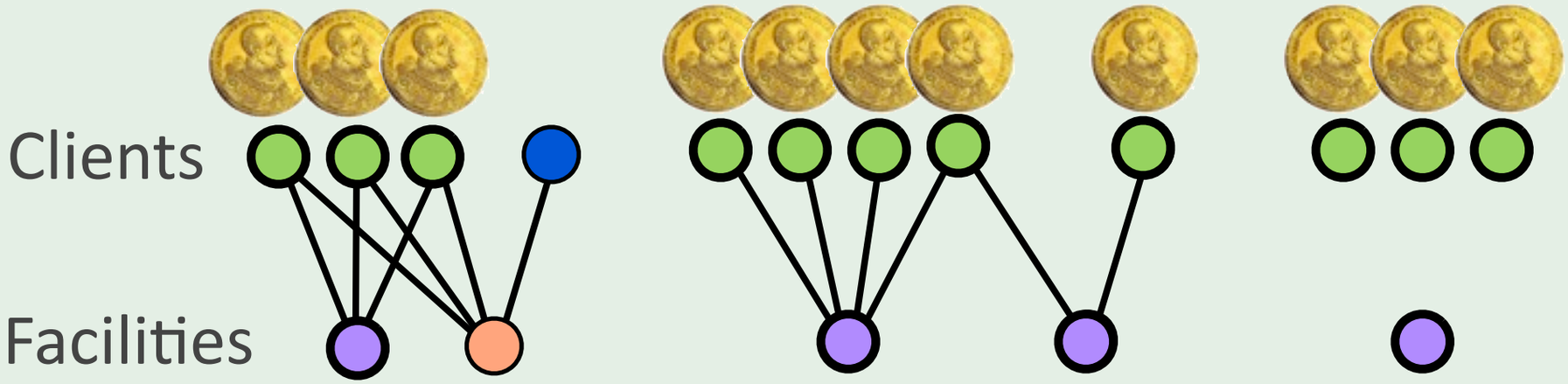

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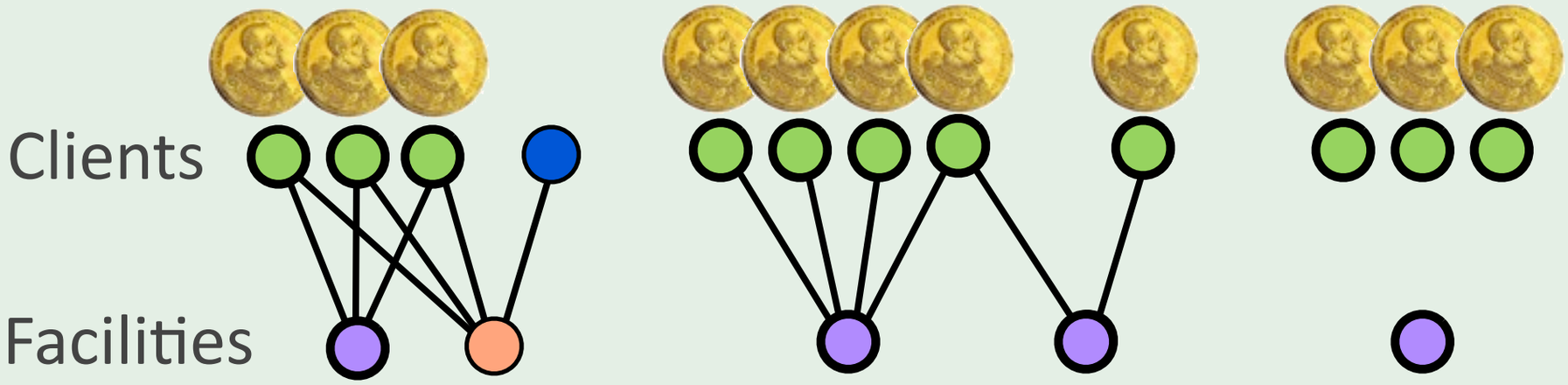

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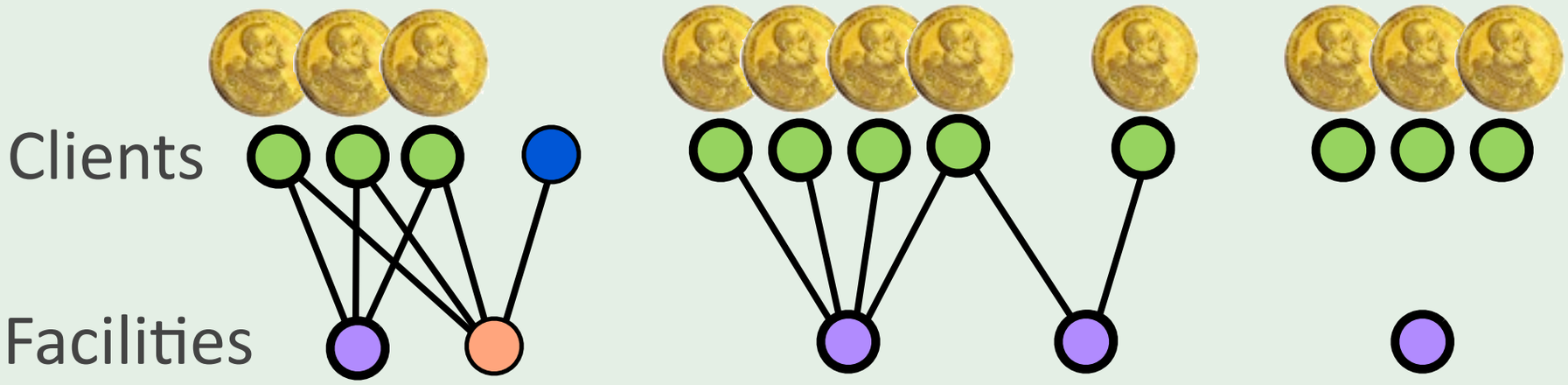

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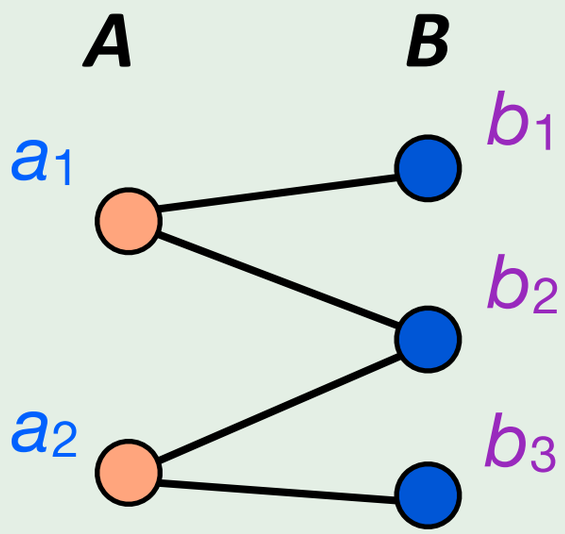
Maximal Nearly Independent Set

Formalizing small overlap and maximality

$N(X)$ = neighbors of X



bipartite graph modeling coverage

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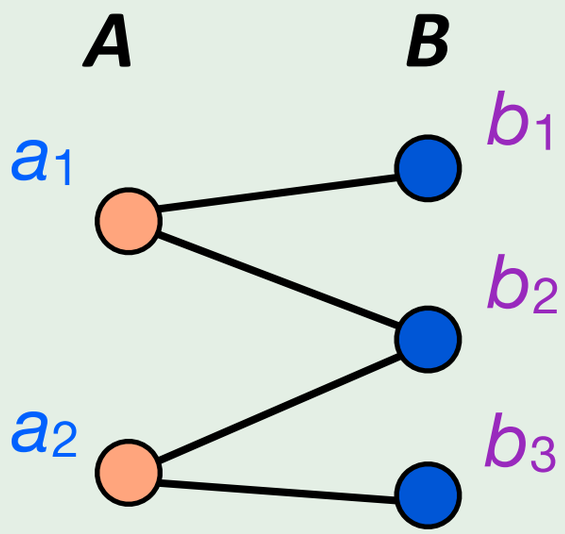

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(ε, δ)**-MaNIS** is $J \subseteq A$ such that

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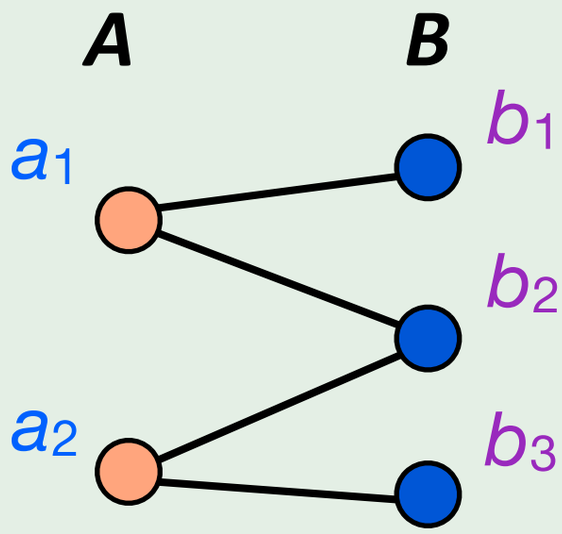

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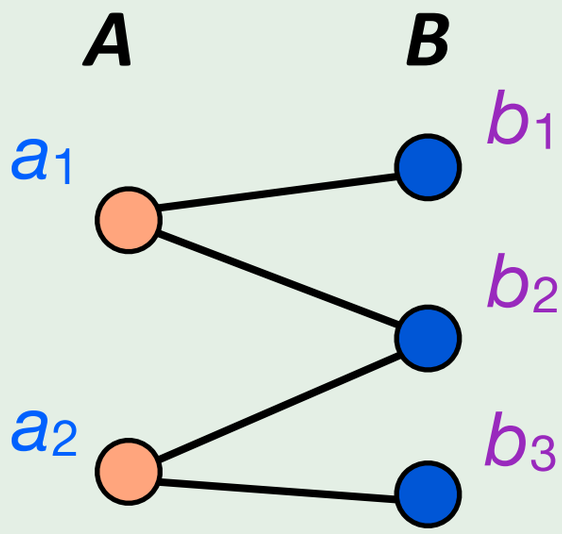

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2 **"maximal"**

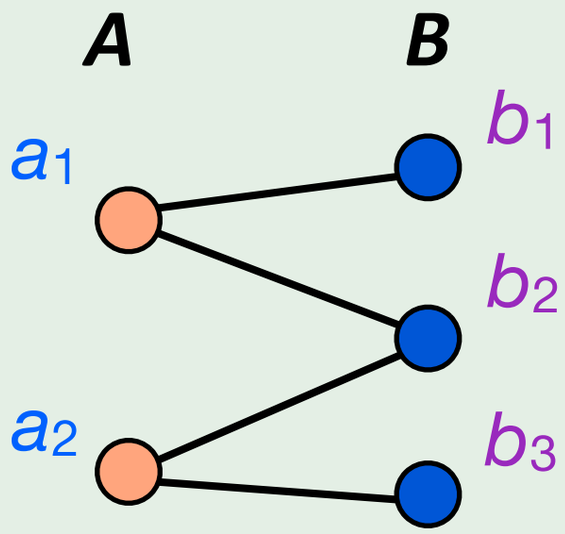
maximality: for all a outside of J,

$$|N(a) \setminus N(J)| < (1-\varepsilon)|N(a)|$$

Maximal Nearly Independent Set

Formalizing small overlap and maximality

$N(X)$ = neighbors of X



bipartite graph modeling coverage

(ε, δ)**-MaNIS** is $J \subseteq A$ such that

1 **small overlaps**

near independence:

$$|N(J)| \geq (1-\delta) \sum_{j \in J} |N(j)|$$

2 **"maximal"**

maximality: for all a outside of J,

$$|N(a) \setminus N(J)| < (1-\varepsilon)|N(a)|$$

1. not unique
2. $\varepsilon = \delta = 0$ no overlap —> maximal set packing
3. simple $O(|E|)$ seq. alg

Back to Facility Location

Lemma: If we can compute (ε, δ)-MaNIS, then we have a $1.861/(1-\varepsilon-\delta)$-approx.

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Lemma 2*: Total #tokens ≤ 1.861**OPT/**$(1-\varepsilon)$

factor-revealing LP + dual fitting

[Jain et al. '03]

+ geometric scaling

How to Compute MaNIS?

Implicit in algorithms from previous work

$$m = |F| \times |C|$$

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Next step: Linear work for any value of ε?

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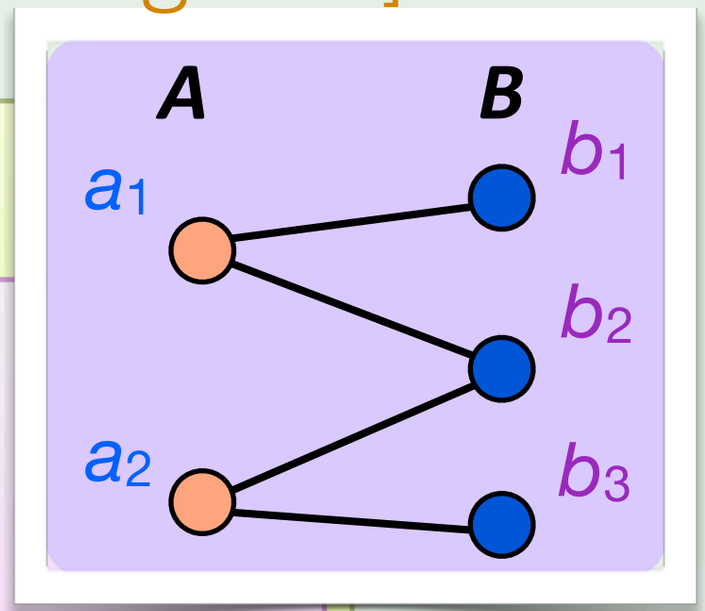
A Simple Linear-Work MaNIS

$O(\log^2 |E|)$-depth MaNIS [Blelloch-Peng-T'11]



While (*A* not empty)

a) Pick a random permutation π of ***A***

b) Each $b \in$ ***B*** joins the highest π-ranked nbr

c) For each $a \in$ ***A***, if (1 - δ) fraction of nbrs joined it, add a to output and remove a's nbr

d) Remove $a \in$ ***A*** if degree less than (1-ε) fraction of its original degree

Idea: random permutation removes a const fraction of edges takes O(log $|E|$) rounds

Parallel Greedy Facility Location

Putting things together

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  outer loop: mimic greedy behavior

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Plus, additional O(log m) depth, O(m) work basic operations in the outer loop.

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Theorem:

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Using MaNIS: Linear-work algorithms for max cover, (weighted) set cover, min-sum set cover

Take-Home Points

Maximal Nearly Independent Set

Pick a maximal collection that has small overlap

... more at SPAA'11

Linear-Work Greedy Parallel Approximation Algorithms for Set Covering and Variants

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Shameless Plug

Near-Linear Work SDD Solver

Solve $Ax = b$ in $\tilde{O}(\#nnz)$-work $O(\#nnz^{1/3})$-depth

if A is symmetric diagonally dominant (SDD)

... more at SPAA'11

Near Linear-Work Parallel SDD Solvers, Low-Diameter Decomposition, and Low-Stretch Subgraphs

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