Chaotic Epidemic Outbreaks: Deterministic or Random?

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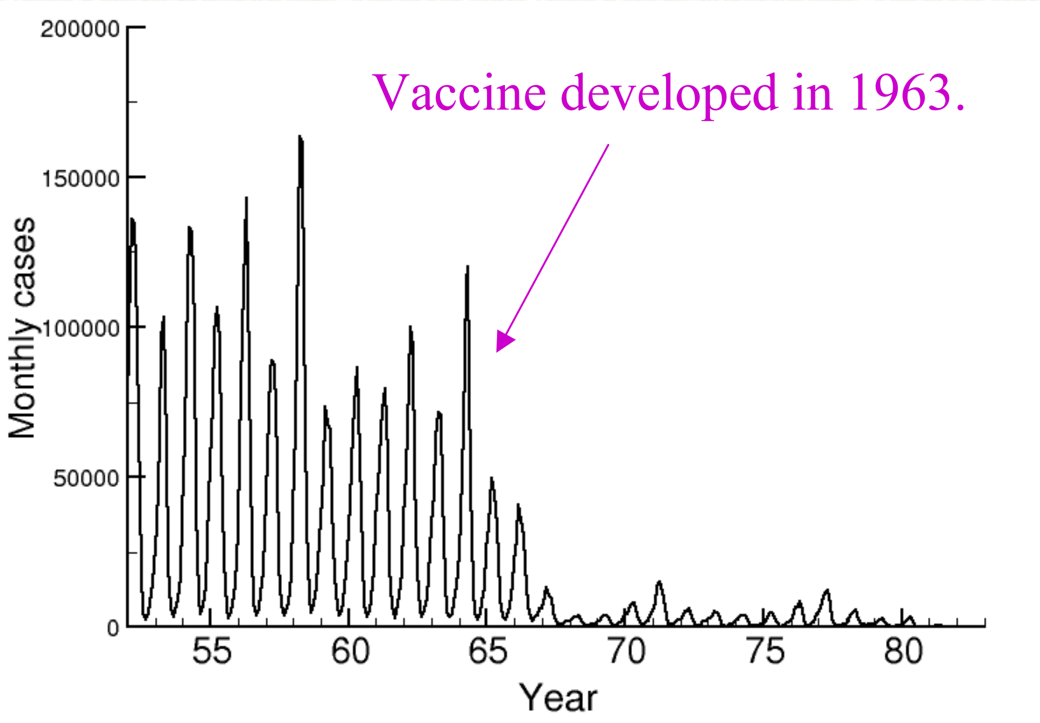
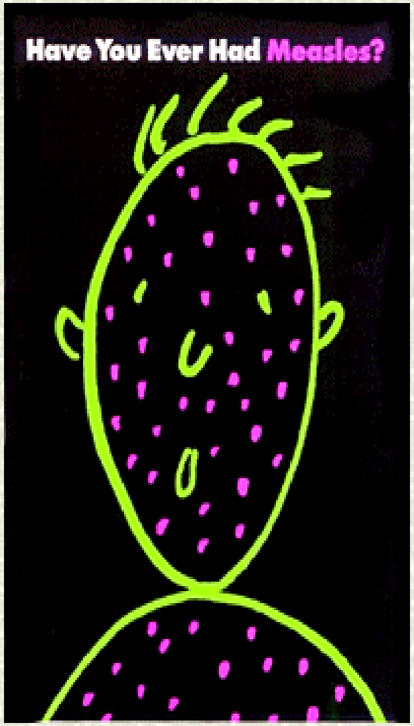
Measles

Total number of U.S. cases



Graph by Alun Lloyd (2002)

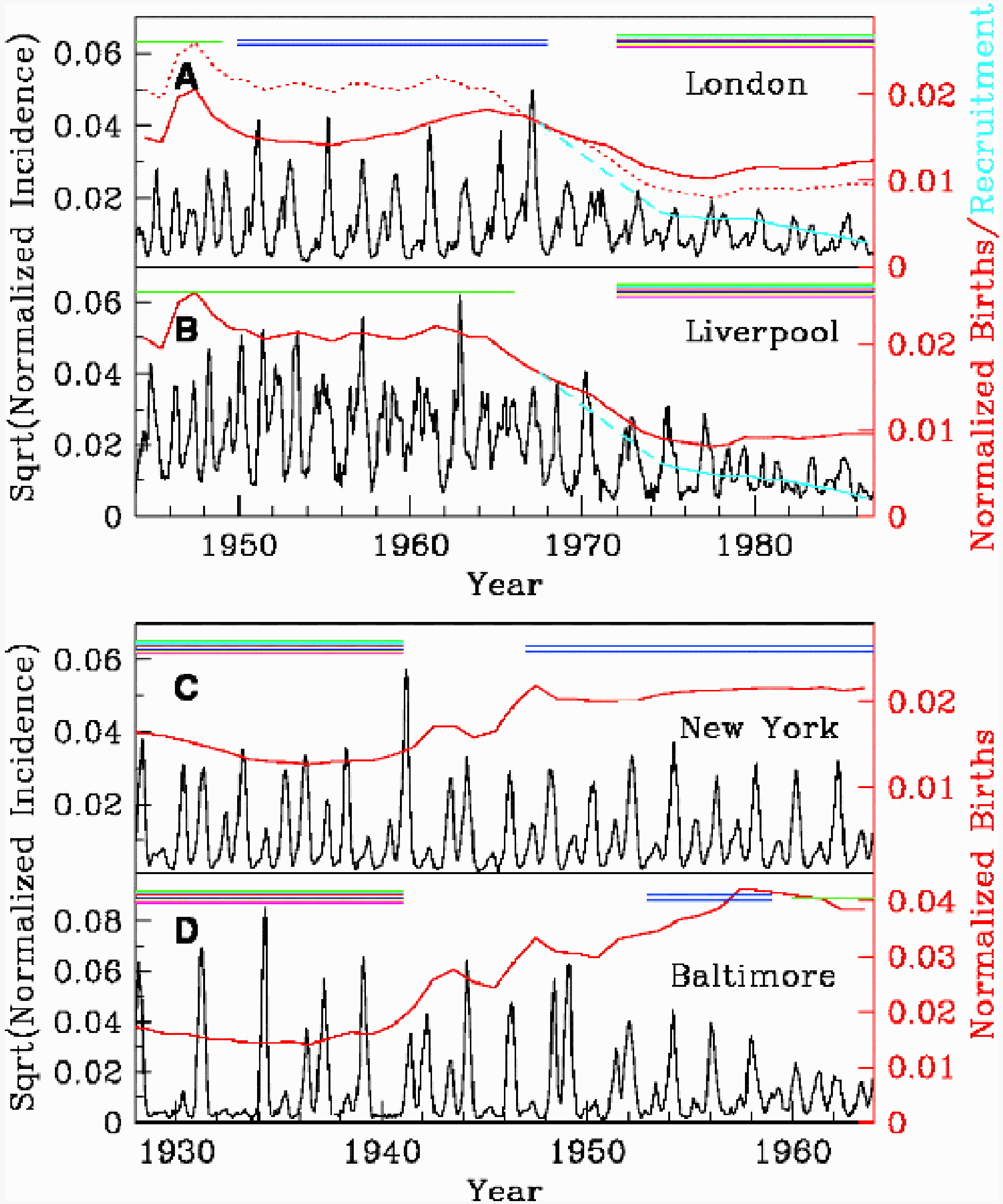


http://science-education.nih.gov

Why do so many people study measles?

- The biological system is fairly simple
  - We can test and improve models (design vaccination strategies)
  - These models can be used for many applications (other diseases, computer viruses, etc.)
- Excellent data is available
  - We can ask detailed questions about spatial and temporal dynamics
  - The data exhibits periodic or more complex behavior
- Some have conjectured that the dynamics could be chaotic

D. Earn, et al. Science, 2000



Question:
Is the pre-vaccine time series chaotic?

Answer:
Undetermined
(Not enough data)

Outline

- SEIR model - a model for epidemics in childhood diseases (Yorke and London (1973); May and Anderson (1979); Schwartz (1983); Grenfell et al. (2000); Hethcote (2000))
- Add stochastic perturbations to represent noise in population size
- Bifurcation to stochastic chaos
- Possible vaccination strategies to control and prevent future outbreaks

Modeling Epidemics: Assumptions

The population:

- Assume the population is large and well mixed.
- Variables and parameters:

S:	Susceptibles	α^{-1}:	mean latent exposed period
E:	Exposed	γ^{-1}:	mean infectious period
I:	Infectives	μ:	birth and death rate
R:	Recovered	β:	contact rate (for S & I)

- Normalize the population: $S + E + I + R = 1$

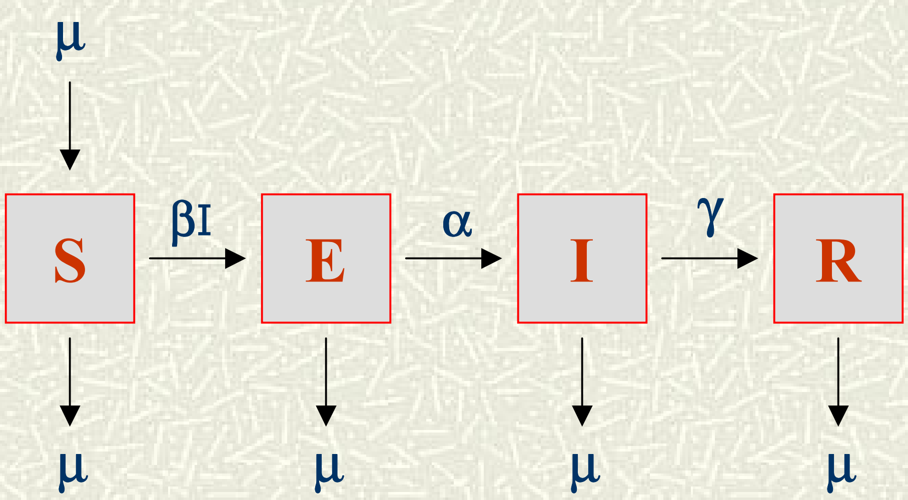
The standard SEIR model

$$\frac{dS}{dt} = \mu - \beta(t) IS - \mu S$$

$$\frac{dE}{dt} = \beta(t) IS - \alpha E - \mu E$$

$$\frac{dI}{dt} = \alpha E - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

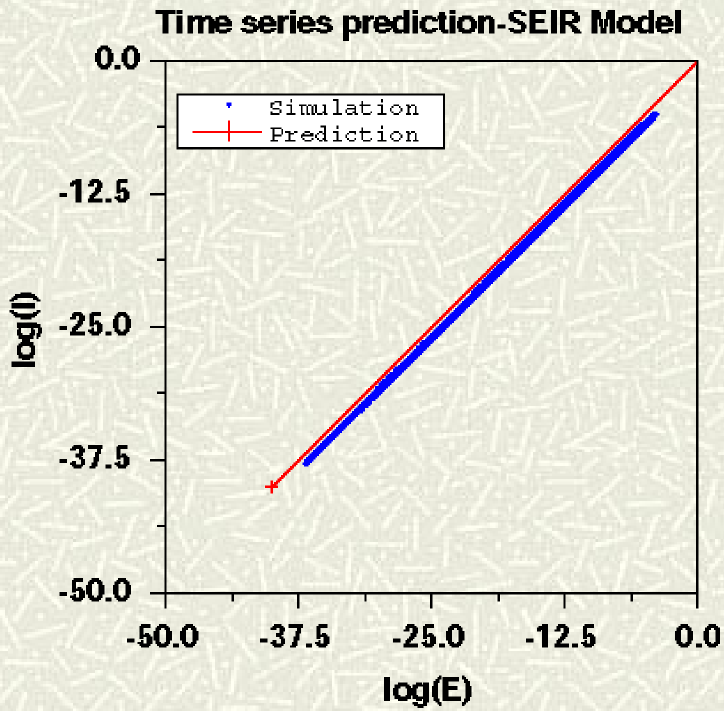


Our flavor

- The contact rate: $\beta(t) = \beta(t+1) = \beta_0(1 + \delta \cos 2\pi t)$

- The infectives are roughly proportional to the exposed [Schwartz, J. Math. Biol. 1985]

$$I(t) \approx \left(\frac{\alpha}{\mu + \gamma}\right) E(t)$$



The model we study

- The modified SI model (MSI)

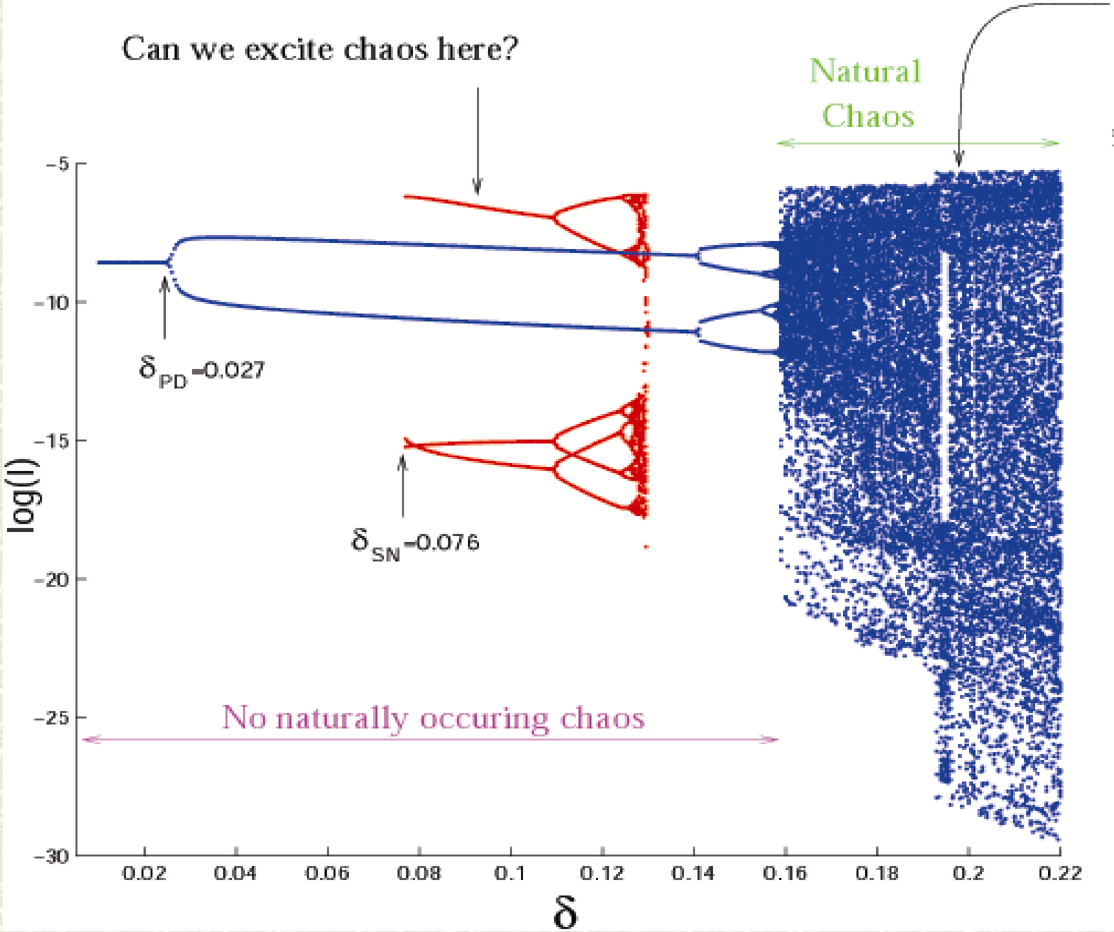
$$\frac{dS}{dt} = \mu - \beta(t)IS - \mu S$$

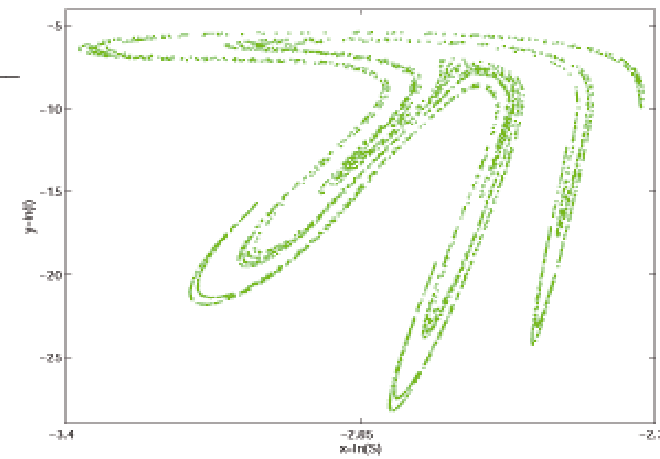
$$\frac{dI}{dt} = \left(\frac{\alpha}{\mu+\gamma}\right)\beta(t)IS - (\mu+\alpha)I$$

$$\beta(t) = \beta_0(1 + \delta\cos(2\pi t))$$

The paremeter we vary is δ

Bifurcation diagram

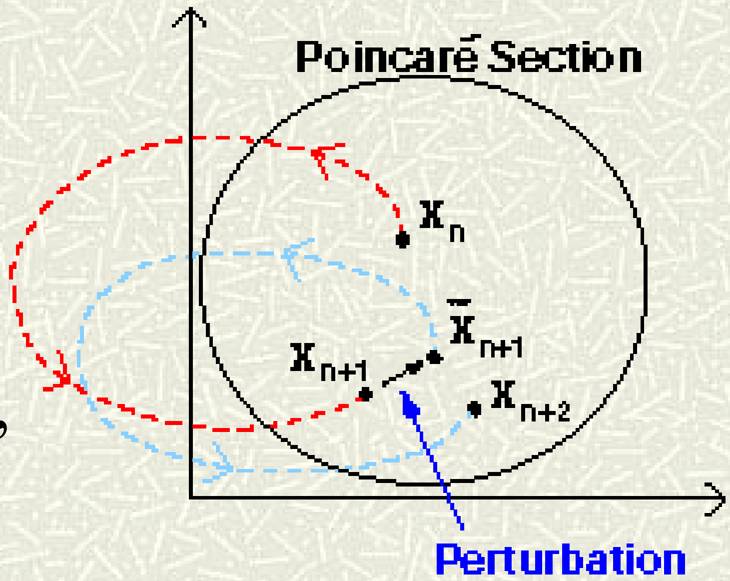




The system is driven periodically, so consider the discrete map on the Poincare section.

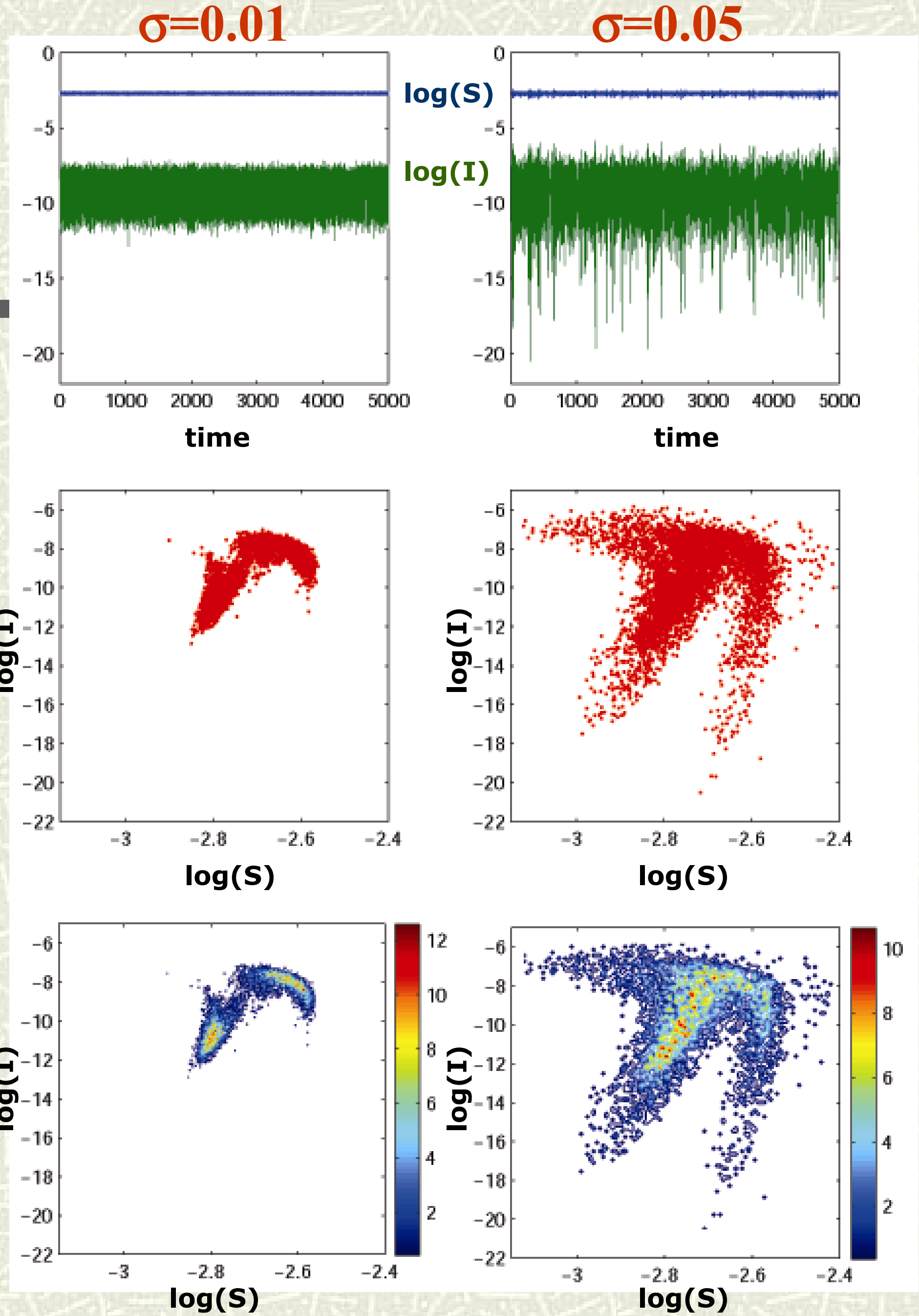
Adding noise

- The system is driven periodically, so add noise as if it is a map. (Additive noise)
- Noise: normal distribution, mean=0, vary the standard deviation (σ)



Noisy dynamics

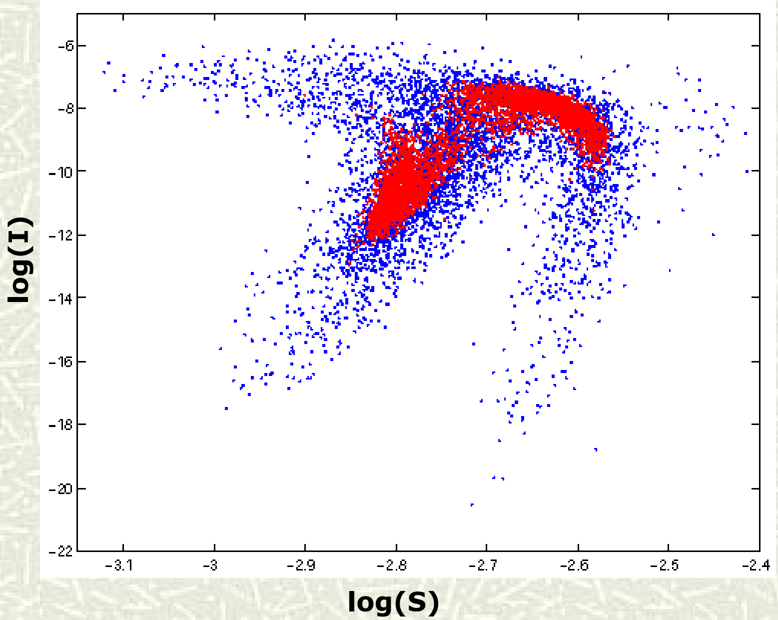
- Time series
- Phase space diagram
- Probability density function



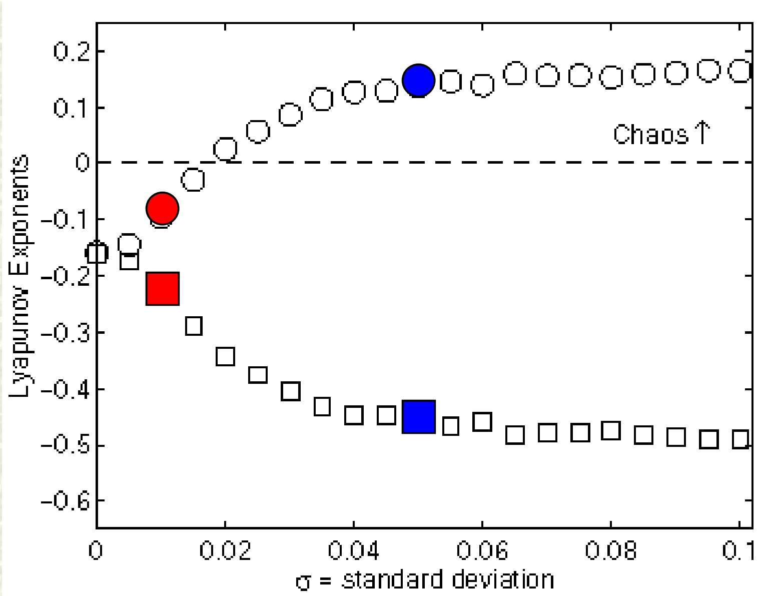
Stochastic Chaos?

- Deterministic definition (numerical)
  - Compact set
  - Positive Lyapunov exponent
  - Not asymptotically periodic
- Stochastic version?
  - Compact set
  - Positive Lyapunov exponent
  - Homoclinic/heteroclinic topology (makes chaotic orbits possible)

Lyapunov exponents



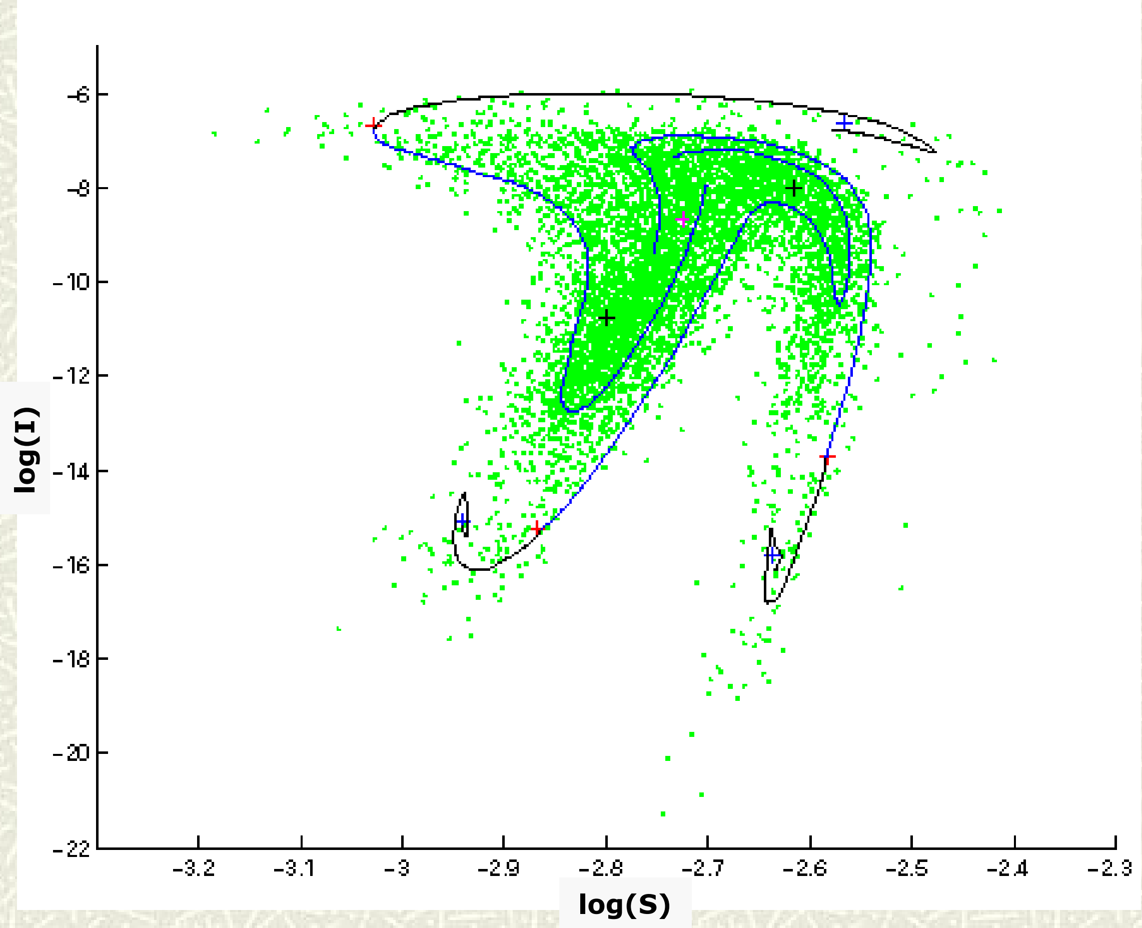
- Red: σ = 0.01 (noisy)
- Blue: σ = 0.05 (chaotic)



But Lyapunov exponents can yield false results

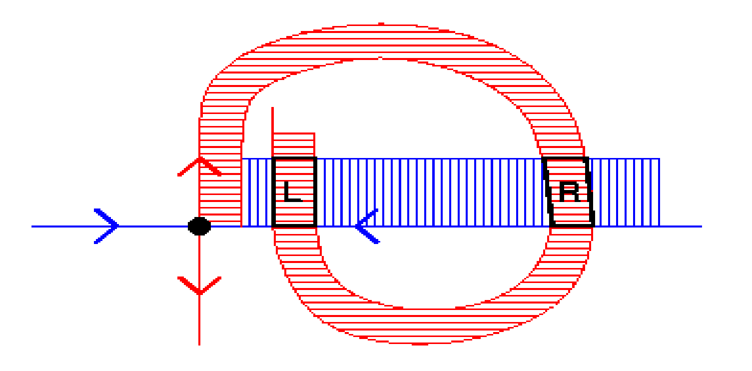
Unstable manifolds

- Random trajectories follow the unstable manifolds of the period three saddle
- What is the role of the manifolds?

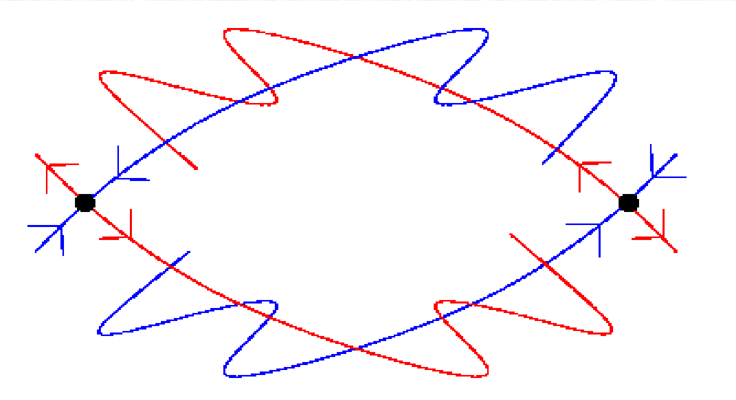


Smale Horseshoe Topology

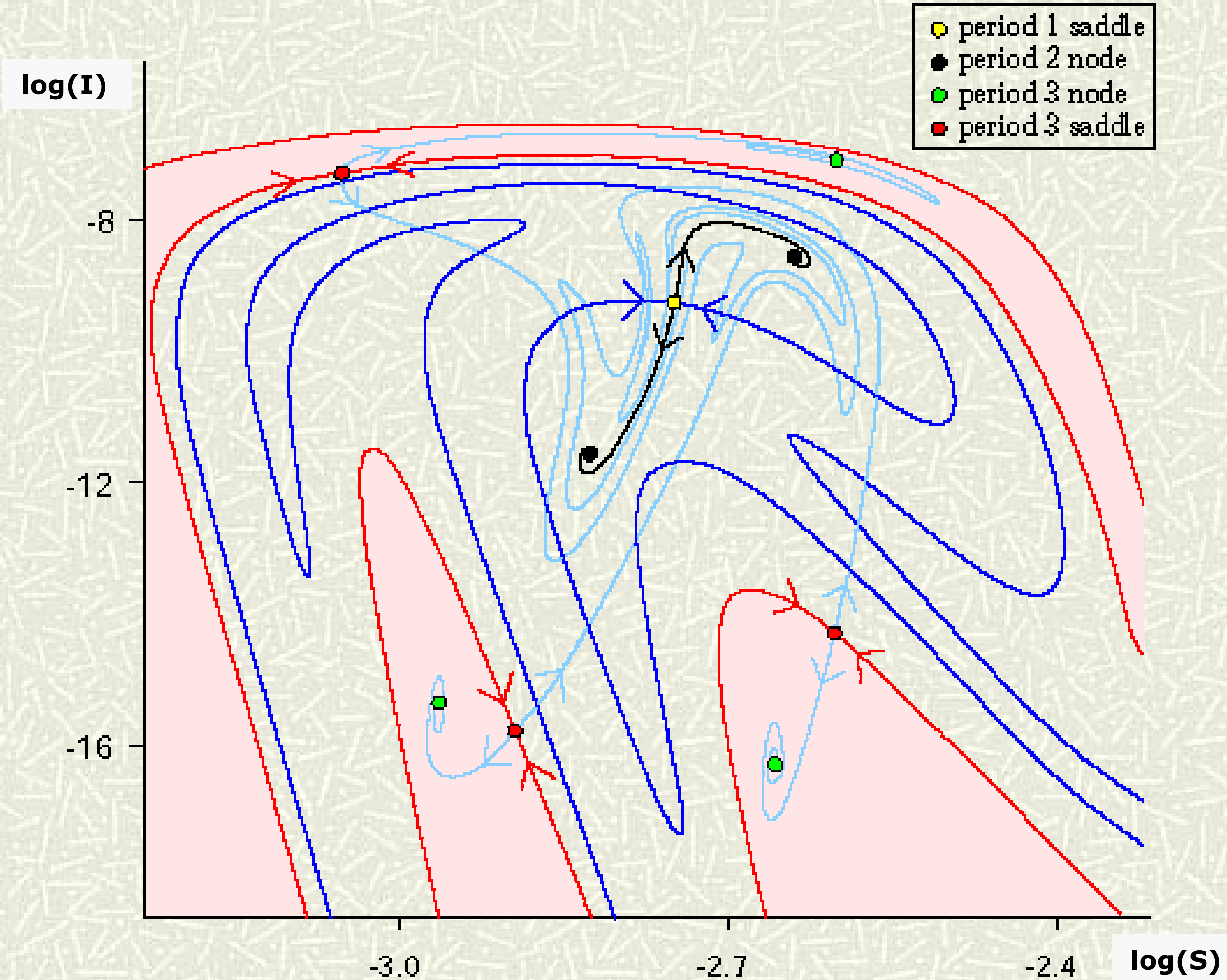
- Homoclinic Orbit



- Heteroclinic orbit

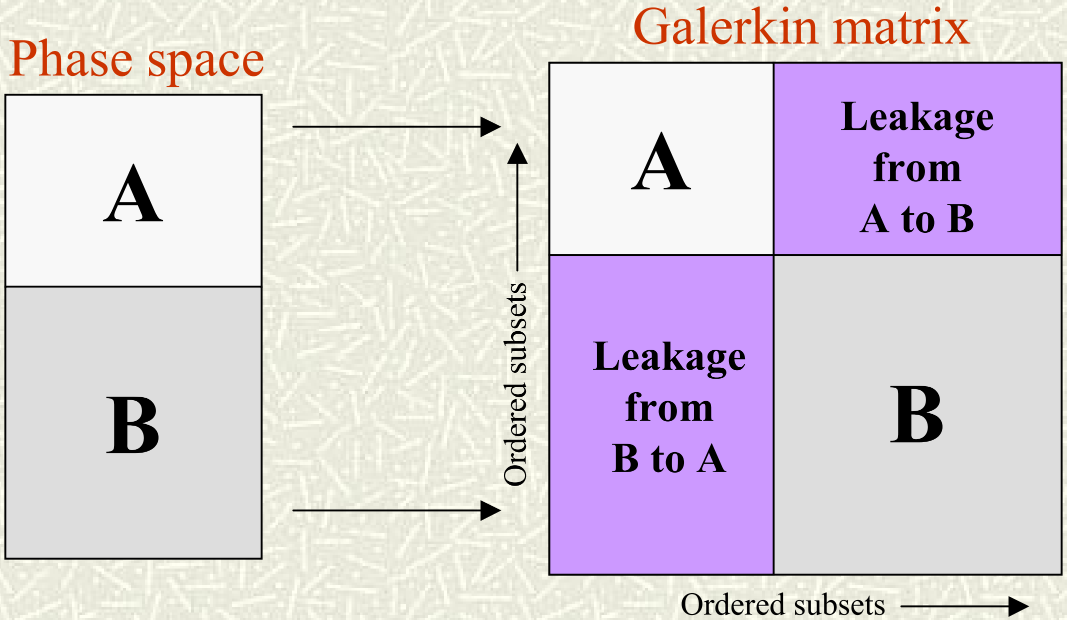


Stochastic Chaotic Saddle

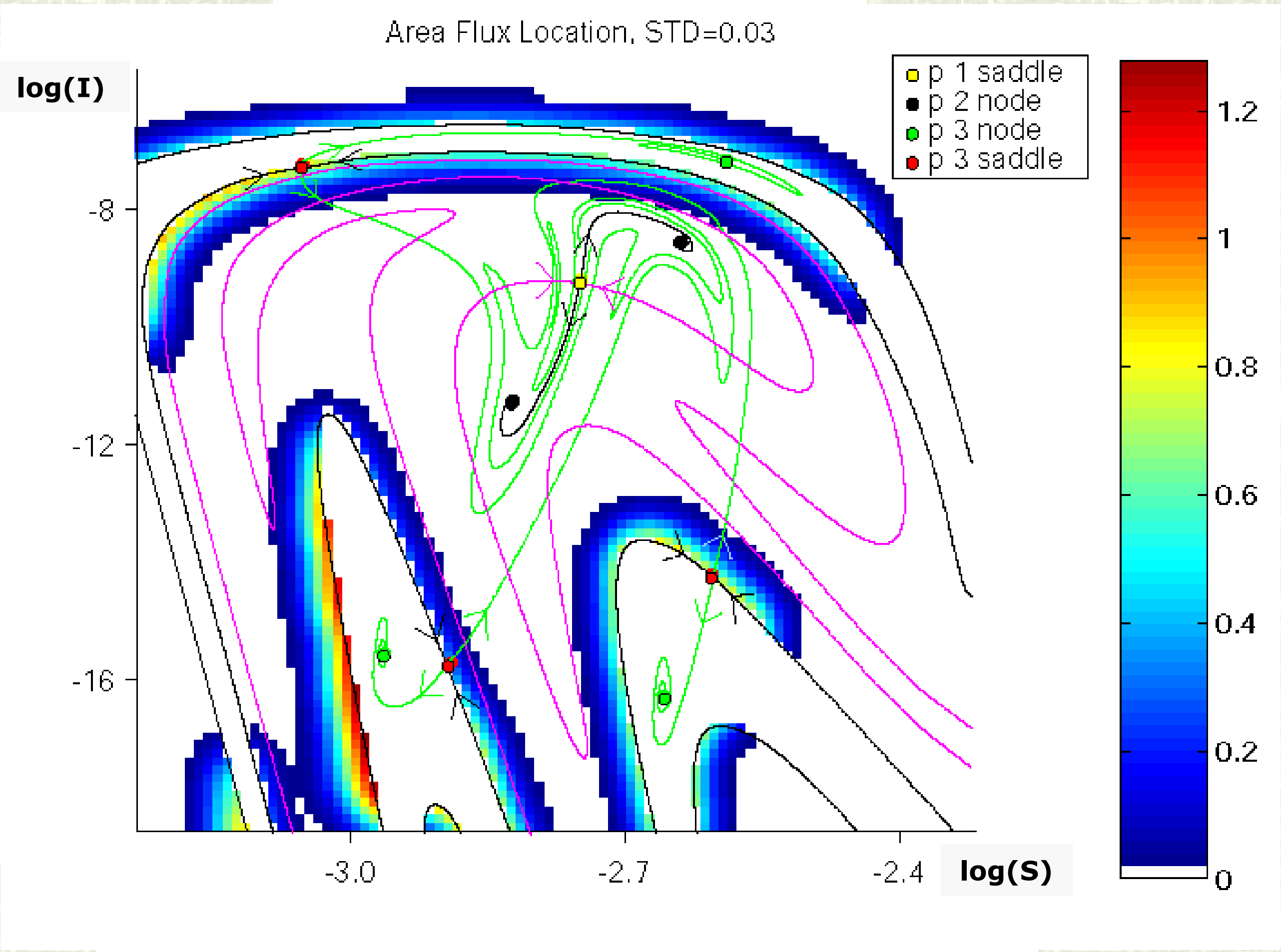


New tool to detect transport

Use a Galerkin approximation of the Stochastic Frobenius-Perron Operator to detect the flux across a basin boundaries and predict the most probable regions of transport created by noise.

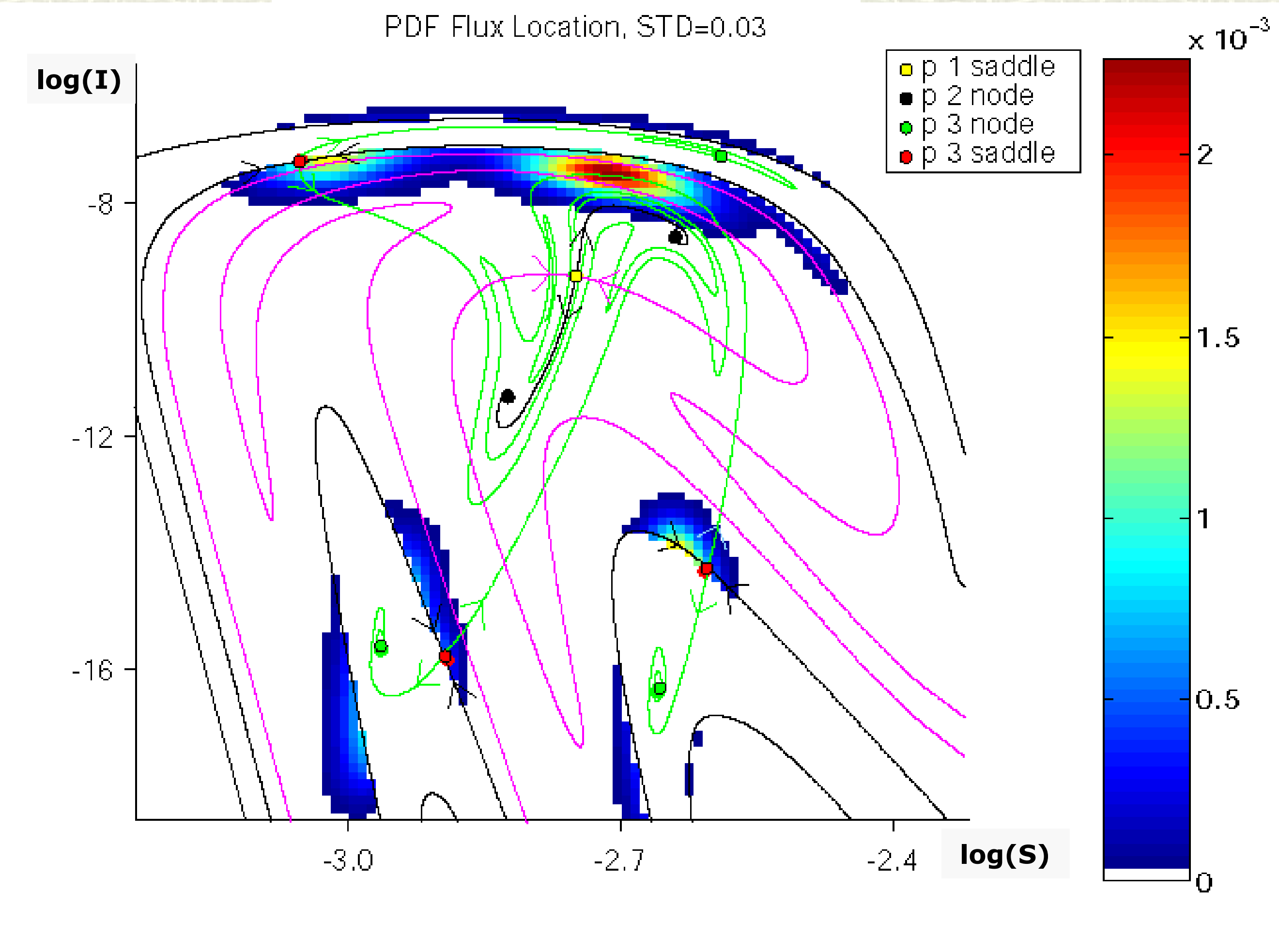


Area Flux



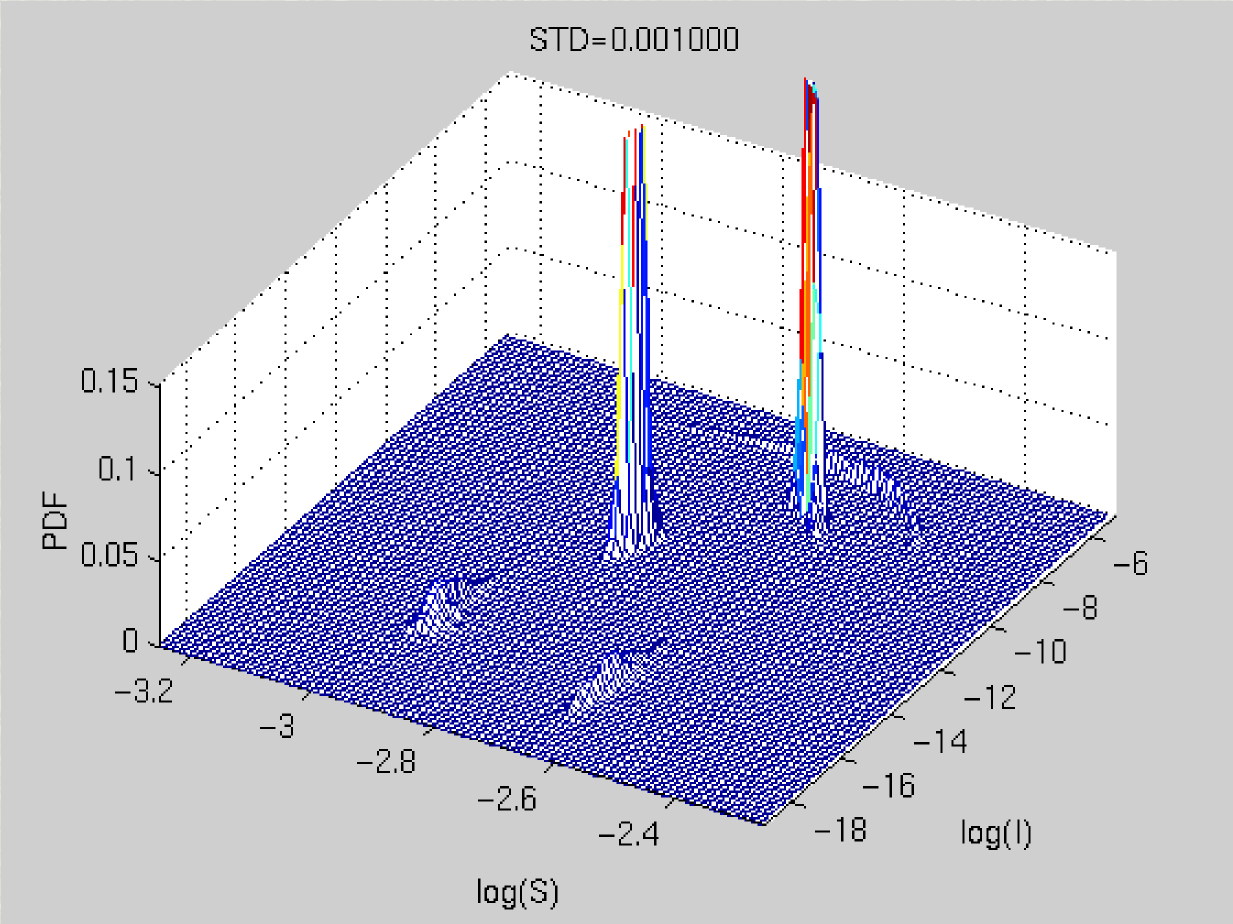
Area Flux Location, STD=0.03
log(I)
-8
-12
-16
-3.0
-2.7
-2.4
log(S)
p 1 saddle
p 2 node
p 3 node
p 3 saddle
1.2
1
0.8
0.6
0.4
0.2
0

PDF Flux



PDF Flux Location, STD=0.03
log(I)
-8
-12
-16
-3.0
-2.7
-2.4
log(S)
p 1 saddle
p 2 node
p 3 node
p 3 saddle
x 10^-3
2
1.5
1
0.5
0

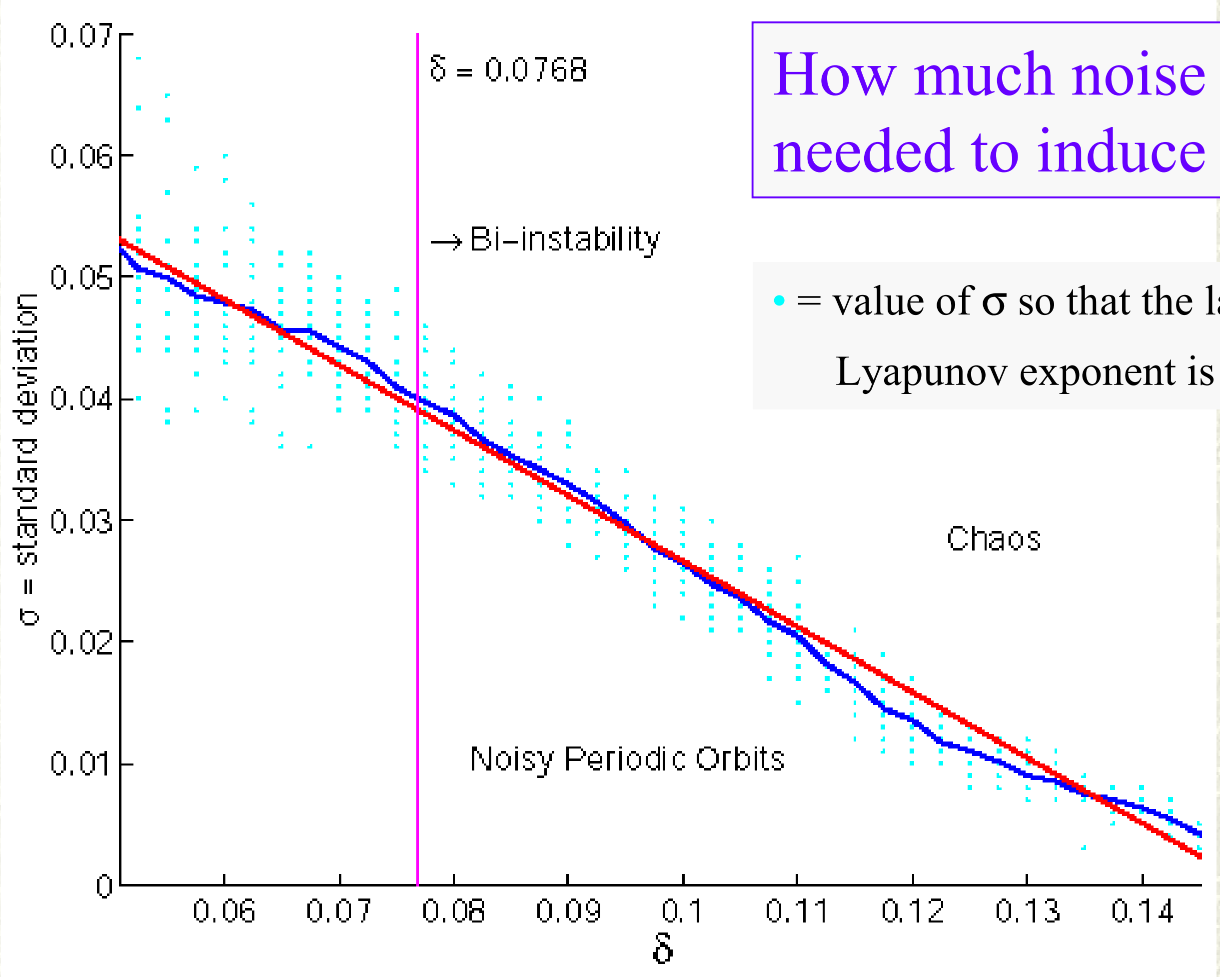
Probability Density Function



How do we use this information?

- Predict the occurrence of chaos
- Control the dynamics/prevent outbreaks

Predicting chaos

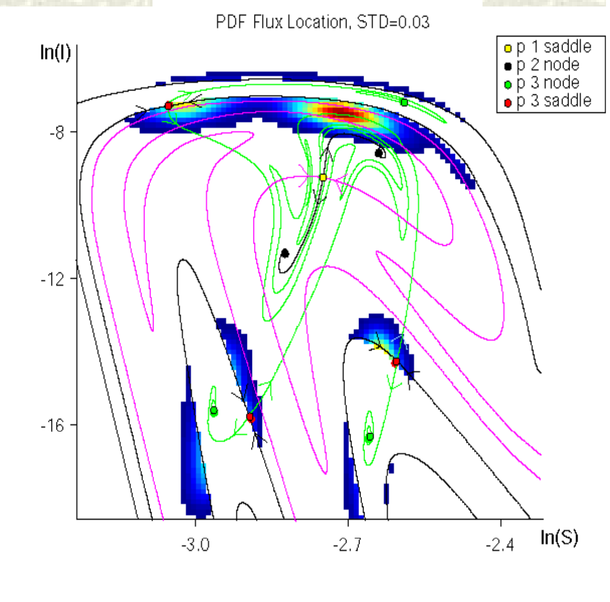


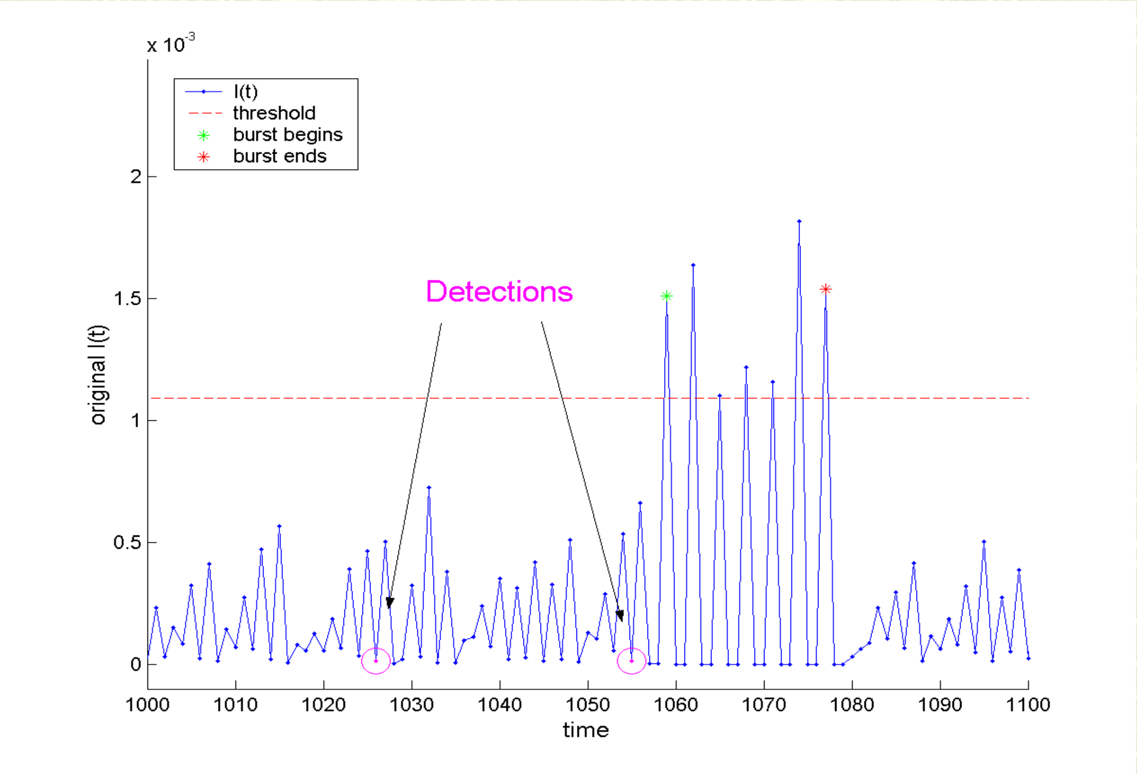
How much noise is needed to induce chaos?

• = value of σ so that the largest Lyapunov exponent is positive (20 trials)

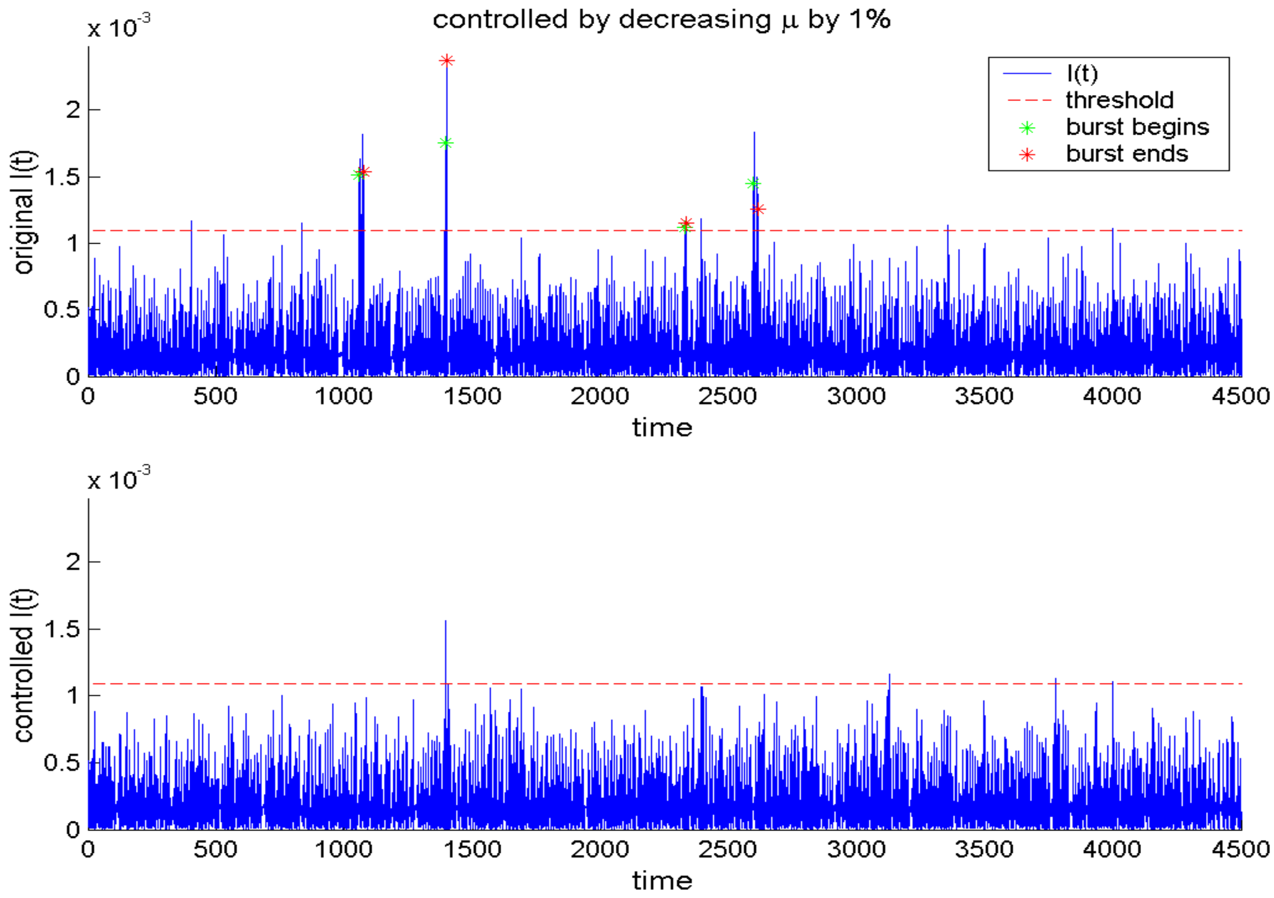
Controlling the dynamics

- If we can identify points in the bull's eye, then we can predict future outbreaks





Controlling the dynamics



Conclusions

- Stochastic perturbations can induce new, emergent dynamics in models
- Chaotic behavior can be induced in models by additive noise
- The topology reveals the mechanism that facilitates these dynamics
- We can use the topology to our advantage and control the system