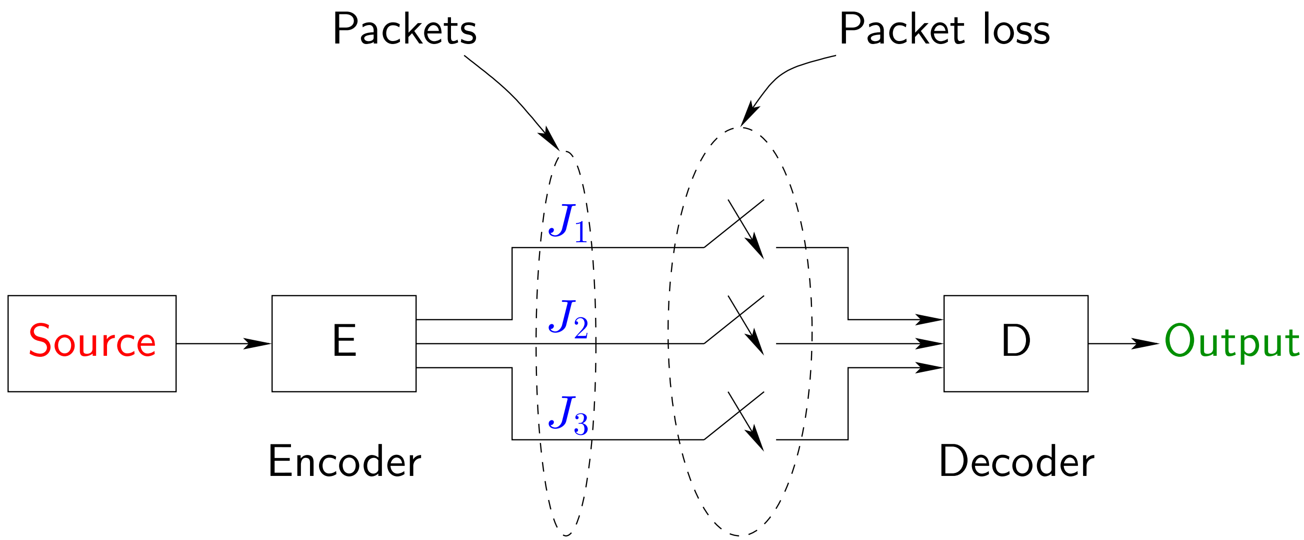
Multiple Description Coding with Many Channels

Raman Venkataramani
Harvard University, Cambridge, MA
raman@deas.harvard.edu

DIMACS workshop on network information theory

Joint work with Gerhard Kramer (Bell Labs) and Vivek Goyal (Digital Fountain)

Packet Networks



Model: Packets are either **lost completely** or received **error-free**.

How do we deal with packet loses?

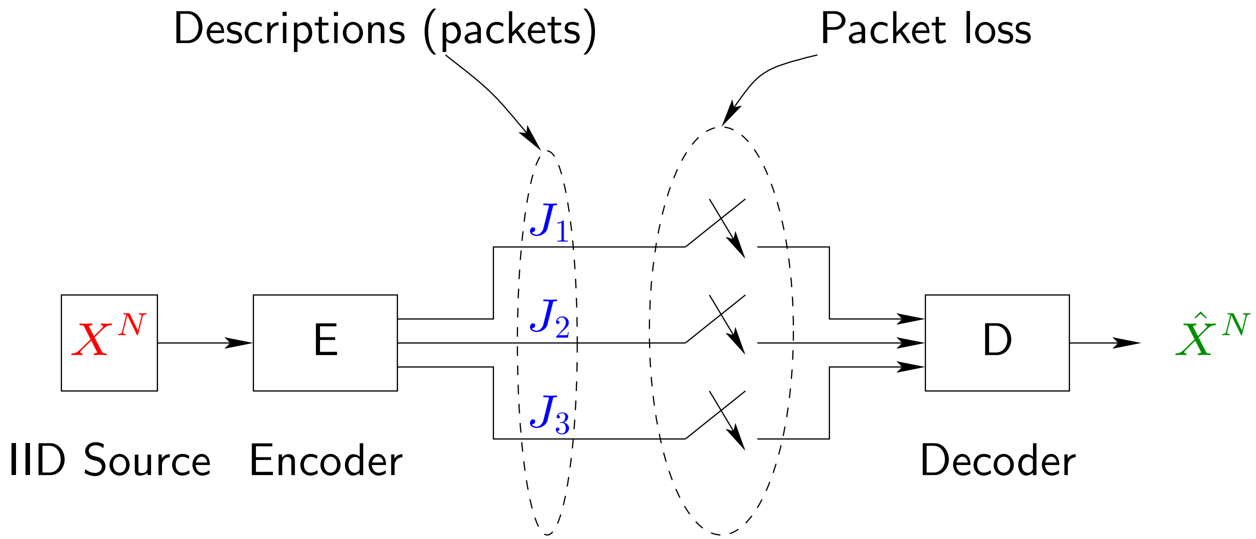
- Request a retransmission.
  - Good for loss-less transmission.
  - Not feasible for **real time** data such as **voice** and **video**.

Alternate Approach:

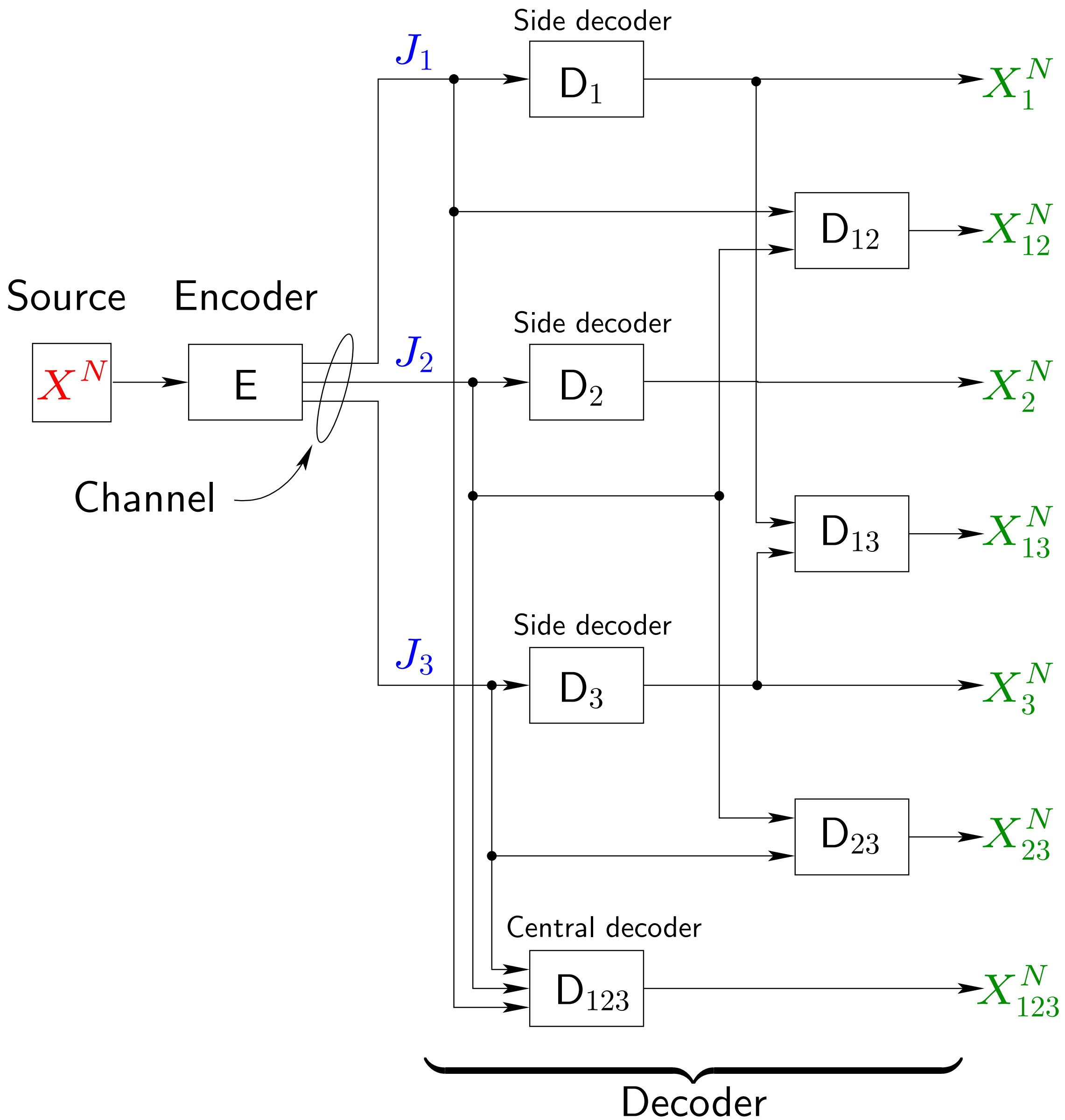
- Reconstruct using available packets.
  - Requires adding **redundancy** to packets (**coding**).
  - **Advantage**: Graceful degradation of output quality when packet losses increase.

The second approach is called **Multiple Description Coding**.

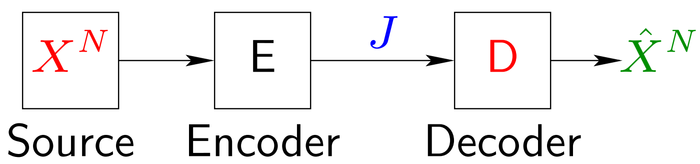
Example: Coding with 3 Descriptions



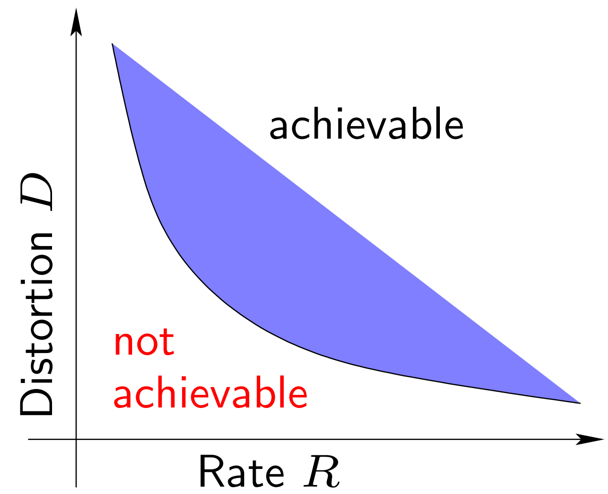
- The source is IID vector X^N (length N).
- The encoder produces L "descriptions" J_1,. . . , J_L of X^N.
- The decoder produces an output $\hat{X}^N$ from the **available** descriptions.


Side decoder
J_1
D_1
X_1^N
D_{12}
X_{12}^N
Source
Encoder
Side decoder
X^N
E
J_2
D_2
X_2^N
Channel
D_{13}
X_{13}^N
Side decoder
J_3
D_3
X_3^N
D_{23}
X_{23}^N
Central decoder
D_{123}
X_{123}^N
Decoder

Review of Rate-Distortion Theory



$\frac{1}{N}H(J) \leq R$
$\frac{1}{N}d(X^N, \hat{X}^N) \leq D$



Theorem 1. [Shannon] *The RD region is the convex set $R \geq R(D)$ where*

$$R(D) = \min_{\hat{X}} I(X; \hat{X}) \quad \textit{s.t.} \quad \mathrm{E}d(X, \hat{X}) \leq D$$

minimized over all $\hat{X}$ jointly distributed with X.

Gaussian Source $X \sim N(0, 1)$: $R(D) = \frac{1}{2}\log(\frac{1}{D})$

Multiple Description (MD) coding

Source: Length N vector X^N of i.i.d. random variables.

Encoder: $X^N \rightarrow \{J_1, \ldots, J_L\}$ which are the L "descriptions" of X^N at rates R_1,. . . R_L per source symbol.

Descriptions: $J_l = f_l(X^N), \quad H(J_l) \leq NR_l, \quad l = 1, \ldots L.$

Decoder: Consists of $2^L - 1$ decoders: one for each non-empty subset of the available descriptions.

Decoder Outputs: $X_{\mathcal{S}}^N = g_{\mathcal{S}}(\{J_l : l \in \mathcal{S}\})$ where $\mathcal{S} \subseteq \{1, \ldots, L\}$, $\mathcal{S} \neq \emptyset$.

In the last example $(L = 3)$:

$$\mathcal{S} = \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \text{ or } \{1,2,3\}$$

Problem Statement

- **Problem: What is the Rate-Distortion (RD) region?**
- The **rates** (L parameters) are

$$R_1, \ldots, R_L.$$

- The **distortions** ($(2^L - 1)$ parameters) are

$$D_{\mathcal{S}} = \frac{1}{N} \mathrm{E} d(X^N, X_{\mathcal{S}}^N), \quad \mathcal{S} \subseteq \{1, \ldots, L\}, \mathcal{S} \neq \emptyset$$

- The RD region is $(L + 2^L - 1)$-dimensional.
- **Remark:** For $L = 1$, it is Shannon's RD region.

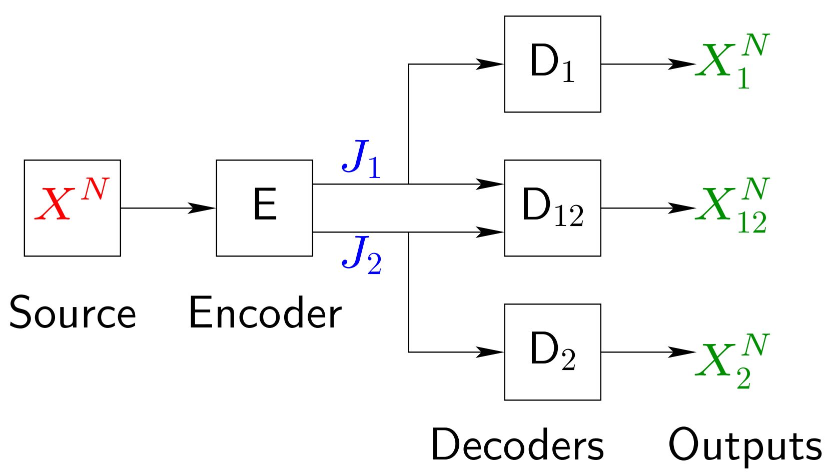
$L = 2$ case

The RD region is the set of possible rates and distortions as $N \to \infty$:

$$R_1 = \frac{1}{N} H(J_1) \qquad D_1 = \frac{1}{N} \mathrm{E} d(X^N, X_1^N)$$

$$R_2 = \frac{1}{N} H(J_1) \qquad D_2 = \frac{1}{N} \mathrm{E} d(X^N, X_2^N)$$

$$D_{12} = \frac{1}{N} \mathrm{E} d(X^N, X_{12}^N)$$

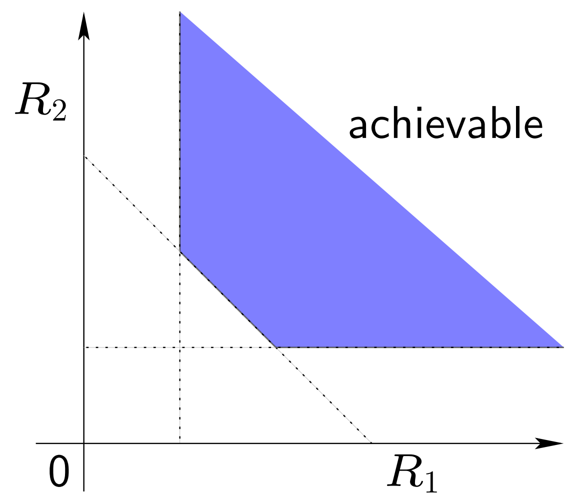


Review of Past Research

El Gamal and Cover (1982) found an achievable region for $L = 2$:

$$R_1 \geq I(X; X_1)$$
$$R_2 \geq I(X; X_2)$$
$$R_1 + R_2 \geq I(X; X_1 X_2 X_{12}) + I(X_1; X_2)$$
$$D_{\mathcal{S}} \geq \mathrm{E}d(X, X_{\mathcal{S}}), \quad \mathcal{S} = 1, 2, 12$$

where X_1, X_2, X_{12} are any r.v's jointly distributed with the source X.



Rate region for fixed X_1, X_2, X_{12} such that $D_{\mathcal{S}} \geq \mathrm{E}d(X, X_{\mathcal{S}})$.

Remark 1: The convex hull of this region is achievable by time-sharing.

Remark 2: Gives the RD region for the **Gaussian source**.

- **Ozarow (1980)** computed an **outer bound** on the RD region for $L = 2$ for the Gaussian source. The bound **meets the inner bound** by El Gamal and Cover.
- **Zhang and Berger (1987)** provided a stronger achievable result than El Gamal and Cover for $L = 2$. For the **binary symmetric source** with Hamming distortion measure, their result provides a **strict improvement**.
- **Wolf, Wyner and Ziv (1980), Witsenhausen and Wyner (1981), Zhang and Berger (1983)** provided some results for the binary symmetric source.

An Achievable Region for $L > 2$

Theorem 2. *The RD region contains the rates and distortions satisfying*

$$\sum_{l \in \mathcal{S}} R_l \geq (|\mathcal{S}| - 1) I(X; X_\emptyset) - H(X_\mathcal{U} : \mathcal{U} \in 2^\mathcal{S} | X)$$

$$+ \sum_{\mathcal{T} \subseteq \mathcal{S}} H(X_\mathcal{T} | X_\mathcal{U} : \mathcal{U} \in 2^\mathcal{T} - \mathcal{T})$$

$$D_\mathcal{S} \geq \mathrm{E} d_\mathcal{S}(X, X_\mathcal{S})$$

for every $\emptyset \neq \mathcal{S} \subseteq \mathcal{L} = \{1, \ldots, L\}$ *and some joint distribution between outputs* $\{X_\mathcal{S}\}$ *and the source* X*.*

Remark: This result generalizes the results of **El Gamal and Cover**, and of **Zhang and Berger**.

Gaussian Source: Outer Bound on the RD Region

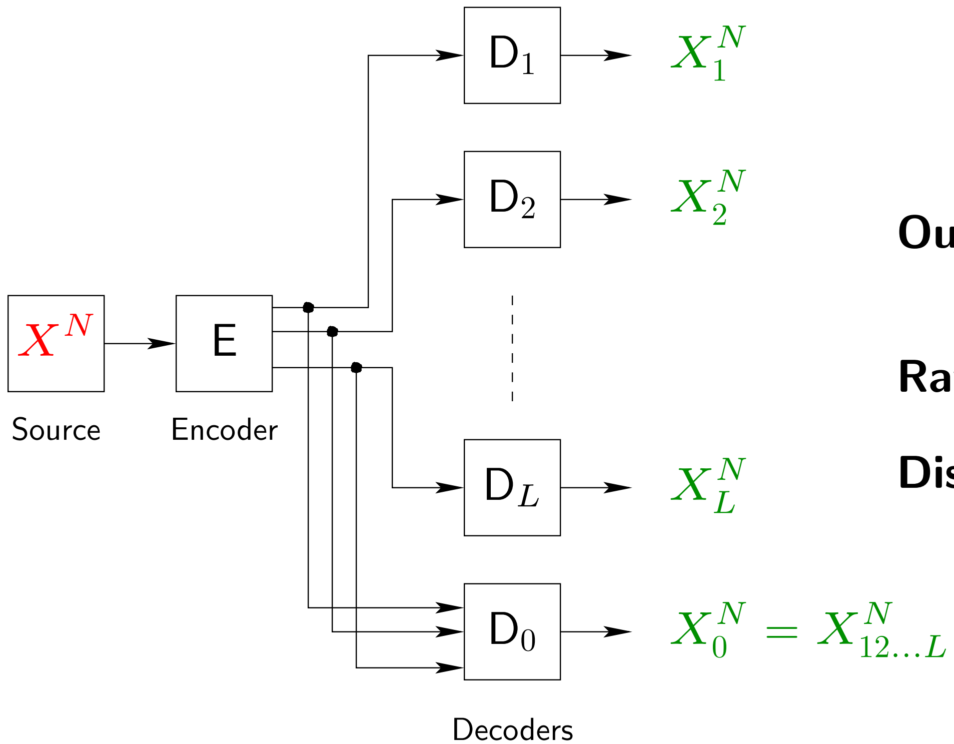
- **Gaussian source**: $X \sim N(0,1)$.
- **Squared-error distortion**: $d(x,y) = |x-y|^2$.
- An **outer bound** on the RD region:

Theorem 3. *The RD region is contained in*

$$\exp\left(-2\sum_{k\in\mathcal{K}} R_k\right) \leq \min_{\{\mathcal{K}_m\}_{m=1}^M} \inf_{\lambda\geq 0} \left(D_{\mathcal{K}} \frac{\prod_{m=1}^M (D_{\mathcal{K}_m}+\lambda)}{(D_{\mathcal{K}}+\lambda)(1+\lambda)^{M-1}}\right), \quad \forall \mathcal{K} \in 2^{\mathcal{L}}$$

minimized over all partitions $\{\mathcal{K}_m\}$ of $\mathcal{K}$.

Special Case: L Channels and $L+1$ Decoders



Outputs: X_1^N, X_2^N,. . . , X_L^N
and $X_0^N = X_{12\ldots L}^N$

Rates: R_1, R_2,. . . , R_L

Distortions: D_1, D_2,. . . , D_L
and D_0

Keep only **side** and **central** decoders. Ignore all other decoders.

Inner and Outer Bounds on the RD region

- **Inner Bound:** Computable from our **achievable region** (Theorem 2).
- **Outer Bound:** Compuatable from Theorem 3 for the **Gaussian source**
- **Tightness of Bounds:** The inner and outer bounds meet for over some range of rates and distortions for the Gaussian source.

Example: 3-Channel 4-Decoder Problem

Take $L = 3$, $D_1 = D_2 = D_3 = 1/2$ and $D_0 = 1/16$.

Outer Bound:

$$R_l \geq 0.5, \quad l = 1, 2, 3$$

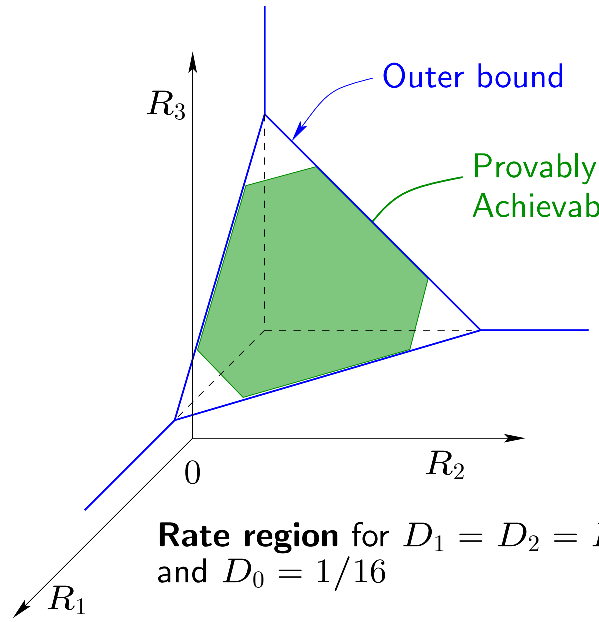
$$R_1 + R_2 + R_3 \geq 2.1755$$

Achievable Rates:

$$R_l \geq 0.5, \quad l = 1, 2, 3$$

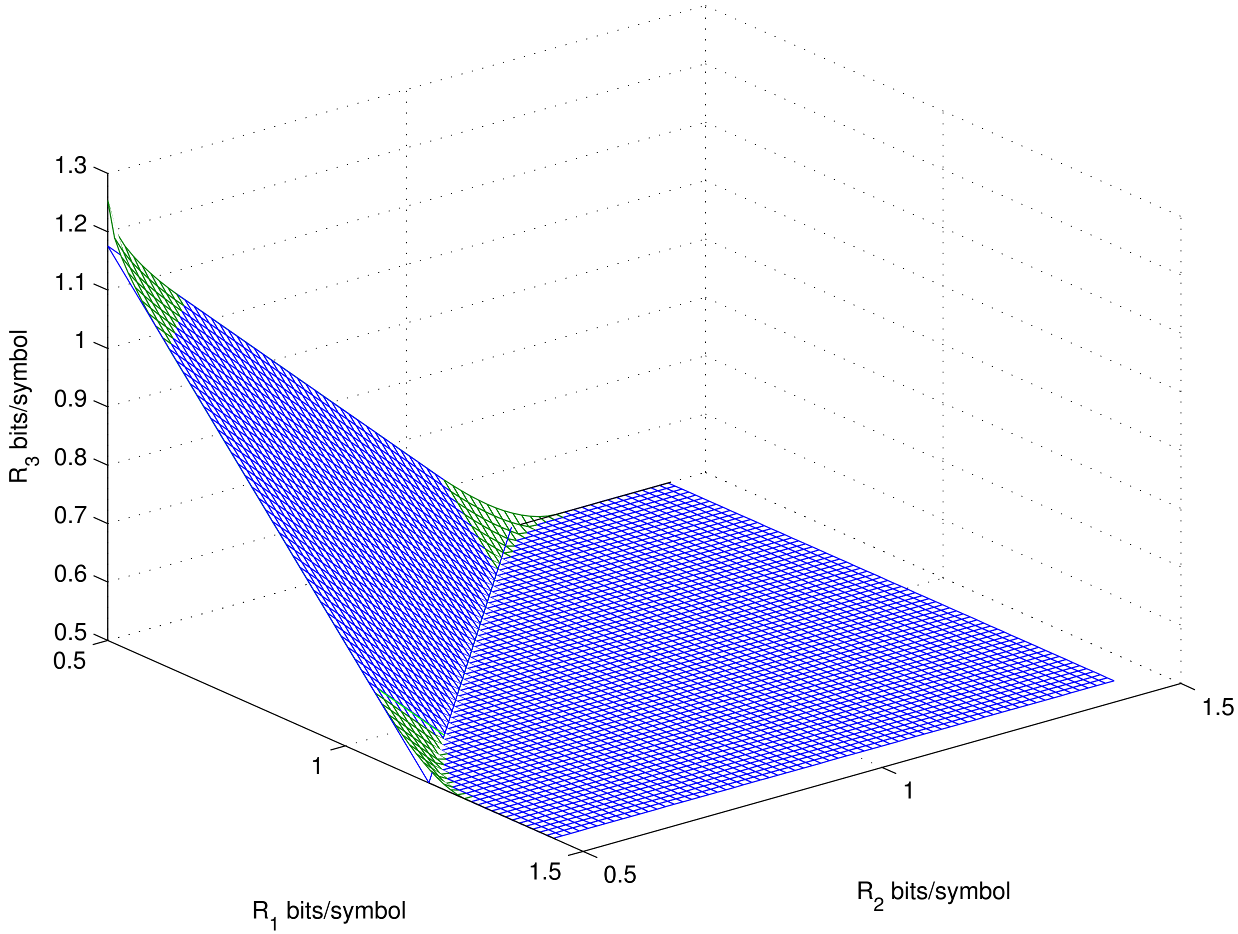
$$R_1 + R_2 + R_3 = 2.1755$$

$$R_l + R_m \geq 1.1258, \quad l < m$$



Rate region for $D_1 = D_2 = D_3 = 1/2$ and $D_0 = 1/16$

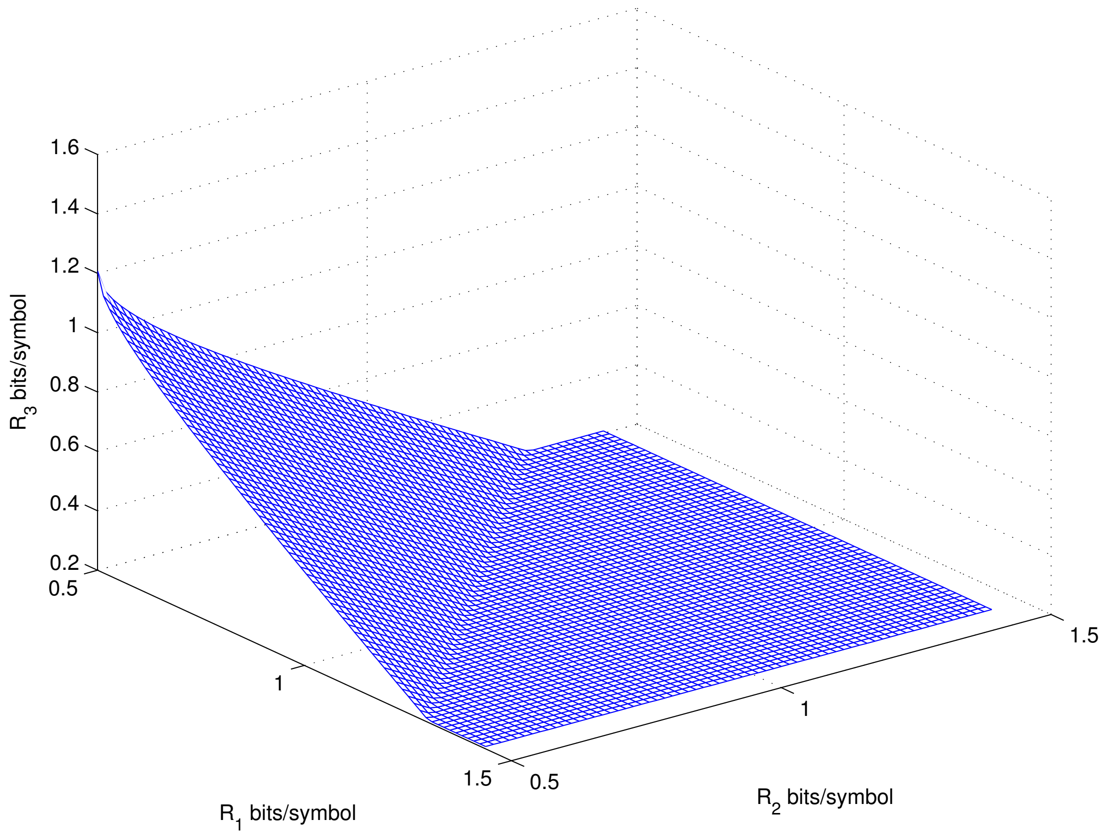
Remark: Excess rate $= 2.1755 - R(D_0) = 0.1755$ bits.



Blue Region: Inner and outer bounds meet on a hexagon.

Green Region: Inner bound (does not meet outer bound).

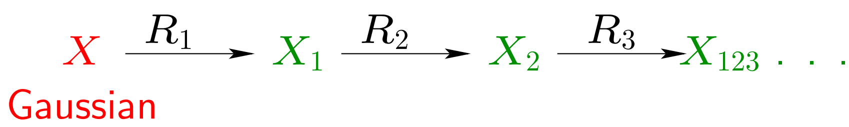
Another Example: $L = 3$, $D_1 = D_2 = 1/2$, $D_3 = 3/4$, and $D_0 = 1/16$:



The RD Region

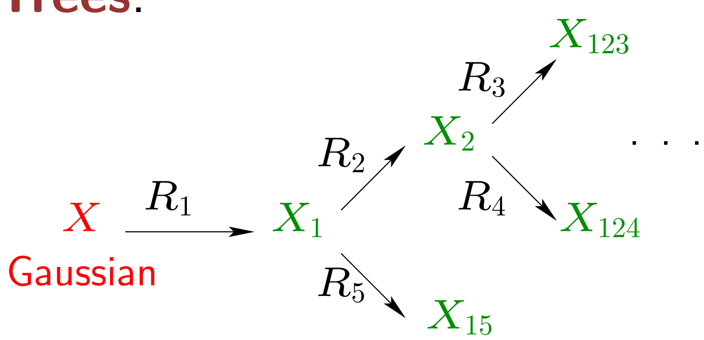
Gaussian Sources are Successively Refinable

Chains:



$$R_1 \geq R(D_1)$$
$$R_1 + R_2 \geq R(D_{12})$$
$$R_1 + R_2 + R_3 \geq R(D_{123})$$

Trees:



$$R_1 \geq R(D_1)$$
$$R_1 + R_2 \geq R(D_{12})$$
$$R_1 + R_2 + R_3 \geq R(D_{123})$$
$$R_1 + R_2 + R_4 \geq R(D_{124})$$
$$R_1 + R_5 \geq R(D_{15})$$

Summary

- **Multiple Description Coding**:
  - Motivation: coding for packet networks.
- **Results**:
  - An achievable region
  - An outer bound for the Gaussian source.
- **Still unsolved**:
  - The Gaussian problem for $L \geq 3$
  - Non-Gaussian sources