Network Routing Capacity

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Detailed results found in:

- R. Dougherty, C. Freiling, and K. Zeger
  "Linearity and Solvability in Multicast Networks"
  IEEE Transactions on Information Theory
  vol. 50, no. 10, pp. 2243-2256, October 2004.

- R. Dougherty, C. Freiling, and K. Zeger
  "Insufficiency of Linear Coding in Network Information Flow"
  IEEE Transactions on Information Theory
  (submitted February 27, 2004, revised January 6, 2005).

- J. Cannons, R. Dougherty, C. Freiling, and K. Zeger
  "Network Routing Capacity"
  IEEE/ACM Transactions on Networking
  (submitted October 16, 2004).

Manuscripts on-line at: *code.ucsd.edu/zeger*

Definitions

- An *alphabet* is a finite set.
- A *network* is a finite d.a.g. with source messages from a fixed alphabet and message demands at sink nodes.
- A network is *degenerate* if some source message cannot reach some sink demanding it.

Definitions - scalar coding

- Each edge in a network carries an alphabet symbol.
- An *edge function* maps in-edge symbols to an out-edge symbol.
- A *decoding function* maps in-edge symbols at a sink to a message.
- A *solution* for a given alphabet is an assignment of edge functions and decoding functions such that all sink demands are satisfied.
- A network is *solvable* if it has a solution for some alphabet.
- A solution is a *routing solution* if the output of every edge function equals a particular one of its inputs.
- A solution is a *linear solution* if the output of every edge function is a linear combination of its inputs (typically, finite-field alphabets are assumed).

Definitions - vector coding

- Each edge in a network carries a vector of alphabet symbols.
- An *edge function* maps in-edge vectors to an out-edge vector.
- A *decoding function* maps in-edge vectors at a sink to a message.
- A network is *vector solvable* if it has a solution for some alphabet and some vector dimension.
- A solution is a *vector routing solution* if every edge function's output components are copied from (fixed) input components.
- A *vector linear solution* has edge functions which are linear combinations of vectors carried on in-edges to a node, where the coefficients are matrices.
- A vector routing solution is *reducible* if it has at least one component of an edge function which, when removed, still yields a vector routing solution.

Definitions - (k, n) fractional coding

- Messages are vectors of dimension k.
  Each edge in a network carries a vector of at most n alphabet symbols.

- A (k, n) *fractional linear solution* has edge functions which are linear combinations of vectors carried on in-edges to a node, where the coefficients are rectangular matrices.

- A (k, n) fractional solution is a *fractional routing solution* if every edge function's output components are copied from (fixed) input components.

- A (k, n) fractional routing solution is *minimal* if it is not reducible and if no (k, n') fractional routing solution exists for any $n' < n$.

Definitions - capacity

- The ratio k/n in a (k, n) fractional routing solution is called an *achievable routing rate* of the network.

- The *routing capacity* of a network is the quantity

$$\epsilon = \sup\{\text{ all achievable routing rates}\}.$$

- Note that if a network has a routing solution, then the routing capacity of the network is at least 1.

Some prior work

- Some solvable networks do not have routing solutions (AhCaLiYe 2000).
- Every solvable multicast network has a scalar linear solution over some sufficiently large finite field alphabet (LiYeCa 2003).
- If a network has a vector routing solution, then it does not necessarily have a scalar linear solution (MéEfHoKa 2003).
- For multicast networks, solvability over a particular alphabet does not imply scalar linear solvability over the same alphabet (RaLe, MéEfHoKa, Ri 2003, DoFrZe 2004).
- For non-multicast networks, solvability does not imply vector linear solvability (DoFrZe 2004).
- For some networks, the size of the alphabet needed for a solution can be significantly reduced using fractional coding (RaLe 2004).

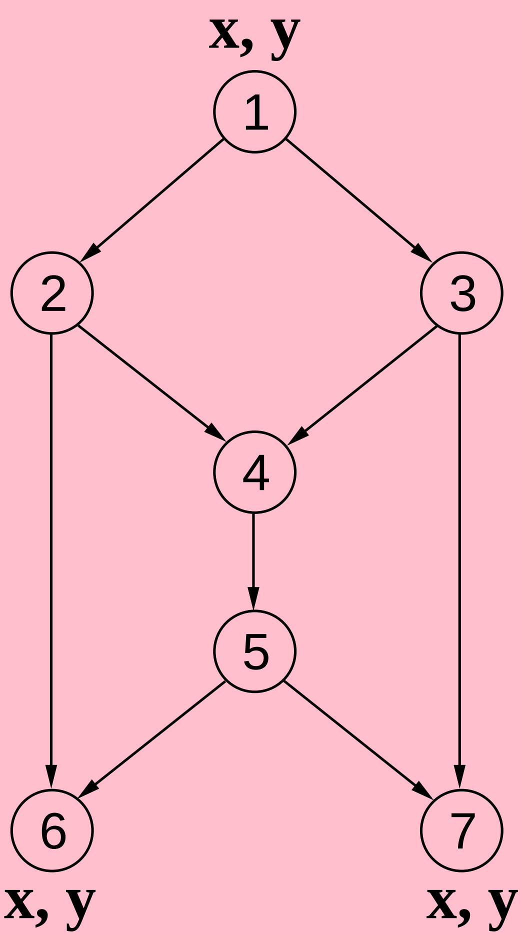
Our results

- Routing capacity definition.
- Routing capacity of example networks.
- Routing capacity is always achievable.
- Routing capacity is always rational.
- Every positive rational number is the routing capacity of some solvable network.
- An algorithm for determining the routing capacity.

Some facts

- Solvable networks may or may not have routing solutions.
- Every non-degenerate network has a (k, n) fractional routing solution for some k and n (e.g. take $k = 1$ and n equal to the number of messages in the network).

Example of routing capacity

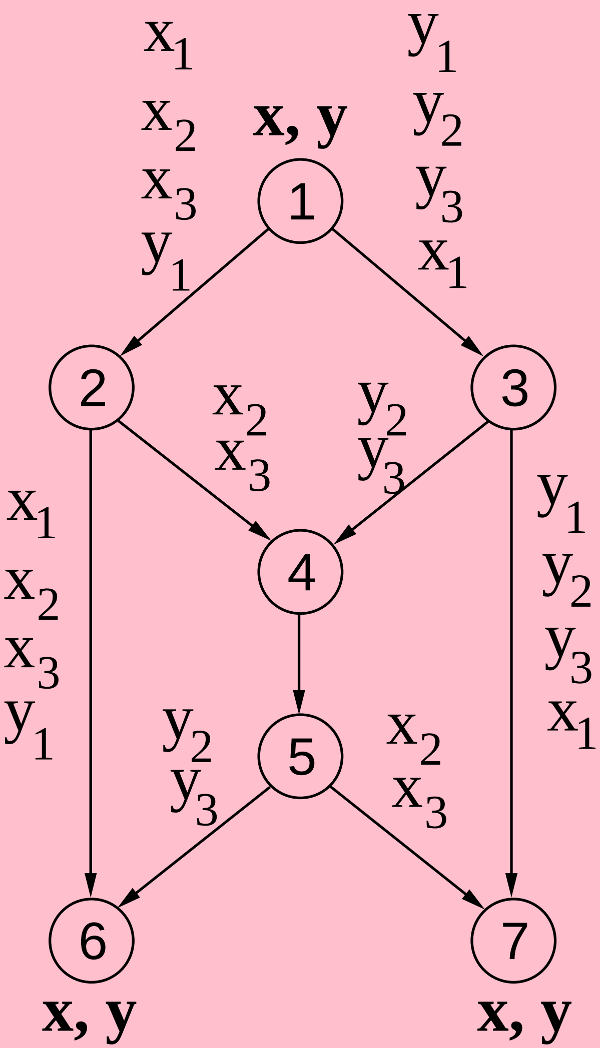


This network has a linear coding solution but no routing solution.

Each of the $2k$ message components must be carried on at least two of the edges $e_{1,2}, e_{1,3}$, $e_{4,5}$. Hence, $2(2k) \leq 3n$, and so $\boxed{\epsilon \leq 3/4}$.

Now, we will exhibit a $(3, 4)$ fractional routing solution...

Example of routing capacity continued...



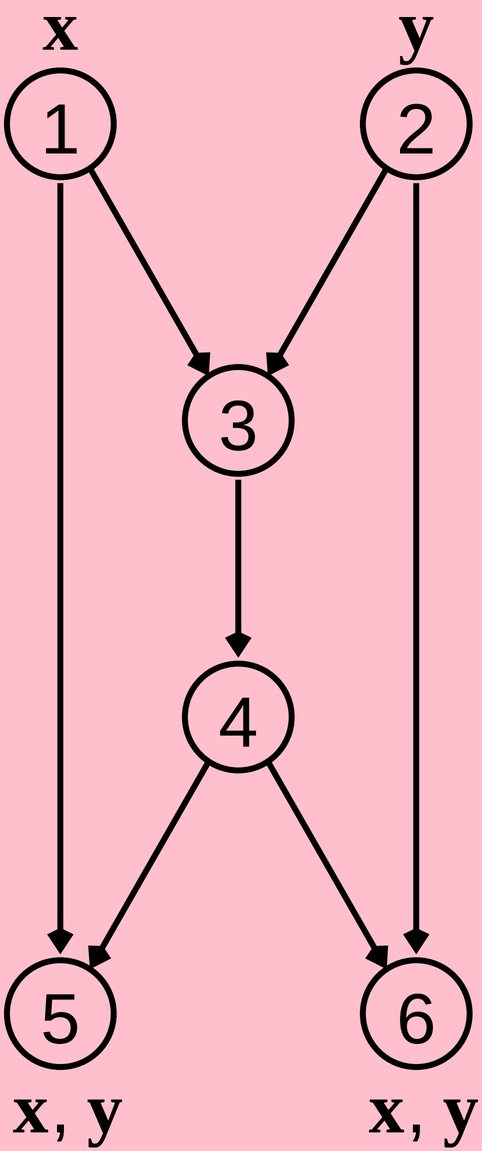
Let $k = 3$ and $n = 4$.

This is a fractional routing solution.

Thus, $3/4$ is an achievable routing rate, so $\boxed{\epsilon \geq 3/4}$.

Therefore, the routing capacity is $\boxed{\epsilon = 3/4}$.

Example of routing capacity



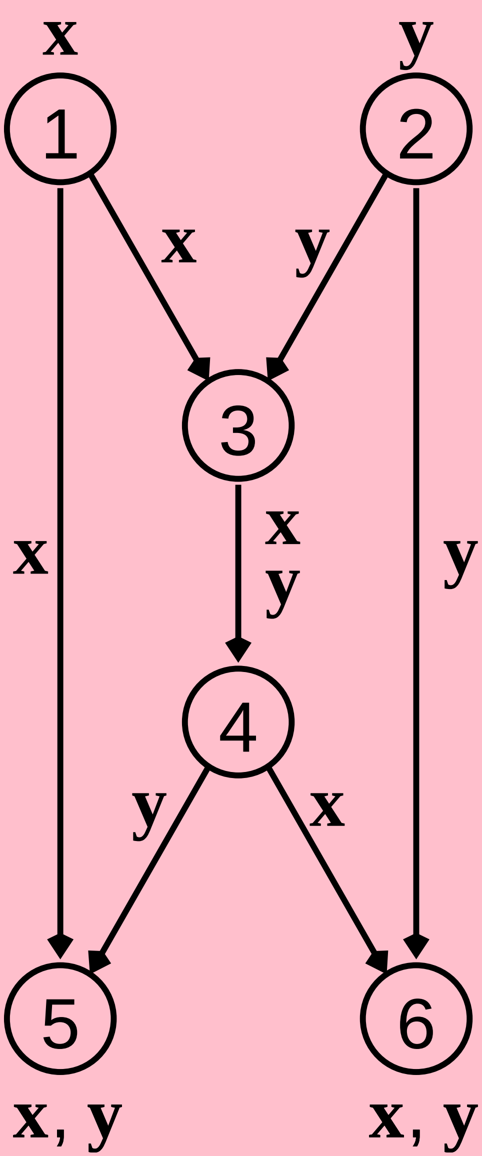
The only way to get **x** to n_6 is $n_1 \to n_3 \to n_4 \to n_6$.

The only way to get **y** to n_5 is $n_2 \to n_3 \to n_4 \to n_5$.

$e_{3,4}$ must have enough capacity for both messages.

Hence, $2k \leq n$, so $\boxed{\epsilon \leq 1/2}$.

Example of routing capacity continued...



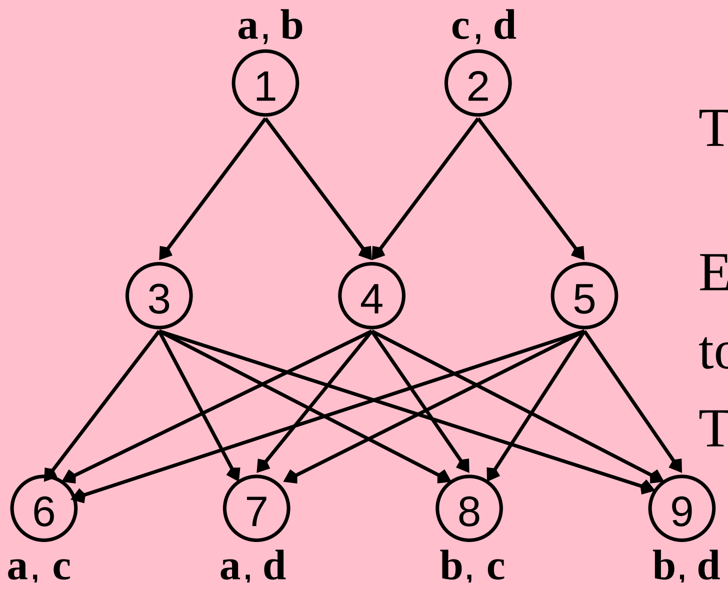
Let $k = 1$ and $n = 2$.

This is a fractional routing solution.

Thus, $1/2$ is an achievable routing rate, so $\boxed{\epsilon \geq 1/2}$.

Therefore, the routing capacity is $\boxed{\epsilon = 1/2}$.

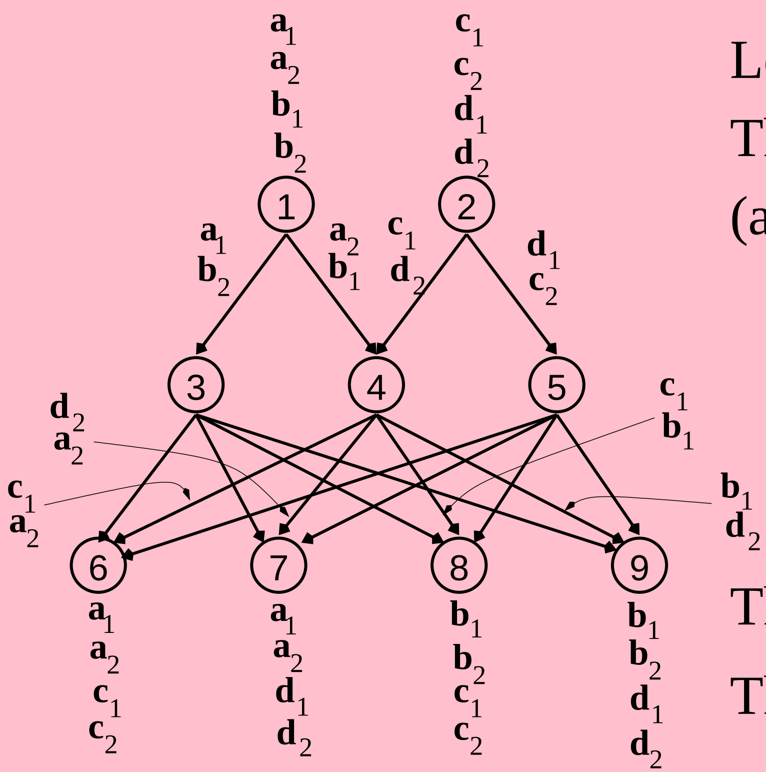
Example of routing capacity



This network is due to R. Koetter.

Each source must emit at least $2k$ components and the total capacity of each source's two out-edges is $2n$. Thus, $2k \leq 2n$, yielding $\boxed{\epsilon \leq 1}$.

Example of routing capacity continued...



Let $k = 2$ and $n = 2$.

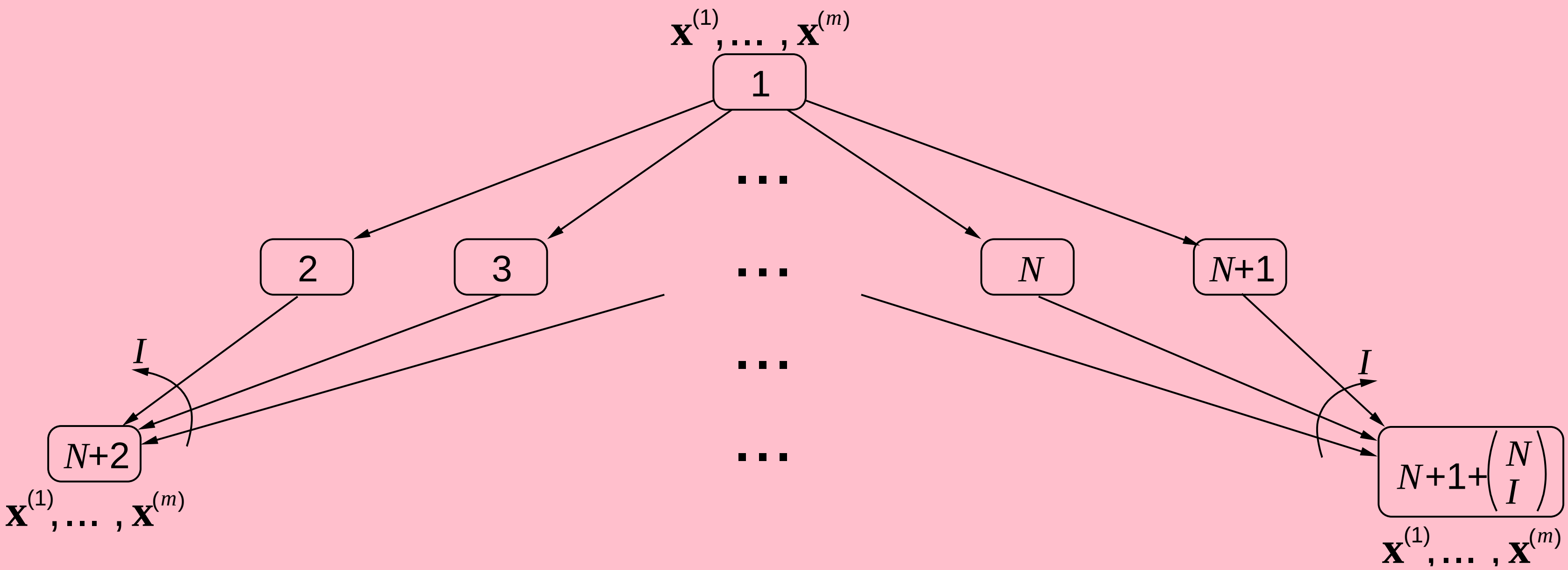
This is a fractional routing solution

(as given in MéEfHoKa, 2003).

Thus, $2/2$ is an achievable routing rate, so $\boxed{\epsilon \geq 1}$.

Therefore, the routing capacity is $\boxed{\epsilon = 1}$.

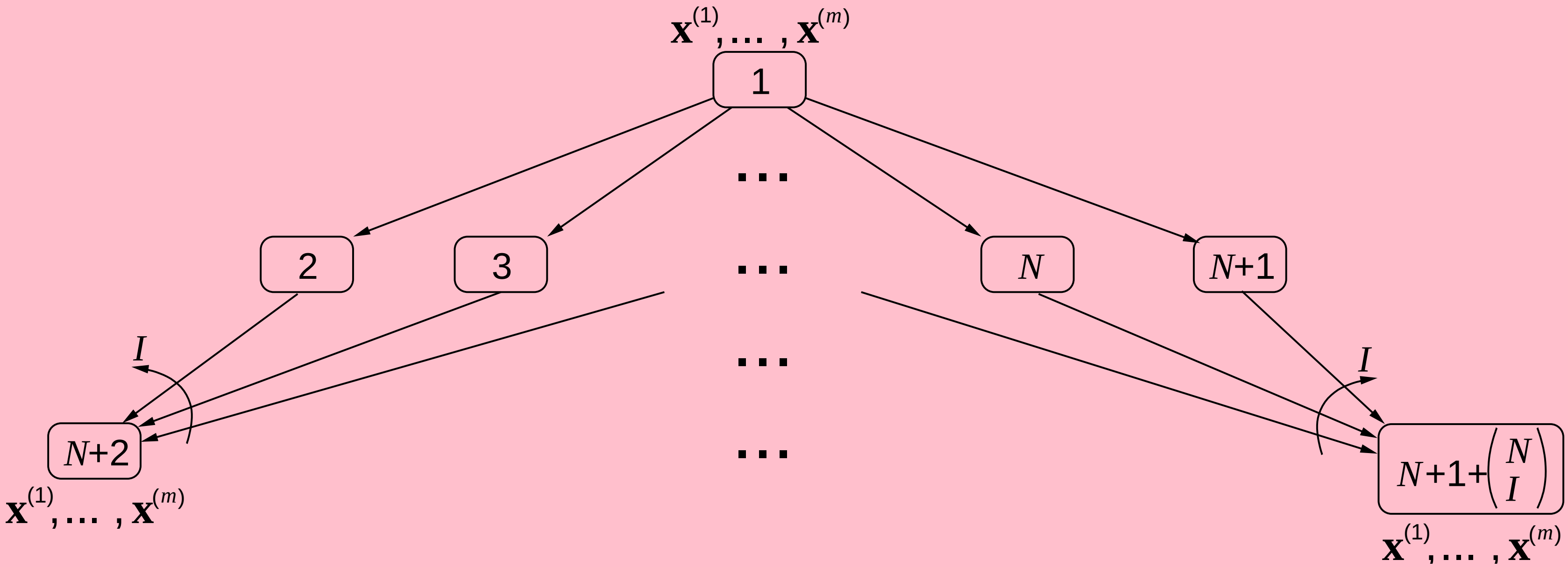
Example of routing capacity



Each node in the 3rd layer receives a unique set of I edges from the 2nd layer.

Every subset of I nodes in layer 2 must receive all mk message components from the source. Thus, each of the mk message components must appear at least $N-(I-1)$ times on the N out-edges of the source. Since the total number of symbols on the N source out-edges is Nn, we must have $mk(N-(I-1)) \leq Nn$ or equivalently $k/n \leq N/(m(N-I+1))$. Hence, $\boxed{\epsilon \leq N/(m(N-I+1))}$.

Example of routing capacity continued...

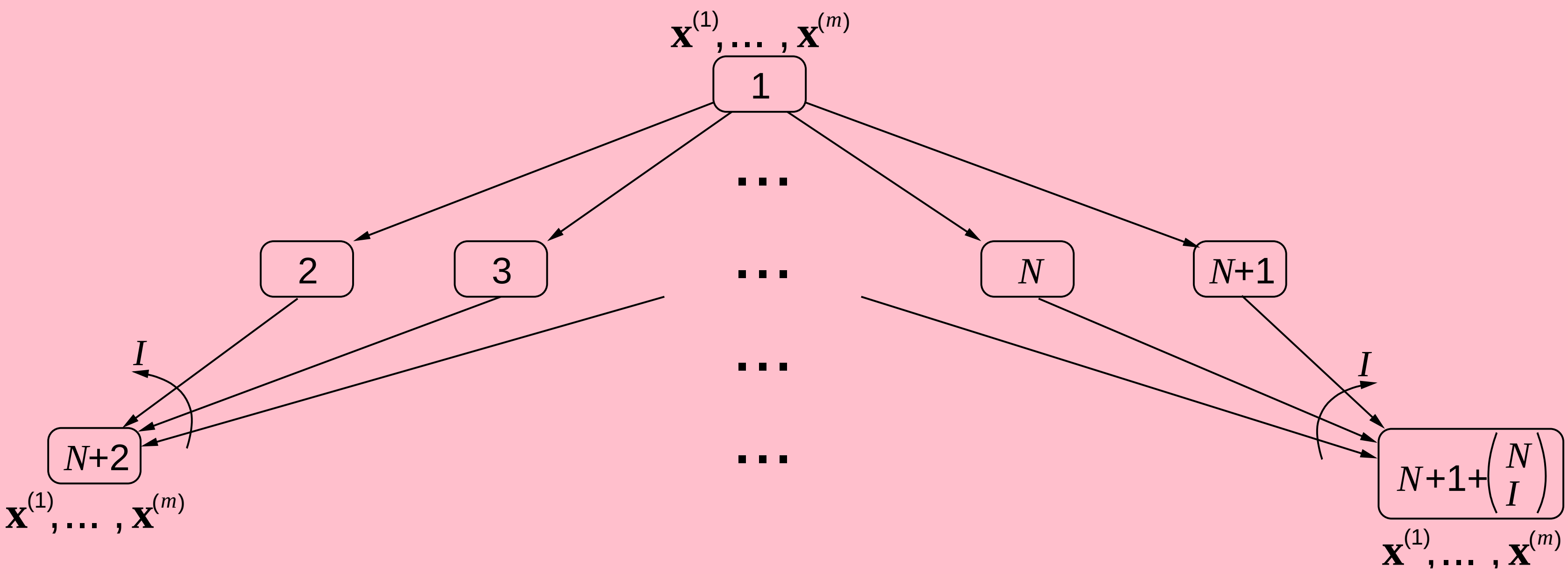


Let $k = N$ and $n = m(N - I + 1)$

There is a fractional routing solution with these parameters (the proof is somewhat involved and will be skipped here).

Therefore, $N/(m(N - I + 1))$ is an achievable routing rate, so $\boxed{\epsilon \geq N/(m(N - I + 1))}$.

Therefore, the routing capacity is $\boxed{\epsilon = N/(m(N - I + 1))}$.

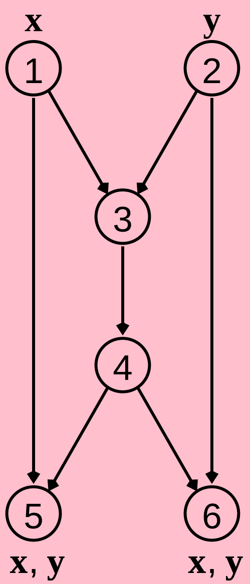


Some special cases of the network:

- $m = 5$, $N = 12$, $I = 8$ (AhRi 2004)
  No binary scalar linear solution exist. It has a non-linear binary scalar solution using a $(5, 12, 5)$ Nordstrom-Robinson error correcting code. We compute that the routing capacity is $\boxed{\epsilon = 12/25}$.

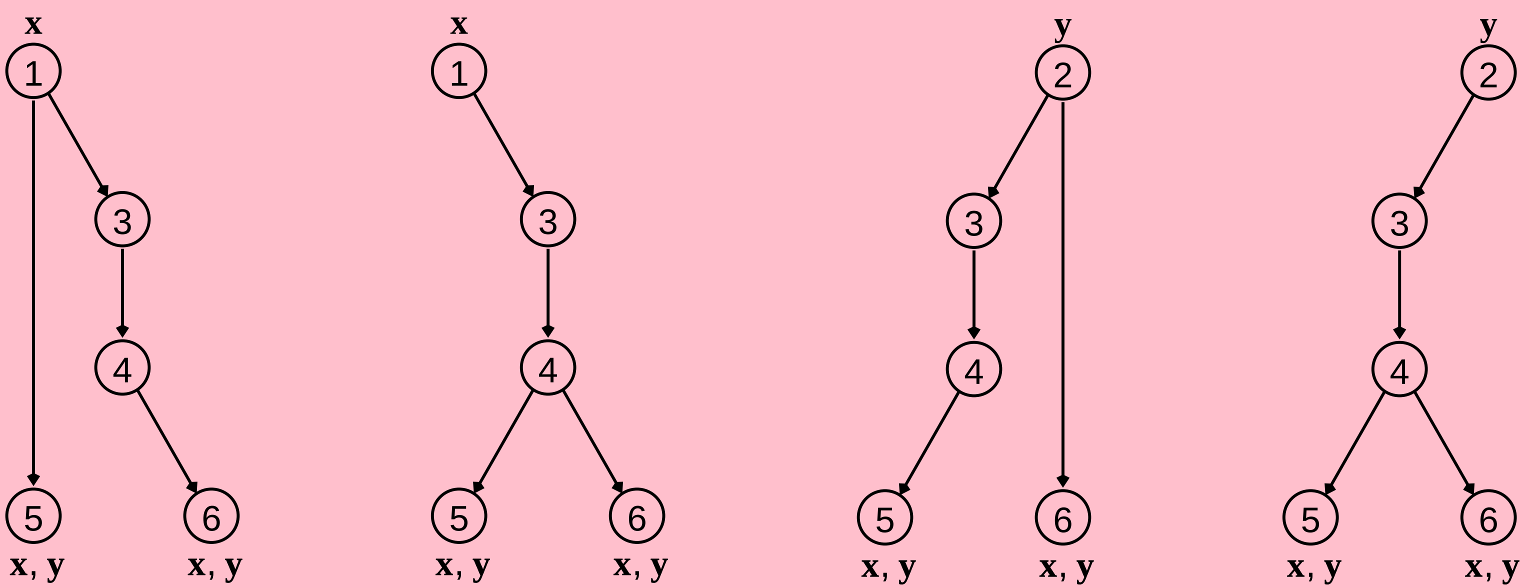
- $m = 2$, $N = p$, $I = 2$ (RaLe 2003)
  The network is solvable, if the alphabet size is at least equal to the square root of the number of sinks. We compute that the routing capacity is $\boxed{\epsilon = p/(2(p-1))}$.

- $m = 2$, $N = I = 3$
  Illustrates that the network's routing capacity can be greater than 1. We obtain $\boxed{\epsilon = 3/2}$.



For each message $\mathbf{m}$, a directed subgraph of G is an <u>$\mathbf{m}$-*tree*</u> if it has exactly one directed path from the source emitting $\mathbf{m}$ to each destination node which demands $\mathbf{m}$, and the subgraph is minimal with respect to this property (similar to directed Steiner trees).

Let $T_1, T_2, \ldots$ be all such $\mathbf{m}$-trees of a network.
e.g., this network has two $\mathbf{x}$-trees and two $\mathbf{y}$-trees:



Define the following index sets:

$$A(\mathbf{m}) = \{i : T_i \text{ is an } \mathbf{m}\text{-tree}\}$$
$$B(e) = \{i : T_i \text{ contains edge } e\}.$$

Denote the total number of trees T_i by t.

For a given network, we call the following 4 conditions the *network inequalities*:

$$\sum_{i \in A(\mathbf{m})} d_i \geq 1 \qquad (\forall \mathbf{m} \in M)$$
$$\sum_{i \in B(e)} d_i \leq \rho \qquad (\forall e \in E)$$
$$0 \leq d_i \leq 1$$
$$0 \leq \rho \leq t$$

where $d_1, \ldots, d_t, \rho$ are real variables. If a solution $(d_1, \ldots, d_t, \rho)$ to the network inequalities has all rational components, then it is said to be a *rational solution*.

(kd_i represents the number of message components carried by T_i.)

Lemma: If a non-degenerate network has a minimal fractional routing solution with achievable routing rate $r > 0$, then the network inequalities have a rational solution with $\rho = 1/r$.

Lemma: If the network inequalities corresponding to a non-degenerate network have a rational solution with $\rho > 0$, then there exists a fractional routing solution with achievable routing rate $1/\rho$.

By formulating a linear programming problem, we obtain:

Theorem: The routing capacity of every non-degenerate network is achievable.

Theorem: The routing capacity of every network is rational.

Theorem: There exists an algorithm for determining the network routing capacity.

Theorem: For each rational $r > 0$ there exists a solvable network whose routing capacity is r.

Network Coding Capacity

- The *coding capacity* is

$$\sup \{k/n \in \mathbb{Q} : \exists (k,n) \text{ fractional coding solution}\}.$$

- routing capacity $\leq$ linear coding capacity $\leq$ coding capacity

- Routing capacity is independent of alphabet size.
  Linear coding capacity is not independent of alphabet size.

- **Theorem**: The coding capacity of a network is independent of the alphabet used.

The End.

Insufficiency of Linear Network Codes

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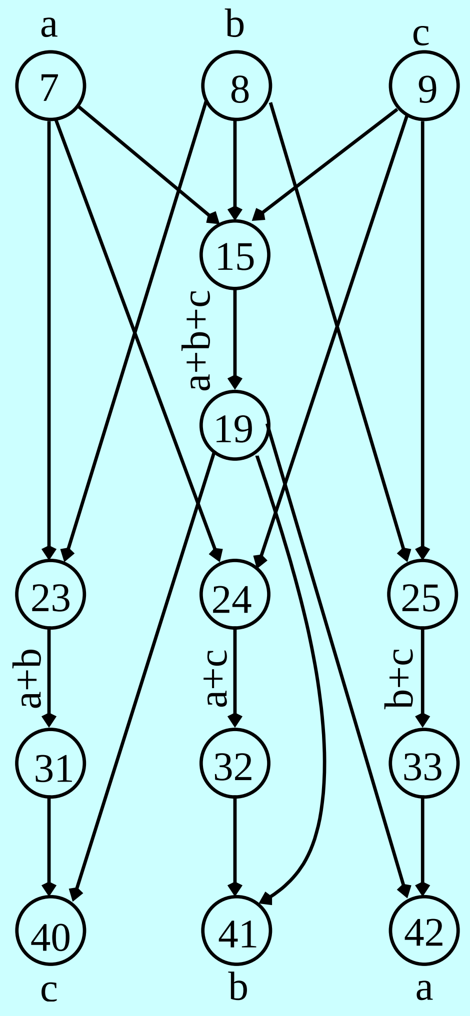
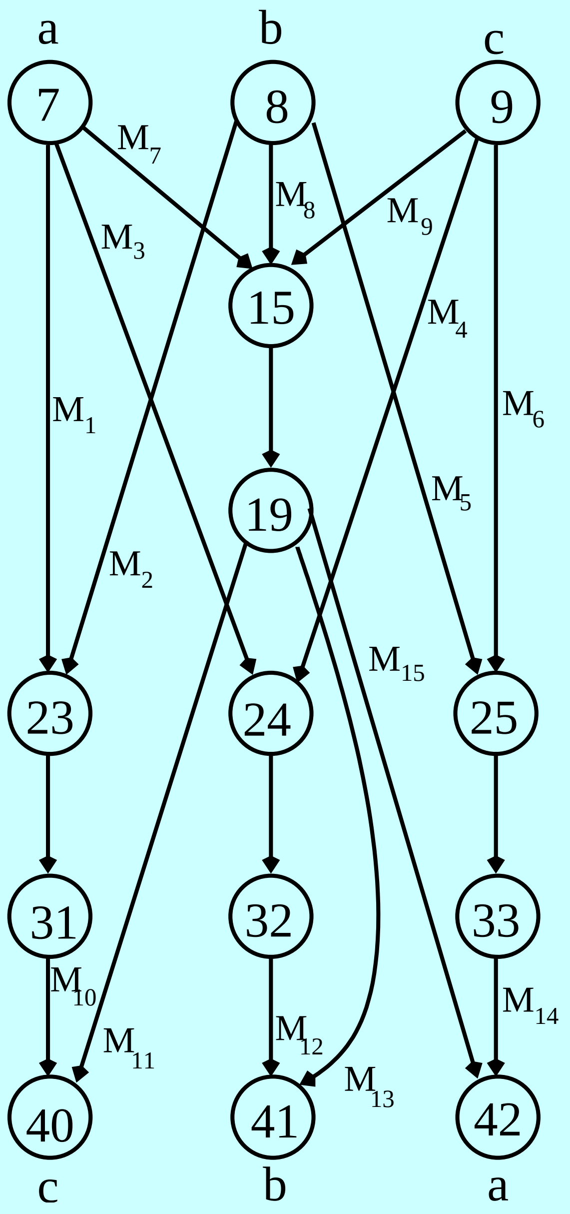


A linearly solvable network.



$$e_{23,31} = M_1 a + M_2 b$$

$$e_{24,32} = M_3 a + M_4 c$$

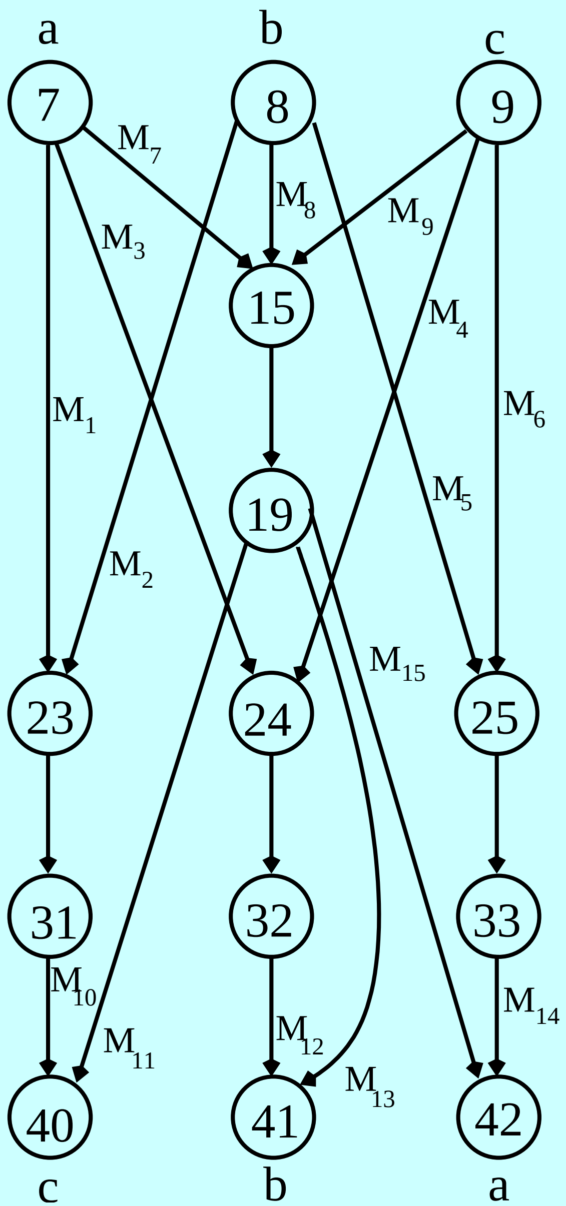
$$e_{25,33} = M_5 b + M_6 c$$

$$e_{15,19} = M_7 a + M_8 b + M_9 c$$

$$c = M_{10}(M_1 a + M_2 b) + M_{11}(M_7 a + M_8 b + M_9 c)$$

$$b = M_{12}(M_3 a + M_4 c) + M_{13}(M_7 a + M_8 b + M_9 c)$$

$$a = M_{14}(M_5 b + M_6 c) + M_{15}(M_7 a + M_8 b + M_9 c)$$



Equating coefficients of a, b, c in the previous equations gives

$$I = M_{11}M_9 = M_{13}M_8 = M_{15}M_7$$

$$M_{10}M_1 = -M_{11}M_7$$

$$M_{10}M_2 = -M_{11}M_8$$

$$M_{12}M_3 = -M_{13}M_7$$

$$M_{12}M_4 = -M_{13}M_9$$

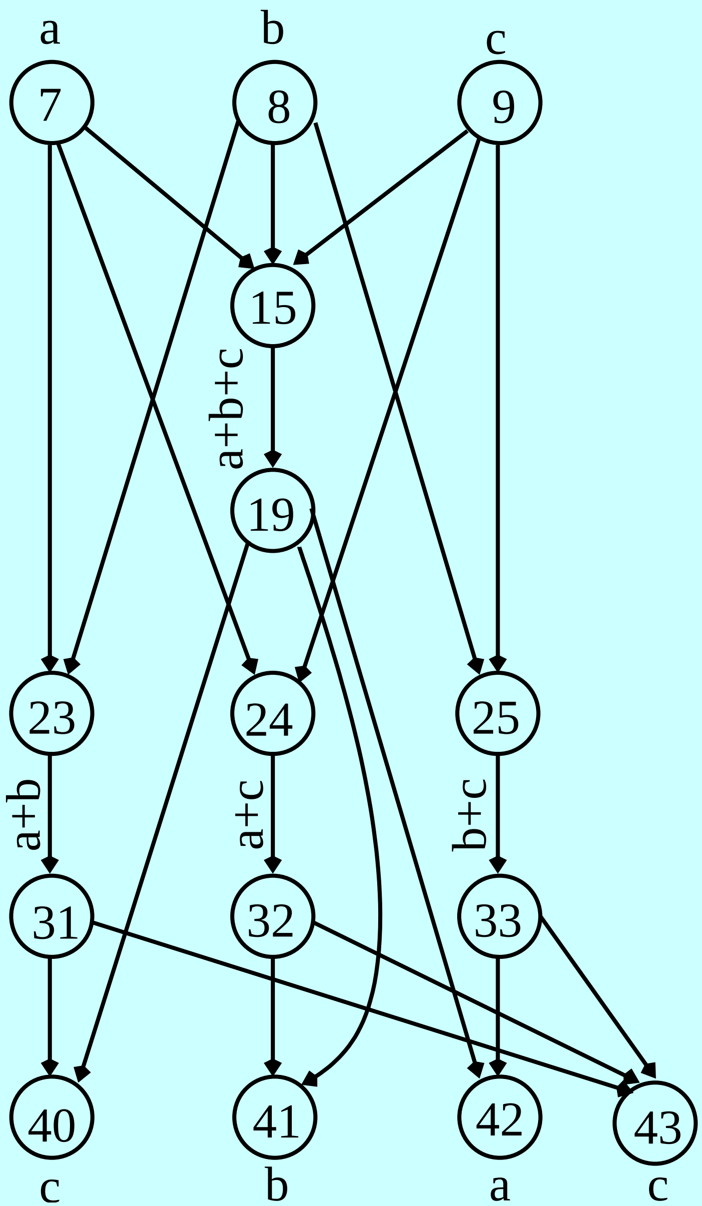
$$M_{14}M_5 = -M_{15}M_8$$

$$M_{14}M_6 = -M_{15}M_9$$

$$M_{10}(M_1a + M_2b) + M_{11}(M_7a + M_8b) = 0$$

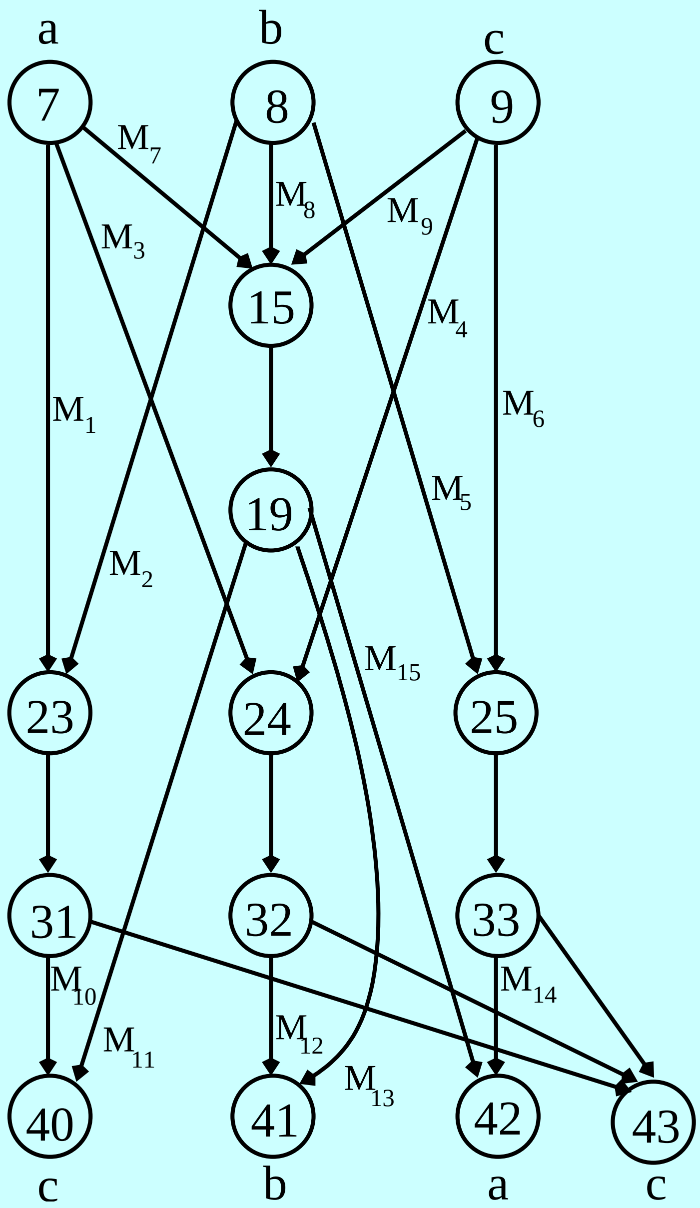
$$M_{12}(M_3a + M_4c) + M_{13}(M_7a + M_9c) = 0$$

$$M_{14}(M_5b + M_6c) + M_{15}(M_8b + M_9c) = 0$$



A network linearly solvable over odd-characteristic fields.

$$c = ((a+c)+(b+c)-(a+b)) \cdot 2^{-1}$$



$$M_7 a \longrightarrow a$$

$$M_8 b \longrightarrow b$$

$$M_9 c \longrightarrow c$$

$$M_7 a + M_8 b \longrightarrow M_1 a + M_2 b$$

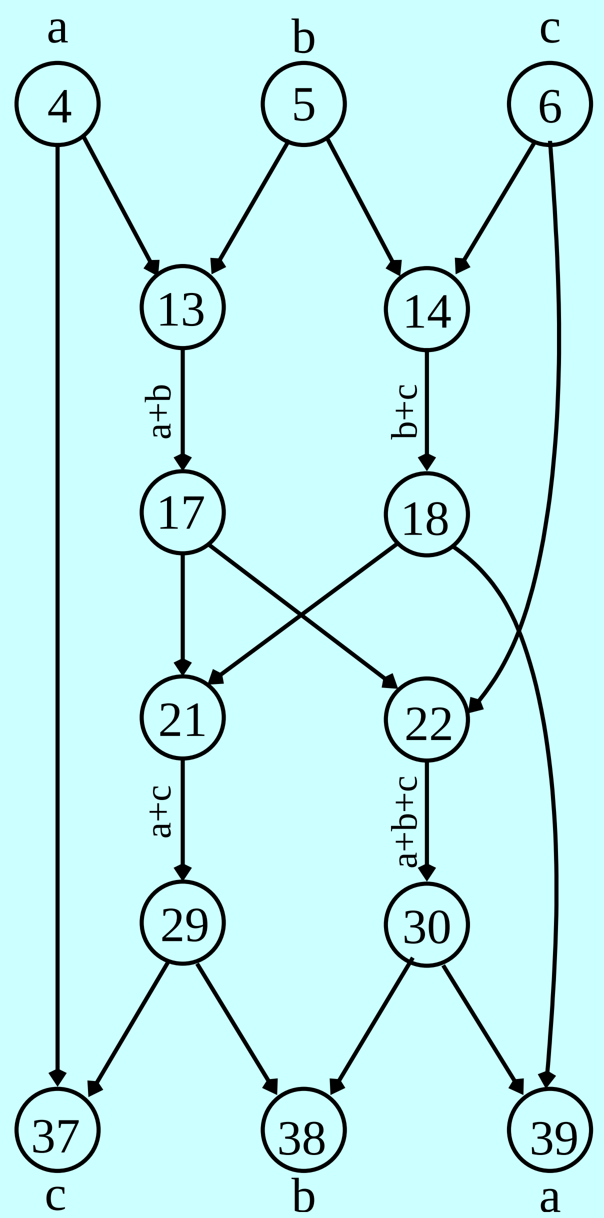
$$M_7 a + M_9 c \longrightarrow M_3 a + M_4 c$$

$$M_8 b + M_9 c \longrightarrow M_5 b + M_6 c$$

In characteristic 2:

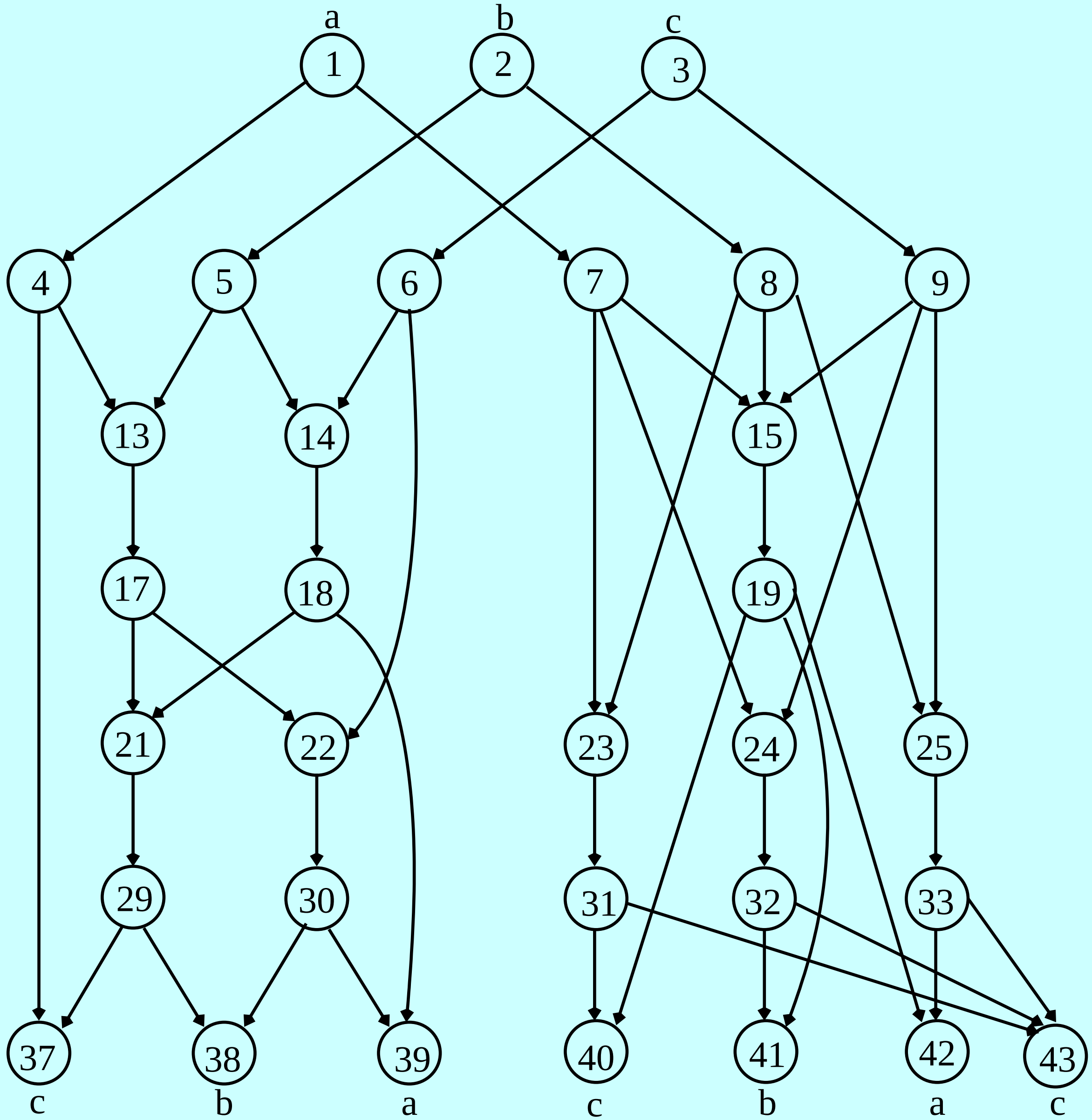
$$M_7 a + M_8 b,$$

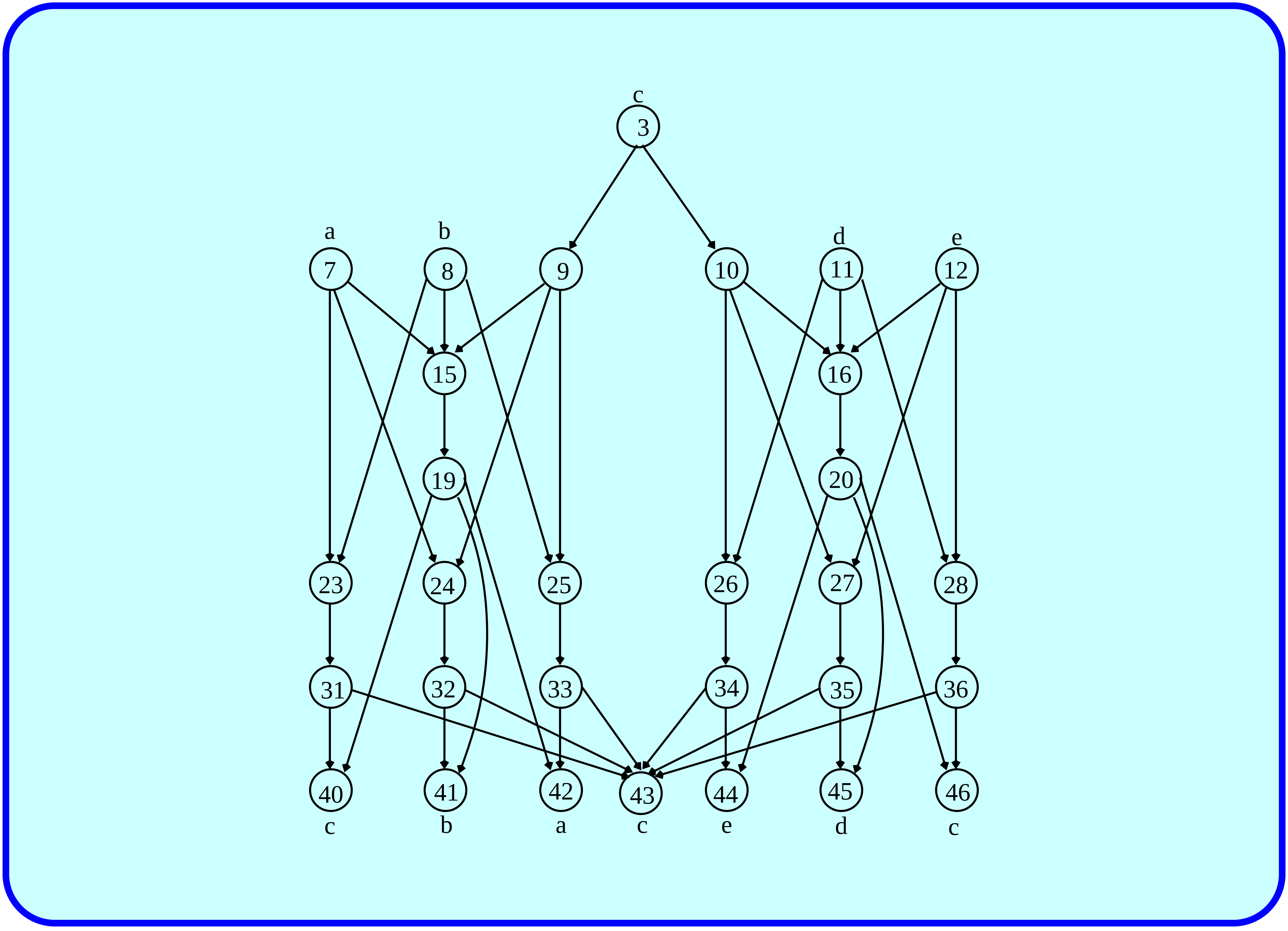
$$M_7 a + M_9 c \longrightarrow a, b, c$$



A network linearly solvable over fields of characteristic 2.

$$(a + c) = (a + b) + (b + c)$$


a b c
1 2 3
4 5 6 7 8 9
13 14 15
17 18 19
21 22 23 24 25
29 30 31 32 33
37 38 39 40 41 42 43
c b a c b a c


c
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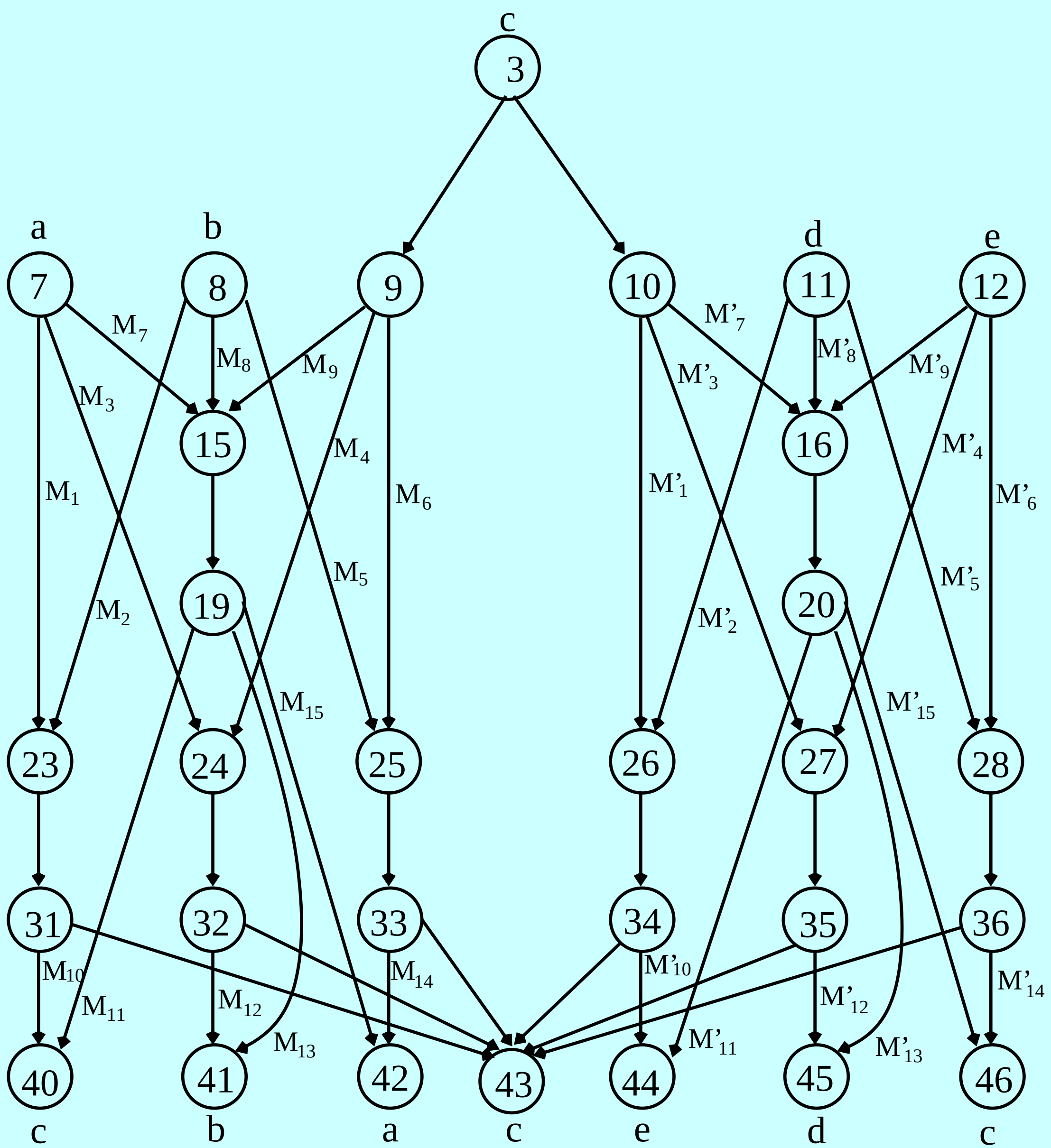


In characteristic 2:

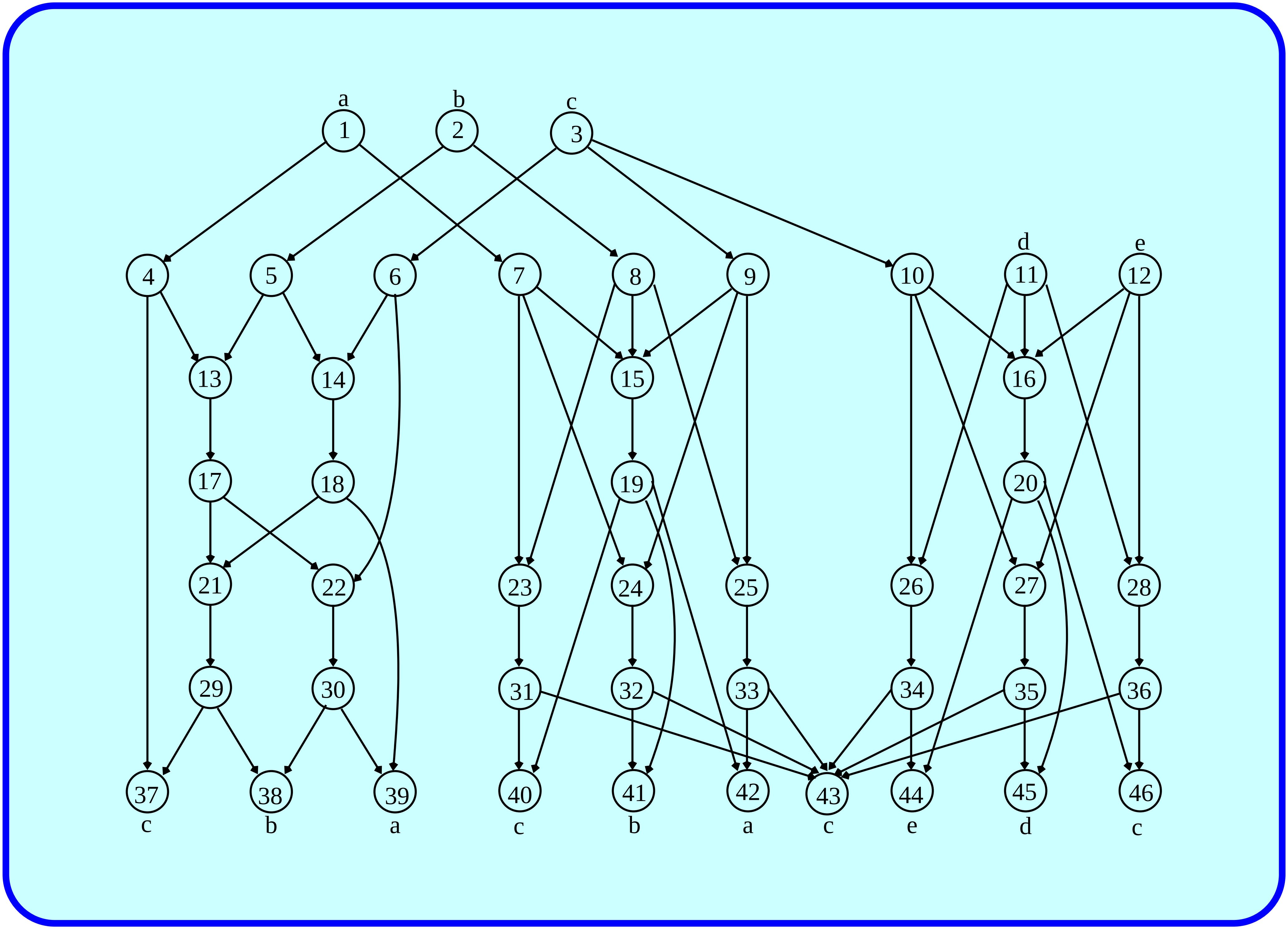
$$M_7 a + M_8 b,$$

$$M_7 a + M_9 c,$$

$$M_7' c + M_8' d,$$

$$M_7' c + M_9' e,$$

$$\longrightarrow a, b, c, d, e$$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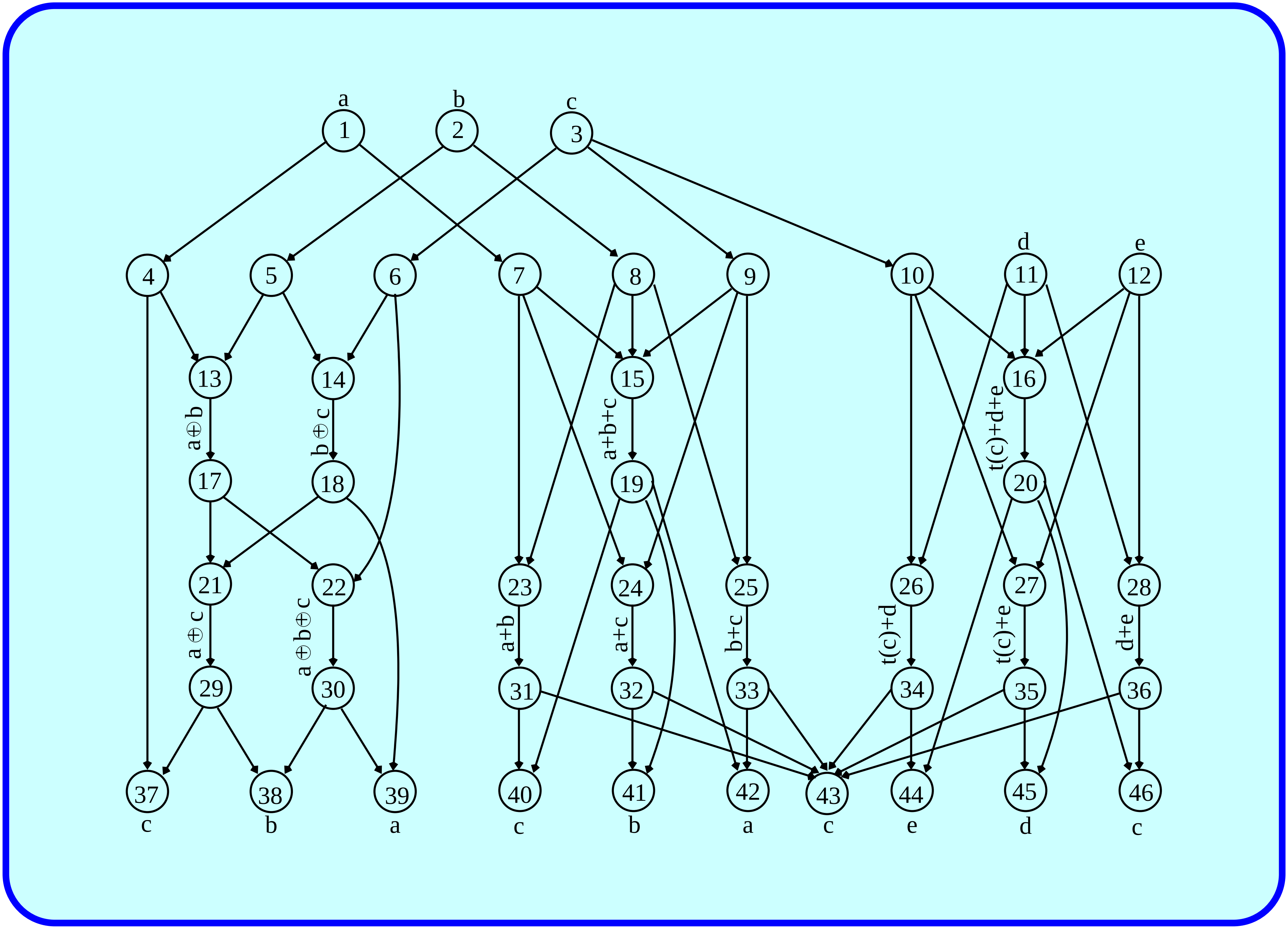
c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a⊕b
b ⊕ c
a+b+c
t(c)+d+e
17
18
19
20
21
22
23
24
25
26
27
28
a ⊕ c
a ⊕b⊕ c
a+b
a+c
b+c
t(c)+d
t(c)+e
d+e
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
c
b
a
c
b
a
c
e
d
c

Definitions

- A (k, n) *fractional linear solution* over F uses linear edge functions and decoding functions, where each source message is a vector of k elements of F and each edge carries a vector of n elements of F.

- The *linear capacity* of a network over F is the supremum of k/n over all pairs (k, n) for which there exists a (k, n) fractional linear solution over F.

- A network is *asymptotically linearly solvable* if its linear capacity is at least 1.

As shown before,

$$\begin{aligned} I &= M_{11}M_9 = M_{13}M_8 = M_{15}M_7 \\ M_{10}M_1 &= -M_{11}M_7 \\ M_{10}M_2 &= -M_{11}M_8 \\ M_{12}M_3 &= -M_{13}M_7 \\ M_{12}M_4 &= -M_{13}M_9 \\ M_{14}M_5 &= -M_{15}M_8 \\ M_{14}M_6 &= -M_{15}M_9. \end{aligned}$$

Notice that:

$$M_1, \ldots, M_9 \text{ are } n \times k$$

$$M_{10}, \ldots, M_{15} \text{ are } k \times n$$

$$M_7, M_8, M_9, M_{11}, M_{13}, M_{15} \text{ have rank } k$$

$$M_{10}, M_{12}, M_{14} \text{ have rank at least } k - (n - k).$$

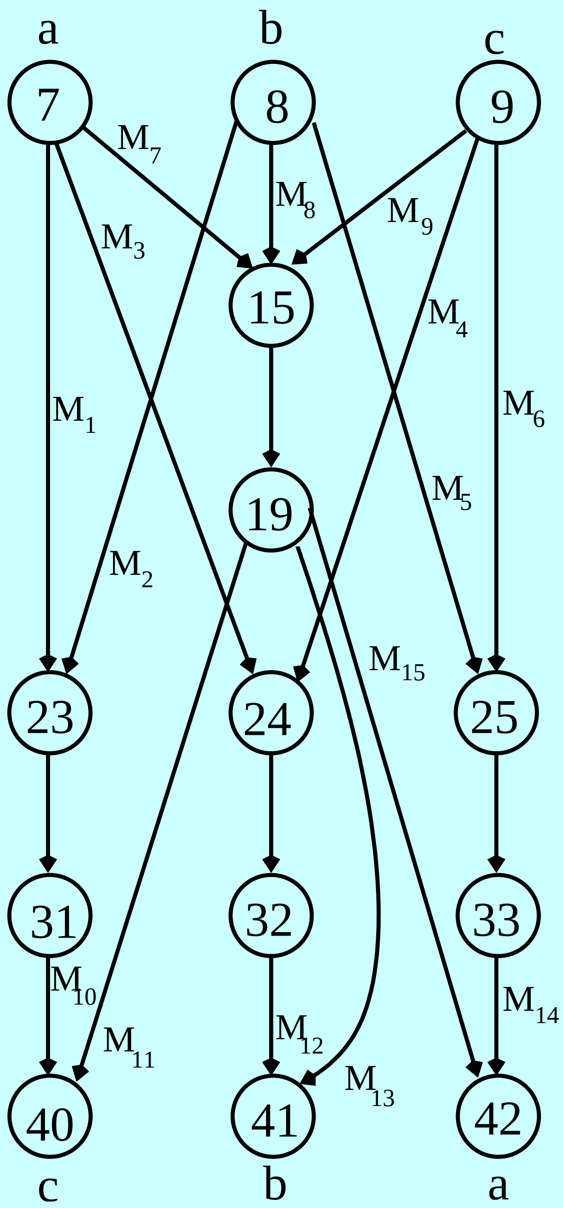
If a $k \times n$ matrix M has rank at least r, then there is an $(n-r) \times n$ matrix Q such that

$$\mathsf{rank}\left(\left[\begin{array}{c} M \\ Q \end{array}\right]\right) = n$$

and hence

$$Mx\ ,\ Qx \longrightarrow x.$$

For M_{10}, M_{12}, M_{14}, the corresponding matrices Q_{10}, Q_{12}, Q_{14} are $2(n-k) \times n$.



From

$$M_{10}(M_1 a + M_2 b) = -M_{11}(M_7 a + M_8 b)$$

we get

$$M_7 a + M_8 b, \;\; Q_{10}(M_1 a + M_2 b) \longrightarrow M_1 a + M_2 b.$$

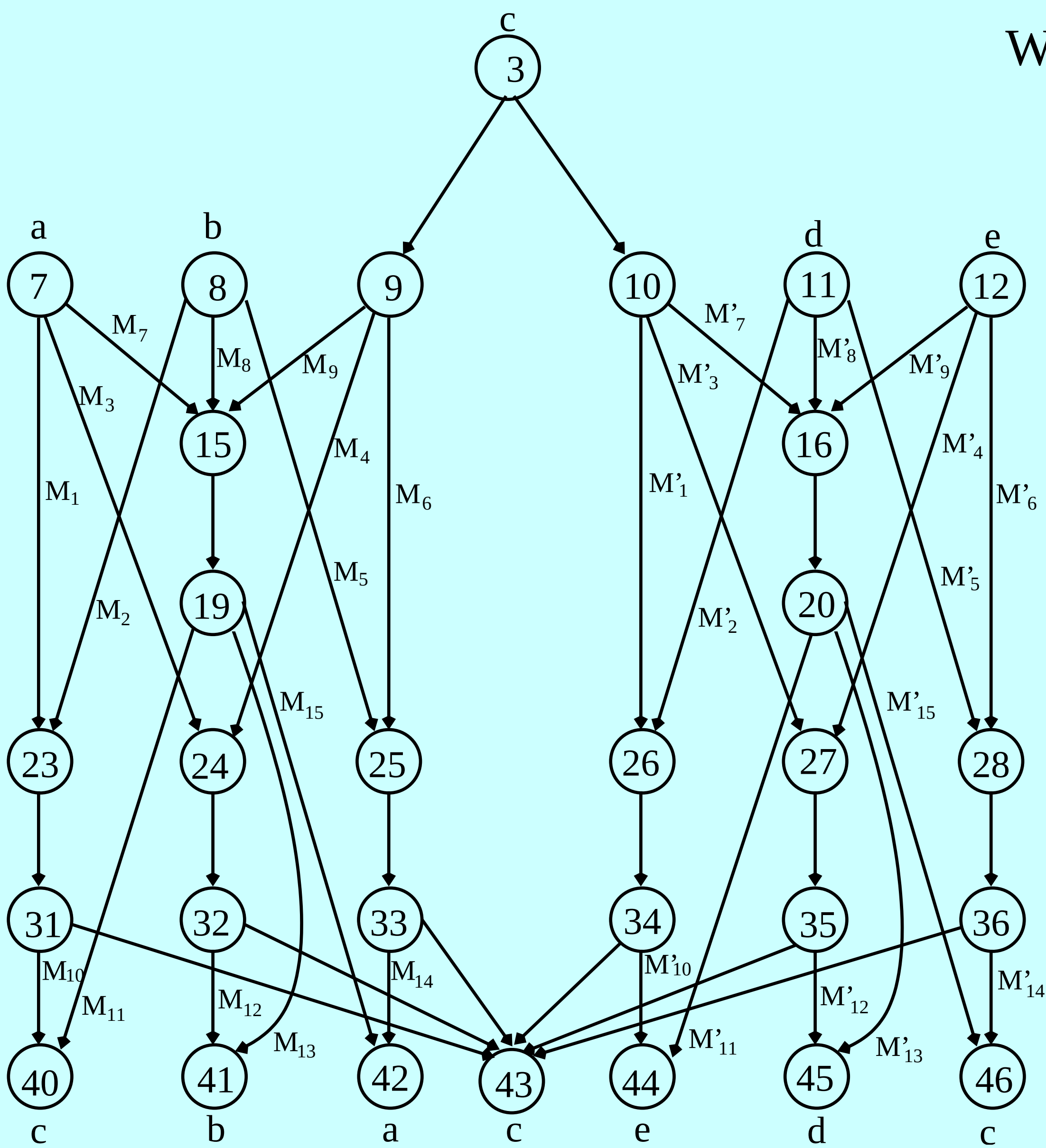
Similarly,

$$M_7 a + M_9 c, \;\; Q_{12}(M_3 a + M_4 c) \longrightarrow M_3 a + M_4 c$$

$$M_8 b + M_9 c, \;\; Q_{14}(M_5 b + M_6 c) \longrightarrow M_5 b + M_6 c.$$

And we still have

$$M_7 a \longrightarrow a, \;\; M_8 b \longrightarrow b \;\; M_9 c \longrightarrow c.$$



We now get in characteristic 2:

$$M_7 a + M_8 b,$$

$$M_7 a + M_9 c,$$

$$Q_{10}(M_1 a + M_2 b),$$

$$Q_{12}(M_3 a + M_4 c),$$

$$Q_{14}(M_5 b + M_6 c),$$

$$M_7' c + M_8' d,$$

$$M_7' c + M_9' e,$$

$$Q_{10}'(M_1' c + M_2' d),$$

$$Q_{12}'(M_3' c + M_4' e),$$

$$Q_{14}'(M_5' d + M_6' e)$$

$$\longrightarrow a, b, c, d, e.$$

From the previous page,
in characteristic 2 we have:

$$
\begin{aligned}
&M_7 a + M_8 b,\\
&M_7 a + M_9 c,\\
&Q_{10}(M_1 a + M_2 b),\\
&Q_{12}(M_3 a + M_4 c),\\
&Q_{14}(M_5 b + M_6 c),\\
&M'_7 c + M'_8 d,\\
&M'_7 c + M'_9 e,\\
&Q'_{10}(M'_1 c + M'_2 d),\\
&Q'_{12}(M'_3 c + M'_4 e),\\
&Q'_{14}(M'_5 d + M'_6 e)\\
&\longrightarrow a, b, c, d, e.
\end{aligned}
$$

There are $5k$ independent components on the right, so there must be at least $5k$ components on the left. So,

$$
\begin{aligned}
4n + 6(2(n-k)) &\geq 5k\\
16n &\geq 17k\\
16/17 &\geq k/n.
\end{aligned}
$$

With substantial additional work, one can show that the complete example network has:

- linear capacity $4/5$ over odd-characteristic fields, and
- linear capacity $10/11$ over even-characteristic fields.

So the network is solvable, but not asymptotically linearly solvable.

Our results

Explicit counterexample network giving:

- Non-linear solution over 4-symbol alphabet.
- No vector linear solution for any dimension or any finite field.
- No R-linear solution over any R-module
(∴ no linear solutions over Abelian groups or arbitrary rings for any dimension).
- Coding capacity is 1.
- Linear coding capacity over finite fields is $4/5$ or $10/11$ depending on parity of alphabet size.
- Linear codes are asymptotically insufficient over finite fields.
- Not solvable by means of convolutional coding or filter-bank coding.

Detailed results found in:

- R. Dougherty, C. Freiling, and K. Zeger
  "Linearity and Solvability in Multicast Networks"
  IEEE Transactions on Information Theory
  vol. 50, no. 10, pp. 2243-2256, October 2004.

- R. Dougherty, C. Freiling, and K. Zeger
  "Insufficiency of Linear Coding in Network Information Flow"
  IEEE Transactions on Information Theory
  (submitted February 27, 2004, revised January 6, 2005).

- J. Cannons, R. Dougherty, C. Freiling, and K. Zeger
  "Network Routing Capacity"
  IEEE/ACM Transactions on Networking
  (submitted October 16, 2004).

Manuscripts on-line at: *code.ucsd.edu/zeger*

The End.