Dissecting and Folding Stacked Geometric Figures

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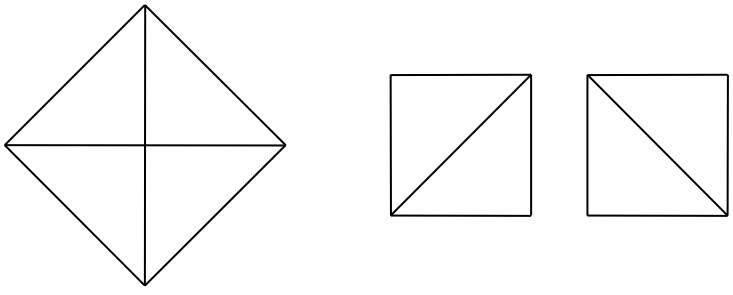
A note about animations and videos

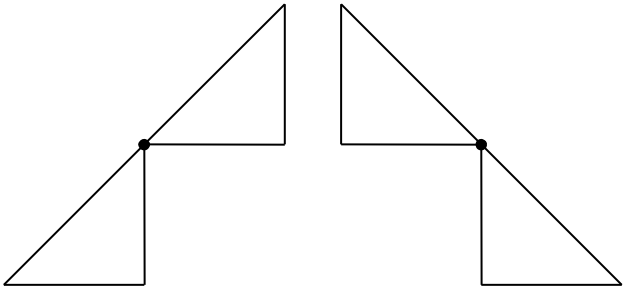
None of the animations and videos were included when my powerpoint file was converted to a pdf file. Any slide that contains a picture with color but no title is the first frame of an animation or video. Selected animations are available from my webpages:
http://www.cs.purdue.edu/homes/gnf/book3/stackfold.html
http://www.cs.purdue.edu/homes/gnf/book3/stackfold2.html
These animations were time-intensive to produce. Please show your appreciation by respecting my copyright.

Dissection of one square to two squares

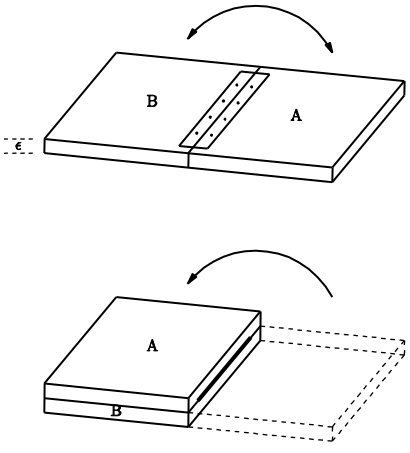
[Plato, 4th century, BCE]



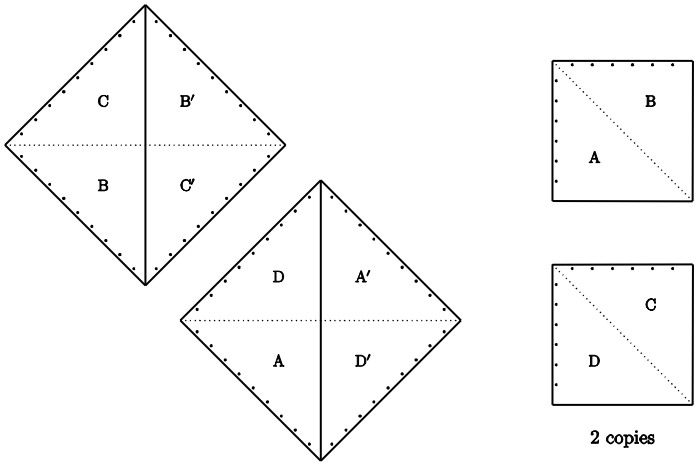
Hinged dissection of one square to two squares
(with 2 swing-hinged assemblages)



Piano-hinge (fold-hinge) brings a piece A
from being next to a piece B to being on top of B



Piano-hinged dissection of 1 square to 2 squares (with 2 assemblages)



Juggling 2 (or more) assemblages is a pain!
So let's change the game:

Instead of having two smaller squares,
let's have just one, but twice as high.

We want a folding dissection of
a 1-high square to a 2-high square,
the latter with smaller cross-section.

General goals

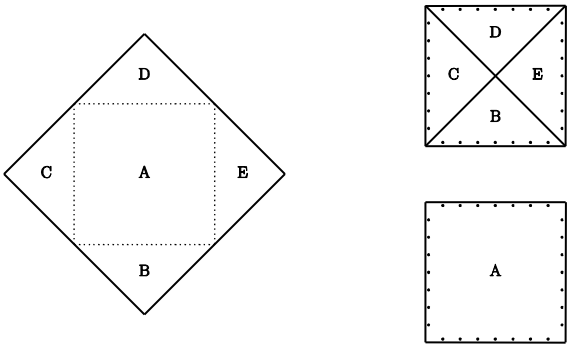
Eliminate multiple assemblages
by increasing the height of figures:

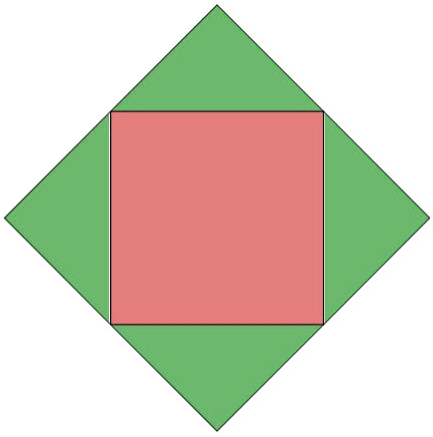
Dissect a *m*-high figure to a *n*-high figure.

Try to dissect, as always, into
the fewest possible number of pieces.

Create a nifty and lovely mathematical object.

Folds for a 1-high square to a 2-high square

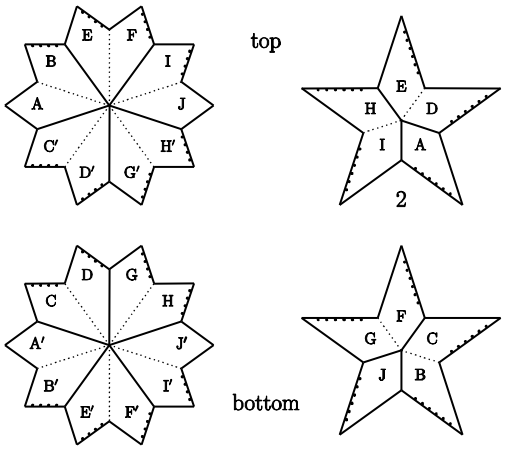


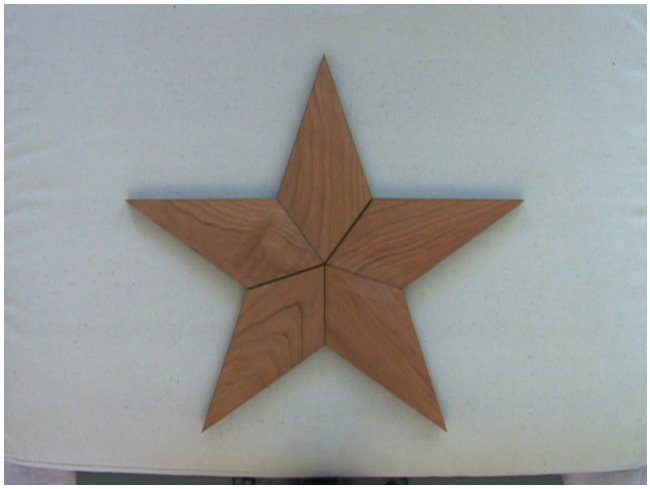


Follow-up to material in:

(published by
A K Peters,
Dec. 2006)

Folds for a 2-high 10-pointed star
to two 2-high 5-pointed stars



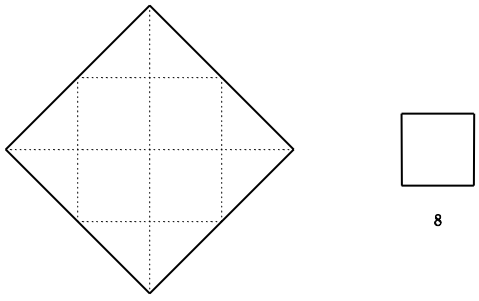


Outline of the talk

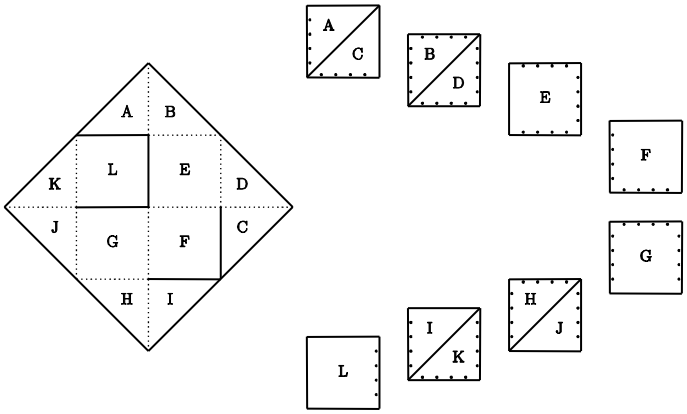
* Introduction
* Squares
* Regular Hexagons
* Hexagrams
* Equilateral Triangles
* Well-formed polyominoes
* 5-pointed and 10-pointed stars

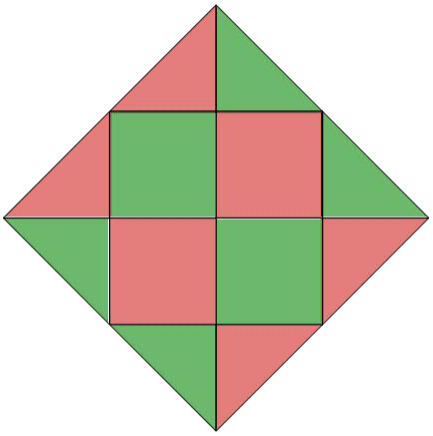
Squares

Structure of 1 large and 8 small squares



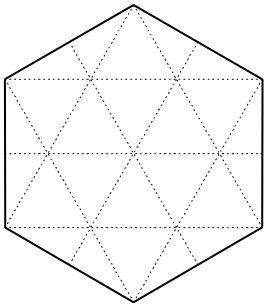
Folds for a 1-high square to an 8-high square

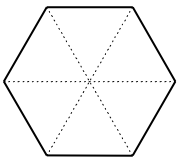




Regular Hexagons

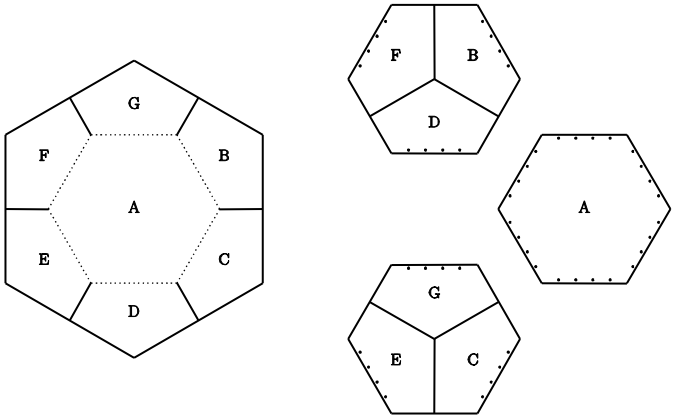
Structure of 1 large and 3 small hexagons

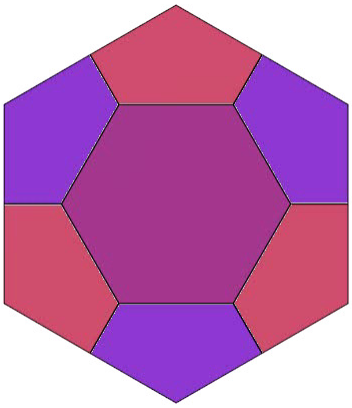




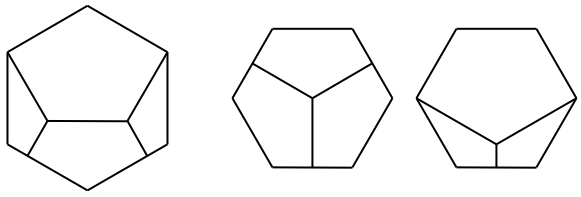
3

Folds for a 1-high hexagon to a 3-high hexagon

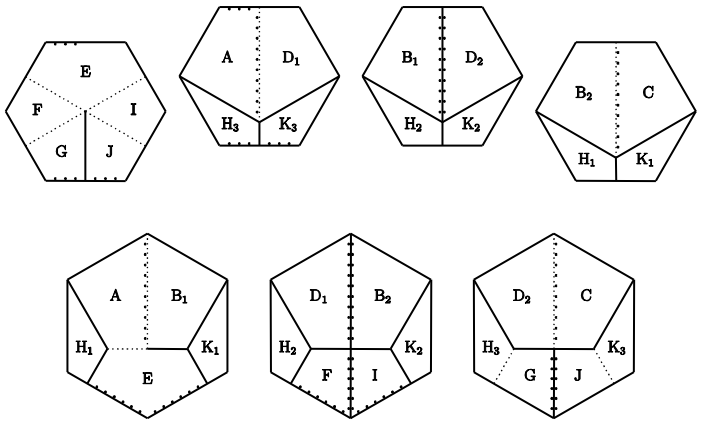


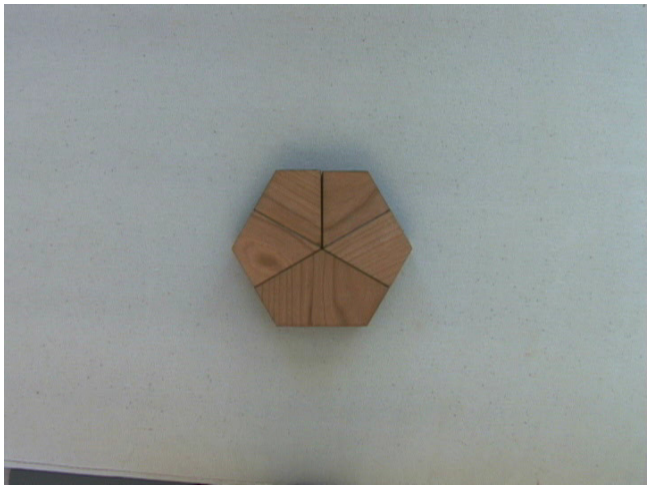


Dissection of 3 hexagons to 4 hexagons
[Ernest Freese, 1957]



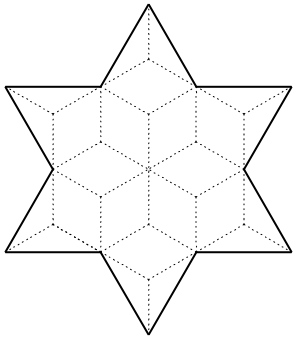
Folds for a 3-high hexagon to a 4-high hexagon

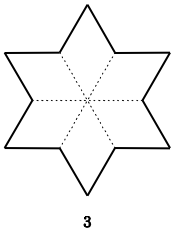




Hexagrams

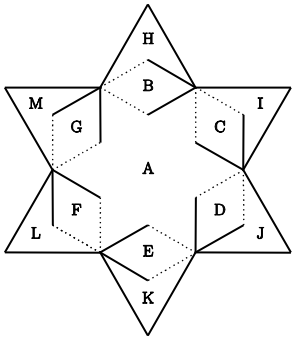
Structure of 1 large and 3 small hexagrams

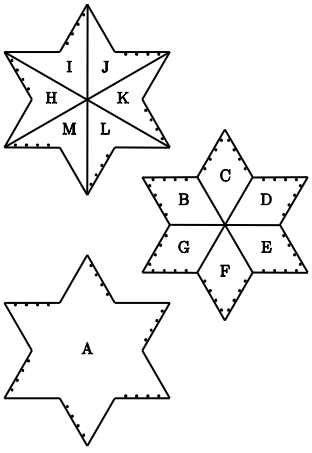


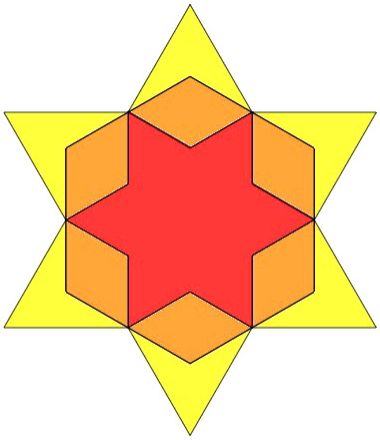


3

Folds for a 1-high hexagram to a 3-high hexagram







Equilateral Triangles

1-high triangle to *n*-high triangle

for *n* a perfect square:

1-high triangle to 4-high triangle

1-high triangle to 9-high triangle

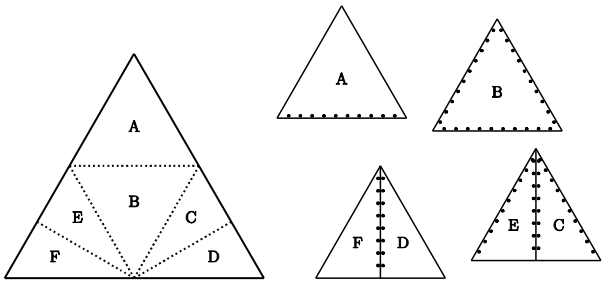
1-high triangle to 16-high triangle

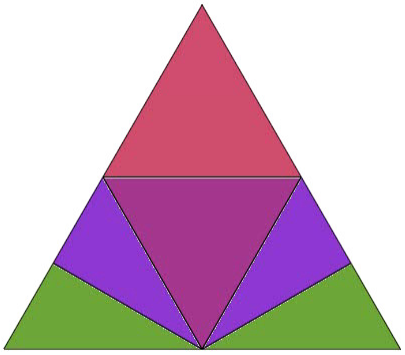
(not shown)

1-high triangle to 25-high triangle

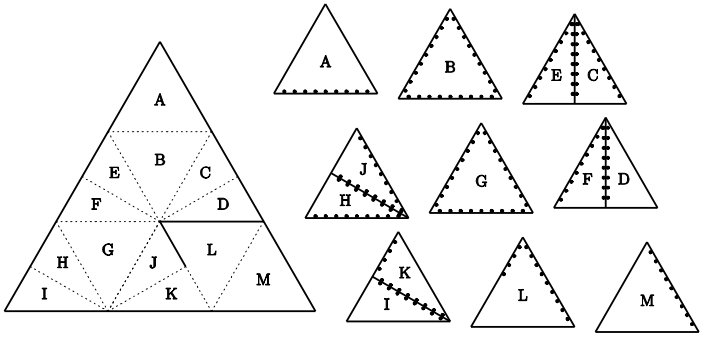
…

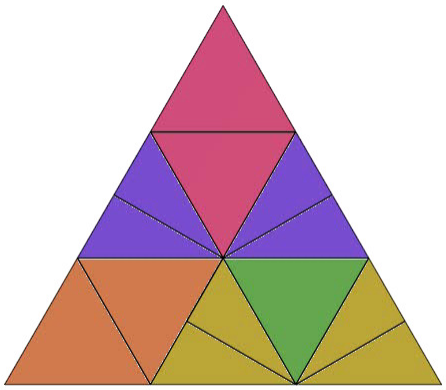
Folds for a 1-high equilateral triangle to a 4-high equilateral triangle



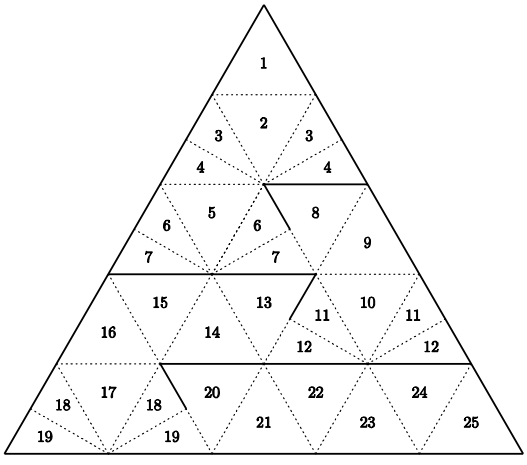


Folds for a 1-high equilateral triangle to a 9-high equilateral triangle





Folds for a 1-high equilateral triangle
to a 25-high equilateral triangle



2-high triangle to *n*-high triangle

for $n = 2(p^2 + p + 1)$:

2-high triangle to 6-high triangle ($p = 1$)

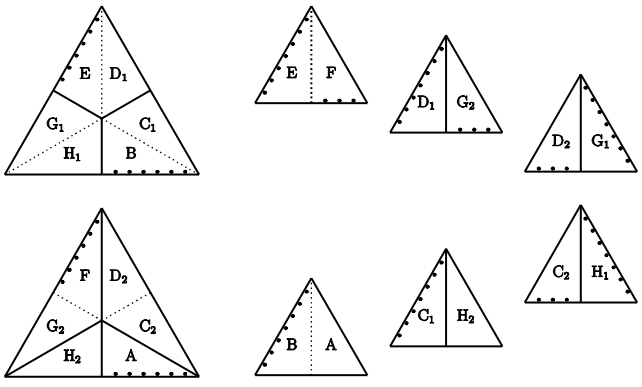
2-high triangle to 14-high triangle ($p = 2$)

2-high triangle to 26-high triangle ($p = 3$)

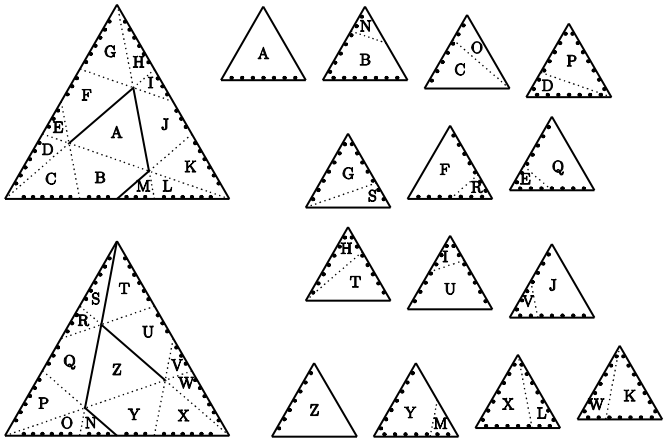
…

(Interior figure for each level of 2-high triangle is a $(p-1)$-triangle.)

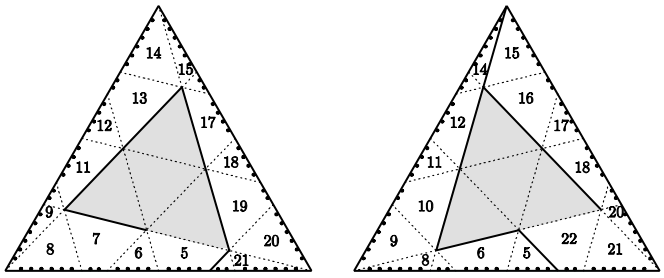
Folds for a 2-high equilateral triangle to a 6-high equilateral triangle



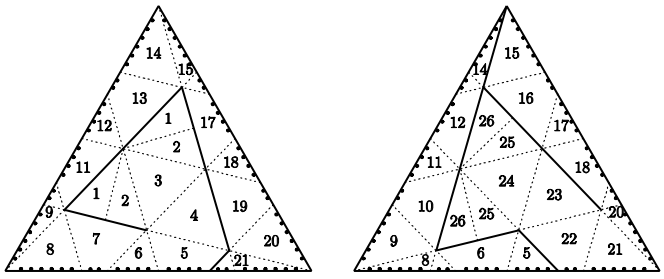
Folds for a 2-high equilateral triangle to a 14-high equilateral triangle



2-high triangle to 26-high triangle will include
two 1-high to 4-high dissections in shaded area.



Folds for a 2-high equilateral triangle
to a 26-high equilateral triangle



2-high triangle to *n*-high triangle

for $n = 2(p^2 + pq + q^2)$:

2-high triangle to 38-high triangle

($p = 3, q = 2$)

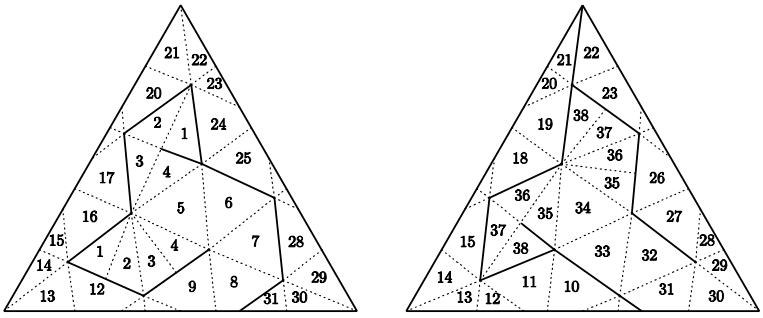
2-high triangle to 74-high triangle

($p = 4, q = 3$)

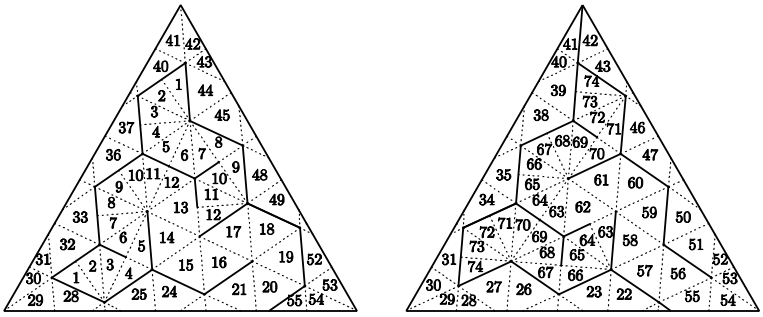
…

(Interior figure for each level of 2-high triangle is no longer a $(p-1)$-triangle when $q > 1$)

Folds for a 2-high equilateral triangle to a 38-high equilateral triangle



Folds for a 2-high equilateral triangle to a 74-high equilateral triangle



m-high triangle to *n*-high triangle

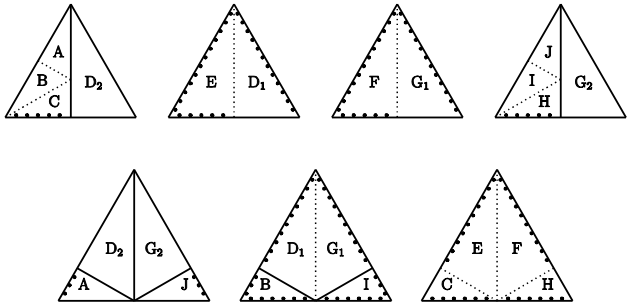
Neither *m* nor *n* is 1 or 2:

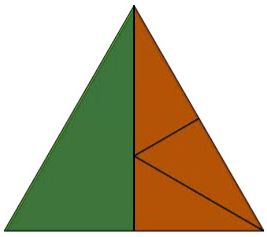
Special case: *m = 3* and *n = 4*

(Relatively easy!)

Many more combinations are possible.

Folds for a 3-high equilateral triangle
to a 4-high equilateral triangle

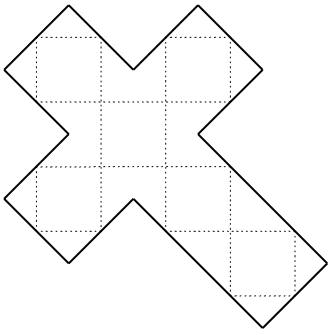


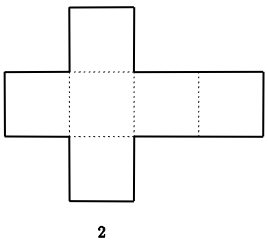


Polyominoes

(not Polynomio !)

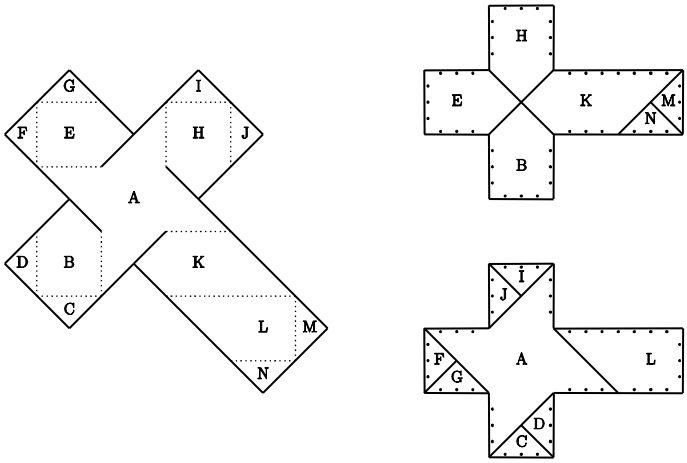
Structure of 1 large and 2 small
Latin Crosses

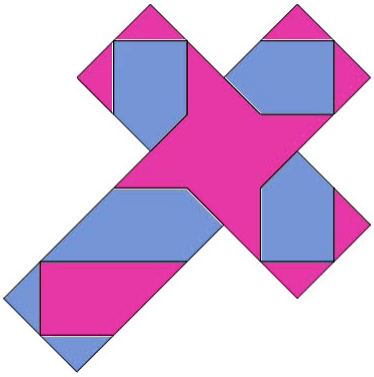




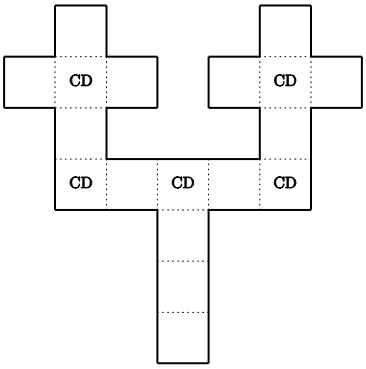
2

Folds for a 1-high Latin Cross to a 2-high Latin Cross





Well-formed Tree-Polyomino
(with “Change-Direction” squares shown)



Well-formed polyominoes

Polyomino – a figure consisting of squares that are glued together edge-to-edge.

Change-direction square – a square in a polyomino where we switch direction between horizontal and vertical.

Well-formed polyomino – a polyomino in which no two change-direction squares are directly adjacent.

Tree polyomino – a polyomino with no cycles.

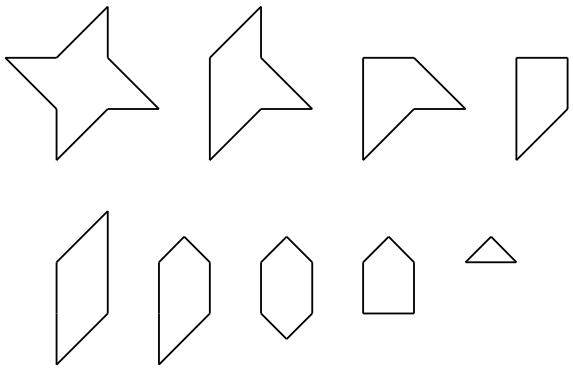
Properties of well-formed polyominoes

There is a dissection of such a 2-high figure to a 1-high figure that uses pieces from a set of just 9 shapes.

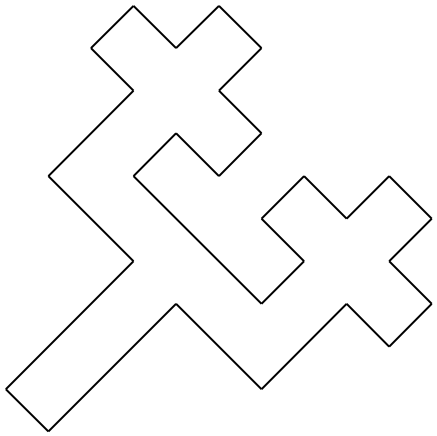
The number of pieces in the dissection equals the number of convex corners in the well-formed polyomino plus the number of squares.

The pieces in the 1-high figure alternate between those from the top level of the 2-high figure and those from the bottom level.

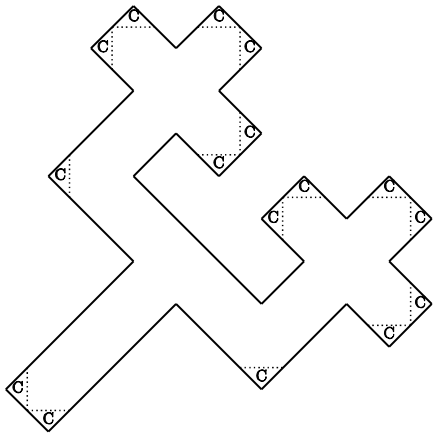
Complete set of piece shapes for dissections of 2-high to 1-high well-formed polyominoes



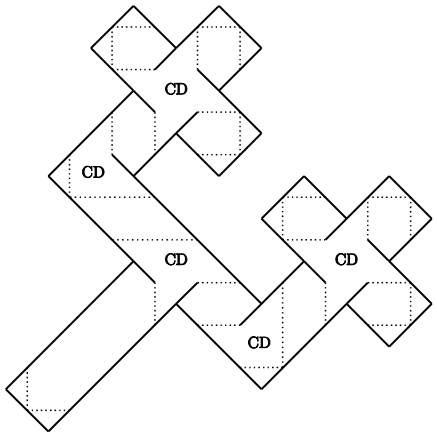
Algorithm to design the folding dissection:
Step 1. Rotate the figure by 45 degrees.



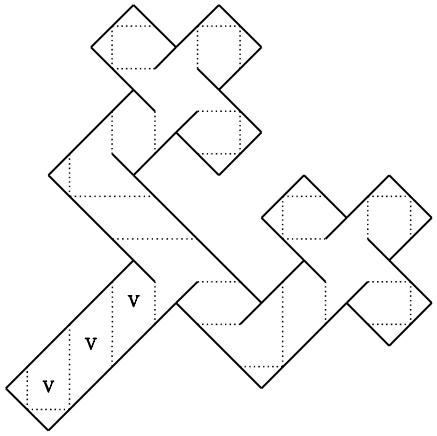
Algorithm to design the folding dissection:
Step 2. Cut isosceles right triangle at each corner.



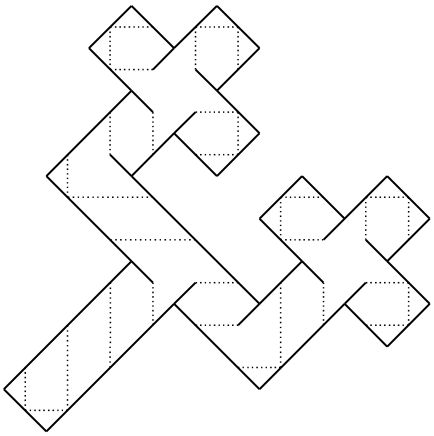
Algorithm to design the folding dissection:
Step 3. For each CD-square use a “CD-piece”.



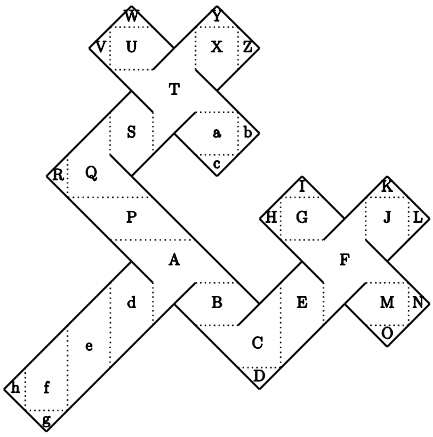
Algorithm to design the folding dissection:
Step 4. Divide up SW-to-NE arms by vertical cuts.



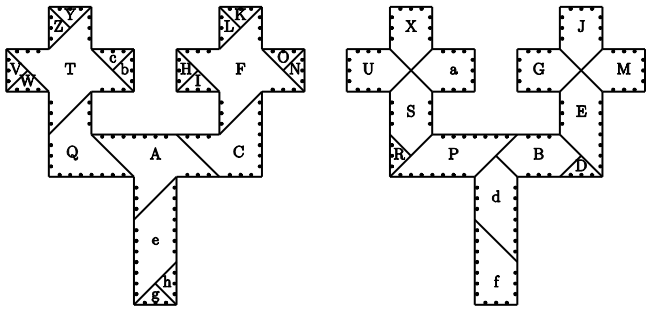
Algorithm to design the folding dissection:
Step 5. Divide up other arms by horizontal cuts.



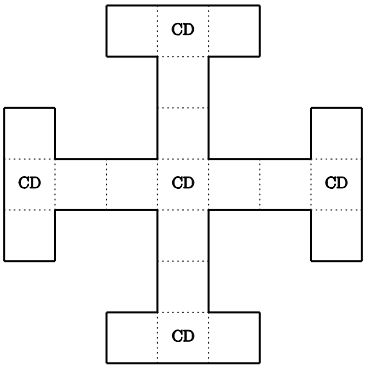
Folds for 1-high tree-polyomino
to a 2-high tree-polyomino (part 1)



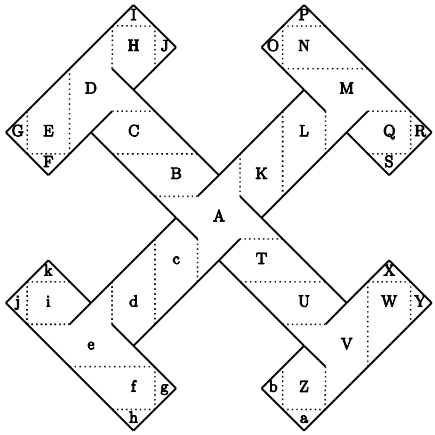
Folds for 1-high tree-polyomino to a 2-high tree-polyomino (part 2)



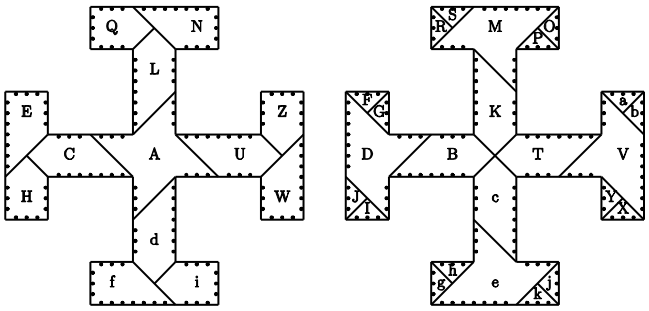
Cross of Jerusalem
(with “Change-Direction” squares shown)

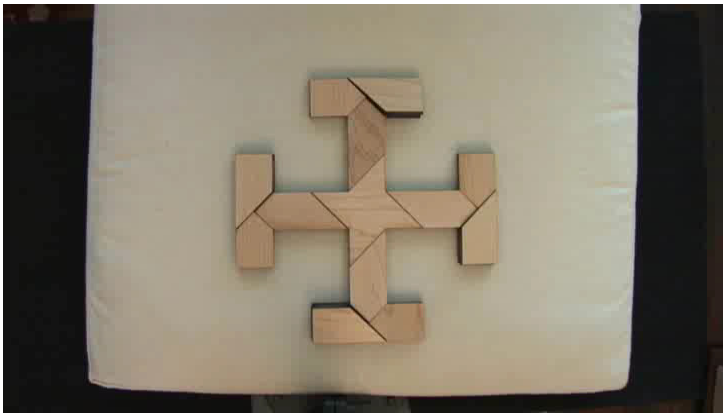


Folds for 1-high Cross of Jerusalem
to a 2-high Cross of Jerusalem (part 1)

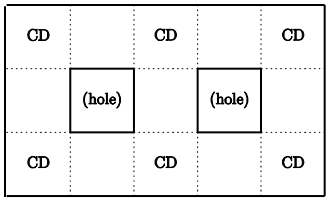


Folds for 1-high Cross of Jerusalem to a 2-high Cross of Jerusalem (part 2)

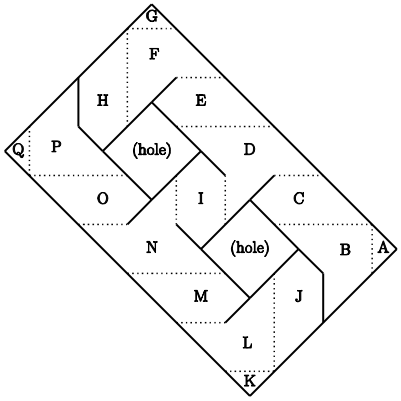


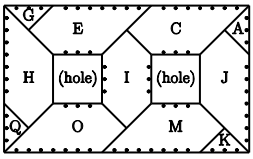


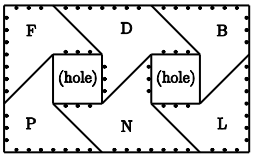
Polyomino that has 2 holes
(with “Change-Direction” squares shown)

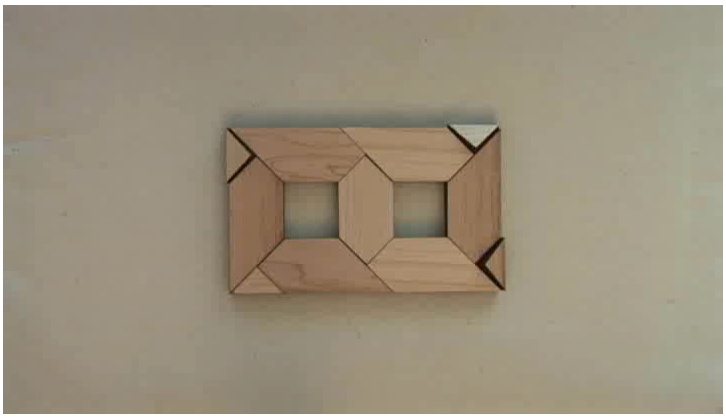


Folds for a 1-high 2-holed polyomino
to a 2-high 2-holed polyomino



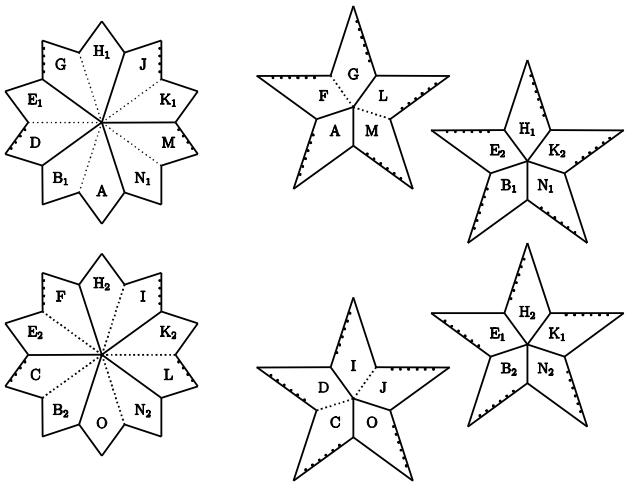


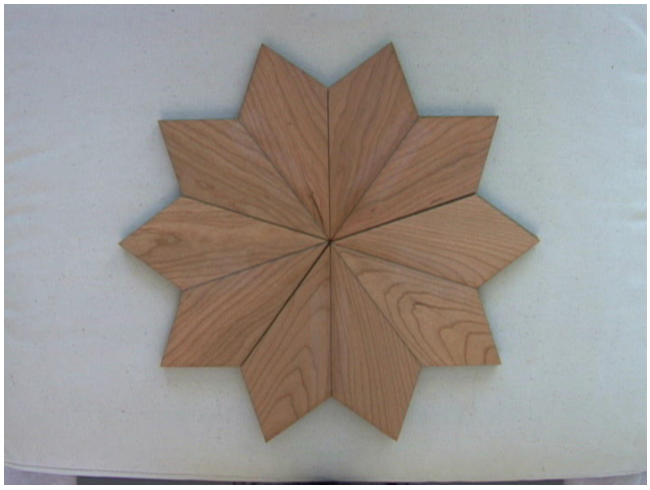




Stars

Folds for a 2-high 10-pointed star to a 4-high 5-pointed star





Conclusion

Folding a *m*-high figure to a *n*-high figure is:

* Nifty variation on piano-hinged dissections.
* At least as challenging than phd's.
* Great basis for manipulation puzzles.
* Math/puzzle/art that is within reach.
* A lot of fun!

May the Folds be with you!