Posets, Lattices and Computer Science

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Outline

- Motivating History
- Basic Structure Theorems
- Applications
- The Case Against Lattices
- Chain-Complete Partial Orders
- A Pet Peeve
- Fixpoint Theorems
- Useful Classes of Posets
- Bases for CPOs

Motivating History

- While working on the structure of B_n I ran into lattice theory
- Join-irreducibles and meet-irreducibles occur naturally in this context
- Seemed to be ignored in lattice theory once they were defined
- Will focus on finite lattices – can generalize to infinite lattices
- To me lattice are very much combinatorial and geometrical objects

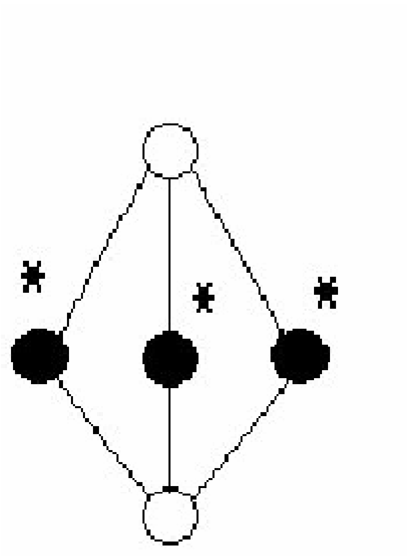
Quick Test for Distributivity

- The following is all that is required (Markowsky 1972)
- Jordan-Dedekind chain condition
- Join-rank = meet-rank = length
- Previously discovered by Avann (1961)

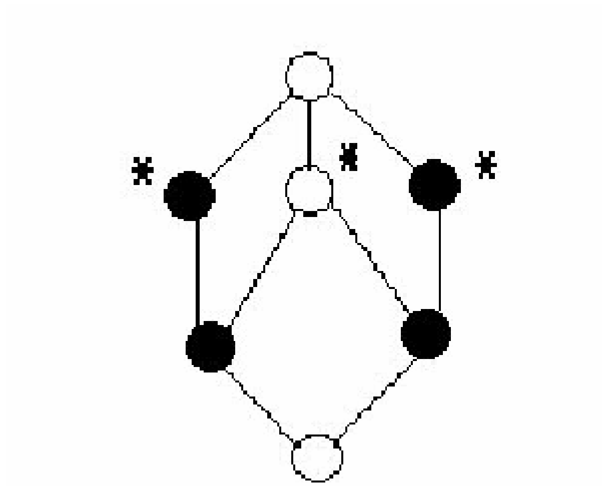
Quick Test for Distributivity

Dark Elements Are Join-irreducibles And * Elements Are Meet-irreducibles

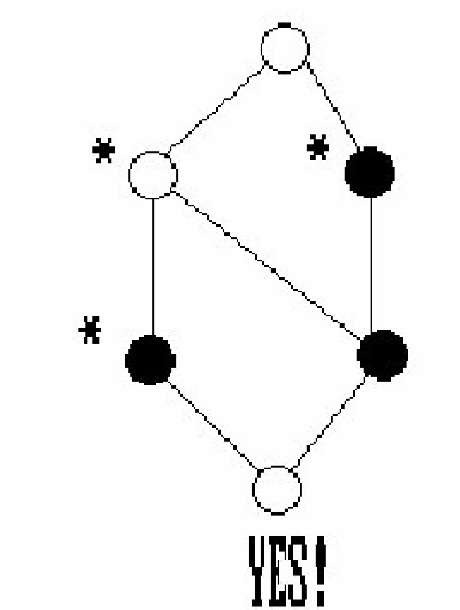
JD-Chain Condition and #JI = #MI = length



No!
Too
Short



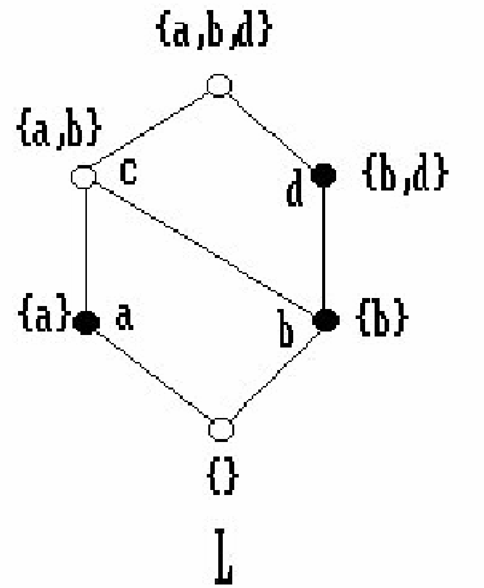
No! Too Many Join-Irreducibles

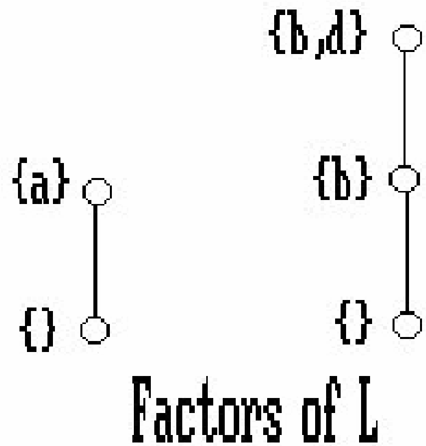


Birkhoff's Theorem

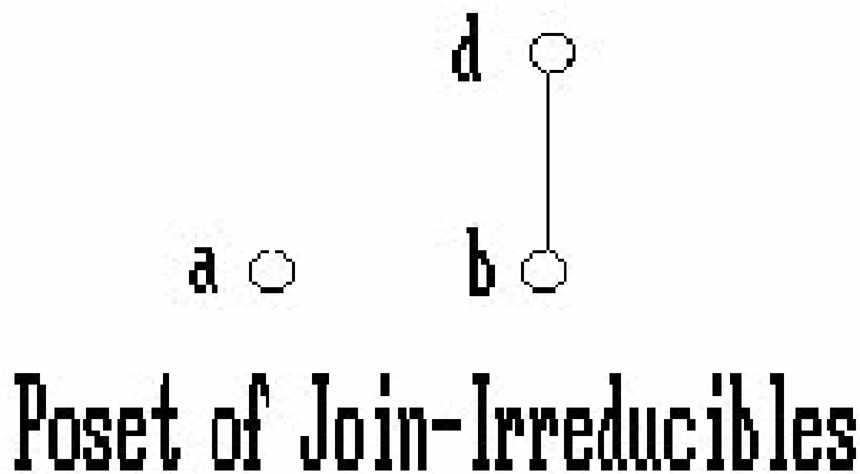
- A finite distributive lattice is isomorphic to the lattice of all closed from below subsets of the poset of join-irreducibles
- Can extend to give direct factorization
- Can extend to give automorphism group
- For distributive lattices poset of meet-irreducibles $\cong$ poset of join-irreducibles

Birkhoff's Theorem





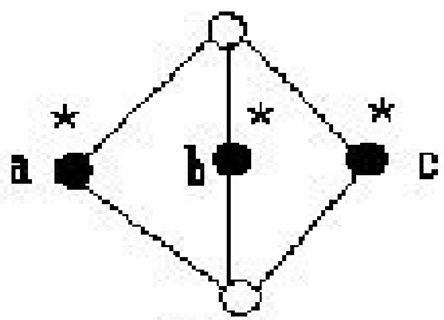
The dark elements are the join-irreducibles



Distributivity is Too Special

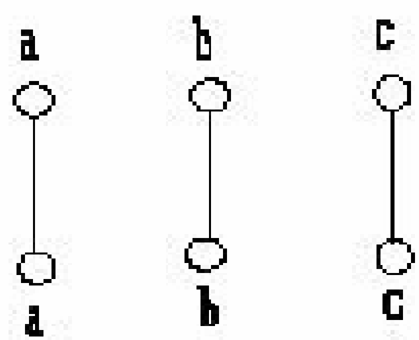
- Must consider join-irreducibles and meet-irreducibles in general
- Since elements can be both join-irreducible and meet-irreducible it seems natural to consider bipartite graphs

Candidates for Poset of Irreducibles

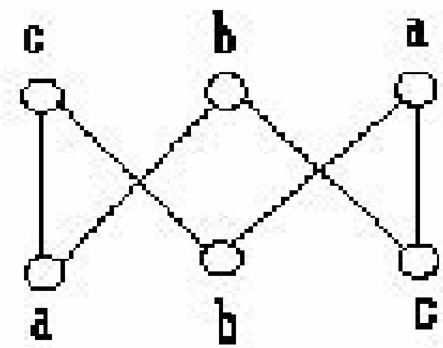


Lattice

Induced Order

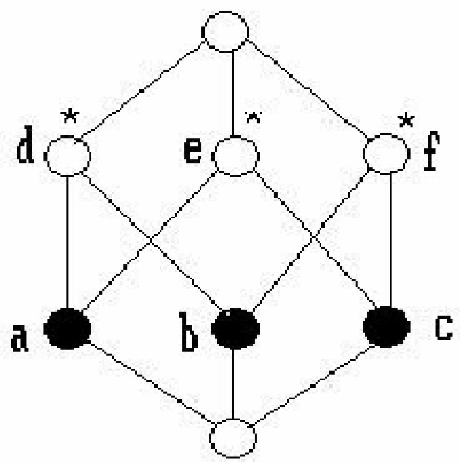


Extended Induced Order

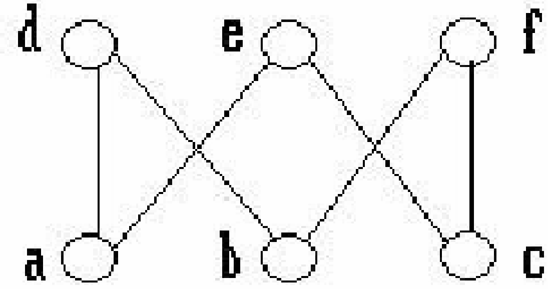


Complementary Extended Induced Order

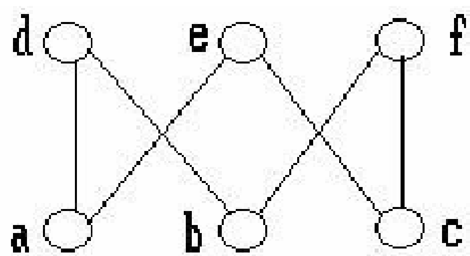
Candidates for Poset of Irreducibles



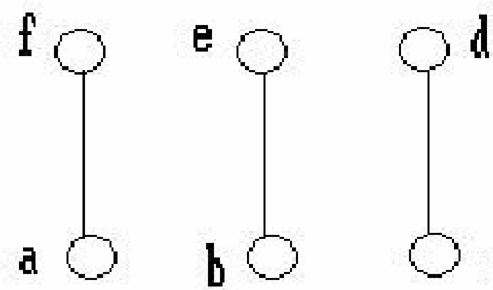
Lattice



Induced Order



Extended Induced Order



Complementary Extended Induced Order

Candidates for Poset of Irreducibles

- Note that the complementary extended induced order shows the direct factorization of the lattice
- Use this as the *Poset of Irreducibles*
- The Poset of Irreducibles was introduced in my thesis in 1972-73

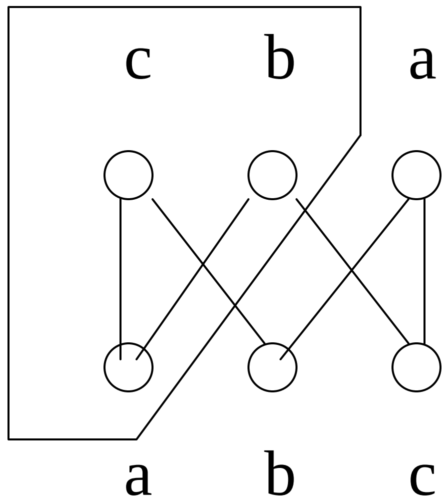
Candidates for Poset of Irreducibles

- Presented as a new approach to analysis of lattices in 1973 at the Houston Lattice Theory conference
- Developed in a series of papers from 1973 through 1994
- The *complement* of the *Poset of Irreducibles* is referred to as the *reduced context* by the Darmstadt school
- Used for data mining and concept analysis

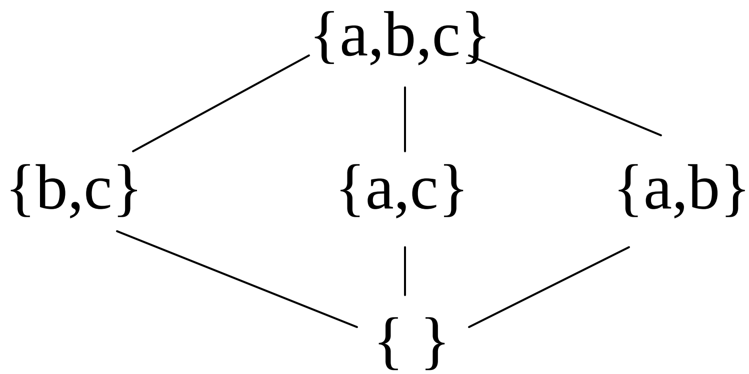
Candidates for Poset of Irreducibles

- The Darmstadt school refuses to reference my work even though it preceded their work and they were aware of it
- In my opinion, the Poset of Irreducibles is a better representation than its dual
- You can get many of their results more simply by working with the Poset of Irreducibles

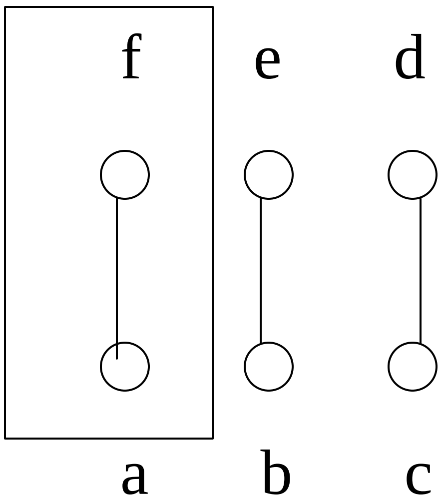
Reconstructing the Lattice



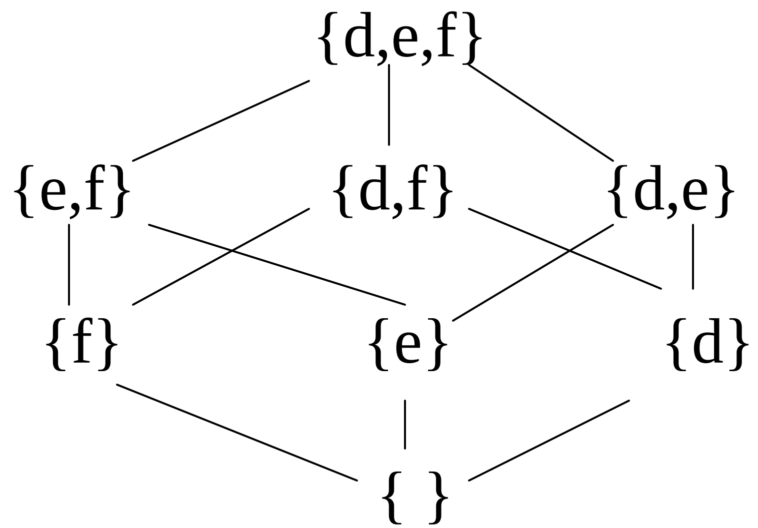
a ⇒ {b,c} **Call this Rep(a)**
b ⇒ {a,c} **Call this Rep(b)**
c ⇒ {a,b} **Call this Rep(c)**



Reconstructing the Lattice



$a \Rightarrow \{f\}$
$b \Rightarrow \{e\}$
$c \Rightarrow \{d\}$

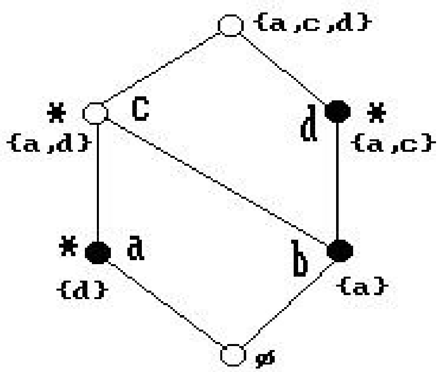


A lot more can be said

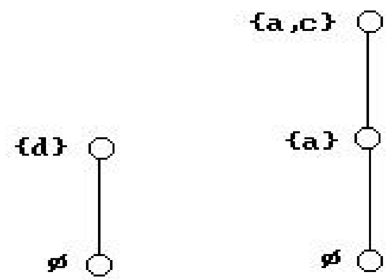
More About the Poset of Irreducibles *P(L)*

- Possibly a compact representation of a lattice (exponentially good in some cases)
- Work with the poset of irreducibles rather than the lattices
- Gives direct factorization
- Gives automorphism group
- Let's use *J(L)* for the set of join-irreducibles and *M(L)* for the set of meet-irreducibles

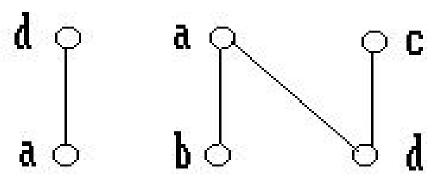
One More Example



L



Factors of L



P(L)

Characterizing Lattices using *P(L)*

- Markowsky (1973)
  - Distributive Lattices
  - Geometric Lattices
- Mario Petrich and I (1975) produced a purely point and hyperplane, numerical-parameter-free, self-dual axiomatization of finite dimensional projective lattices

Characterizing Lattices using *P(L)*

- Avann (1961), Greene & Markowsky (1974)
- Upper Locally Distributive:
  - Jordan-Dedekind
  - Meet-rank = length
- Lower Locally Distributive:
  - Jordan-Dedekind
  - Join-rank = length

Removing the Jordan-Dedekind Chain Condition

- Clearly, *length(L)* $\leq$ |*J(L)*|, |*M(L)*|
- Some definitions
- *Join-extremal: length(L)* = |*J(L)*|
- *Meet-extremal: length(L)* = |*M(L)*|
- *Extremal: length(L)* = |*J(L)*| = |*M(L)*|
- *P-extremal* means you can substitute any of the previous three definitions
- **Theorem:** *A Cartesian product of lattices is p-extremal iff each factor is p-extremal*

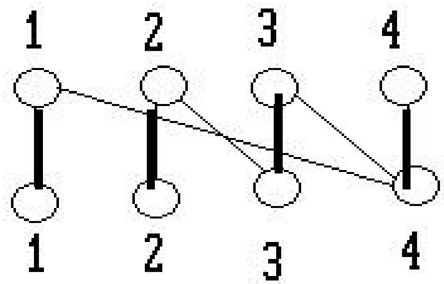
P-Extremal Lattices

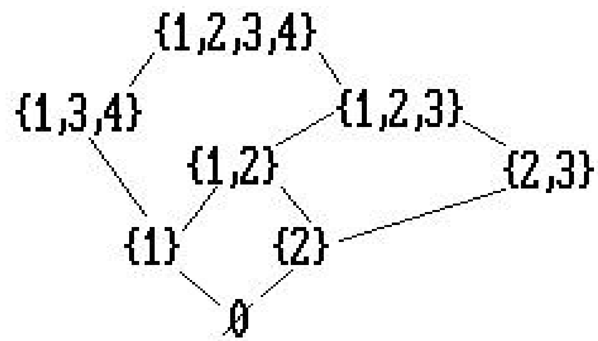
- Many interesting properties and generalize decompositions of finite Boolean algebras
- **Cannot** be categorized algebraically
- Strong retracts for distributive and Tamari lattices
- Structure theorems for distributive and locally-distributive lattices

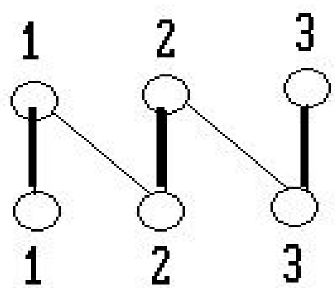
P-Extremal Lattices

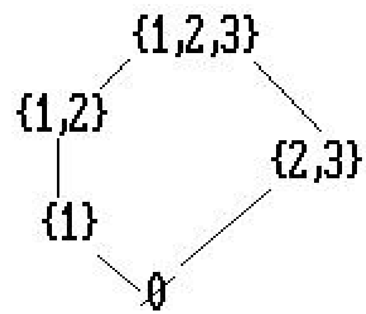
- Include distributive, locally distributive and Tamari Associativity lattices
- **Theorem:** *A bidigraph (X, Y, Arcs) is P(L) for an extremal lattice iff:*
  - $|X| = |Y| = n$
  - *Can number X and Y from 1 to n such that*
    - (x_i, y_i) *is an arc for all i*
    - *if* (x_i, y_j) *is an arc,* $i \geq j$

P(L) for Extremal Lattices





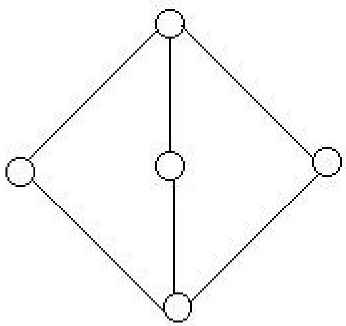




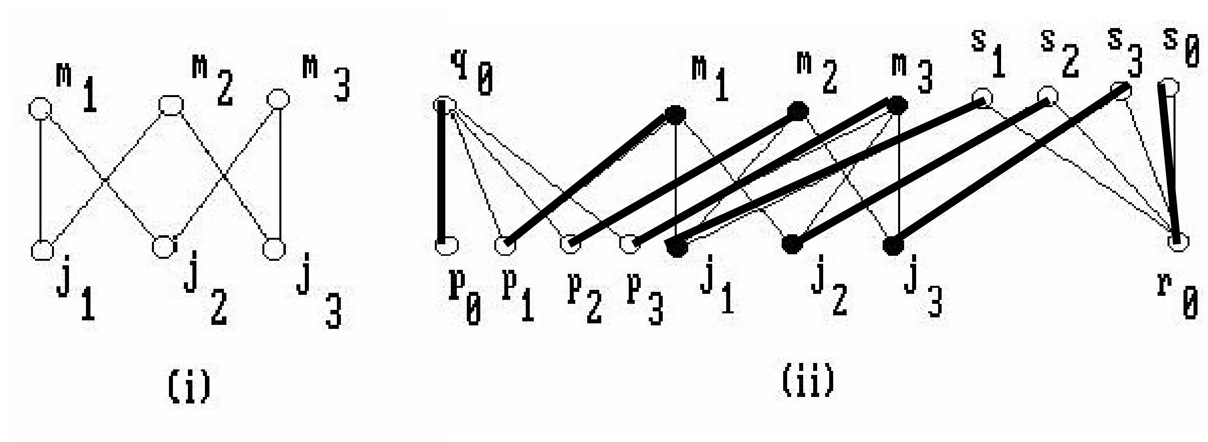
Embeddings of Lattices

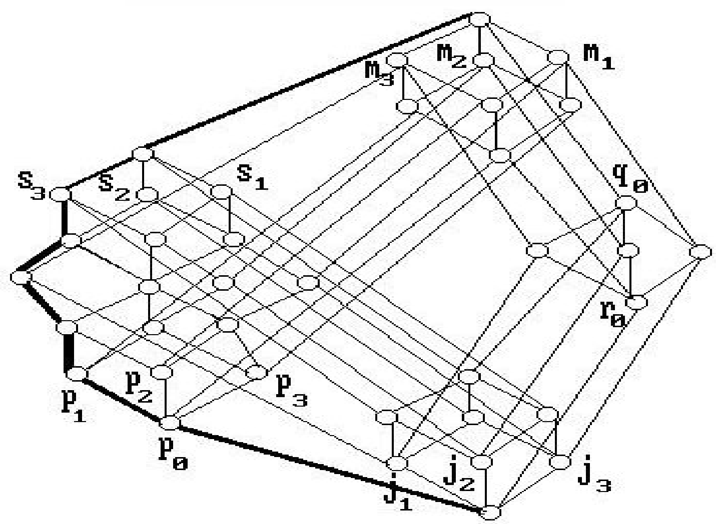
- **Theorem:** *Any finite lattice is isomorphic to an interval of some finite extremal lattice*
- **Corollary:** *Extremal lattices cannot be characterized algebraically*

Embeddings of Lattices



M_5





Coprimes and Primes

- **Definition:** An element $a \neq O$ in L is called *coprime* if for all x and y in L, $x \vee y \geq a$ implies that $x \geq a$ or $y \geq a$.
- **Definition:** An element $a \neq I$ in L is called *prime* if for all x and y in L, $x \wedge y \leq a$ implies that $x \leq a$ or $y \leq a$.
- Coprimes are special kinds of join-irreducibles
- Primes are special kinds of meet-irreducibles

Coprimes and Primes

- **Theorem:** *The following are equivalent*
  - *L is distributive*
  - *All join-irreducibles are coprime*
  - *All meet-irreducibles are prime*
- **Theorem:** *L is meet-pseudocomplemented iff each atom is coprime*
- **Theorem:** *In a Cartesian product elements are coprime iff one component is coprime and the others are O*

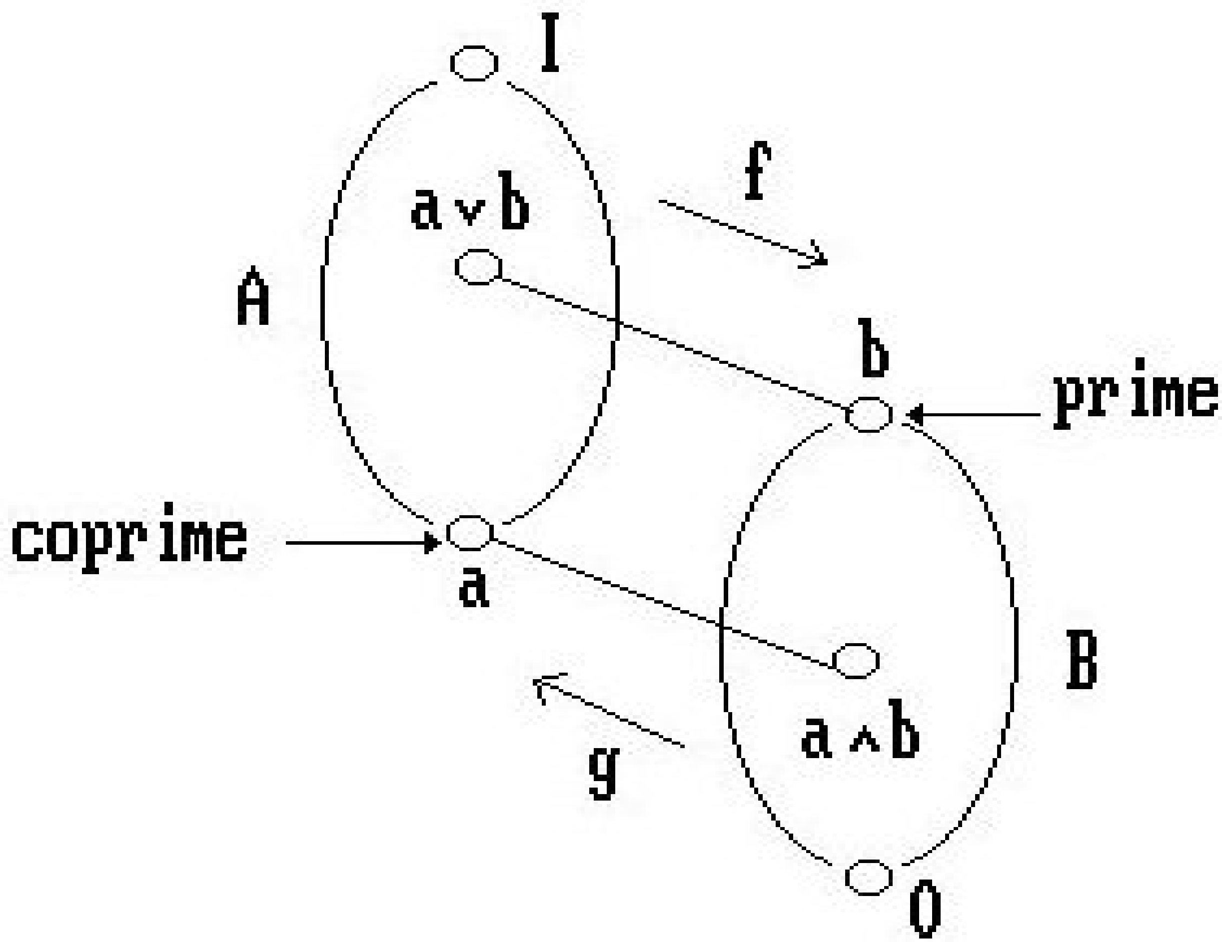
Coprimes and Primes

- **Theorem:** *In any lattice the subposet of coprimes is isomorphic to the subposet of primes*
- **Corollary:** *In a distributive lattice J(L) is isomorphic to M(L)*
- ***Extremal lattices are the combinatorial generalization of distributive lattices***

Coprime/Prime Decompositions

- **Theorem:** *The following are equivalent:*
  - *L contains a coprime a*
  - *L contains a prime b*
  - *L = [O,b] $\oplus$ [a,I] (disjoint union)*

Coprime/Prime Decomposition Summary



Additional Applications

- Checking posets for being lattices
- Analysis of the Permutation Lattices
- Concept Lattices
- Tamari Associativity Lattices
- Various lattice decompositions
- Semigroup of Binary Relations
- Biological Applications
  - Anti-body/Antigen Systems
  - Specificity Covers
  - Factor-Union Systems

The Case Against Lattices

- Early on I got interested in Scott's Theory of Continuous Lattices
- Bothered by the fact that many structures of interest in computer science were not naturally lattices
- Let Str(A) be the set of all strings over the alphabet A, and let s ≤ t iff s is a prefix of t.
- Thus, *sta* ≤ *star* ≤ *start*, etc.

The Case Against Lattices

- However, there is no natural element *x* such that $a \leq x$ and $b \leq x$, where a and b are letters
- In general, for two different words there is no natural way to find a third word which has both of them as prefixes
- Similarly, if you let Pfun(X,Y) be the set of partial functions from X into Y, with f ≤ g iff f(x) defined means g(x) = f(x).

The Case Against Lattices

- This is the order of more definition, but again there is no natural way to bound two functions that have different values at the same point
- The usual solution was to create a lattice by adding $\top$ and calling it the "overdefined" element

The Case Against $\top$

- One problem with using $\top$ is that it tends to breed!
- In Dana Scott's work he made extensive use of repeated Cartesian products.
- This would result in many elements having $\top$ in at least one component
- In fact, if you use (n+1) elements instead of n you quickly run across the following famous theorem:

$$\lim_{k\to\infty} \frac{n^k}{(n+1)^k} = 0$$

Conclusion: almost all elements are eventually bogus!

What is the Solution?

- Abandon the requirement for a lattice!
- What should we replace it with?
- The minimal requirements seemed to be that you needed a poset in which chains had sups
- *Definition: A poset is* ***chain-complete*** *iff every chain has a sup.*
  - There was some confusion about whether you should require directed sets to have sups and not just chains.

Chain-Complete Posets

- I got interested in seeing how far I could get with CPOs
- First, it turns out that if ***every*** chain has a supremum, then every directed set does as well. (CPOs have bottom elements)
- This is not as simple to establish as it appears
- I wrote a paper laying out a variety of properties of CPOs, including fixpoint theorems

Chain-Complete Posets

- Another nice feature of the definition of chain-completeness, is that if a lattice happens to be chain-complete, then it is a complete lattice.
- CPOs have a nice chain-completion.
- CPOs have lots of nice categorical properties – better than complete lattices with chain-*complete maps
    - These are maps that preserve sups of arbitrary chains including the empty chain

A Pet Peeve

- This is probably a vain hope, but I would be a happier man if people would use ***isotone*** when they mean order-preserving instead of ***monotone***, which can be either increasing or decreasing
- Birkhoff has had isotone in his ***Lattice Theory*** for quite some type and once straightened me out about using the right term.

CPO Fixpoint Theory

- For CPO with chain-continuous maps it is easy to construct fixpoints:
- $0 \leq f(0) \leq f(f(0)) \leq \ldots$
- $\lim_{n\to\infty} f^n(0) = x$ and $f(x) = x$
- It turns out that continuity is not needed for the basic fixpoint result

CPO Fixpoint Theory

- Abian and Brown proved that every isotone self-map on a CPO has a fixpoint
- I proved that the set of fixpoints forms a CPO in the induced order and has a least fixpoint
- Proof does not require the axiom of choice

Useful Classes of Posets

- A poset has ***bounded joins*** iff every *finite* subset that has an upper bound, has a sup.
- If a poset has bounded joins and is a CPO, then every set that has an upper bound has a sup.

Useful Classes of Posets

- A poset is ***coherent*** iff every set which is pairwise bounded has a sup
- Coherence → Bounded Joins, CPO
- Many posets of computational interest are coherent:
  - Partial functions
  - Strings

Basis for a Poset

- Poset of irreducible focused on a basis of sorts for lattices
- Want to explore this concept for posets
- In general, a basis incorporates two features
- Independence of its elements
- Generation of the total set

Basis for a Poset

- Barry Rosen and I came up with the following definition
- A subset B of a CPO P is a ***basis*** for P iff for every CPO Q and isotone f:B→Q there is a **unique** extension of f to a continuous function g:P→Q
- Notice how this captures the ideas of generation and independence

Basis for a Poset

- How does this translate into more concrete terms?
- **Definition:** An element, x, in a poset, P, is called ***compact*** iff $x \leq \sup D$, for some directed subset of P implies that $\exists\, d \varepsilon D$ such that $x \leq d$
- In other words, the only way a sup of a directed set can get above a compact element is if some element of D is above that element

Fundamental Basis Theorem

- Let P be a CPO and C its subset of compact elements
- P has a basis iff
- For each x in P, the set $C_x = \{ y \varepsilon C \mid y \leq x\}$ is directed **and**
- sup $C_x = x$
- Note the unique basis is C

Recursively Based Posets

- Since want to have posets that are useful in computer science, need to have a basis which you can grasp computationally
- This leads to the idea of a ***recursively based CPO***.
- Will skip the details, but basically can computationally answer certain questions about basis elements and their bounds and sups

Connection with Scott's Work

- These bases for CPOs do generalize Scott's concept of basis
- One chief goal of Scott's work is to construct domains that have the property that $D \cong [D \rightarrow D]$ where $[D \rightarrow D]$ is some appropriate set of mappings from D to D
- Scott used "continuous lattices" and continuous maps
- Can use CPOs and chain continuous maps

Connection with Scott's Work

- Have results like the following
- If P and Q are coherent, recursively based posets, then [P→Q] is a coherent, recursively based poset
- The variaties of CPOs seem like the natural environment for the theory of computation.

Contact Information

- **http://www.cs.umaine.edu/~markov**

- All papers will be available on-line soon – many are already available on-line