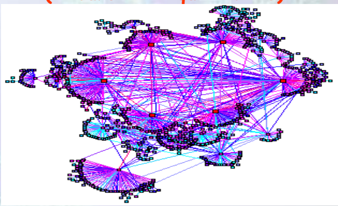
Large Complex Networks: Deteministic Models

(Recursive Clique-Trees)



http://www.caida.org/tools/visualization/plankton/

Francesc Comellas
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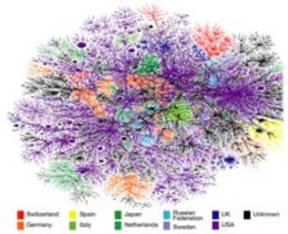
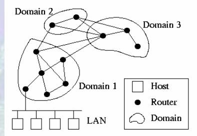
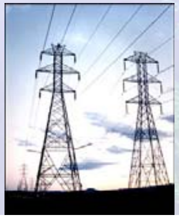


Internet

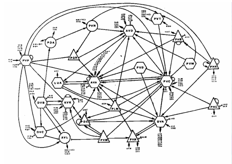


WWW

Air routes

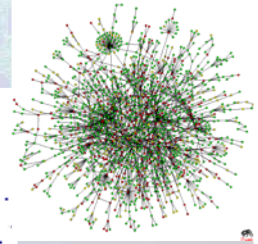


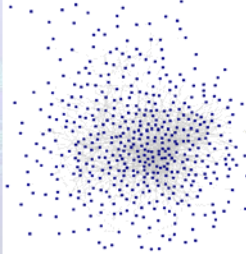
Power grid



C. Elegans

Erdös number





Proteins

Complex systems

Different elements (nodes)

Interaction among elements (links)

Complex networks

Mathematical model: Graphs

Real networks very often are

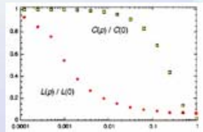
- Large
- Small-world
  - small diameter log(|V|), large clustering
- Scale-free
  - power law degree distribution ("hubs")
- Self-similar / fractal

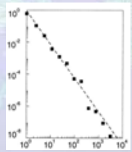
Deterministic models

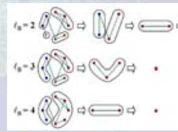
- Based on cliques
  - (hierarchical graphs, recursive clique-trees, Apollonian graphs)

Most "real" networks are

small-world scale-free self-similar







Small diameter (logarithmic)
Milgram 1967
High clustering
Watts & Strogatz 1998

Power law (degrees)
Barabási & Albert 1999

Fractal
Song, Havlin & Makse 2005,2006

Main parameters (invariants)

Diameter – average distance

Degree
Δ degree.
P(k): Degree distribution.

Clustering
Are neighbours of a vertex also neighbours among them?

Small-world networks

small diameter (or average dist.)
high clustering

---

6 degrees of separation ! Stanley Milgram (1967)

160 letters Omaha -Nebraska- -> Boston

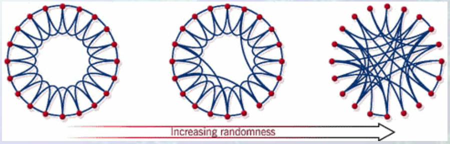
Small world phenomenon in social networks

What a small-world !

Structured graph
- high *clustering*
- large diameter
- regular

Small-world graph
- high *clustering*
- small diameter
- almost regular

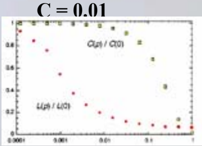
Random graph
- small *clustering*
- small diameter



$|V| = 1000$ $\Delta = 10$
$D = 100$ $d = 49.51$
$C = 0.67$

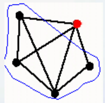
$|V| = 1000$ $\Delta = 8\text{-}13$
$D = 14$ $d = 11.1$
$C = 0.63$

$|V| = 1000$ $\Delta = 5\text{-}18$
$D = 5$ $d = 4.46$
$C = 0.01$



Watts & Strogatz,
Collective dynamics of "small-world" networks,
Nature 393, 440-442 (1998)

Network characteristics



Clustering $C(v) = \frac{\text{\# of links among neighbors}}{n(n-1)/2}$

Diameter or Average distance
Maximum communication delay

Degree distribution
Resilience

Real life networks are clustered, large C(p), but have small average distance L(p). Very often they are also scale-free

	L	L_{rand}	C	C_{rand}	N
www	3.1	3.35	0.11	0.00023	153127
Actors	3.65	2.99	0.79	0.00027	225226
Power Grid	18.7	12.4	0.080	0.005	4914
C. Elegans	2.65	2.25	0.28	0.05	282

Erdös number

http://www.oakland.edu/enp/

1- 509
2- 7494

N= 268.000 Jul 2004
(connected component)

$D=23$ $R=12$ $D_{avg} = 7.64$
$\delta=1$ $\Delta = 509$ $\Delta_{avg} = 5.37$
$C = 0.14$

```
Erdös number 0 ---      1 person
Erdös number 1 ---    504 people
Erdös number 2 ---   6593 people
Erdös number 3 ---  33605 people
Erdös number 4 ---  83642 people
Erdös number 5 ---  87760 people
Erdös number 6 ---  40014 people
Erdös number 7 ---  11591 people
Erdös number 8 ---   3146 people
Erdös number 9 ---    819 people
Erdös number 10 ---   244 people
Erdös number 11 ---    68 people
Erdös number 12 ---    23 people
Erdös number 13 ---     5 people

(MathSciNet Jul 2004)
```

Notable Erdös coathors:

Frank Harary (257 coautors)
Noga Alon (143 coautors)
Saharon Shela (136)
Ronald Graham (120)
Charles Colbourn (119)
Daniel Kleitman (115)
A. Odlyzko (104)

Erdös did not write a joint paper with his PhD advisor,
Leopold Fejér

Some other Erdös coautors

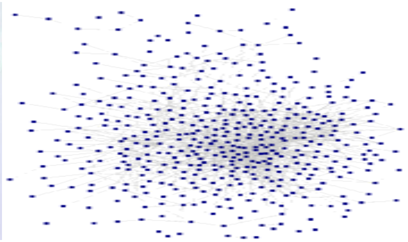
articles with Erdös

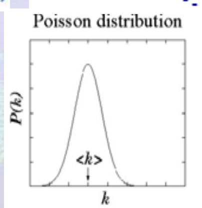
András Sárközy	57
András Hajnal	54
Ralph Faudree	45
Richard Schelp	38
Vera Sós	34
Alfréd Rényi	32
Cecil C. Rousseau	32
Pál Turán	30
Endre Szemerédi	29
Ronald Graham	27
Stephan A. Burr	27
Joel Spencer	23
Carl Pomerance	21
Miklos Simonovits	21
Ernst Straus	20
Melvyn Nathanson	19
Richard Rado	18
Jean Louis Nicolas	17
Janos Pach	16
Béla Bollobás	15
Eric Milner	15
John L. Selfridge	13
Harold Davenport	7
Nicolaas G. de Bruijn	6
Ivan Niven	7
Mark Kac	5
Noga Alon	4
Saharon Shela	3
Arthur H. Stone	3
Gabor Szegö	2
Alfred Tarski	2
Frank Harary	2
Irving Kaplansky	2
Lee A. Rubel	2

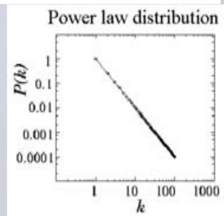
Fields medals

Lars Ahlfors	1936	Finland	4
Jesse Douglas	1936	USA	4
Laurent Schwartz	1950	France	4
Atle Selberg	1950	Norway	2
Kunihiko Kodaira	1954	Japan	2
Jean-Pierre Serre	1954	France	3
Klaus Roth	1958	Germany	2
Rene Thom	1958	France	4
Lars Hormander	1962	Sweden	3
John Milnor	1962	USA	3
Michael Atiyah	1966	Great Britain	4
Paul Cohen	1966	USA	5
Alexander Grothendieck	1966	Germany	5
Stephen Smale	1966	USA	4
Alan Baker	1970	Great Britain	2
Heisuke Hironaka	1970	Japan	4
Serge Novikov	1970	USSR	3
John G. Thompson	1970	USA	3
Enrico Bombieri	1974	Italy	2
David Mumford	1974	Great Britain	2
Pierre Deligne	1978	Belgium	3
Charles Fefferman	1978	USA	2
Gregori Margulis	1978	USSR	4
Daniel Quillen	1978	USA	3
Alain Connes	1982	France	3
William Thurston	1982	USA	3
Shing-Tung Yau	1982	China	2
Simon Donaldson	1986	Great Britain	4
Gerd Faltings	1986	Germany	4
Michael Freedman	1986	USA	3
Valdimir Drinfeld	1990	USSR	4
Vaughan Jones	1990	New Zealand	4
Shigemufi Mori	1990	Japan	3
Edward Witten	1990	USA	3
Pierre-Louis Lions	1994	France	4
Jean Christophe Yoccoz	1994	France	3
Jean Bourgain	1994	Belgium	2
Efim Zelmanov	1994	Russia	3
Richard Borcherds	1998	S Afr/Gt Brtn	2
William T. Gowers	1998	Great Britain	4
Maxim L. Kontsevich	1998	Russia	4
Curtis McMullen	1998	USA	3
Vladimir Voevodsky	2002	Russia	4
Laurent Lafforgue	2002	France	inf
Andrei Okounkov	2006	USA	3
Terence Tao	2006	USA	3
Wendelin Werner	2006	France	3

Pál Erdős
(1913-1996)







Scale-free networks

Scalability vs Fractality

1	1227
2	1656
3	1060
4	401
5	252
6	137
7	84
8	46
9	27
10	26
11	11
12	5
13	5
14	3
15	0
16	0
17	0
18	1
19	1

Power grid

|V|=4491 δ =1 Δ =19

D = 46 d = 34.54

Small World

C = 0.08

7	0
8	5
9	93
10	806
11	90
12	5
13	1
14	0

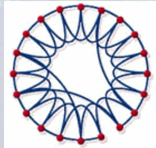
SWCirculant

|V|=1000 Δ =8-13

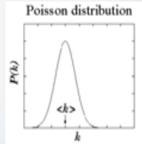
D = 14 d = 11.1

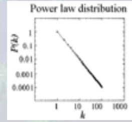
Small World

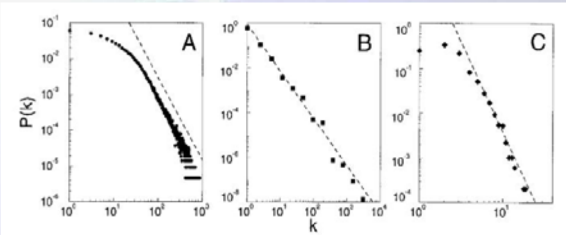
C = 0.63



A-L. Barabási i R. Albert,
Emergence of scaling in random networks.
Science 286, 509-510 (1999)







$P(k) = k^{-\gamma}$

A: actors	N=212.250 k=28.78 γ =2.3
B: WWW	N=325.729 k=5.46 γ =2.67
C: power grid	N= 4.94 k=2.67 γ =4

Real networks for which we know the topology:
$P(k) \sim k^{-\gamma}$

NON BIOLOGICAL γ > 2

www (in) γ = 2.1 www (out) γ =2.45
actors γ = 2.3
citations γ = 3
power grid γ = 4

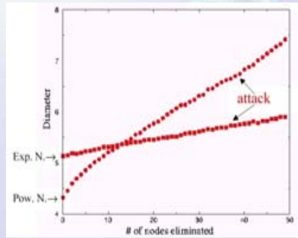
BIOLOGICAL γ < 2

yeast protein-protein net γ =1.5, 1.6, 1.7, 2.5
E. Coli metabolic net γ = 1.7, 2.2
yeast gene expression net γ = 1.4-1.7
gene functional interaction γ = 1.6

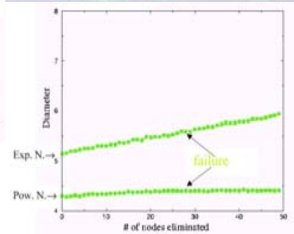
But, usual random models give: $P(k) \sim e^{-k}$

Interest on **scale-free** nets:
Resilience / Survival of the WWW

Albert, Jeong, Barabási
Nature 406, 378 (2000)



What happens when nodes fail randomly ?



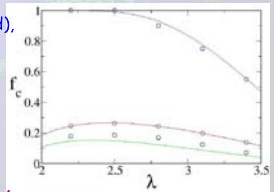
And when there are intentionate attacks to the best connected nodes ?

Epidemics spreading / “vaccination”
WWW, social networks

R. Cohen, D. ben-Avraham, S. Havlin;
Efficient immunization of populations and computers
Phys. Rev. Lett. 91, 247901 (2003)

f_c threshold
λ power law exponent
upper– totally random
lower- acquaintance immunisation (red),
double acq. imm. (green)
method:
* select a node at random
* ask it to select a high degree node
and immunize it

“vaccinate” high degree nodes !



Search in power-law networks
Adamic, Lukose, Puniyani, Huberman;
Phys. Rev. E **64**, 046135 (2001)

Even the Inquisition knew about *scale-free* networks!!

From random “vaccination”

Arnau d'Amaurí 1209. Besièrs

Caedite eos.
Nobit enim Dominus qui sunt eius
Kill them all, God will know his own

to selection (see figure)

P. Ormerod, A.P. Roach;
The Medieval inquisition: scale-free networks and the suppression of heresy.
Physica A 339 (2004) 645-652

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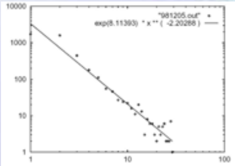
spectral properties

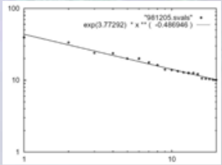
- **Connectivity and vulnerability (diameter, cut sets, distances between subsets)**
- **Scalability, expansion (Cheeger constants)**
- **Routings (spanning trees)**
- **Load balancing**
- **Clustering (triangles)**
- **Reconstruction** (Ipsen & Mikhaliov, 2001)
- **Dynamical aspects (interlacing theorem)**

Experimental and simulation results

WWW / Internet eigenvalues

Faloutsos, Faloutsos & Faloutsos, 1999

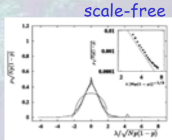


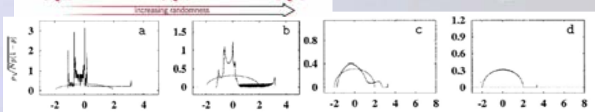


Adjacency matrix eigenvalues
(Watts & Strogatz model, Scale-free models)

Farkas, Derenyi, Barabási & Vicsek *Phys. Rev E* 2001
Goh, Kahng & Kim 2001
Fan & Chen *IEEE Circ. Sys.* 2002

scale-free

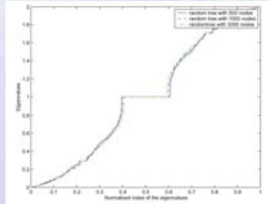


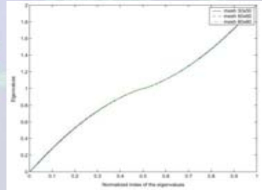


Watts-Strogatz

Normalized Laplacian eigenvalues
(meshes, random trees)

Vukadinovic, Huang, Erlebach 2002





How to model real networks ?

Erdös-Rény,
Watts-Strogatz
Barabási-Albert

other models ?

Why appears a power law?

1. Networks grow continously by addition of new nodes

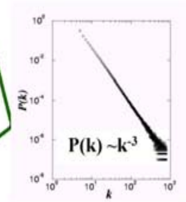
2. Growth is NOT uniform: A new node will join, with high probability, an old well connected node

WWW: New documents point to "classic" references
Erdös: I would prefer to publish with a well known mathematician

"Standard" model: Barabási, Albert; Science 286, 509 (1999)

Preferential attachment : At each time unit a new node is added with m links which connect to existing nodes . The probability P to connect to a node i is proportional to its degree k_i

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

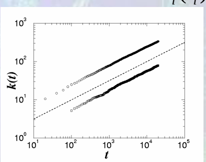


A.-L.Barabási, R. Albert, Science **286,** 509 (1999)

Mean Field Theory

$$\frac{\partial k_i}{\partial t} \propto \Pi(k_i) = A\frac{k_i}{\sum_j k_j} = \frac{k_i}{2t}$$, with initial condition $k_i(t_i) = m$

$$k_i(t) = m\sqrt{\frac{t}{t_i}}$$



$$P(k_i(t) < k) = P_t(t_i > \frac{m^2 t}{k^2}) = 1 - P_t(t_i \le \frac{m^2 t}{k^2}) = 1 - \frac{m^2 t}{k^2(m_0 + t)}$$

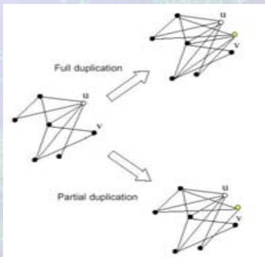
$$\therefore P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^2 t}{m_o + t}\frac{1}{k^3} \sim k^{-3}$$

$$\gamma = 3$$

A.-L.Barabási, R. Albert and H. Jeong, *Physica A* **272,** 173 (1999)

Duplication models:

Fan Chung, Lu, Dewey, Galas; (2002)
Nodes are duplicated together with all (or part) of their edges.
can produce γ <2 as in biological networks
keep some network properties

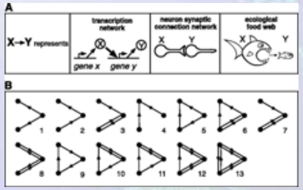


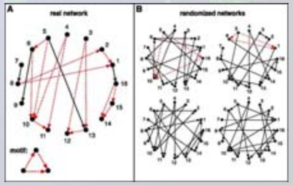
Fractal networks

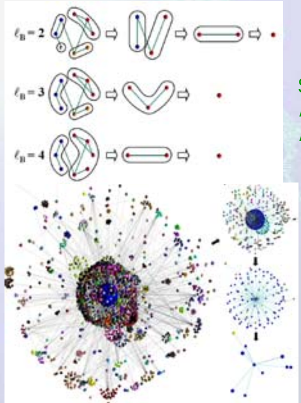
Motifs, graphlets

Milo, Shen-Orr, Itzkoviz, Kashtan, Chkovskii, Alon
Science 298, 824-827 (2002)

Pržulj, Corneil, Jurisica
Bioinformatics 20, 3508-3515 (2004)



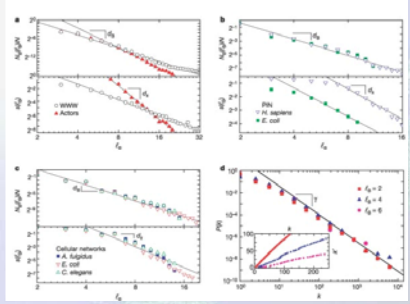




Song, Havlin, Makse
Nature 433, 392-396 (2005)
Nature Physics 2, 275-281 (2006)

Self-similarity of complex networks

Origins of fractality in the growth of complex networks



$$N_B \approx \ell_B^{-d_B}$$

$$k \rightarrow k' = s(\ell_B)k$$

$$s(\ell_B) \approx \ell_B^{-d_k}$$

$$\gamma = 1 + d_B/d_k$$

WWW, protein interaction networks are fractal

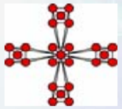
Internet (AS) is not fractal

Barabási-Albert is not fractal

Real complex networks: self-organized criticality (SOC) by some optimization process ! !

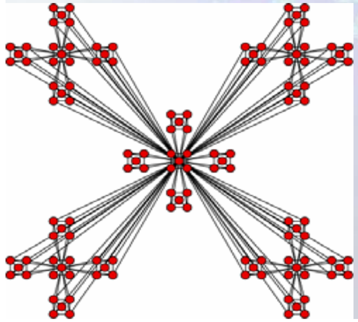
Cliques-trees, as **deterministic models** for real networks.

Hierarchical graphs
Recursive clique-trees
Apollonian graphs

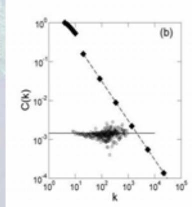


Hierarchical graphs

Ravasz, Barabasi.
Hierarchical organization in complex networks
Phys. Rev. E (2003).

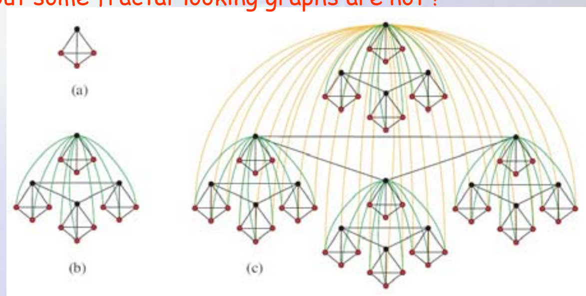


$$C(k) \sim k^{-1}$$



$$\gamma = 1 + (\ln 5/ \ln 4)$$

Real-life networks are fractal (Song, Havlin, Makse) but some fractal-looking graphs are not !



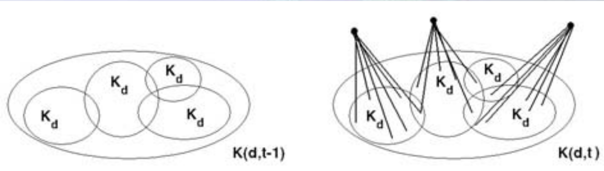
Barrière, Comellas, Dalfó
(2006)

$$\gamma \approx 1 + \frac{\ln(d+1)}{\ln d}$$

Recursive clique-trees

SN Dorogovtsev, AV Gotsev, JFF Mendes., *Phys. Rev. E (2002)*

F. Comellas, Guillaume Fertin, André Raspaud, *Phys. Rev. E (2004)*



Recursive clique-trees

F.Comellas, G. Fertin, A. Raspaud, *Phys. Rev. E*

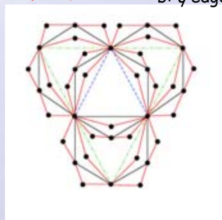
Initial graph: K_q -the complete graph with q vertices-.

Operation: t>=0, obtain K(q,t+1) from K(q,t) by adding for every clique K_q of K(q,t+1):

a: A new vertex *u*

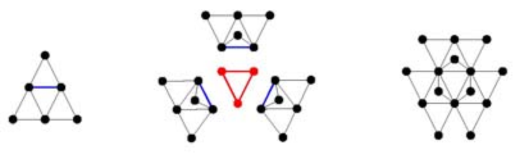
b: *q* edges joining *u* with the vertices of this clique

example q=2



example q=3

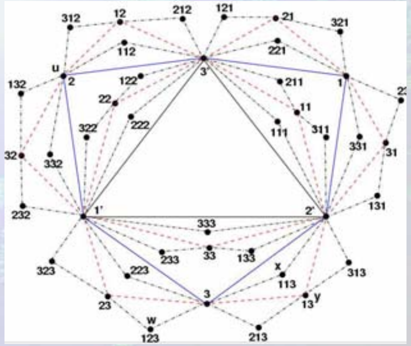
Recursive equivalent operation



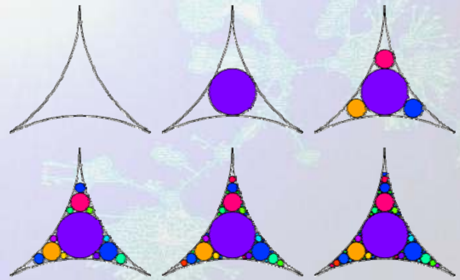
- Order, size
- Degree distribution $\gamma \approx 1 + \frac{\ln(d+1)}{\ln d}$ $2 < \gamma < 2.58496$
- Clustering 0.8 <= C <= 1
- Diameter logarithmic

Distance-labeling and routing (example q=2)

F.Comellas, G. Fertin, A. Raspaud, *Sirocco 2003*



Apollonian graphs

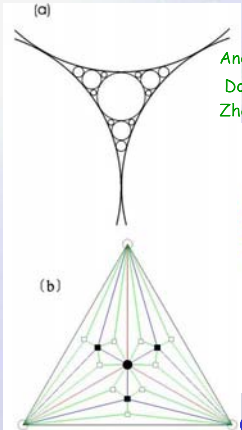


Apollonian packings

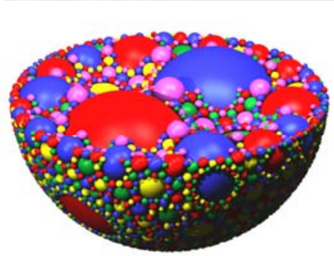
And let us not confine our cares
To simple circles, planes and spheres,
But rise to hyper flats and bends
Where kissing multiple appears,
In n-ic space the kissing pairs
Are hyperspheres, and Truth declares -
As n+2 such osculate
Each with an n+1 fold mate
The square of the sum of all the bends
Is n times the sum of their squares.

Thorold Gosset, The Kiss Precise, Nature 139 (1937) 62.

Apollonian graphs



Andrade et al. *Phys. Rev. Lett. (2005)*
Doye, Massen *Phys. Rev. E (2005)*
Zhang, Rong, Comellas, Fertin *J. Phys.A (2006)*



http://graphics.ethz.ch/~peikert/personal/packing/images/apoll3d.png

d=2 d=3

Step (t)	New edges	Number of K_{d+1}
0	$\frac{d(d+1)}{2}$	1
1	$d+1$	$d+1$
2	$(d+1)^2$	$(d+1)^2$
3	$(d+1)^3$	$(d+1)^3$
$\cdots$	$\cdots$	$\cdots$
i	$(d+1)^i$	$(d+1)^i$
$i+1$	$(d+1)^{i+1}$	$(d+1)^{i+1}$
$\cdots$	$\cdots$	$\cdots$

$$N_t = (d+1) + \sum_{j=0}^{t-1}(d+1)^j = \frac{(d+1)^t - 1}{d} + d + 1$$

$$|E|_t = \frac{d(d+1)}{2} + \sum_{j=1}^{t}(d+1)^j = \frac{d(d+1)}{2} + \frac{(d+1)^{t+1} - d - 1}{d} \quad (1)$$

TABLE II: Distribution of vertices and degrees for $A(d,t)$ at each step t.

Step(t)	Num. vert.	Degree
0	$d+1$	d
1	$d+1$	$d+1$
	1	$d+1$
2	$d+1$	$d+1+d$
	1	$(d+1)+(d+1)$
	$d+1$	$d+1$
3	$d+1$	$d^2+d+1+d$
	1	$(d+1)d+(d+1)+(d+1)$
	$d+1$	$(d+1)+(d+1)$
	$(d+1)^2$	$d+1$
4	$d+1$	$d^3+d^2+d+1+d$
	1	$(d+1)d^2+(d+1)d+(d+1)+(d+1)$
	$d+1$	$(d+1)d+(d+1)+(d+1)$
	$(d+1)^2$	$(d+1)+(d+1)$
	$(d+1)^3$	$d+1$
$\cdots$	$\cdots$	$\cdots$
i	$d+1$	$d^{i-1}+d^{i-2}+\cdots+d+1+d$
	1	$(d+1)d^{i-2}+(d+1)d^{i-3}\cdots +(d+1)d+(d+1)+(d+1)$
	$d+1$	$(d+1)d^{i-3}+(d+1)d^{i-4}\cdots +(d+1)d+(d+1)+(d+1)$
	$\cdots$	
	$(d+1)^{i-2}$	$(d+1)+(d+1)$
	$(d+1)^{i-1}$	$d+1$
$\cdots$	$\cdots$	$\cdots$

Discrete degree spectrum (with larger and larger jumps)

$P_{cum}(k) \equiv \sum_{k' \geq k} N(k',t)/N_t \sim k^{1-\gamma}$

$$\gamma \approx 1 + \frac{\ln(d+1)}{\ln d}$$

$$2 < \gamma < 2.58496$$

Random Apollonian graphs

Instead of adding simultaneously a new vertex to each clique (never used before), we add an unique vertex to a random clique.

Initially $A(d,0)$ is K_{d+2}

Step t choose clique K_{d+1} NEVER USED and add a node (and the corresponding edges)

Order increments by 1 at each step
$N_t = t+d+2$

Degree distribution (self-averaging)

Given a vertex, when its degree increases by 1, the number of K_{d+1} which contains it increases by d-1. Thus, when the vertex attains degree k_i the number of K_{d+1} is $(d+1)+(k_i-d-1)(d-1)=(d-1)k_i-d^2+d+2$

$$\frac{\partial k_i}{\partial t} = \frac{(d-1)k_i - d^2 + d + 2}{dt + d + 2}$$

with initial condition $k_i(t_i) = d+1$ we obtain

$$k_i(t) = \frac{d^2-d-2}{d-1} + \frac{d+1}{d-1}\left(\frac{dt+d+2}{dt_i+d+2}\right)^{\frac{d-1}{d}}$$

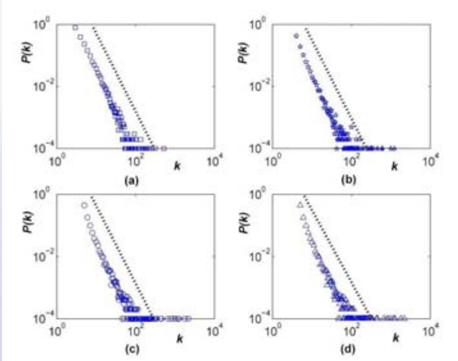
$$P(k_i(t) < k) = P\left(t_i > \frac{(dt+d+2)\left(\frac{d+1}{d-1}\right)^{\frac{d}{d-1}}}{d\left(k-\frac{d^2-d-2}{d-1}\right)^{\frac{d}{d-1}}} - \frac{d+2}{d}\right)$$

$$P(k) = d(d+1)^{\frac{d}{d-1}}\left((d-1)k-(d^2-d-2)\right)^{\frac{1-2d}{d-1}}$$

If $k \gg d$ we have $P(k) \sim k^{-\gamma}$ with $\gamma(d) = \frac{2d-1}{d-1}$

γ=3 (for d=2, random seq.)

vs **γ=2.58496** (d=2 parallel)

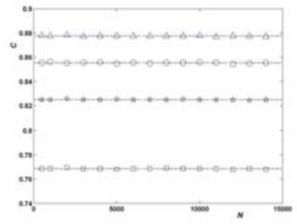


Degree distribution when N=10000, d=2,3,4,5

$$C(k) = \frac{\frac{d(d+1)}{2} + d(k-d-1)}{\frac{k(k-1)}{2}} = \frac{d(2k-d-1)}{k(k-1)}$$

Clustering

$$C = \int_{d+1}^{\infty} C(k)P(k)\,dk = \int_{d+1}^{\infty} \frac{d^2(2k-d-1)(d+1)^{\frac{d}{d-1}}}{k(k-1)}\left((d-1)k-(d^2-d-2)\right)^{\frac{1-2d}{d-1}}\,dk$$



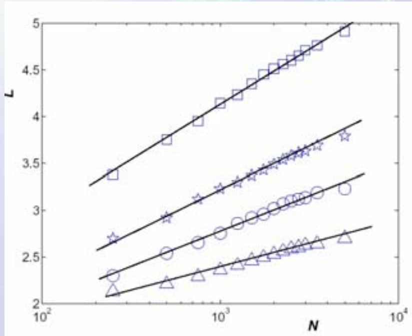
$$C = \tfrac{46}{3} - 36\ln\tfrac{3}{2} = 0.7366$$

$$C = 18+36\sqrt{2}\arctan\sqrt{2}+\tfrac{9}{2}\pi-18\sqrt{2}\,\pi = 0.8021$$

HDRAN clustering

N=10000, d=2,3,4,5

Average path length



- Zhongzhi Zhang, Lili Rong, F. Comellas, Guillaume Fertin,
  High dimensional Apollonian networks
  J. Phys. A (2006)
- Zhongzhi Zhang, Lili Rong, F. Comellas,
  High dimensional random Apollonian networks
  Physica A (2006),

Deterministic recursive clique-trees

	Adding at the same time a vertex to each d-clique **with** repetition	Adding at the same time a vertex to each d-clique **without** repetition
Case $d=2$	*Pseudofractal scale-free* Dorogotsev, Goltsev, Mendes Phys.Rev.E 65 (2002) 066122	*Deterministic SW network* Zhang, Rong, Guo Physica A cond-mat/0503637
Case $d=3$		*Apollonian network* Andrade, Herrmann, Andrade, Silva Phys.Rev.Lett. 94 (2005) 018702 Doye, Massen Phys. Rev. E 71 (2005) 016128.
General case $d=2\ldots\infty$ (includes cases d=2,3)	*Recursive clique-trees* Comellas, Fertin, Raspaud Phys.Rev.E 69 (2004) 037104.	*High dimensional Apollonian network* Zhang, Comellas, Fertin, Rong J. Phys. A. 39 (2006) 1811 (introduced by Doye and Massen, Phys. Rev. E 71 (2005) 016128.)

Random recursive clique-trees

	Adding a single vertex to a random clique **with** repetition	Adding a single vertex to a random clique **without** repetition
Case $d=2$		*Random SW network* Ozik, Hunt, Ott Phys.Rev.E 69 (2004) 02618
Case $d=3$		*Random Apollonian network* Zhou, Yan, Wang Phys.Rev.E 71 (2005) 046141
General case $d=2\ldots\infty$ (includes cases d=2,3)	*Random recursive clique-tree* see Appendix	*HD random Apollonian network* Zhang, Comellas, Rong Physica A. cond-mat/0502591

Deterministic vs Random

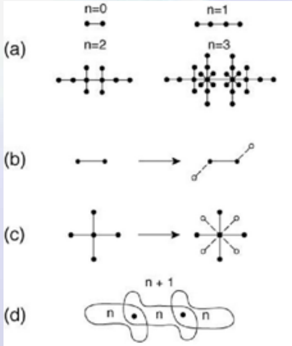
Graph family	$P(k)$ or γ-exponent	Clustering
Deterministic SW [78]	$2^{-\frac{k}{2}}$	$0.69 = \ln 2$
Random SW [77]	$\frac{3}{4}(\frac{2}{3})^{-k}$	$0.65(=\frac{3}{2}\ln 3 - 1)$
Apollonian [7,34]	$2.58(=1+\frac{\ln 3}{\ln 2})$	0.83
Random Apollonian [82]	$\frac{3N-5}{N} \approx 3$	$0.74(=\frac{46}{3} - 36\ln\frac{3}{2})$
High-Dim. Apollonian [81]	$1+\frac{\ln(d+1)}{\ln d}$ (2 to 2.58)	0.83 to 1
High-Dim. Random Apollonian [80]	$\frac{2d-1}{d-1}$ 2 to 3	0.74 to 1
Pseudo fractal scale-free [29]	$1+\frac{\ln 3}{\ln 2} = 2.58$	$0.80(=\frac{4}{5})$
Random pseudo fractal scale-free	$\frac{5}{2} = 2.5$	
Determ. recursive clique-trees [22]	$1+\frac{\ln(d+1)}{\ln d}$ (2 to 2.58)	0.80 to 1
Random rec. clique-trees [see Appendix]	$\frac{2d-1}{d-1}$ (2 to 3)	0.74 to 1

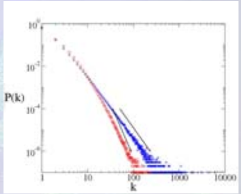
Why the random approach produces a different distribution

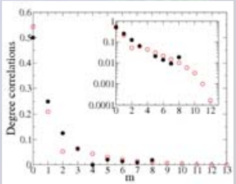
F. Comellas, Hernan D. Rozenfeld, Daniel ben-Avraham
Synchronous and asynchronous recursive random scale-free nets
Phys. Rev. E (2005),

In many simulations choosing an edge might be biased.

It is not the same to choose edge e from |E| edges than choose a node and then and adjacent node.







Present and future work in SW-SF networks

How to construct a better WWW (new topologies -Akamai) ?

How to analyse very large graphs ?
- mean field and other statistical methods
- fractal techniques
- spectral theory
- new invariants
- this workshop

How to deal with dynamical networks ?

New communication protocols