Hyperbolic Polynomials Approach to Van der Waerden/Schrijver-Valiant like Conjectures : Sharper Bounds , Simpler Proofs and Algorithmic Applications

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Contents

Doubly Stochastic matrices and matrix tuples , Permanent , Mixed Discriminant

Doubly Stochastic $n \times n$ Matrix :

$$\Omega_n = \{A = A(i,j) : A(i,j) \geq 0, 1 \leq i,j \leq n; Ae = A^T e = e,$$

$\Omega_n =$ The set of $n \times n$ Doubly Stochastic matrices.

Doubly Stochastic n-tuple $\mathbf{A} = (A_1, \cdots, A_n)$:

$A_i \succeq 0$ (PSD $n \times n$ complex hermitian) , $trA_i = 1, 1 \leq i,j \leq n; \sum_{i=1}^n A_i = I$.

$D_n =$ The set of Doubly Stochastic n-tuples.

The permanent :$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n A(i, \sigma(i))$

The mixed discriminant :

$$D(A_1, A_2, \cdots, A_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \det(t_1 A_1 + \cdots + t_n A_n)$$

Determinantal Polynomial :

$DET_{\mathbf{A}}(t_1, ..., t_n) = \det(\sum_{1 \leq i \leq n} t_i A_i)$.

Multilinear Polynomial :

$Mul_A(t_1, ..., t_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} A(i, j) t_j$.

$per(A) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} Mul_A(t_1, ..., t_n)$

($per(A) = 2^{-n} \sum_{b_i \in \{-1,1\}, 1 \leq i \leq n} Mul_A(b_1, ..., b_n)$: Ryser's formula .)

Multilinear is commutative(solvable) case of Determinantal .

Van der Waerden Conjecture

The famous Van der Waerden Conjecture states that

$$min_{A \in \Omega_n} D(A) = \frac{n!}{n^n} \textbf{ (VDW-bound)}$$

and the minimum is attained uniquely at the matrix J_n in which every entry equals $\frac{1}{n}$.

Van der Waerden Conjecture was posed in 1926 and proved only in 1981 :

D.I. Falikman proved the lower bound **(VDW-bound)** and the full conjecture , i.e. the uniqueness part , was proved by G.P. Egorychev . They shared Fulkerson Prize , 1982 .

Aleksandrov-Fenchel inequalities and many other ingredients , about 25 years of research.

Was used by N. Linial, A. Samorodnitsky and A. Wigderson (1998) to approximate the permanent of nonnegative matrices :

$A = Diag(a_1, ..., a_n) B Diag(b_1, ..., b_n), B \in \Omega_n$

Sinkhorn's Scaling .

As $\frac{n!}{n^n} \leq per(B) \leq 1$ thus

$$f(A) =: \prod_{1 \leq i \leq n} a_i b_i \Rightarrow 1 \leq \frac{f(A)}{per(A)} \leq (\frac{n!}{n^n})^{-1} \approx e^n.$$

Strongly polynomial algorithms .

Bapat's Conjecture (Van der Waerden Conjecture for mixed discriminants)

One of the problems posed by R.V.Bapat (1989) is to determine the minimum of mixed discriminants of doubly stochastic tuples : $min_{A \in D_n} D(A) =?$

Quite naturally, R.V.Bapat conjectured that
$min_{A \in D_n} D(A) = \frac{n!}{n^n}$ **(Bapat-bound)**

and that it is attained uniquely at $\mathbf{J}_n =: (\frac{1}{n}I, ..., \frac{1}{n}I)$.

The original conjecture was formulated for real symmetric PSD matrices.

L.G. had proved it (1999 , 2006 in *Advances in Mathematics* for the complex case, i.e. for complex positive semidefinite and, thus, hermitian matrices .

Was motivated by the ellipsoid algorithm to approximate (deterministically) mixed discriminants/mixed volumes .

Schrijver-Valiant Conjecture

Let $\Lambda(k,n)$ denote the set of $n \times n$ matrices with nonnegative integer entries and row and column sums all equal to k (k-regular bipartite graphs) .

We define the following subset of rational doubly stochastic matrices : $\Omega_{k,n} = \{k^{-1}A : A \in \Lambda(k,n)\}$.

Define
$\lambda(k,n) = \min\{per(A) : A \in \Omega_{k,n}\} =$
$k^{-n} \min\{per(A) : A \in \Lambda_{k,n}\}$;
$\theta(k) = \lim_{n\to\infty}(\lambda(k,n))^{\frac{1}{n}}$.

$\lambda(2,n) = 2^{-n+1}$, Erdos-Renyi (1968) : $\theta(k) =?$, even the case $k = 3$ was open until 1979-1980 .

M. Voorhoeve in (1979) : $\lambda(k,n) \geq (\frac{2}{3})^{2(n-3)}\frac{2}{9}$.

Schrijver-Valiant (1980) $\theta(k) \leq g(k) = (\frac{k-1}{k})^{k-1}$, which gives $\theta(3) = \frac{4}{9}$.

Schrijver-Valiant Conjecture (1980) : $\theta(k) = g(k) = (\frac{k-1}{k})^{k-1}$.

Settled by Lex Schrijver in 1998 : $\min\{per(A) : A \in \Omega_{k,n}\} \geq (\frac{k-1}{k})^{(k-1)n}$ **(Schrijever-bound)** .

remarkable result — unpassable proof .

I will present a vast and unifying generalization of those three results .

Homogeneous polynomials with nonnegative coefficients

Let $Hom(m,n)$ be a linear space of homogeneous polynomials $p(x), x \in R^m$ of degree n in m varibles ; correspondingly $Hom_+(m,n)(Hom_{++}(m,n))$ be a subset of homogeneous polynomials $p(x), x \in R^m$ of degree n in m varibles and nonnegative(positive) coefficients .

Let

$p \in Hom_+(n,n), p(x_1,...,x_n) =$

$= \sum_{r_1+...+r_n=n} a_{(r_1,...,r_n)} \prod_{1 \le i \le n} x_i^{r_i}$.

The support :

$supp(p) = \{(r_1,...,r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$.

The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p .

For a subset $A \subset \{1, ..., n\}$ we define

$S_p(A) = \max_{(r_1,...,r_n) \in supp(p)} \sum_{i \in A} r_i$.

Given a vector $(a_1, ..., a_n)$ with positive real coordinates , consider univariate polynomials

$D_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{1 \leq j \leq n} a_j e_j)$,

$V_A(t) = p(t(\sum_{i \in A} e_i) + \sum_{j \in A'} a_j e_j)$.

$S_p(A)$ can be expressed as an univariate degree :

$$S_p(A) = deg(D_A) = deg(V_A)$$

Homogeneous polynomials with nonnegative coefficients

The following linear differential operator maps $Hom(n,n)$ onto $Hom(n-1,n-1)$:

$$p_{x_1}(x_2,...,x_n) = \frac{\partial}{\partial x_1} p(0,x_2,...,x_n).$$

We define $p_{x_i}, 2 \leq i \leq n$ in the same way for all polynomials $p \in Hom(n,n)$. Notice that

$$p(x_1,...,x_n) = x_i p_{x_i}(x_2,...,x_n) + q(x_1,...,x_n); q_{x_i} = 0.$$

The following inequality follows straight from the definition :

$$S_{p_{x_1}}(A) \leq \min(n-1, S_p(A)) :$$
$$A \subset \{2,...,n\}, p \in Hom_+(n,n).$$

Consider $p \in Hom_+(n,n)$ We define the **Capacity** as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \le i \le n} x_i = 1} p(x_1, ..., x_n).$$

It follows that if $p \in Hom_+(n,n)$ then

$$Cap(p) \ge \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0)$$

$(p(x_1, ..., x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0) x_1 ... x_n +$ nonnegative stuff .)

Notice that

$$\log(Cap(p)) = inf_{\sum_{1 \le i \le n} y_i = 0} \log(p(e^{y_1}, ..., e^{y_n})),$$

and if $p \in Hom_+(n,n)$

then the functional $\log(p(e^{y_1}, ..., e^{y_n}))$ is convex .

EXAMPLE

Let $A = \{A(i,j) : 1 \le i \le n\}$ be $n \times n$ matrix with nonnegative entries . Assume that $\sum_{1 \le j \le n} A(i,j) > 0$ for all $1 \le i \le n$. Define the following homogeneous polynomial :

$Mul_A(t_1, ..., t_n) = \prod_{1 \le i \le n} \sum_{1 \le j \le n} A(i,j) t_j$.

$Mul_A \in Hom_+(n,n)$ and $Mul_A \neq 0$.

It is easy to check that

$S_{Mul_A}(\{j\}) = |\{i : A(i,j) \neq 0\}|$

($S_{Mul_A}(\{j\})$ is equal to the number of non-zero entries in the jth column of A) .

Notice that if $A \in \Lambda(k,n)$ **(or** $A \in \Omega(k,n)$**) then** $S_{Mul_A}(\{j\}) \le k, 1 \le j \le n$ **.**

More generally , consider a n-tuple $\mathbf{A} = (A_1, A_2, ...A_n)$, where the complex hermitian $n \times n$ matrices are positive semidefinite and $\sum_{1 \leq i \leq n} A_i \succ 0$ (their sum is positive definite).

Then the homogeneous polynomial

$DET_{\mathbf{A}}(t_1, ..., t_n) = \det(\sum_{1 \leq i \leq n} t_i A_i) \in Hom_+(n, n)$ and $DET_{\mathbf{A}} \neq 0$.

Similarly to polynomials Mul_A :

$S_{DET_{\mathbf{A}}}(\{j\}) = Rank(A_j), 1 \leq j \leq n.$

As Van Der Waerden conjecture on permanents as well Bapat's conjecture on mixed discriminants can be eqiuvalently stated in the following way (notice the absence of doubly stochasticity):

$$\frac{n!}{n^n} Cap(q) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} q(0, ..., 0) \leq Cap(q)(*)$$

The van der Waerden conjecture on the permanents corresponds to polynomials $Mul_A \in Hom_+(n, n)$: $A \geq 0$, the Bapat's conjecture on mixed discriminants corresponds to $DET_{\mathbf{A}} \in Hom_+(n, n) : \mathbf{A} \succeq 0$. The connection between inequality (*) and the standard forms of the van der Waerden and Bapat's conjectures is established with the help of the scaling .

Notice that the functional $\log(p(e^{y_1}, ..., e^{y_1}))$ is convex if $p \in Hom_+(n, n)$. Thus the inequality (*) allows a convex relaxation of the permanent of nonnegative matrices and the mixed discriminant of semidefinite tuples . This observation was implicit in [LSW, 1998] and crucial in [GS 2000 , 2002] .

VDW-FAMILIES

Consider a stratified set of homogeneous polynomials : $F = \bigcup_{1 \leq n < \infty} F_n$, where $F_n \in Hom_+(n,n)$. We call such set **VDW-FAMILY** if it satisfies the following properties :

1. If a polynomial $p \in F_n, n > 1$ then for all $1 \leq i \leq n$ the polynomials $p_{x_i} \in F_{n-1}$.

2.
$$Cap(p_{x_i}) \geq g(S_p(\{i\}))Cap(p) :$$
$p \in F_j, 1 \leq i \leq j; g(k) = (\frac{k-1}{k})^{k-1}, k \geq 1.$

Meta-Theorem , main idea :

Let $F = \bigcup_{1 \leq n < \infty} F_n$ *be a* **VDW-FAMILY** *and the homogeneous polynomial* $p \in F_n$*. Then the following inequality holds :*

$$\prod_{1 \leq i \leq n} g(\min(S_p(\{i\})), i)) Cap(p) \leq$$

$$\leq \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \leq Cap(p).$$

Corollaries :

1. If the homogeneous polynomial $p \in F_n$ then
$$\frac{n!}{n^n} Cap(p) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \leq Cap(p).$$
2. If the homogeneous polynomial $p \in F_n$ and $S_p(\{i\})) \leq k \leq n, 1 \leq i \leq n$ then
$(\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k} Cap(p) \leq \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0) \leq$
$\leq Cap(p)$.

What is left now is to present a **VDW-FAMILY** which contains all polynomials $DET_{\mathbf{A}}$, where the n-tuple $\mathbf{A} = (A_1, ..., A_n)$ consists of positive semidefinite hermitian matrices (and thus contains all polynomials Mul_A , where A is $n \times n$ matrix with nonnegative entries).

If such **VDW-FAMILY** set exists than Van der Waerden , Bapat , Schrijver-Valiant conjectures would follow (without any extra work , see Example) from Meta-Theorem and its Corollaries .

One of such **VDW-FAMILY** , consisiting of POS-hyperbolic polynomials , is defined below.

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Hyperbolic polynomials

The following concept of hyperbolic polynomials was originated in the theory of partial differential equations **I.G. Petrowsky (1937 , in german) , L. Garding (1950) , L. Hormander ....**

It recently became "popular" in the optimization literature.

$$p \in Hom(m, n), X, e \in R^m : p(X - te) = 0$$

The polynomial p is e-hyperbolic

If all the roots $\lambda_n(X) \geq ... \geq \lambda_1(X)$ are real .

Hyperbolic (convex) Cones :

$N_e(p) = \{X \in R^m : \lambda_1(X) \geq 0$ (closed) ,

$C_e(p) = \{X \in R^m : \lambda_1(X) > 0$ (open) .

$p \in Hom(m,n)$ is POS-hyperbolic if it is $(1,1,...,1) = e$-hyperbolic , $p(e) > 0$
and the nonnegative orthant $R_+^m \subset N_e(p)$.

Equivalent definitionS :
$|p(x_1 + iy_1, ..., x_m + iy_m)| > 0$ if $x_i > 0, 1 \leq i \leq n$

So called wide sense stability in **CONTROL THEORY** .

Or : $p(1,...,1) > 0$ and all the roots of the univariate equation $p(x_1 - t, ..., x_n - t) = 0$
are real positive numbers if $x_i > 0, 1 \leq i \leq n$.

bf p-Mixed Forms

Let $p \in Hom(m,n)$ Khovanskii defined the p-mixed form of an n-vector tuple $\mathbf{X} = (X_1, .., X_n) : X_i \in C^m$ as

$$M_p(\mathbf{X}) =: M_p(X_1, .., X_n) = \frac{\partial^n}{\partial\alpha_1 ... \partial\alpha_n} p(\sum_{1 \leq i \leq n} \alpha_i X_i)$$

The following polarization identity is well known

$$M_p(X_1, .., X_n) = 2^{-n} \sum_{b_i \in \{-1,1\}, 1 \leq i \leq n} p(\sum_{1 \leq i \leq n} b_i X_i) \prod_{1 \leq i \leq n} b_i$$

Associate with any vector $r = (r_1, ..., r_n) \in I_{n,n}$ an n-tuple of m-dimensional vectors $\mathbf{X}_r$ consisting of r_i copies of $x_i (1 \leq i \leq n)$. It follows from the Taylor's formula that

$$p(\sum_{1 \leq i \leq n} \alpha_i X_i) = \sum_{r \in I_{n,n}} \prod_{1 \leq i \leq n} \alpha_i^{r_i} M_p(\mathbf{X}_r) \frac{1}{\prod_{1 \leq i \leq n} r_i!}$$

POS-Hyperbolic polynomials , basic facts

FACT 1 . $p(X) = p(e)\prod_{1 \leq i \leq n} \lambda_i(X)$.

FACT 2 . If p is e-hyperbolic polynomial and $p(e)$ is a real nonzero number then the coefficients of p are real.
If p is e-hyperbolic polynomial and $p(e) > 0$ then $p(X) > 0$ for all e-positive vectors $X \in C_e(p) \subset R^m$.

FACT 3 . Let $p \in Hom(m, n)$ be e-hyperbolic polynomial and $d \in C_e(p) \subset R^m$. Then p is also d-hyperbolic and $C_d(p) = C_e(p), N_d(p) = N_e(p)$.

FACT 4 . Let $p \in Hom(m, n)$ be e-hyperbolic polynomial . Then the polynomial $p_e(X) =: \frac{d}{dt}p(X + te)|_{(t=0)}; p_e \in Hom(m, n-1)$ is also e- hyperbolic and $C_e(p) \subset C_e(p_e)$ (Rolle's theorem).

FACT 5 . Let $p \in Hom(m,n)$. Then the p-mixed form $M_p(X_1, .., X_n)$ is linear in each vector argument $X_i \in C^m$. Let $p \in Hom(m,n)$ be e- hyperbolic and $p(e) > 0$. Then $M_p(X_1, .., X_n) > 0$ if the vectors $X_i \in R^m, 1 \leq i \leq n$ are e-positive (proved by induction using FACT 4) .

POS-Hyperbolic polynomials form VDW-FAMILY

Define $Rank_q(X)$ as $|\{i : \lambda_i(X) \neq 0\}|$.
Then $S_q(A) = Rank_q(\sum_{i \in A} e_i)$.

Theorem

1. Let $q \in Hom_+(n,n)$ be POS-hyperbolic polynomial . If $1 \leq Rank_q(e_1) = k \leq n$ then
$$Cap(q_{x_1}) \geq g(k) Cap(q), g(k) = (\frac{k-1}{k})^{k-1}.$$
2. Let $q(x_1, x_2, ..., x_n)$ be a POS-hyperbolic (homogeneous) polynomial of degree n . Then either the polynomial $q_{x_1} = 0$ or q_{x_1} is POS-hyperbolic . If $Cap(q) > 0$ then q_{x_1} is (nonzero) POS-hyperbolic .

Corollary

Let $PHP(n) \subset Hom_{+}(n,n)$ be a set of POS-hyperbolic polynomial of degree n in n variables ; define $PHP_{+}(n) = \{p \in PHP(n) : Cap(p) > 0\}$. Then as $\cup_{n \geq 1}(PHP(n) \cup \{0\})$ as well $\cup_{n \geq 1} PHP_{+}(n)$ is **VDW-FAMILY** .

Second Part was almost known : Rolle's theorem + a bit of pertubrations .

First is ... a particularly easy case of the Van der Waerden Conjecture .

$q(t, x_2, ..., x_n) = q(0, x_2, ..., x_n) + tq_{x_1}(x_2, ..., x_n) + ...c_k t^k = R(t)$.

Fix a positive $n-1$-dim. vector $(x_2, ..., x_n), x_2...x_n = 1$.

We know that $R(t) \geq Cap(q)t, t \geq 0$ and want to prove
that $q_{x_1}(x_2, ..., x_n) = R'(0) \geq (\frac{k-1}{k})^{k-1} Cap(q)$.

Tha fact that q is POS-hyperbolic implies that

$$R(t) = \prod_{1 \leq i \leq k} (a_i t + b_i) : a_i, b_i > 0.$$

Consider $k \times k$ matrix

$A = [a|d|d|, ..., |d], a = (a_1, ..., a_n)^T, d = \frac{1}{n-1}(b_1, ..., b_n)^T$

.

Then $\frac{(k-1)!}{(k-1)^{k-1}} R'(0) = per(A)$ and $Cap(Mul_A) \geq Cap(q)$.

Sinkhorn's Scaling : $A = Diag_1 B Diag_2, B \in \Omega_n$: $Cap(q) \leq Cap(Mul_A) = \det(Diag_1 Diag_2)$,

$per(A) = \det(Diag_1 Diag_2) per(B) \geq \frac{(k)!}{(k)^k}$

Which gives that

$R'(0) = (\frac{(k-1)!}{(k-1)^{k-1}})^{-1} per(A) \geq$

$\geq \frac{(k)!}{(k)^k} (\frac{(k-1)!}{(k-1)^{k-1}})^{-1} Cap(q) = (\frac{k-1}{k})^{k-1} Cap(q)$

Another proof based on the Newtons inequalities : Let $R(t) = \sum_{0 \leq i \leq n} d_i t^i$ be an univariate polynomial with real coefficients. If such polynomial R has all real roots then its coefficients satisfy the following Newton's inequalities :

$$NIs : d_i^2 \geq d_{i-1} d_{i+1} \frac{\binom{n}{i}^2}{\binom{n}{i-1}\binom{n}{i+1}} : 1 \leq i \leq n-1.$$

The following weak Newton's inequalities $WNIs$ follow from NIs if the coefficients are nonnegative:

$$WNIs : d_i d_0^{i-1} \leq \frac{d_1}{n}^i \binom{n}{i} : 2 \leq i \leq n.$$

Lemma

Let $R(t) = \sum_{0 \leq i \leq n} d_i t^i$ be an univariate polynomial with real nonnegative coefficients satisfying weak Newton's inequalities $WNIs$ .If for some positive real number C the inequality $R(t) \geq Ct$ holds for all $t \geq 0$ then

$$d_1 \geq C((\frac{n-1}{n})^{n-1}).$$

Proof:

If $d_0 = 0$ then $d_1 \geq C > C((\frac{n-1}{n})^{n-1})$. Thus we can assume that $d_0 = 1$. It follows from weak Newton's inequalities $WNIs$ that

$$d_i \leq (\frac{d_1}{n})^i \binom{n}{i} : 2 \leq i \leq n.$$

Therefore for nonnegative values of $t \geq 0$ we get the inequality

$$R(t) \leq 1+(\frac{d_1 t}{n})\binom{n}{1}+(\frac{d_1 t}{n})^2\binom{n}{2}+...(\frac{d_1 t}{n})^n\binom{n}{n} = (1+\frac{d_1 t}{n})^n.$$

Which gives the inequality $(1 + \frac{d_1 t}{n})^n \geq Ct$. The inequality $d_1 \geq C((\frac{n-1}{n})^{n-1})$ follows now easily. Indeed consider the following optimization problem $\min_{t>0} \log((1+\frac{d_1 t}{n})^n) - \log(t)$. Its only minimizer is $t = \frac{n}{d_1(n-1)}$. Which gives the next inequality :

$$d_1(\frac{n}{n-1})^{n-1} = \min_{t>0}(1 + \frac{d_1 t}{n})^n t^{-1} \geq$$

$$\geq \inf_{t>0} \frac{R(t)}{t} \geq C.$$

We finally get that $d_1 \geq C((\frac{n-1}{n})^{n-1})$.

Consider the class of the Minkowski polynomials :

$Vol_{\mathbf{C}}(x_1, ..., x_n) = Vol(x_1C_1 + ... + x_nC_n)$, where C_i are convex compact subsets of R^n.
The Minkowski polynomials are not generally hyperbolic if $n \geq 3$. But the previous Lemma allows to prove that there exists a **VDW-FAMILY** containing the Minkowski polynomials .

This leads to a randomized (we need to evaluate $Vol(x_1C_1 + ... + x_nC_n)$) poly-time algorithm to approximate the mixed volume

$M(C_1, ..., C_n) = \frac{\partial^n}{\partial x_1 ... \partial x_n} Vol_{\mathbf{C}}(0, ..., 0)$
within a multiplicative factor e^n . The best result up to date .

Algorithmic Applications

Theorem

1. Let $p \in Hom_+(n,n)$ be POS-hyperbolic polynomial . Then the function $Rank_p(\sum_{i \in A} e_i) = S_p(A)$ is submodular , i.e. $S_p(A \cup B) \leq S_p(A) + S_p(B) - S_p(A \cap B) : A, B \subset \{1, 2, ..., n\}$.
2. Consider a nonnegative integer vector
   $r = (r_1, ..., r_n), \sum_{1 \leq i \leq n} r_i = n$.
   Then
   $r \in supp(p)$ iff $r(S) = \sum_{i \in S} r_i \leq S_p(S) : S \subset \{1, 2, ..., n\}$.

Corollary

Let $p \in Hom_+(n,n)$ be POS-hyperbolic polynomial. Associate with this polynomial p the following bounded convex polytope :

$SUB_p = \{(x_1, ..., x_n) : \sum_{i \in S} x_i \leq S_p(S) : S \subset \{1, 2, ..., n\}$;

$\sum_{1\leq i\leq n} x_i = n; x_i \geq 0, 1 \leq i \leq n\}$

$\sum_{1\leq i\leq n} x_i = n; x_i \geq 0, 1 \leq i \leq n\}$.

Then SUB_p is equal to the Newton polytope of p , i.e. $SUB_p = CO(supp(p)$.

Corollary

Given POS-hyperbolic polynomial $p \in Hom_+(n, n)$ as an oracle , there exists strongly polynomial-time oracle algorithm for the membership problem as for $supp(p)$ as well for the Newton polytope $CO(supp(p))$.

The membership problep for$supp(p)$ is **NP-HARD** for general $p \in Hom_+(n, n)$:

Define $Bar(x_1, ..., x_n) = tr((Diag(x_1, ..., x_n)A)^n)$, then

$\frac{1}{n}\frac{\partial^n}{\partial x_1 ... \partial x_n} Bar(0, ..., 0) =$

the number of Hamiltonian circuits in the graph defined by a boolean matrix A .

Or , let $F = \{S_1, ..., S_m\} : S_i \subset \{1, 2, ..., n\}, |S_i| = k; \frac{n}{k} \in Z.$
Define $COV_F(x_1, ..., x_n) = (\sum_{S_j \in F} \prod_{i \in S_j} x_i)^{\frac{n}{k}}$.
Then $(k!)^{-1}\frac{\partial^n}{\partial x_1 ... \partial x_n} COV_F(0, ..., 0) =$

the number of exact coverings .

There exists a deterministic polynomial-time oracle algorithm which computes for given as an oracle indecomposable POS-hyperbolic polynomial $p(x_1, ..., x_n)$ a number $F(p)$ satisfying the inequality

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \leq F(p) \leq$$

$$\leq 2\left(\prod_{1 \leq i \leq n} g(\min(S_p(\{i\})), i))\right)^{-1} \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \leq$$

$$\leq 2 \frac{n^n}{n!} \frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \ .$$

The prev. result can be (slightly) improved . I.e. it can be applied to the polynomial

$$p_k(x_{k+1}, ..., x_n) = \frac{\partial^k}{\partial x_1 ... \partial x_k} p(0, .., 0, x_{k+1}, ..., x_n).$$

Notice that the polynomial p_k is a homogeneous polynomial of degree $n-k$ in $n-k$ variables .

If $p = p_0$ is POS-hyperbolic and $Cap(p) > 0$ then for all $0 \leq k \leq n$ the polynomials p_k are also POS-hyperbolic and $Cap(p_k) > 0$.

Also , if $p = p_0$ is indecomposable then p_k is indecomposable as well (Theorem 4.6).

The trick is that if $k = m \log_2(n)$ then (using the polarizational formula) the polynomial p_k can be evaluated using $O(n^{m+1})$ oracle calls of the (original) polynomial $p = p_0$.

This observations allows to decrease the worst case multiplicative factor from e^n to $\frac{e^n}{n^m}$ for any fixed m . If the polynomial $p = p_0$ can be explicitly evaluated in deterministic polynomial time , this observation results in deterministic polynomial time algorithms to approximate $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0)$ within multiplicative factor $\frac{e^n}{n^m}$ for any fixed m . Which is an improvement of results in [LSW] (permanents , p is a multilinear polynomial) and in [GS] (mixed discriminants, p is a determinantal polynomial) .