Branch Flow Model

relaxations, convexification, equivalence

Masoud Farivar Steven Low
Subhonmesh Bose Mani Chandy

Computing + Math Sciences
Electrical Engineering
Caltech

February 2013

Acks and refs

Collaborators

- S. Bose, M. Chandy, L. Chen, L. Gan, D. Gayme, J. Lavaei, L. Li, U. Topcu
- SCE: R. Sherick, Juan Castaneda, C. Clark, A. Auld, J. Gooding, M. Montoya

BFM references

- Branch flow model: relaxations and convexification

  M. Farivar and S. H. Low, April 2012
- Equivalence of BF and BI models

  Bose, Low and Chandy, Allerton, Oct 2012

Other references

- Lavaei, L, TPS, 2012
- Bose, Gayme, Chandy, L., 2012
- Li, Chen, L, SGC, 2012
- Gan, Li, Topcu, L, CDC 2012

Outline

Two power flow models

- Bus injection model
- Branch flow model

OPF in BI model

- Semidefinite relaxation

OPF in BF model

- SOCP relaxation
- Convexification using phase shifters

Equivalence relationship

how to overcome nonconvexity in OPF

Why is convexity important

Foundation of LMP

- Convexity justifies the use of Lagrange multipliers as various prices
- Critical for efficient market theory

Efficient computation

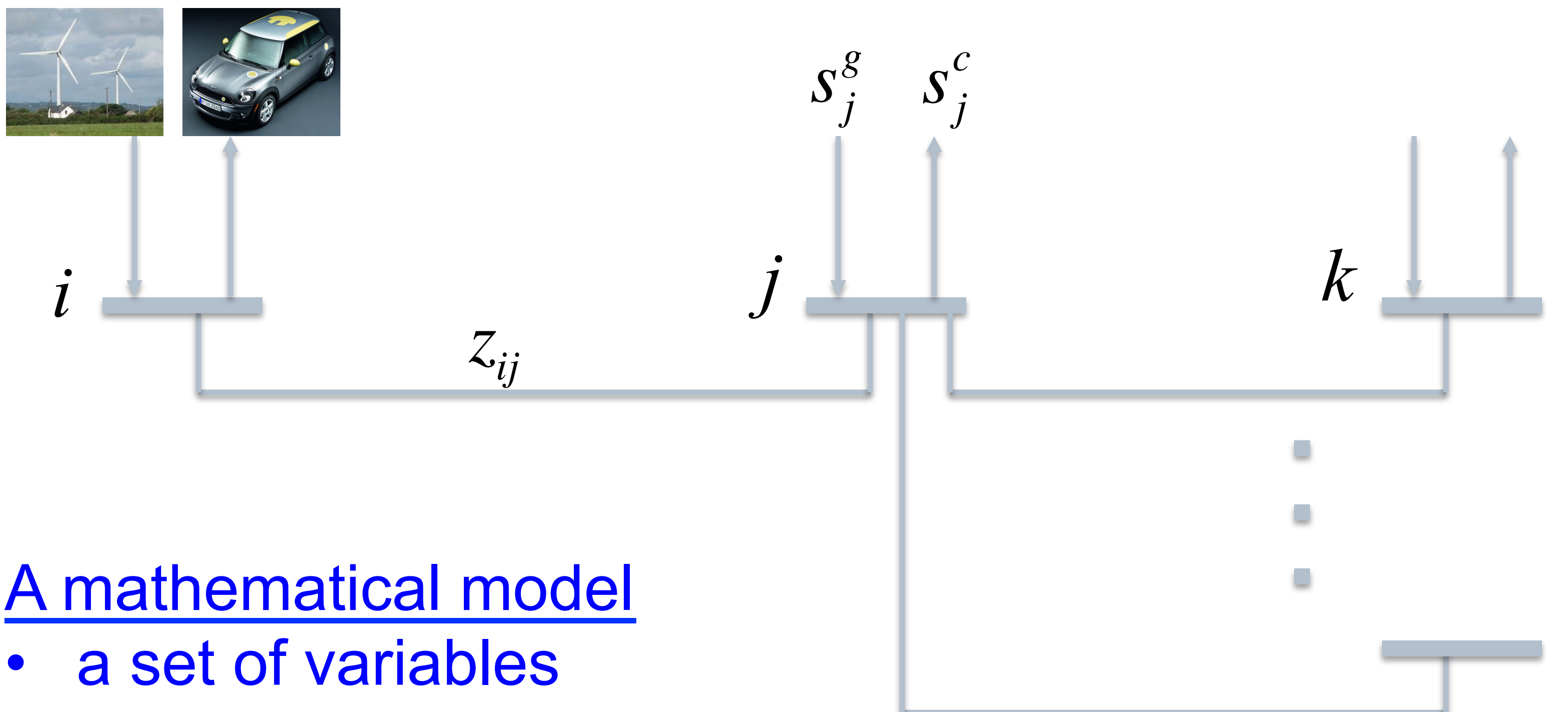
- Convexity delineates computational efficiency and intractability

A lot rides on (assumed) convexity structure

- engineering, economics, regulatory

two models

Power network



A mathematical model
- a set of variables
- a set of equations

… that describe
- Kirchhoff law
- power definition
- power balance

Bus injection model

$$\tilde{I} = Y\tilde{V}$$ Kirchhoff law

$$\tilde{S}_j = \tilde{V}_j \tilde{I}_j^* \qquad \text{for all } j$$ power definition

$$\tilde{S}_j = s_j \qquad \text{for all } j$$ power balance

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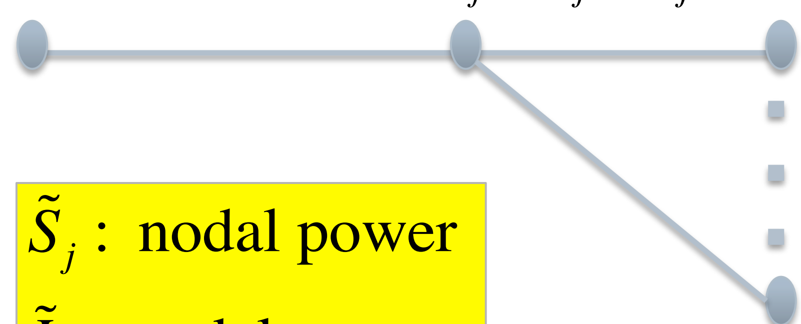
admittance matrix:

$$Y_{ij} := \begin{cases} \sum_{k \sim i} y_{ik} & \text{if } i = j \\ -y_{ij} & \text{if } i \sim j \\ 0 & \text{else} \end{cases}$$

$$s_j = s_j^g - s_j^c$$



$\tilde{S}_j$: nodal power

$\tilde{I}_j$: nodal current

$\tilde{V}_j$: voltage

Bus injection model

$$\tilde{I} = Y\tilde{V}$$ Kirchhoff law

$$\tilde{S}_j = \tilde{V}_j \tilde{I}_j^* \qquad \text{for all } j$$ power definition

$$\tilde{S}_j = s_j \qquad \text{for all } j$$ power balance

Given (Y, s) find $(\tilde{S}, \tilde{I}, \tilde{V})$

$\mathbf{W}_1$

BIM is self-contained (e.g. no branch vars)

Bus injection model

Can reduce to $\tilde{V}$:

$$s_j = \operatorname{tr}\left(Y^* e_j e_j^T \tilde{V}\tilde{V}^*\right) \qquad \text{for all } j$$

Given (Y, s) find $\tilde{V}$

$\mathbf{W}_1$

BIM is self-contained (e.g. no branch vars)

Branch flow model

$$V_i - V_j = z_{ij} I_{ij} \qquad \text{for all } i \rightarrow j$$ Kirchhoff law

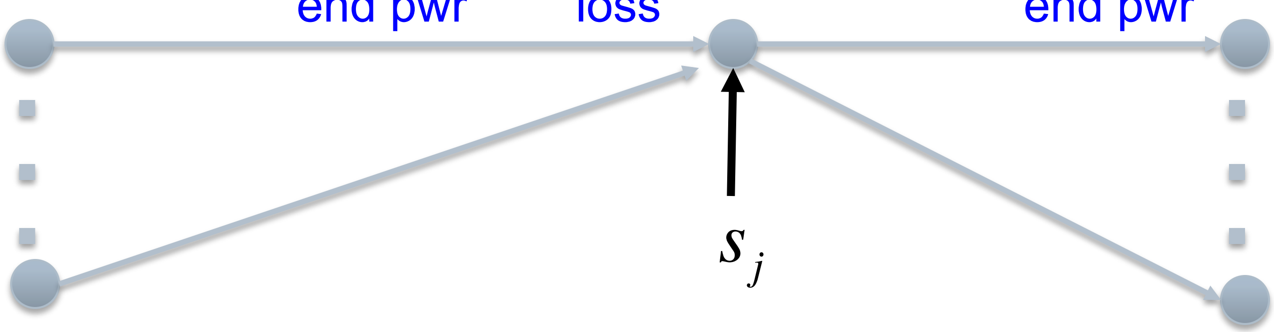
$$S_{ij} = V_i I_{ij}^* \qquad \text{for all } i \rightarrow j$$ power definition

$$\sum_{i \rightarrow j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j = \sum_{j \rightarrow k} S_{jk} \qquad \text{for all } j$$ power balance

sending end pwr

loss

sending end pwr



S_{ij} : branch power

I_{ij} : branch current

V_j : voltage

Branch flow model

$$V_i - V_j = z_{ij} I_{ij} \qquad \text{for all } i \to j$$ Kirchhoff law

$$S_{ij} = V_i I_{ij}^* \qquad \text{for all } i \to j$$ power definition

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) + s_j = \sum_{j \to k} S_{jk} \qquad \text{for all } j$$ power balance

---

Given (z, s) find (S, I, V)

$\mathbf{X}_c$

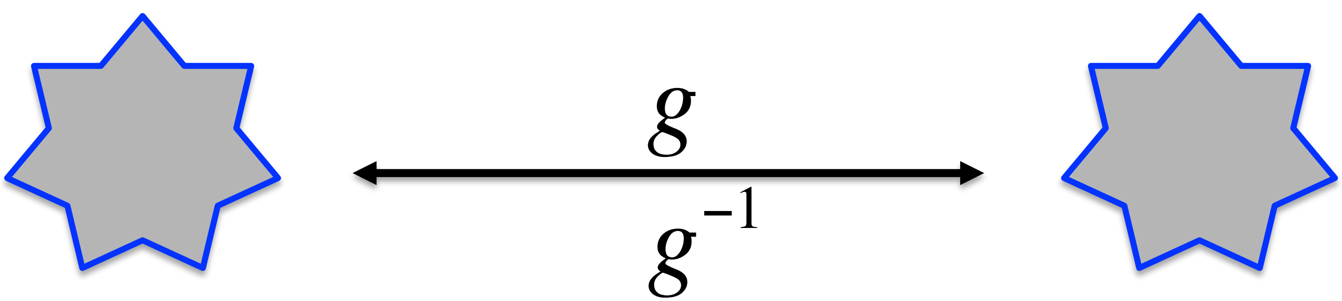
BFM is self-contained (e.g. no nodal pwr/currents)

Equivalence

<u>Theorem</u>

The branch flow and bus injection models are equivalent

- There is bijection between solution sets



$\left(\tilde{S},\tilde{I},\tilde{V}\right)$ $\left(S,I,V\right)$

Equivalence

$$\tilde{V} = V$$

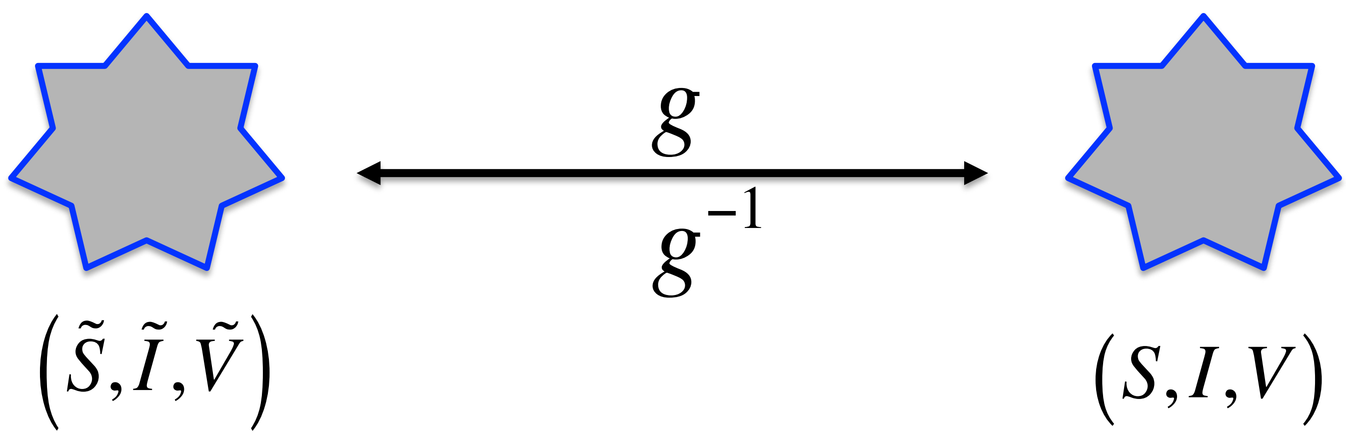
$$\tilde{I}_j = \sum_{j \to k} I_{jk} - \sum_{i \to j} I_{ij}$$

$$\tilde{S}_j = \sum_{j \to k} S_{jk} - \sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right)$$

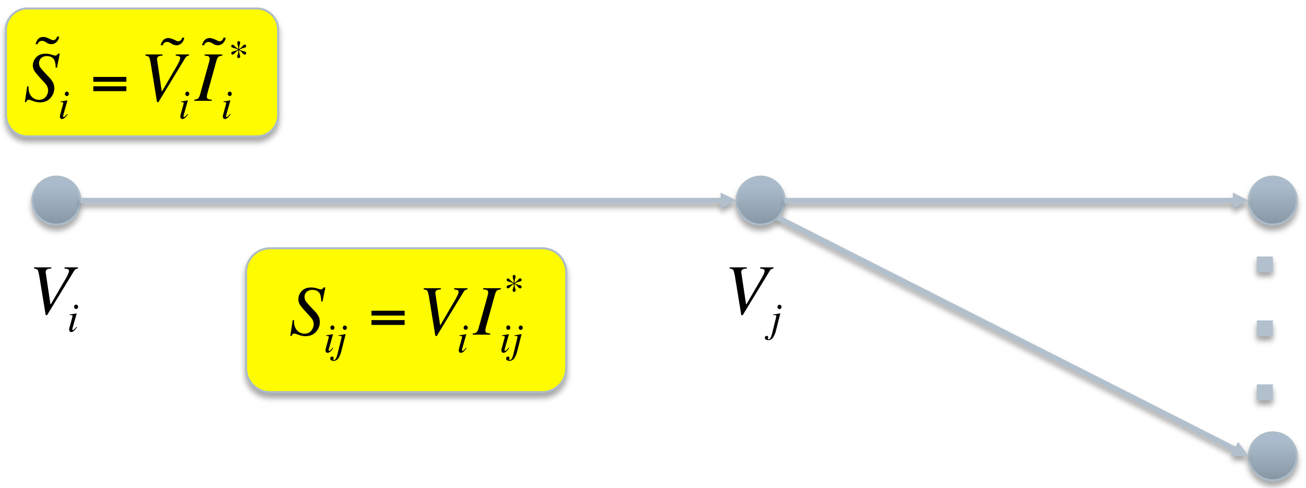
$$V = \tilde{V}$$

$$I_{ij} = y_{ij} \left(\tilde{V}_i - \tilde{V}_j \right)$$

$$S_{ij} = y_{ij}^* \left(\left| \tilde{V}_i \right|^2 - \tilde{V}_i \tilde{V}_j^* \right)$$



Recap: two models



Equivalent models of Kirchhoff laws

- Bus injection model focuses on nodal vars
- Branch flow model focuses on branch vars

Outline

Two power flow models

- Bus injection model
- Branch flow model

OPF in BI model

- Semidefinite relaxation

OPF in BF model

- SOCP relaxation
- Convexification using phase shifters

Equivalence relationship

OPF: bus injection model

min $\sum_j f_j(\tilde{x})$ e.g. quadratic gen cost

over $\tilde{x} := \left(\tilde{S}, \tilde{I}, \tilde{V}, s\right)$

subject to

OPF: bus injection model

$$\min \quad \sum_j f_j(\tilde{x}) \qquad \text{e.g. quadratic gen cost}$$

$$\text{over} \quad \tilde{x} := \left(\tilde{S}, \tilde{I}, \tilde{V}, s\right)$$

$$\text{subject to} \quad \boxed{\underline{s}_j \leq s_j \leq \overline{s}_j \qquad \underline{V}_k \leq |\tilde{V}_k| \leq \overline{V}_k}$$

OPF: bus injection model

min $\quad \sum_j f_j(\tilde{x})$ $\quad$ e.g. quadratic gen cost

over $\quad \tilde{x} := \left(\tilde{S}, \tilde{I}, \tilde{V}, s\right)$

subject to $\quad \underline{s}_j \le s_j \le \overline{s}_j \qquad \underline{V}_k \le |\tilde{V}_k| \le \overline{V}_k$

$\tilde{I} = Y\tilde{V}$ $\quad$ Kirchhoff law

$\tilde{S}_j = s_j \qquad \tilde{S}_j = \tilde{V}_j \tilde{I}_j^*$ $\quad$ power balance

nonconvex, NP-hard

Semidefinite relaxation

In terms of V:

$$P_k = \mathrm{tr}\, \Phi_k V V^*$$

$$Q_k = \mathrm{tr}\, \Psi_k V V^*$$

$$\Phi_k := \left(\frac{Y_k^* + Y_k}{2} \right)$$

$$\Psi_k := \left(\frac{Y_k^* - Y_k}{2\mathrm{i}} \right)$$

$$\begin{array}{ll}
\min & \sum_{k \in G} \mathrm{tr}\, M_k V V^* \\
\text{over} & V \\
\text{s.t.} & \underline{P}_k^g - P_k^d \le \mathrm{tr}\, \Phi_k V V^* \le \overline{P}_k^g - P_k^d \\
 & \underline{Q}_k^g - Q_k^d \le \mathrm{tr}\, \Psi_k V V^* \le \overline{Q}_k^g - Q_k^d \\
 & \underline{V}_k^2 \le \mathrm{tr}\, J_k V V^* \le \overline{V}_k^2
\end{array}$$

Key observation [Bai et al 2008]:
OPF = rank constrained SDP

Semidefinite relaxation

$$\min \quad \sum_{k \in G} \operatorname{tr} M_k W$$

$$\text{over} \quad W \ \text{positive semidefinite matrix}$$

$$\text{s.t.} \quad \underline{P}_k \le \operatorname{tr} \Phi_k W \le \overline{P}_k$$

$$\underline{Q}_k \le \operatorname{tr} \Psi_k W \le \overline{Q}_k$$

$$\underline{V}_k^2 \le \operatorname{tr} J_k W \le \overline{V}_k^2$$

$$W \ge 0, \quad \cancel{\operatorname{rank} W = 1}$$

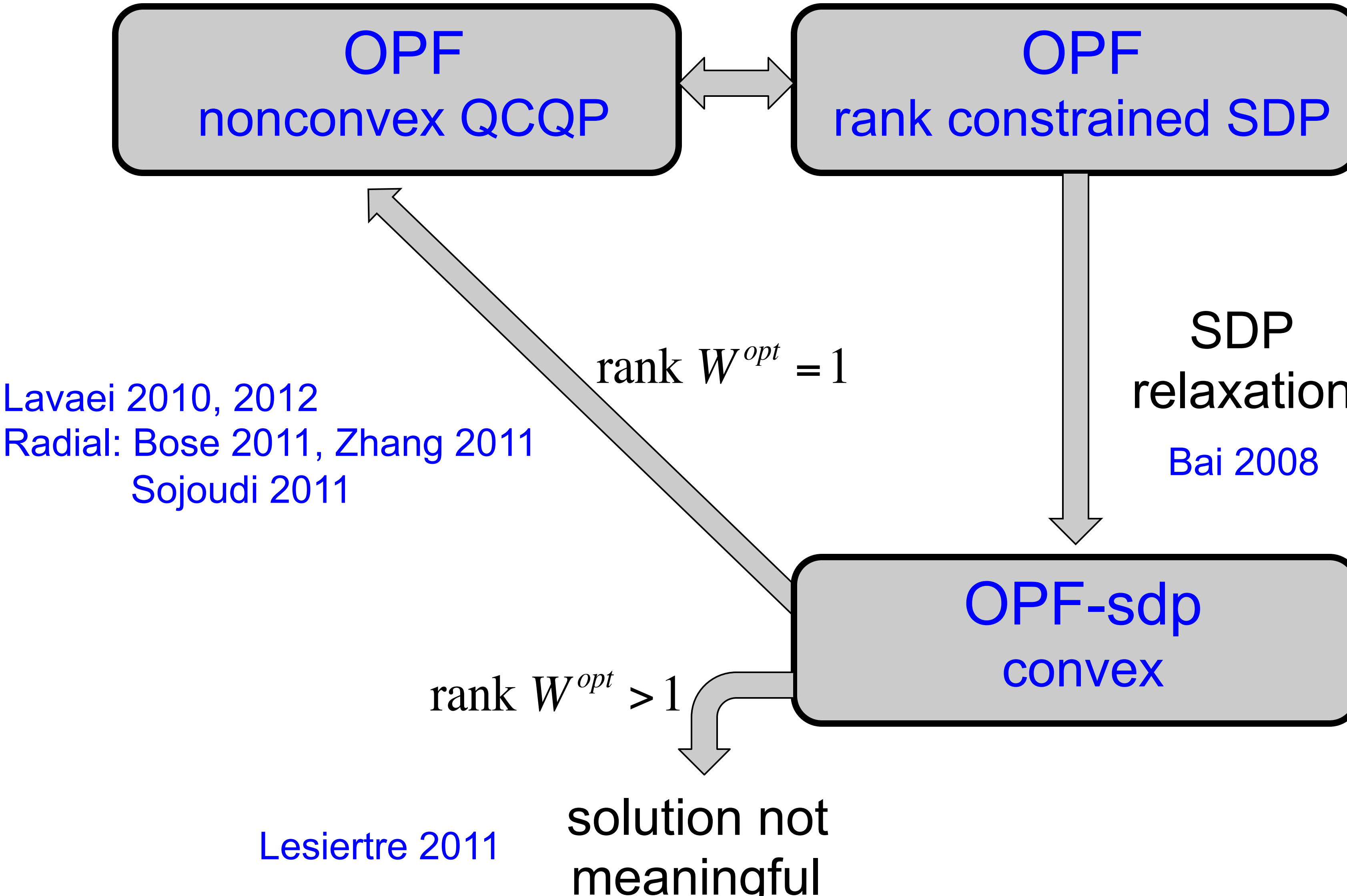
convex relaxation: SDP
polynomial

General QCQP

$$\begin{array}{ll} \min & \operatorname{tr} C_0 W \\ \text{over} & W \text{ matrices} \\ \text{s.t.} & \operatorname{tr} C_k W \le b_k \\ & W \ge 0 \\ & \cancel{\operatorname{rank} W = 1} \end{array}$$

convex relaxation: SDP
polynomial

Solution strategy

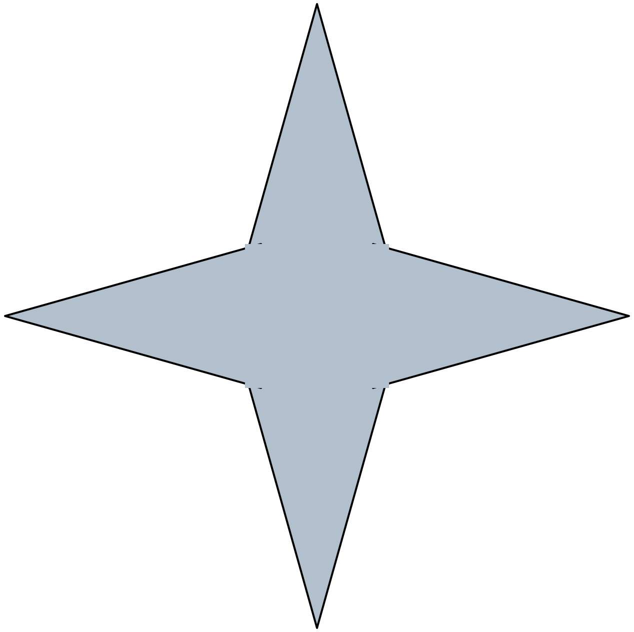


Feasible set: OPF

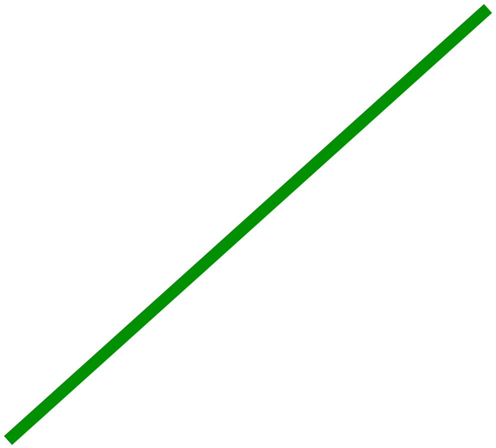
power flow solutions $\tilde{V}$:

$$s_j = \operatorname{tr}\left(Y^* e_j e_j^T \tilde{V}\tilde{V}^*\right)$$

$$\mathbf{N} := \left\{ W \geq 0 \,\middle|\, \operatorname{rank} W = 1 \right\}$$

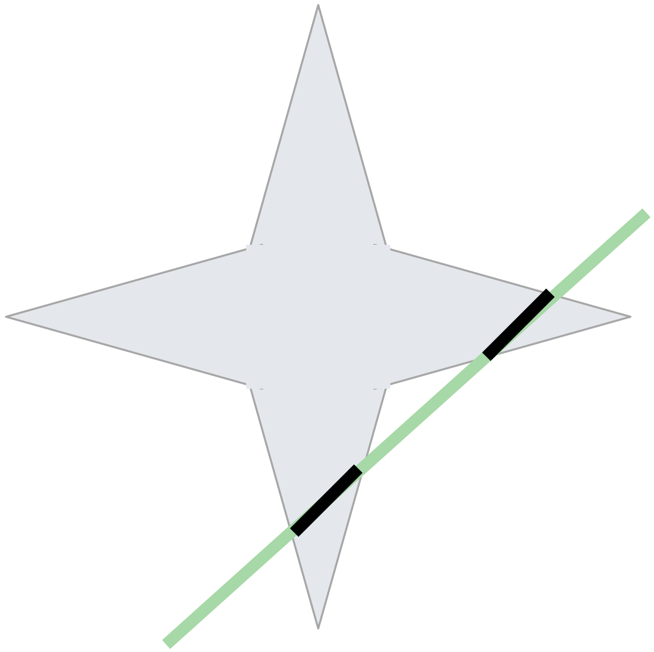


$$\mathbf{L} := \left\{ W \,\middle|\, s_j = \operatorname{tr}\left(Y^* e_j e_j^T W\right) \right\}$$



Feasible set: OPF

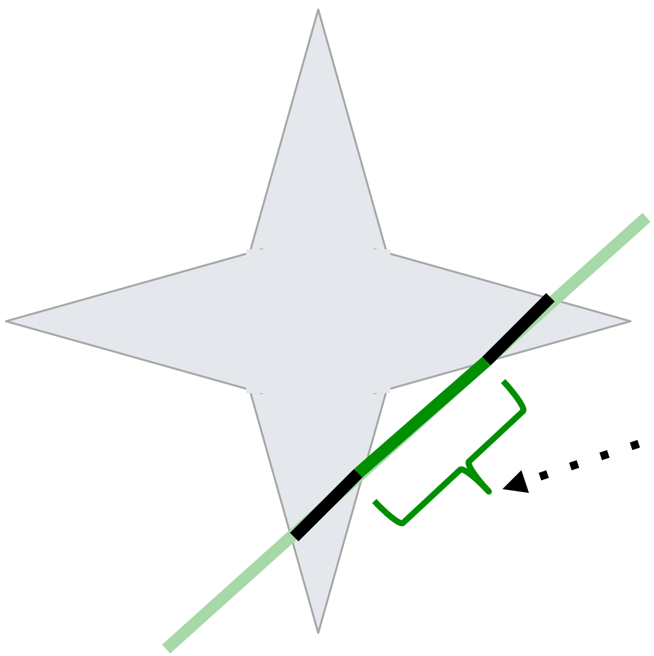
power flow solutions $\tilde{V}$: $$s_j = \text{tr}\left(Y^* e_j e_j^T \tilde{V}\tilde{V}^*\right)$$



$$\mathbf{N} \cap \mathbf{L}$$

Feasible set: convex hull

power flow solutions $\tilde{V}$: $$s_j = \mathrm{tr}\left(Y^* e_j e_j^T \tilde{V}\tilde{V}^*\right)$$



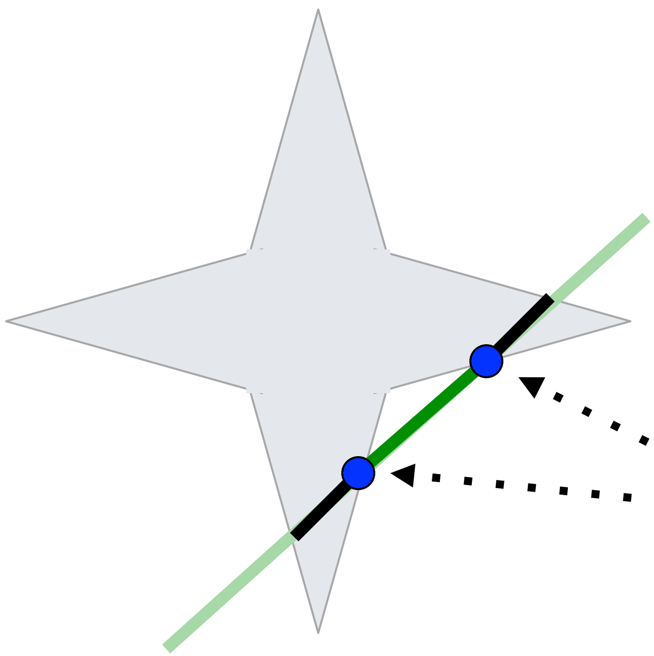
Any of these rank-2 W's are optimal iff

$$\mathrm{conv}\left(\mathbf{N} \cap \mathbf{L}\right)$$

Feasible set: convex hull

power flow solutions $\tilde{V}$:

$$s_j = \mathrm{tr}\left(Y^* e_j e_j^T \tilde{V}\tilde{V}^*\right)$$



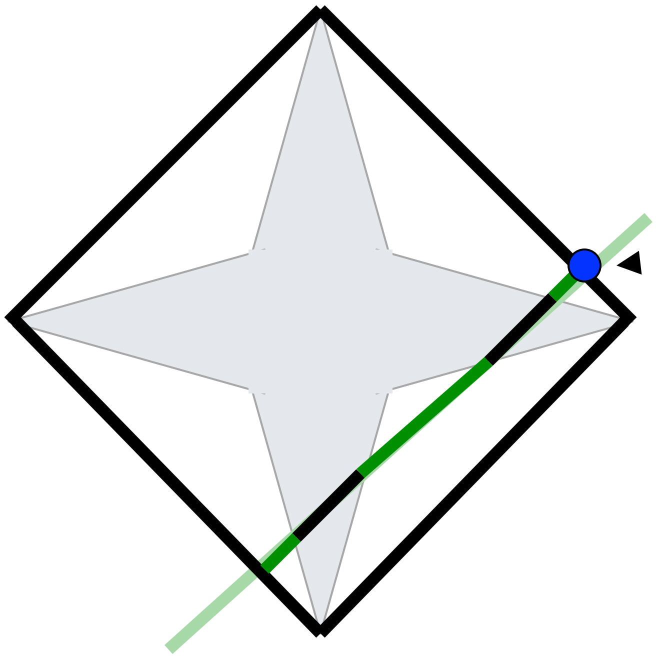
$\mathrm{conv}\left(\mathbf{N} \cap \mathbf{L}\right)$

these rank-1 W's are optimal

This relaxation always works
but is **not** SDP

Feasible set: SDP

SDP solution may not have
Feasible rank-1 decomposition



rank-2 SDP solution that
is not physically meaningful

SDP: $\text{conv}\mathbf{N} \cap \mathbf{L} \supseteq \text{conv}\left(\mathbf{N} \cap \mathbf{L}\right)$

QCQP over tree

QCQP $\left(C, C_k\right)$

$$
\begin{array}{lll}
\min & x^* C x & \\
\text{over} & x \in \mathbf{C}^n & \\
\text{s.t.} & x^* C_k x \le b_k & k \in K
\end{array}
$$

graph of QCQP

$$G\left(C, C_k\right) \text{ has edge } (i, j) \quad \Leftrightarrow$$

$$C_{ij} \neq 0 \text{ or } \left[C_k\right]_{ij} \neq 0 \text{ for some } k$$

QCQP over tree

$$G\left(C, C_k\right) \text{ is a tree}$$

QCQP over tree

QCQP $\left(C, C_k\right)$

$$\begin{array}{lll} \min & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \le b_k & k \in K \end{array}$$

Semidefinite relaxation

$$\begin{array}{lll} \min & \operatorname{tr} CW & \\ \text{over} & W \ge 0 & \\ \text{s. t.} & \operatorname{tr} C_k W \le b_k & k \in K \end{array}$$

QCQP over tree

QCQP (C, C_k)

$$\begin{array}{lll} \min & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \ \leq \ b_k & k \in K \end{array}$$

Key condition

$$i \sim j: \quad 0 \ \notin \ \text{int conv hull} \left(C_{ij}, [C_k]_{ij}, \ \forall k \right)$$

<u>**Theorem**</u>

semidefinite relaxation is exact for QCQP over tree

S. Bose, D. Gayme, S. H. Low and M. Chandy, March 2012

QCQP over tree

Remarks

- condition implies constraint patterns
- full AC: inc real and reactive powers
- allows constraints on line flows, loss

Key condition

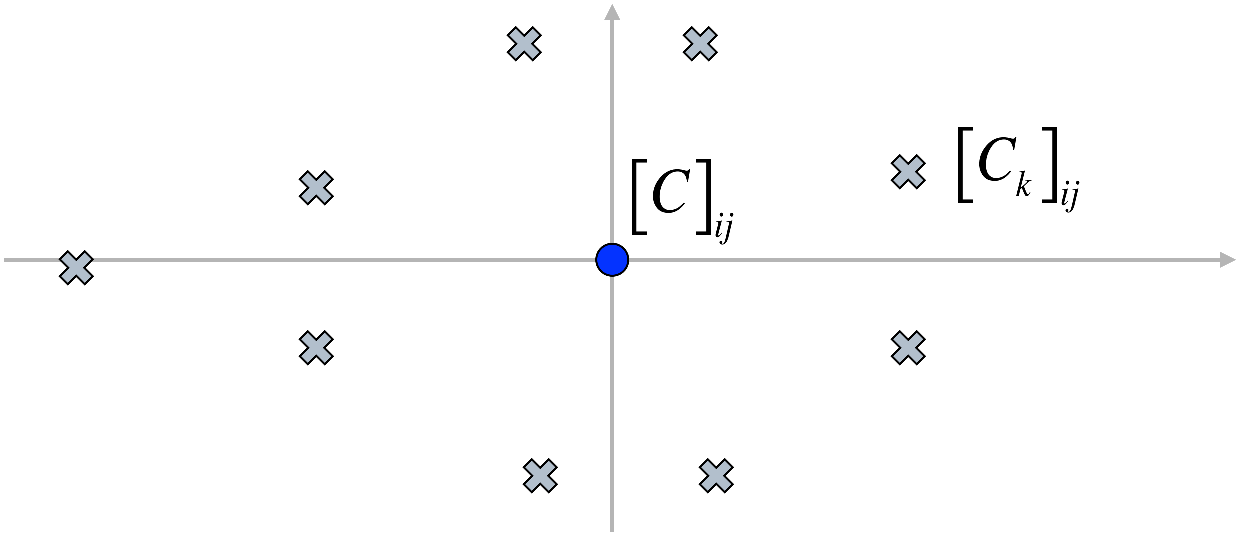
$$i \sim j: \quad 0 \notin \text{ int conv hull } \left(C_{ij}, [C_k]_{ij}, \ \forall k \right)$$

Theorem

semidefinite relaxation is exact for QCQP over tree

S. Bose, D. Gayme, S. H. Low and M. Chandy, March 2012

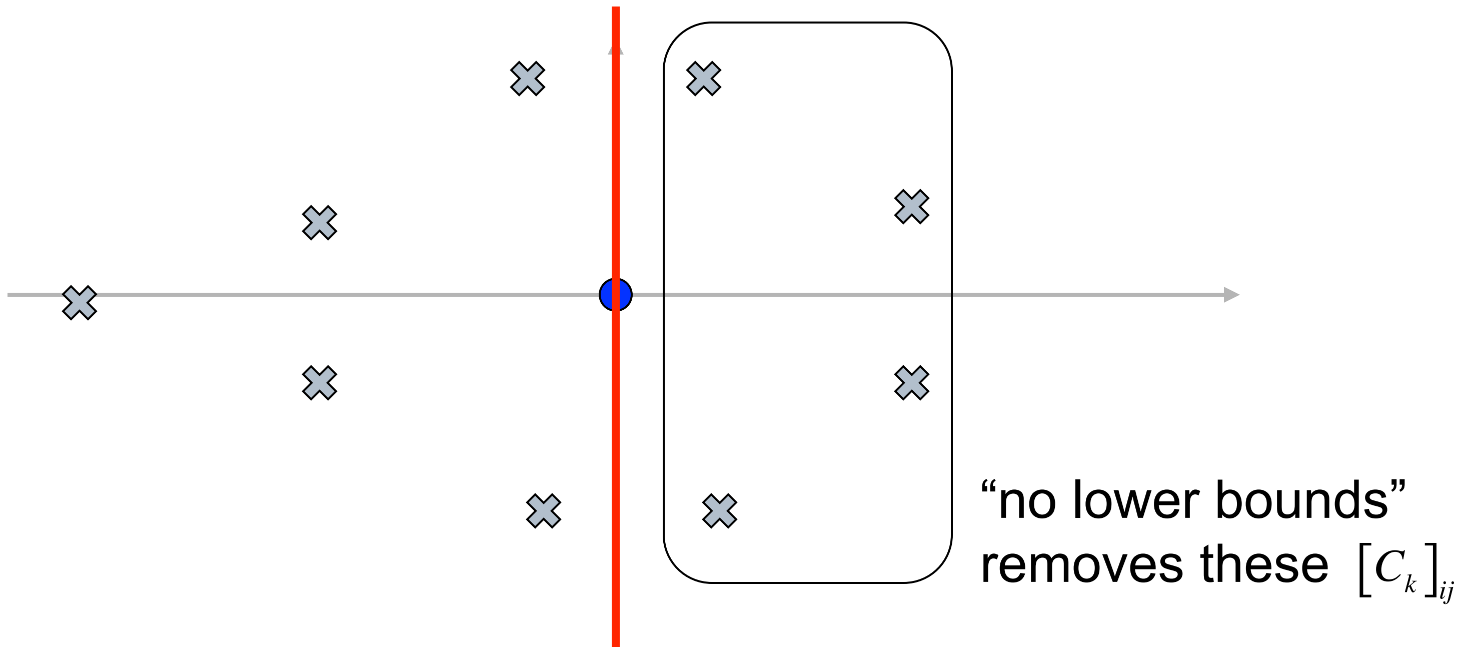
OPF over radial networks



full constraints violate key condition

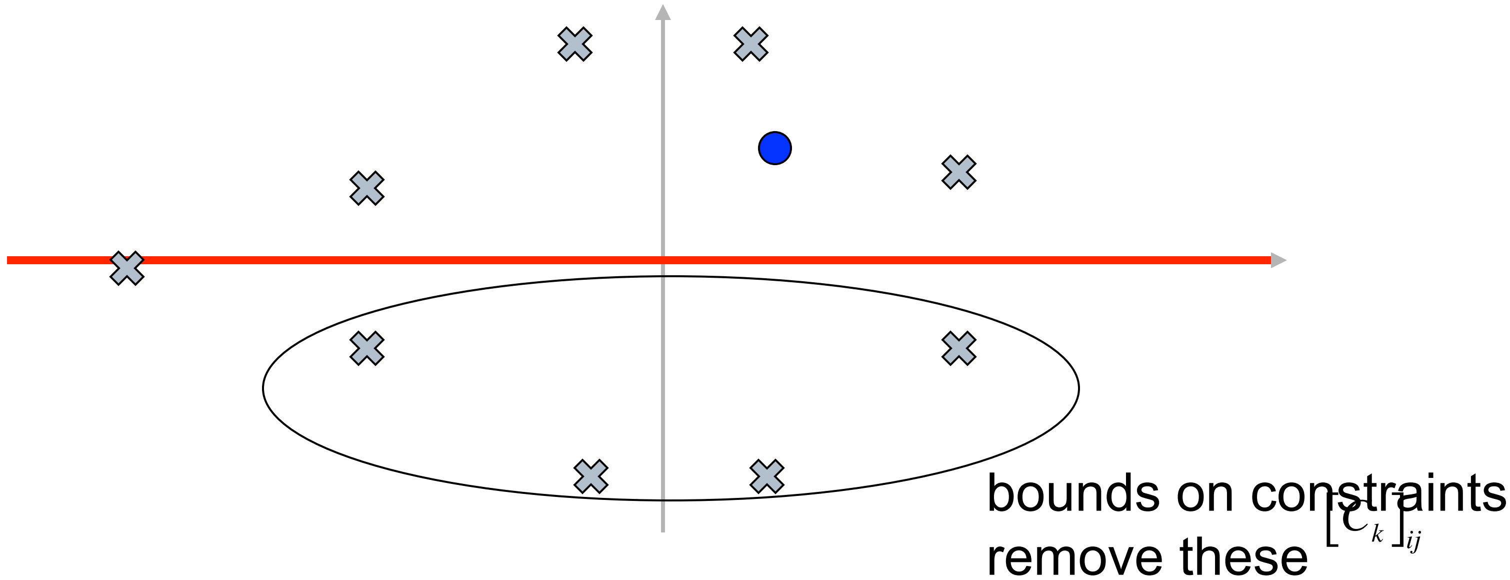
$$i \sim j: \quad 0 \notin \text{int conv hull}\left(C_{ij}, [C_k]_{ij}, \ \forall k\right)$$

OPF over radial networks



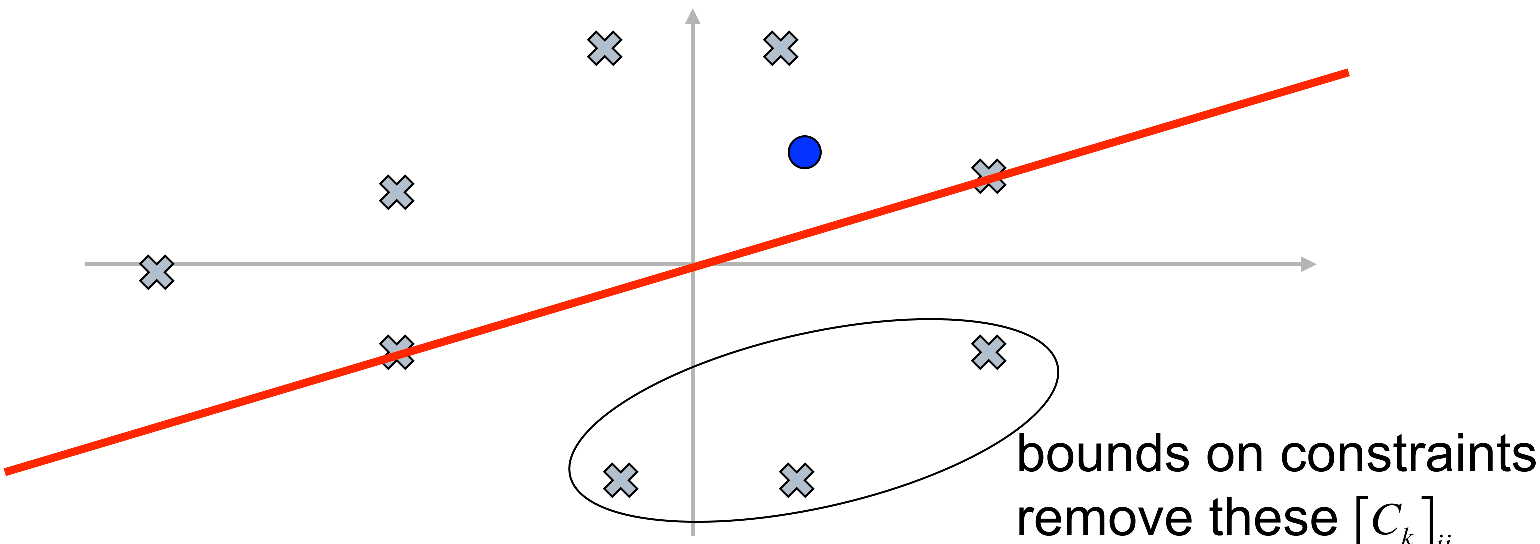
P_i	Q_i	P_j	Q_j	line flow	loss
UB	UB	UB	UB	any	any
UB	LB	LB	UB	no P_{ij}	any
LB	UB	UB	LB	no $\overline{P}_{ji}$	any

OPF over radial networks



P_i	Q_i	P_j	Q_j	line flow	loss
UB	UB	UB	UB	any	any
UB	LB	LB	UB	no $\overline{P}_{ij}$	any
LB	UB	UB	LB	no $\overline{P}_{ji}$	any

OPF over radial networks



P_i	Q_i	P_j	Q_j	line flow	loss
UB	UB	UB	UB	any	any
UB	LB	LB	UB	no $\overline{P}_{ij}$	any
LB	UB	UB	LB	no $\overline{P}_{ji}$	any

Bus injection model: summary

OPF = rank constrained SDP

Sufficient conditions for SDP to be exact

- Whether a solution is globally optimal is always easily checkable
- Mesh: must solve SDP to check
- Tree: depends only on constraint pattern or r/x ratios

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Equivalence relationship

OPF: branch flow model

$$\min \quad \sum_{i \sim j} r_{ij} \left| I_{ij} \right|^2 + \sum_i \alpha_i \left| V_i \right|^2$$

real power loss

CVR (conservation voltage reduction)

OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

s. t.

OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \boxed{\underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i}$$

OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i$$

generation,
VAR control

branch flow model

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

Branch flow model is more convenient for applications

OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \boxed{\underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i}$$

demand response

branch flow model

$$\sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$V_j = V_i - z_{ij} I_{ij} \qquad\qquad S_{ij} = V_i I_{ij}^*$$

OPF: branch flow model

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \boxed{\underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i}$$

demand response

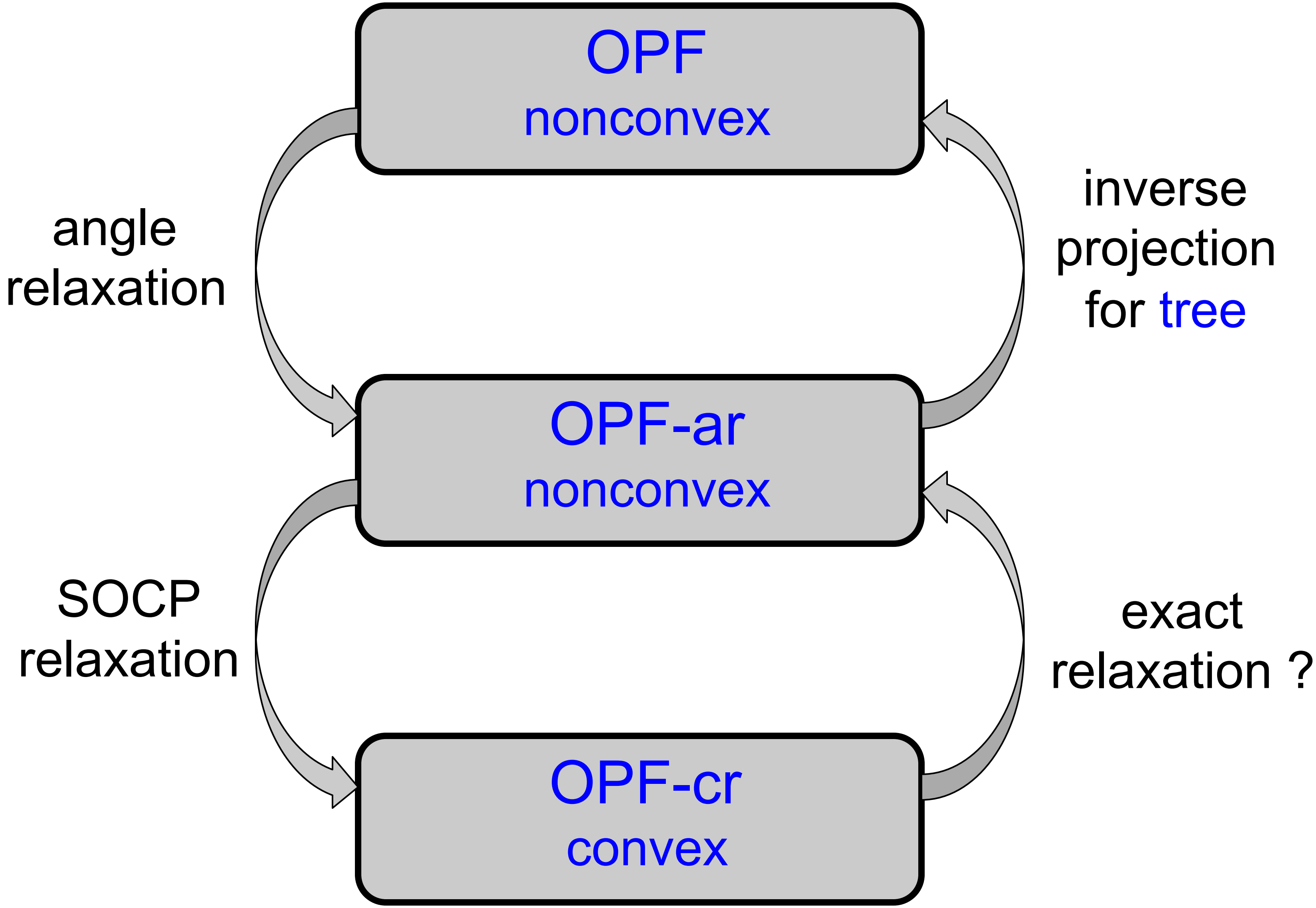
branch flow model

$$\sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

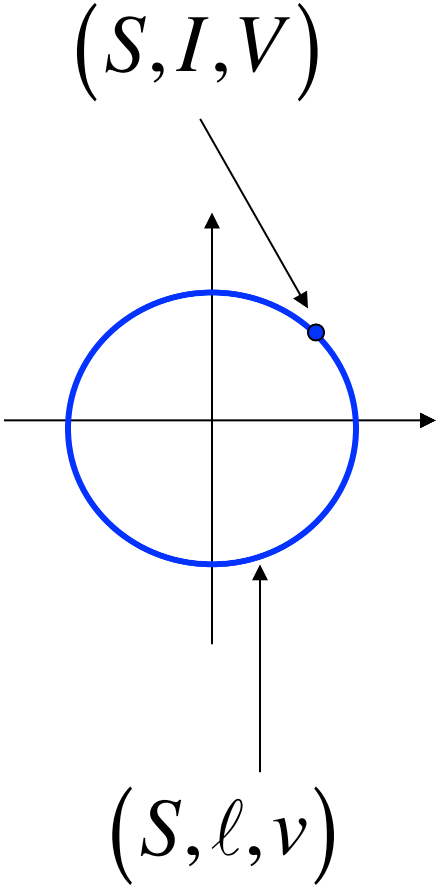
$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

Challenge: nonconvexity !

Solution strategy



BFM: power flow solutions

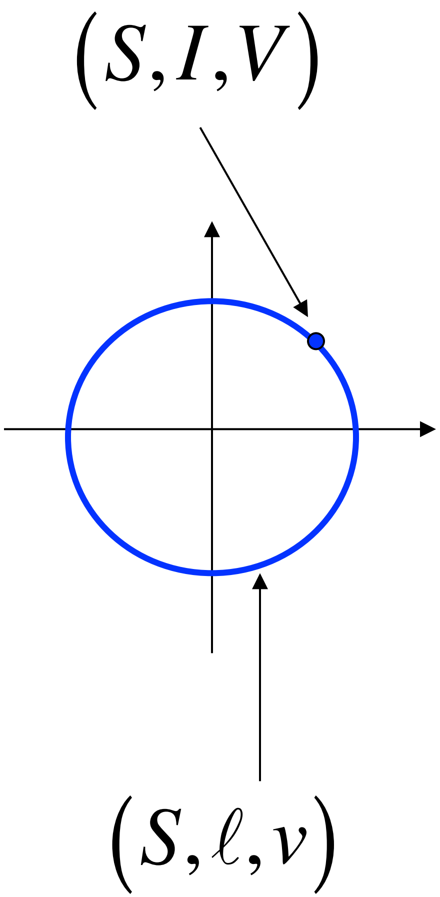


$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

Relaxed BF model

relaxed branch flow solutions: (S, ℓ, v, s) satisfy



$$\sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$v_i = v_j + 2\,\mathrm{Re}\left(z_{ij}^* S_{ij} \right) - \left| z_{ij} \right|^2 \ell_{ij}$$

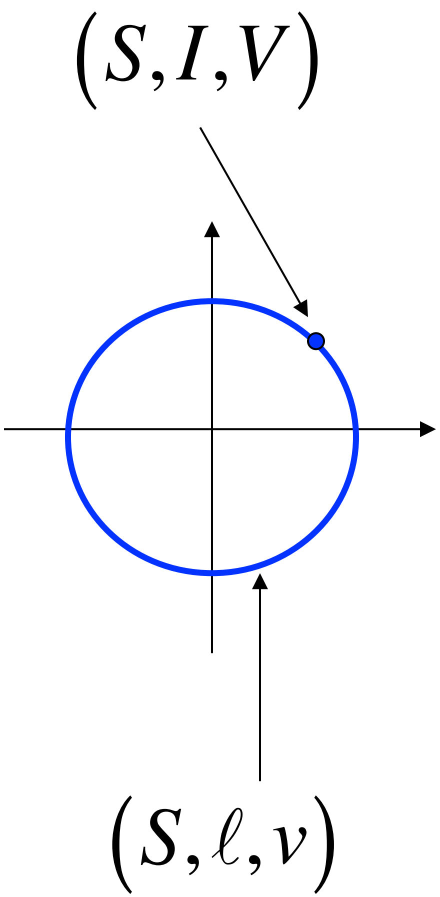
$$\ell_{ij} = \frac{\left| S_{ij} \right|^2}{v_i}$$

$$\boxed{\begin{aligned} \ell_{ij} &:= \left| I_{ij} \right|^2 \\ v_i &:= \left| V_i \right|^2 \end{aligned}}$$

Baran and Wu 1989
for radial networks

Relaxed BF model

relaxed branch flow solutions: (S, ℓ, v, s) satisfy



$$\sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$v_i = v_j + 2 \operatorname{Re}\left(z_{ij}^* S_{ij} \right) - \left| z_{ij} \right|^2 \ell_{ij}$$

$$\ell_{ij} \geq \frac{\left| S_{ij} \right|^2}{v_i}$$

second order cone !

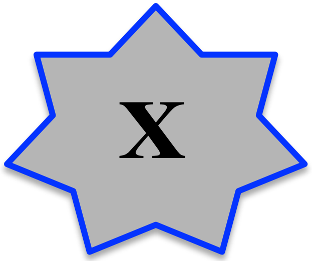
$$\ell_{ij} := \left| I_{ij} \right|^2$$

$$v_i := \left| V_i \right|^2$$

OPF

$$\begin{aligned} \min \quad & f(x) \\ \text{over} \quad & x := (S, I, V, s^g, s^c) \\ \text{s. t.} \quad & \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i \end{aligned}$$

$$x \in \mathbf{X}$$



OPF-ar

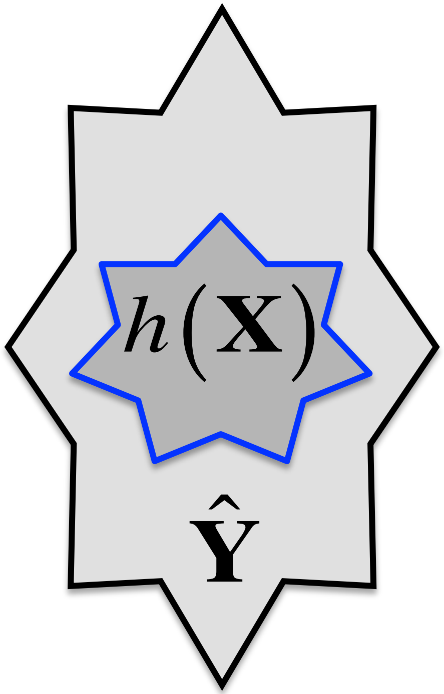
$$\min \quad f(\hat{y})$$

$$\text{over} \quad \hat{y} := (S, \ell, v, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i^c \qquad \underline{v}_i \le v_i \le \overline{v}_i$$

$$\hat{y} := h(x) \in \hat{\mathbf{Y}}$$

relax each voltage/current from a point in complex plane into a circle



OPF-cr

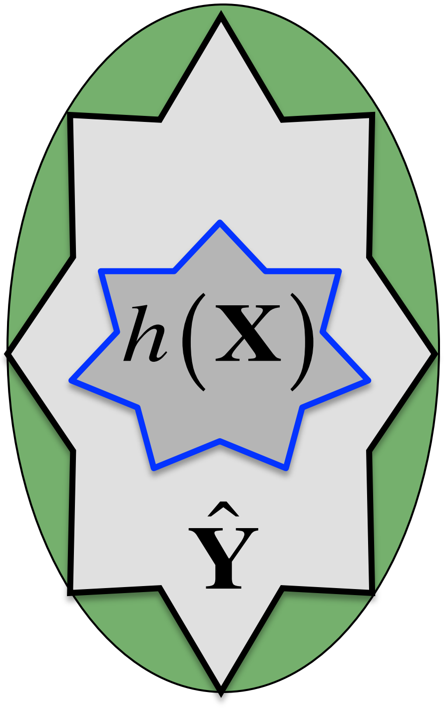
$$\min \quad f(\hat{y})$$

$$\text{over} \quad \hat{y} := (S, \ell, v, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i^c \qquad \underline{v}_i \le v_i \le \overline{v}_i$$

$$\hat{y} \in \operatorname{conv} \hat{\mathbf{Y}}$$

relax to convex hull
(SOCP)



OPF

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i$$

branch flow model

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \left| I_{ij} \right|^2 \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

OPF-ar

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i$$

$$\sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$v_i = v_j + 2\,\mathrm{Re}\left(z_{ij}^* S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

source of nonconvexity

$$\ell_{ij} = \frac{|S_{ij}|^2}{v_i}$$

OPF-cr

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c \le \overline{s}_i \qquad \underline{v}_i \le v_i \le \overline{v}_i$$

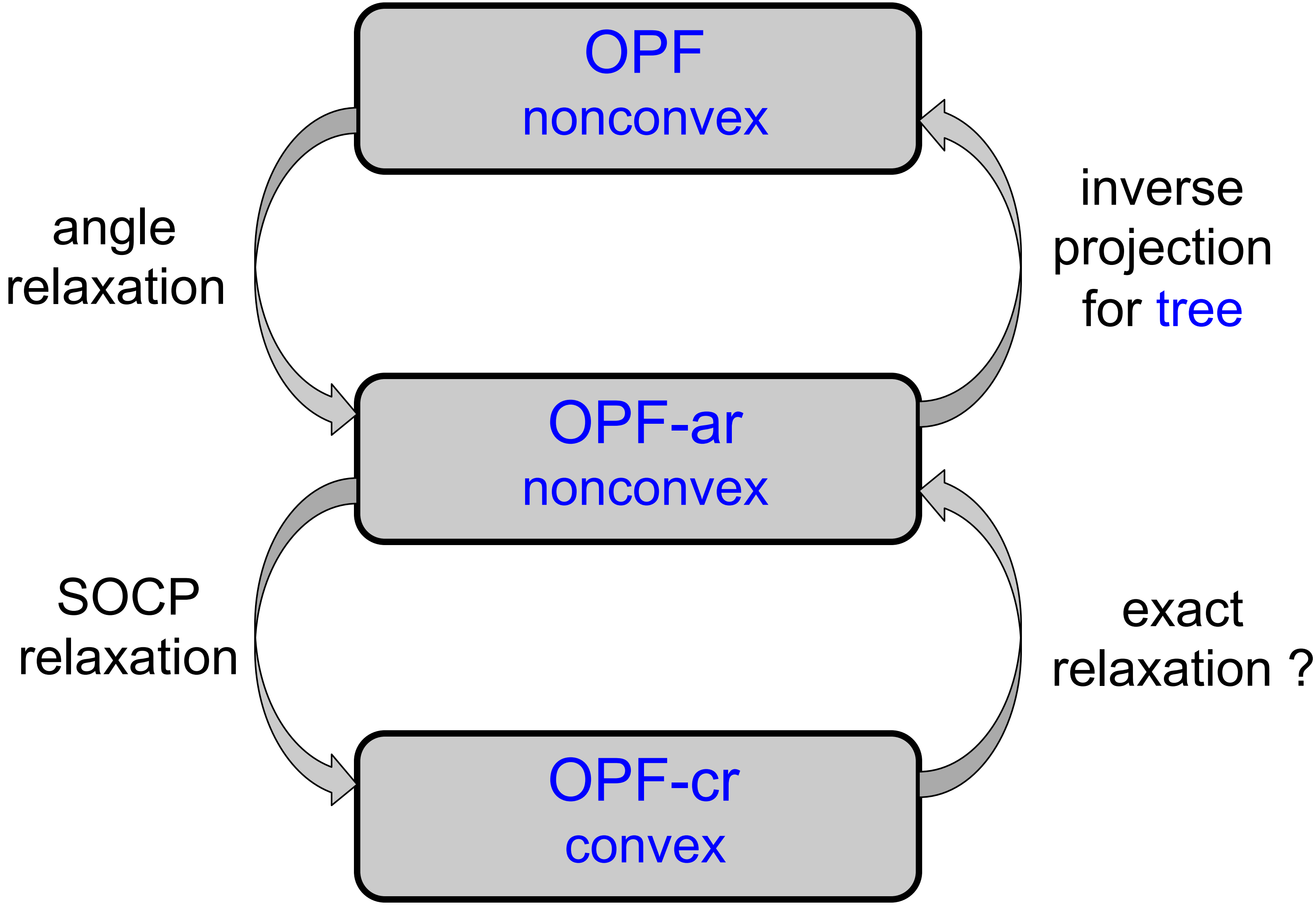
$$\sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) - \sum_{j \to k} S_{jk} = s_j^c - s_j^g$$

$$v_i = v_j + 2\,\mathrm{Re}\left(z_{ij}^* S_{ij} \right) - \left| z_{ij} \right|^2 \ell_{ij}$$

$$\boxed{\ell_{ij} \ge \frac{\left| S_{ij} \right|^2}{v_i}}$$

SOCP relaxation

Recap so far …


OPF
nonconvex
angle relaxation
inverse projection for tree
OPF-ar
nonconvex
SOCP relaxation
exact relaxation ?
OPF-cr
convex

OPF-cr is exact relaxation

Theorem

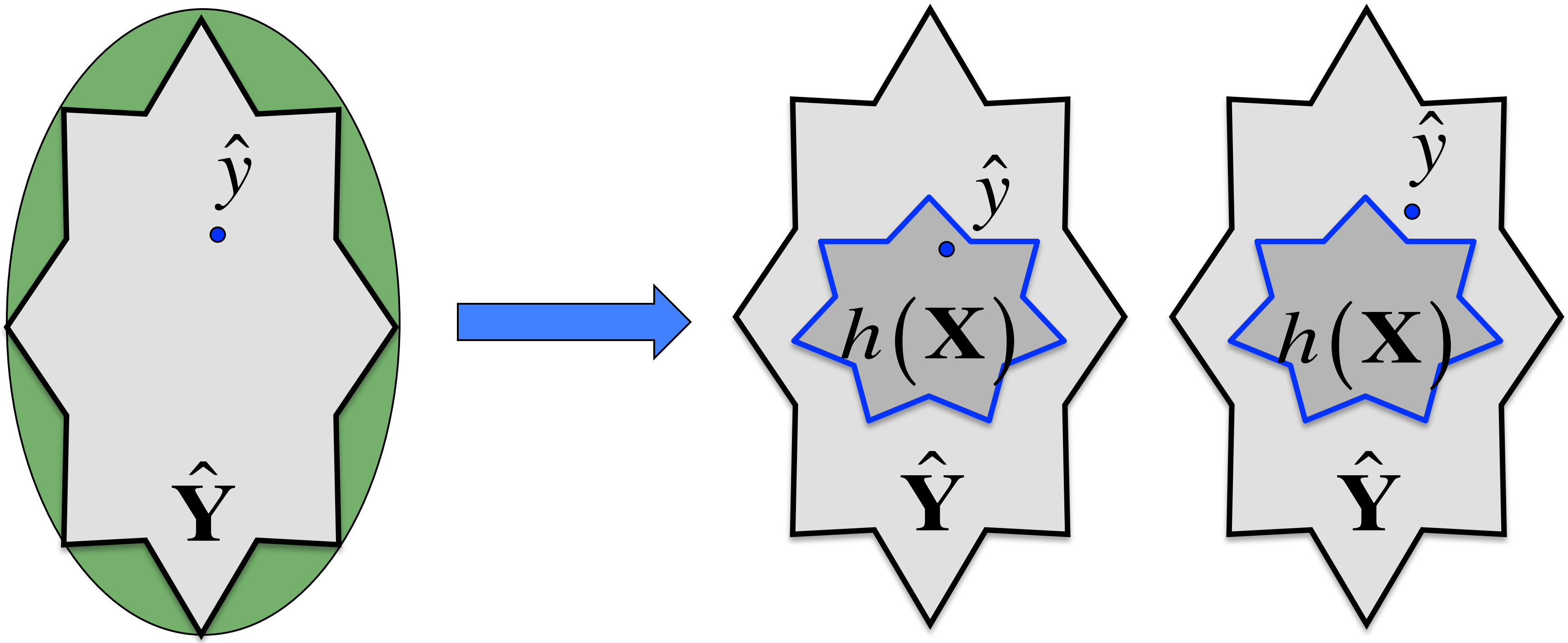
OPF-cr is SOCP

- when objective is linear
- SOCP much simpler than SDP

OPF-cr is exact

- optimal of OPF-cr is also optimal for OPF-ar
- for mesh as well as radial networks
- … if no upper bounds on loads
- (or alternative conds for radial networks)

Angle recovery



OPF-ar

does there exist θ s.t. $h_\theta^{-1}(\hat{y}) \in \mathbf{X}$?

Angle recovery

<u>**Theorem**</u>

solution x to OPF recoverable from $\hat{y}$ iff inverse projection exist iff $\exists! \theta$ s.t.

$$B\theta = \beta\left(\hat{y}\right) \quad \mod 2\pi$$

incidence matrix;
depends on topology

depends on
OPF-ar solution

Angle recovery

Theorem

solution x to OPF recoverable from $\hat{y}$ iff inverse projection exist iff $\exists!\theta$ s.t.

$$B\theta = \beta\left(\hat{y}\right) \quad \mod 2\pi$$

implied phase angle differences sum to 0 (mod 2π) around each cycle

Two simple angle recovery algorithms

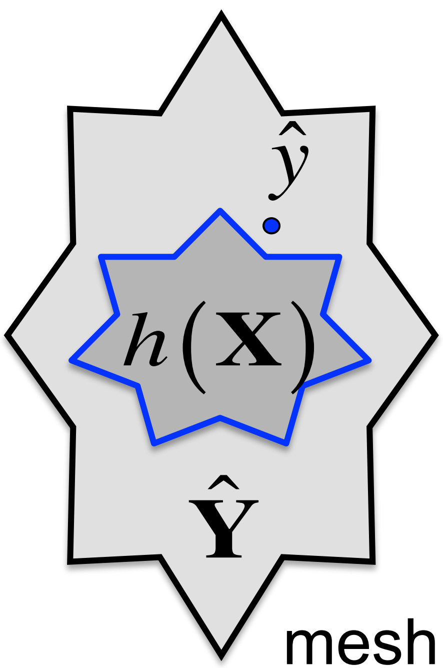
- centralized: explicit formula
- decentralized: recursive alg

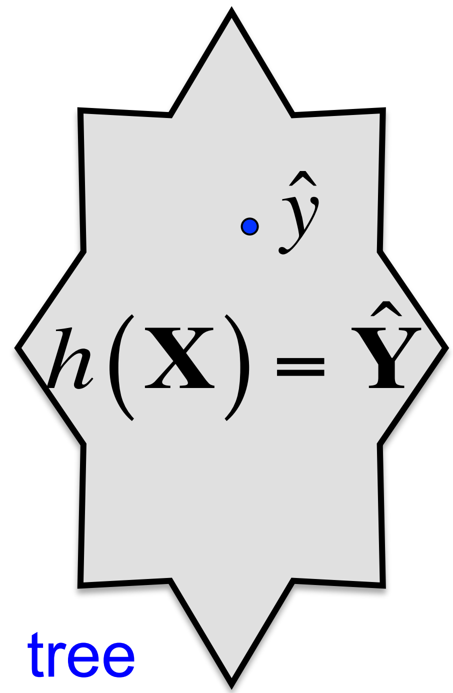
Angle recovery

Theorem

For radial network: $\exists ! \theta$

$$B\theta = \beta\left(\hat{y}\right) \mod 2\pi$$





Angle recovery

$$\begin{array}{r} \\ \text{\#lines in T} \\ \text{\#lines outside T} \end{array} \begin{array}{c} \text{\#buses - 1} \\ \begin{bmatrix} B_T \\ B_\perp \end{bmatrix} \end{array} \theta = \begin{bmatrix} \beta_T \\ \beta_\perp \end{bmatrix}$$

<u>Theorem</u>

Inverse projection exist iff $B_\perp \left(B_T^{-1} \beta_T \right) = \beta_\perp$

Unique inverse given by $\theta^* = B_T^{-1} \beta_T$

For radial network: $B_\perp = \beta_\perp = 0$

OPF solution

Solve OPF-cr

SOCP

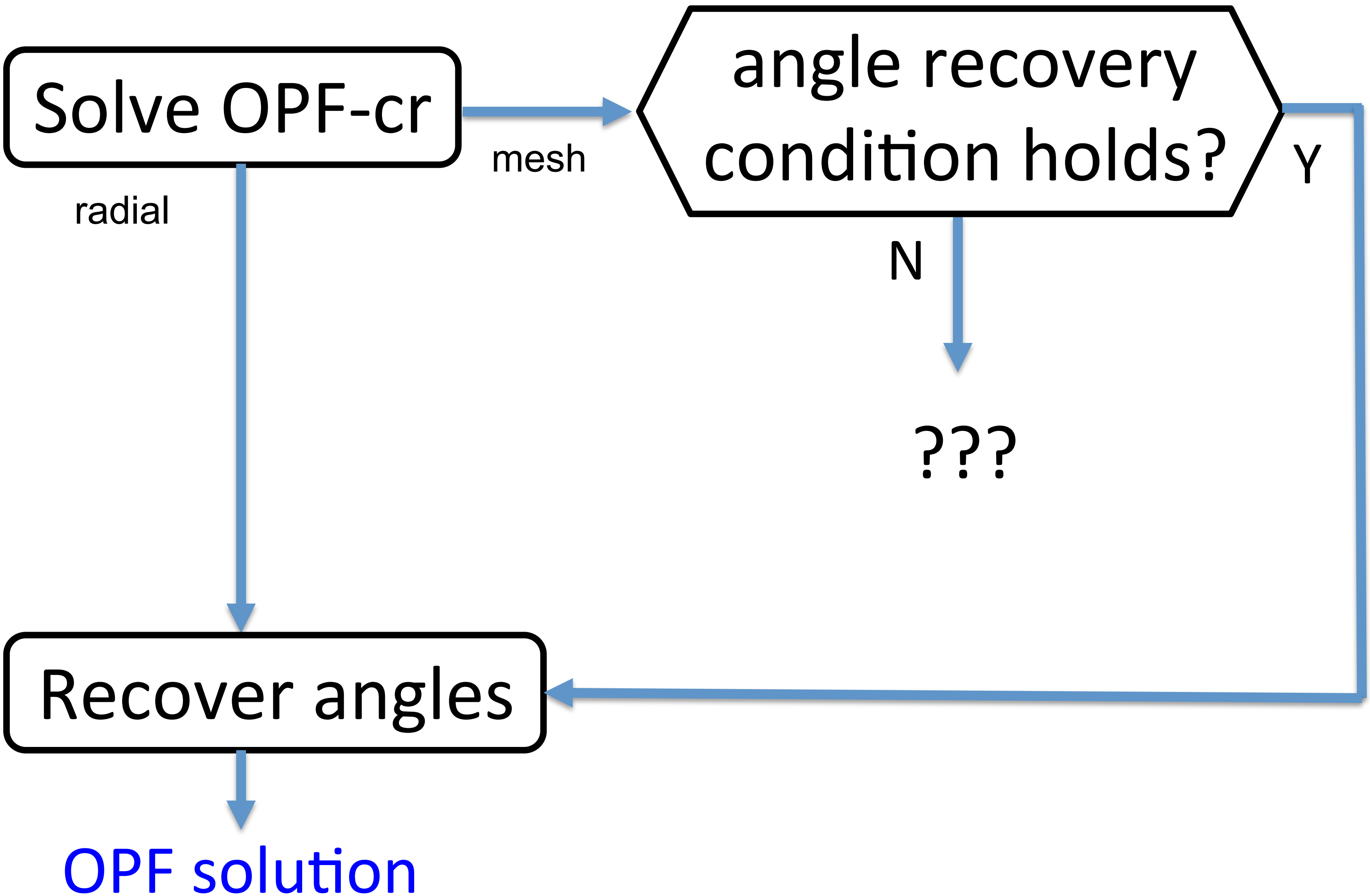
radial

Recover angles

- explicit formula
- distributed alg

OPF solution

OPF solution



Outline

Two power flow models

- Bus injection model
- Branch flow model

OPF in BI model

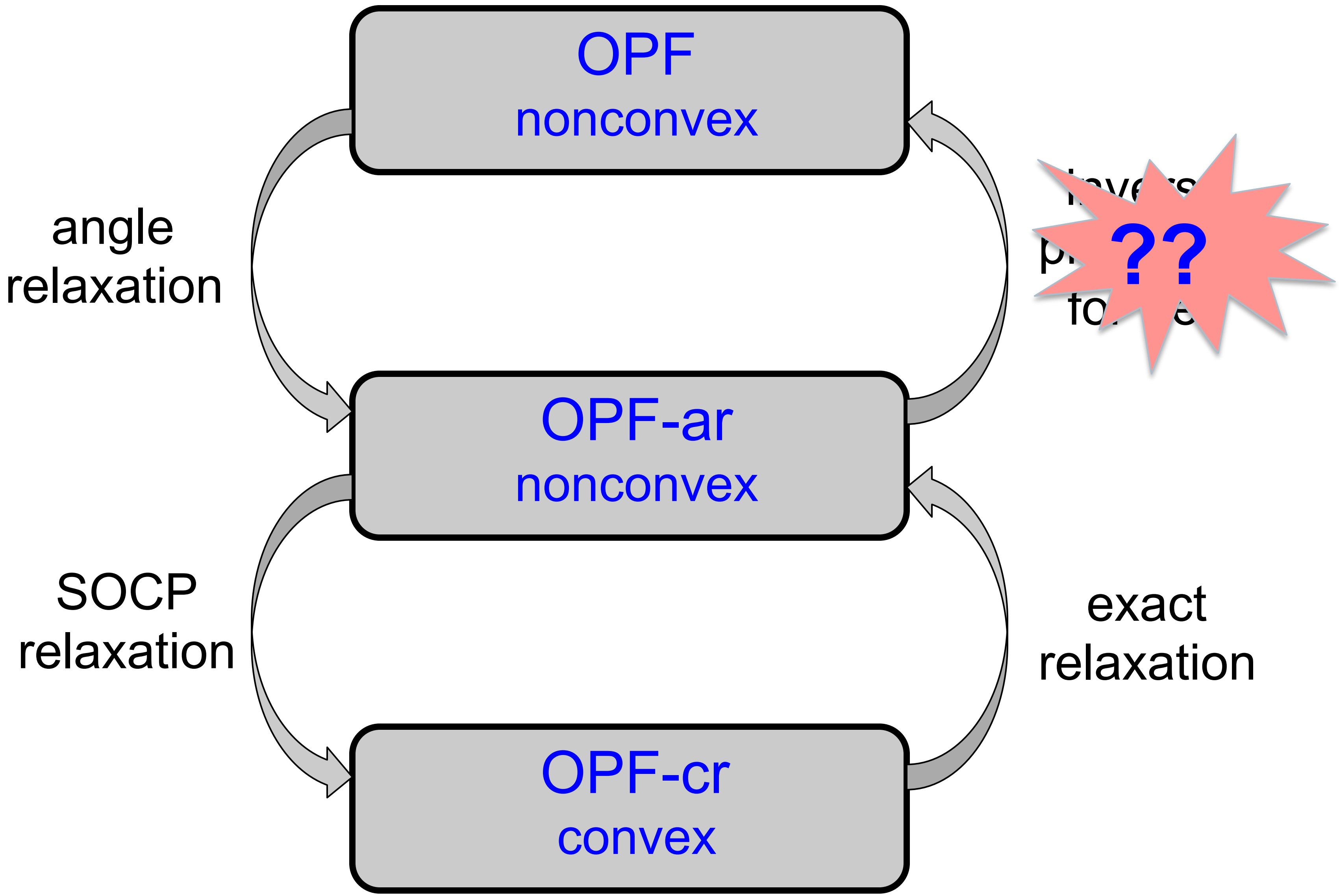
- Semidefinite relaxation

OPF in BF model

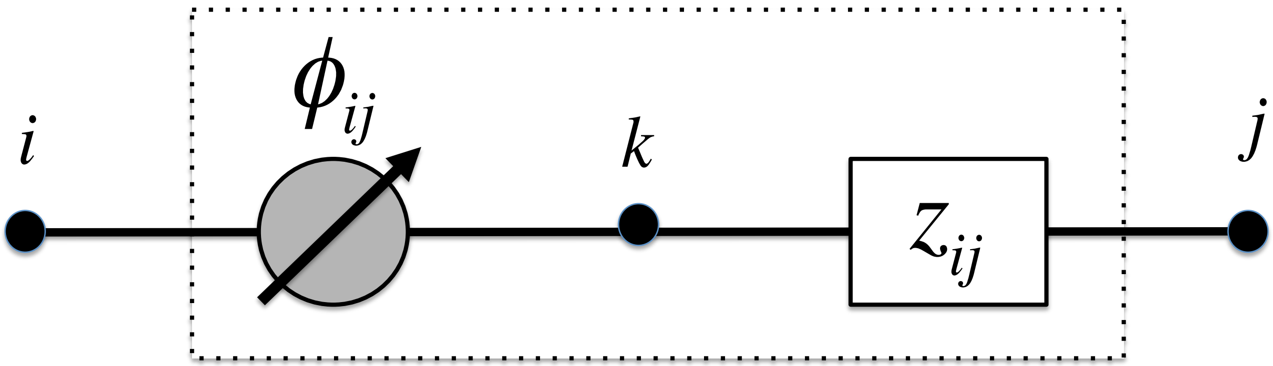
- SOCP relaxation
- Convexification using phase shifters

Equivalence relationship

Recap: solution strategy



Phase shifter



ideal phase shifter

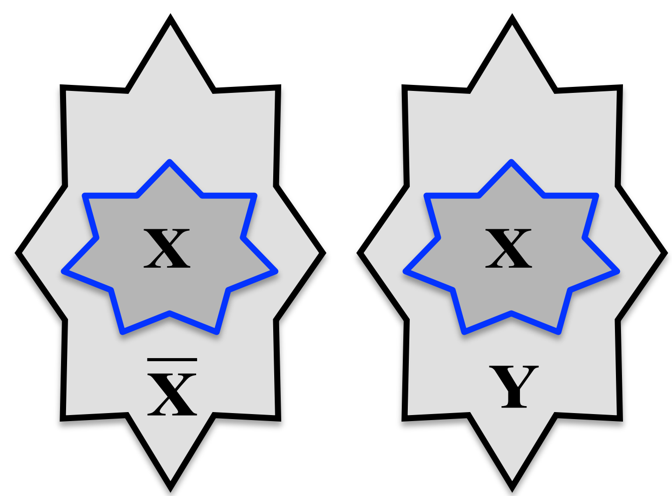
Convexification of mesh networks

OPF $\min_{x} \; f\left(h(x)\right) \quad \text{s.t.} \quad x \in \mathbf{X}$

OPF-ar $\min_{x} \; f\left(h(x)\right) \quad \text{s.t.} \quad x \in \mathbf{Y}$

OPF-ps $\min_{x,\phi} \; f\left(h(x)\right) \quad \text{s.t.} \quad x \in \overline{\mathbf{X}}$

optimize over phase shifters as well



<u>Theorem</u>

- $\overline{\mathbf{X}} = \mathbf{Y}$
- Need phase shifters only outside spanning tree

Angle recovery with PS

$$\begin{bmatrix} B_T \\ B_\perp \end{bmatrix} \theta = \begin{bmatrix} \beta_T \\ \beta_\perp \end{bmatrix} - \begin{bmatrix} 0 \\ \phi_\perp \end{bmatrix}$$

Theorem

Inverse projection always exists

Unique inverse given by $\theta^* = B_T^{-1} \beta_T$

Don't need PS in spanning tree $\phi_\perp^* = 0$

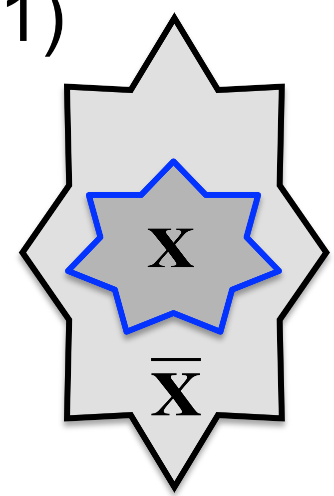
Convexification of mesh networks

OPF-ps $\quad \min_{x,\phi} \; f\left(h(x)\right) \quad \text{s.t.} \quad x \in \overline{\mathbf{X}}$

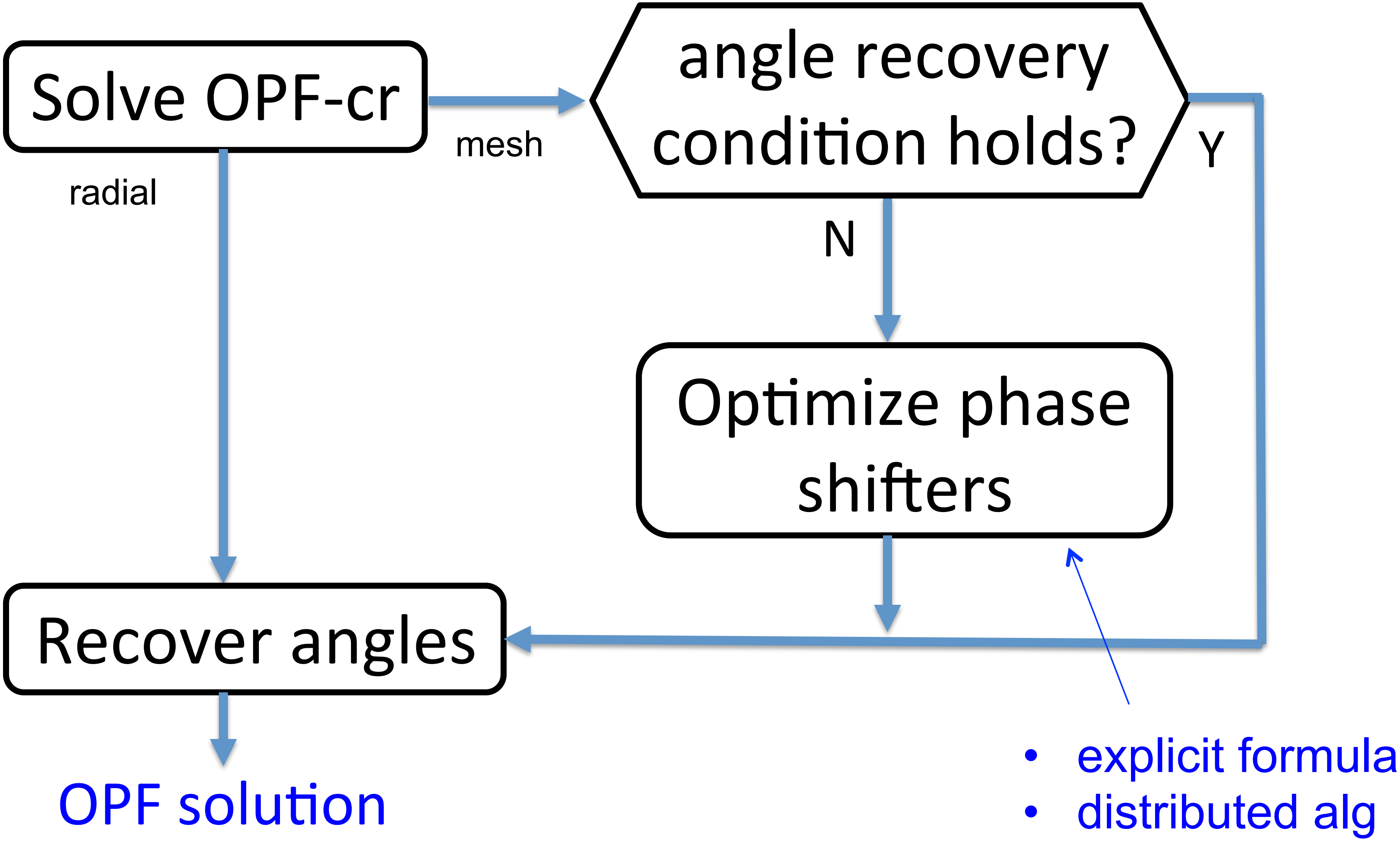
optimize over phase shifters as well

<u>Optimization of ϕ</u>

- Min # phase shifters (#lines - #buses + 1)
- Min $\|\phi\|_2$: NP hard (good heuristics)
- Given existing network of PS, min # or angles of additional PS



OPF solution



Examples

Test cases	# links (m)	No PS Min loss (OPF, MW)	With PS Min loss (OPF-cr, MW)
IEEE 14-Bus	20	0.546	0.545
IEEE 30-Bus	41	1.372	1.239
IEEE 57-Bus	80	11.302	10.910
IEEE 118-Bus	186	9.232	8.728
IEEE 300-Bus	411	211.871	197.387
New England 39-Bus	46	29.915	28.901
Polish (case2383wp)	2,896	433.019	385.894
Polish (case2737sop)	3,506	130.145	109.905

Examples

Test cases	# links (m)	# active PS $\lvert\phi_i\rvert > 0.1^\circ$	Min #PS ($^\circ$) $[\phi_{\min}, \phi_{\max}]$
IEEE 14-Bus	20	2 (10%)	$[-2.09,\ 0.58]$
IEEE 30-Bus	41	3 (7%)	$[-0.20,\ 4.47]$
IEEE 57-Bus	80	19 (24%)	$[-3.47,\ 3.15]$
IEEE 118-Bus	186	36 (19%)	$[-1.95,\ 2.03]$
IEEE 300-Bus	411	101 (25%)	$[-13.3,\ 9.40]$
New England 39-Bus	46	7 (15%)	$[-0.26,\ 1.83]$
Polish (case2383wp)	2,896	373 (13%)	$[-19.9,\ 16.8]$
Polish (case2737sop)	3,506	395 (11%)	$[-10.9,\ 11.9]$

Examples

Test cases	# links (m)	Min #PS ($^\circ$) $[\phi_{\min}, \phi_{\max}]$	Min $\|\phi\|^2$ ($^\circ$) $[\phi_{\min}, \phi_{\max}]$
IEEE 14-Bus	20	$[-2.09,\ 0.58]$	[-0.63, 0.12]
IEEE 30-Bus	41	$[-0.20,\ 4.47]$	[-0.95, 0.65]
IEEE 57-Bus	80	$[-3.47,\ 3.15]$	[-0.99, 0.99]
IEEE 118-Bus	186	$[-1.95,\ 2.03]$	[-0.81, 0.31]
IEEE 300-Bus	411	$[-13.3,\ 9.40]$	[-3.96, 2.85]
New England 39-Bus	46	$[-0.26,\ 1.83]$	[-0.33, 0.33]
Polish (case2383wp)	2,896	$[-19.9,\ 16.8]$	[-3.07, 3.23]
Polish (case2737sop)	3,506	$[-10.9,\ 11.9]$	[-1.23, 2.36]

Key message

Radial networks computationally simple

- Exploit tree graph & convex relaxation
- Real-time scalable control promising

Mesh networks can be convexified

- Design for simplicity
- Need few (?) phase shifters (sparse topology)

Outline

Big picture

Two power flow models

- Bus injection model
- Branch flow model

OPF in BI model

- Semidefinite relaxation

OPF in BF model

- SOCP relaxation
- Convexification using phase shifters

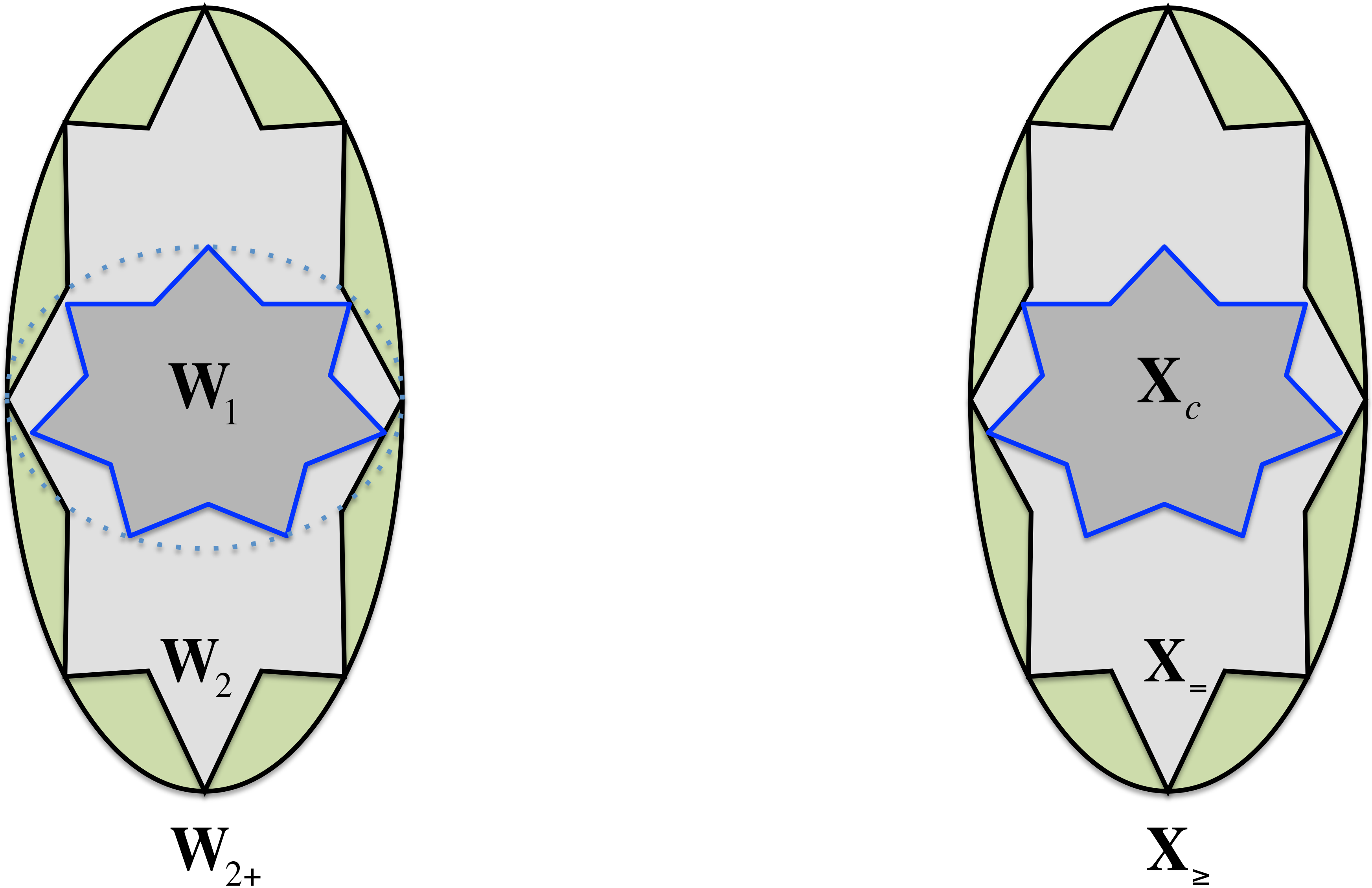
Equivalence relationship

Summary

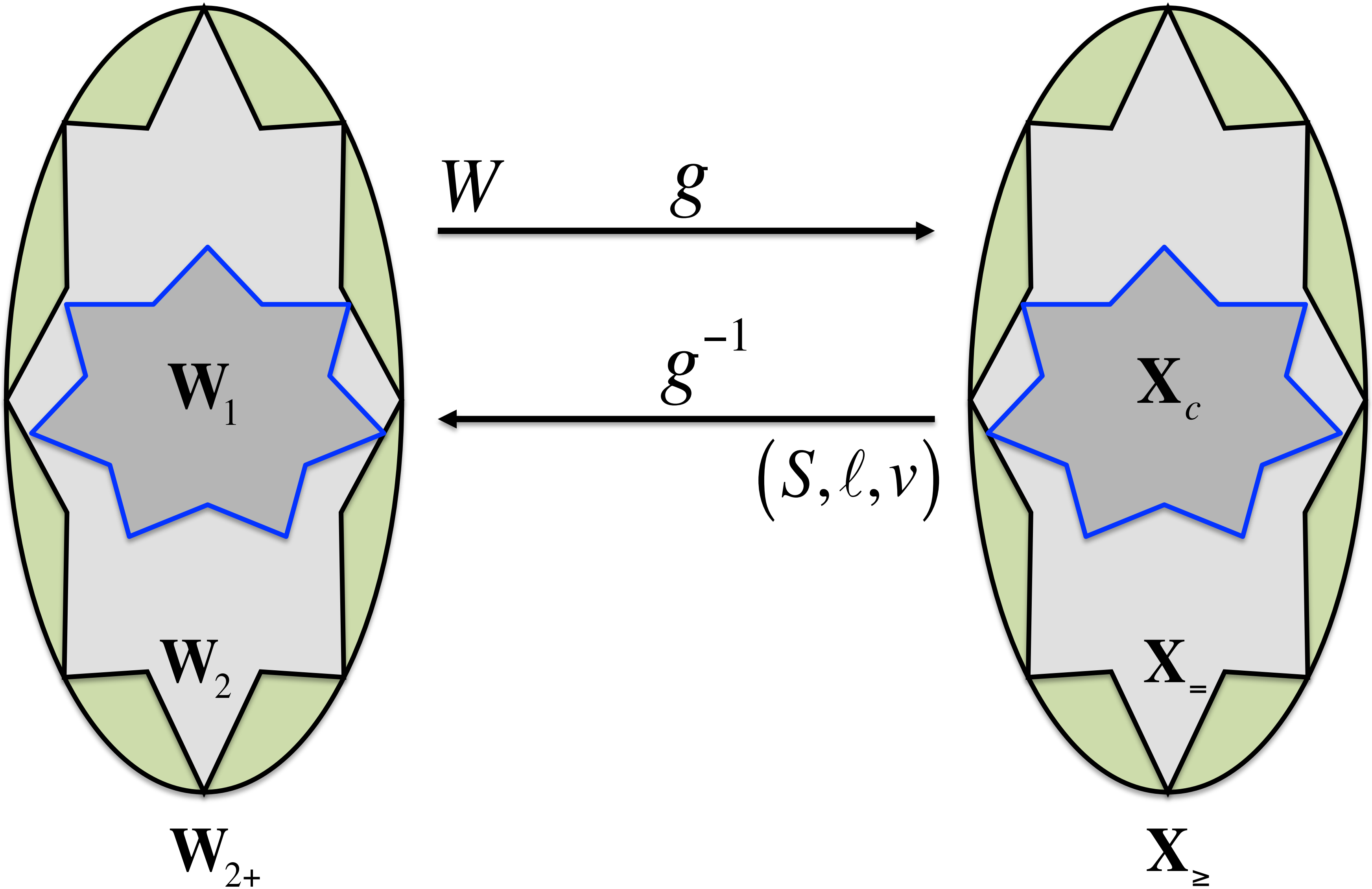
$\mathbf{W}_1$

$\mathbf{X}_c$

Summary

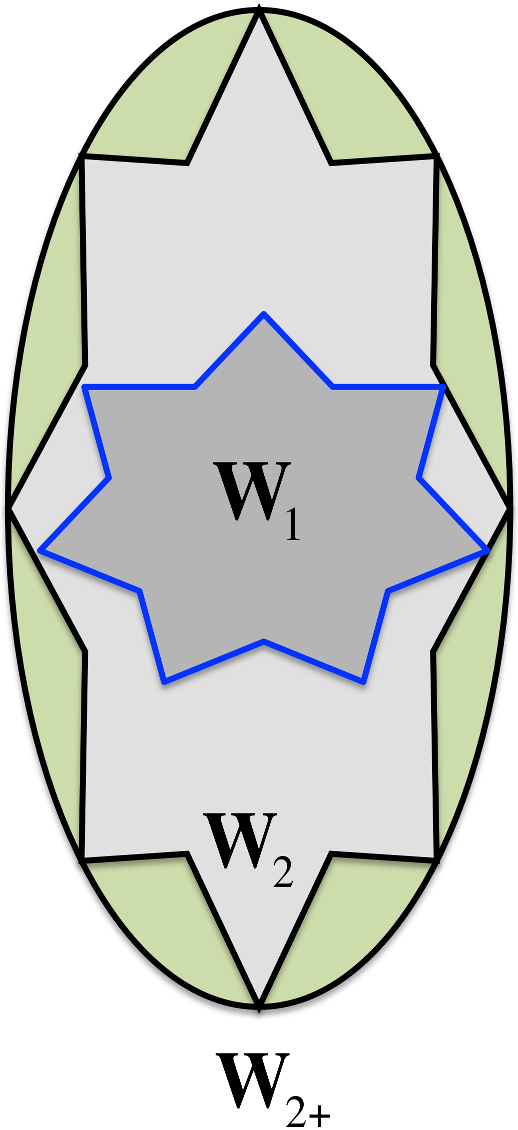


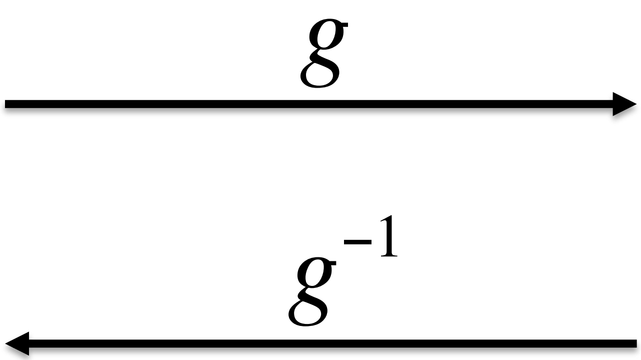
Summary

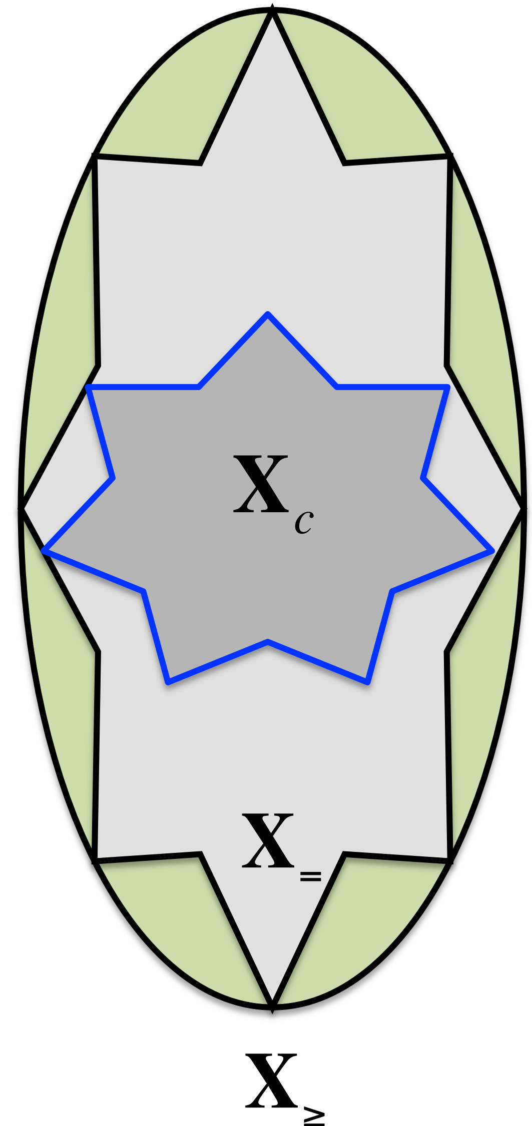


W
g
g^{-1}
(S, ℓ, v)
$\mathbf{W}_1$
$\mathbf{W}_2$
$\mathbf{W}_{2+}$
$\mathbf{X}_c$
$\mathbf{X}_=$
$\mathbf{X}_\geq$

Summary





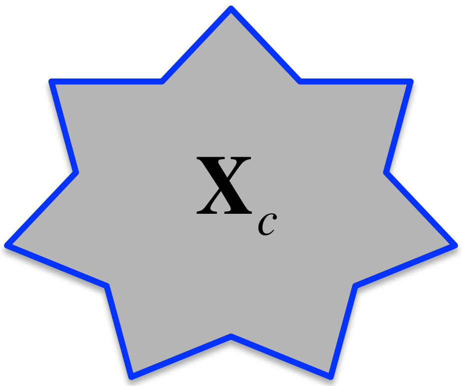


Theorem

g is a bijection

(on eq classes of $\mathbf{W}_{2+}$)

BFM: solution set



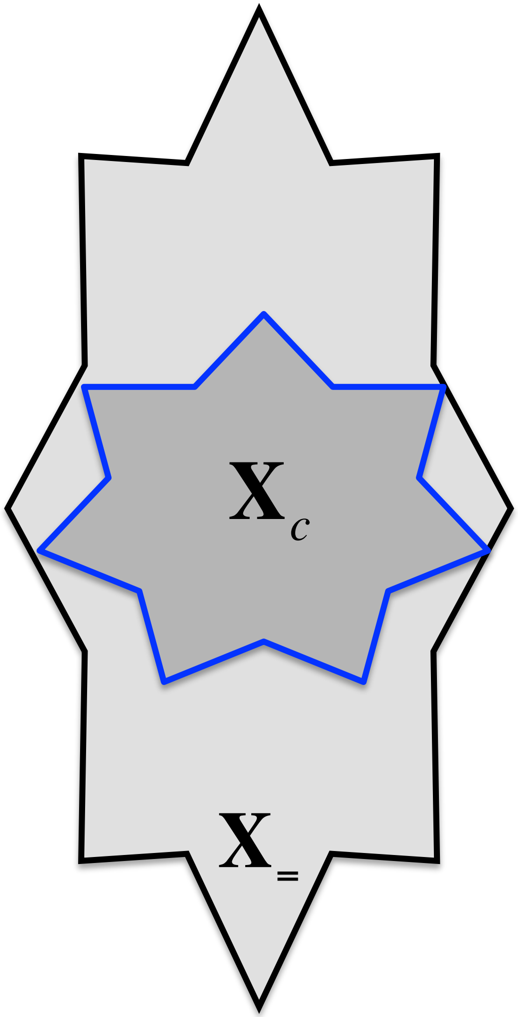
$\mathbf{X}_c$: BFM solution

nonconvex

$\exists\theta$ s.t. $h_\theta\left(S,\ell,v\right) =: (S,I,V)$

satisfies BFM

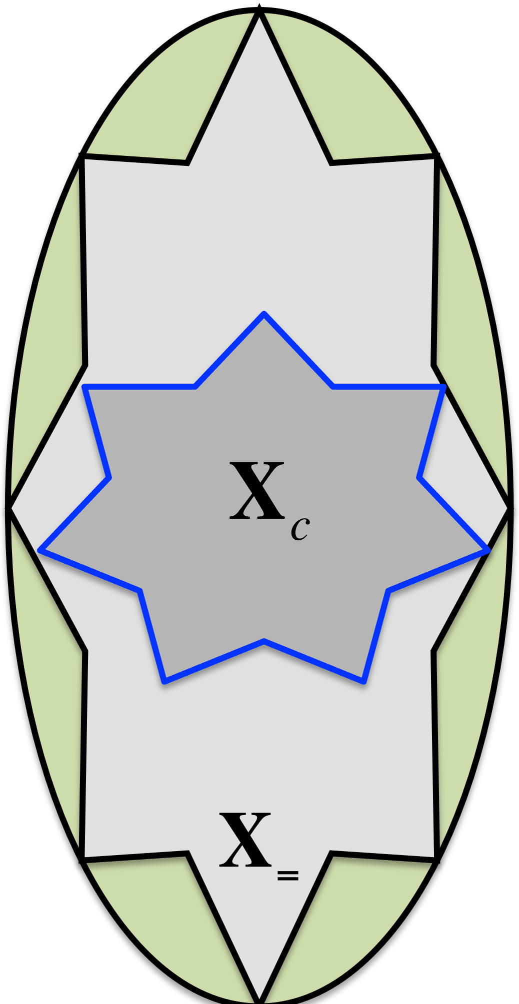
BFM: angle relaxation



$\mathbf{X}_c$: BFM solution — nonconvex

$\mathbf{X}_=$: no phase angles — nonconvex

BFM: SOC relaxation

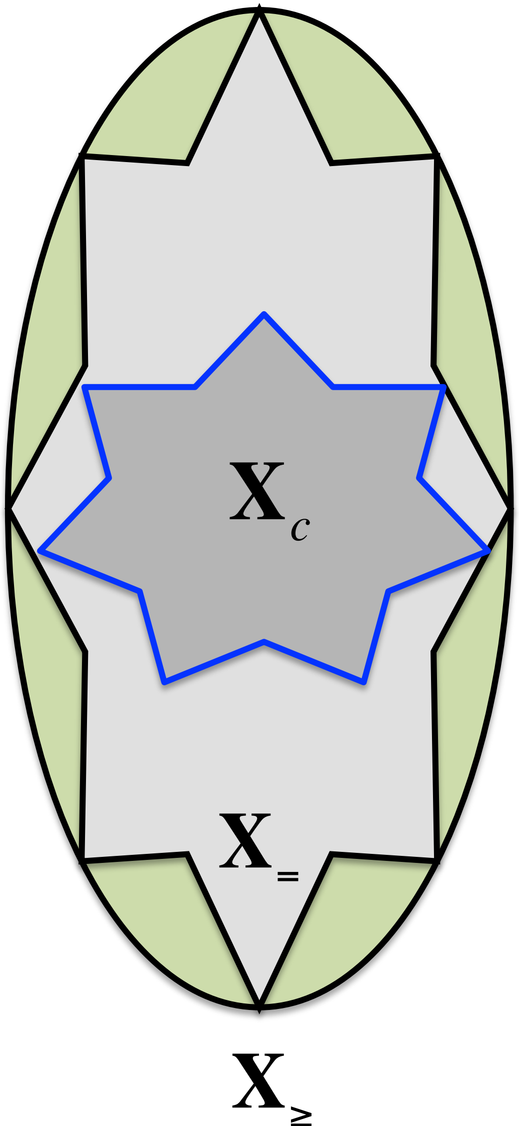


$\mathbf{X}_c$: BFM solution — nonconvex

$\mathbf{X}_=$: no phase angles — nonconvex

$\mathbf{X}_\geq$: $\ell_{ij} \geq \dfrac{|S_{ij}|^2}{v_i}$ — convex

BFM: exactness



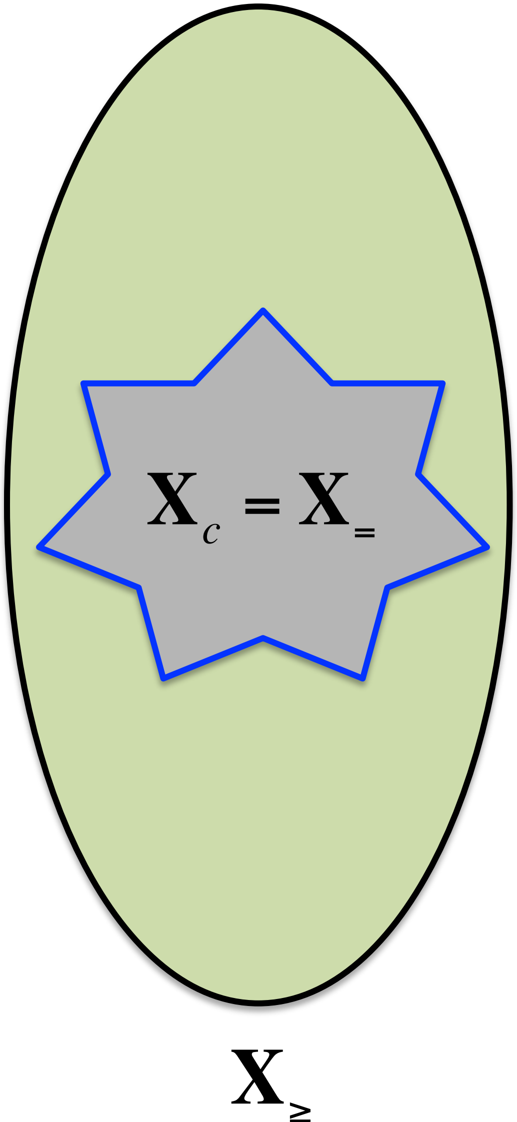
Theorem

$$\left(S,\ell,v\right) \in \mathbf{X}_c \quad \text{iff}$$

$$\exists\theta \;\; \text{s.t.} \;\; B\theta = \beta\left(S,\ell,v\right)$$

Moreover $\left(S,I,V\right) := h_\theta\left(S,\ell,v\right)$ is unique

BFM: exactness



Theorem

For radial networks:

$$\mathbf{X}_c = \mathbf{X}_=$$

BIM: solution set

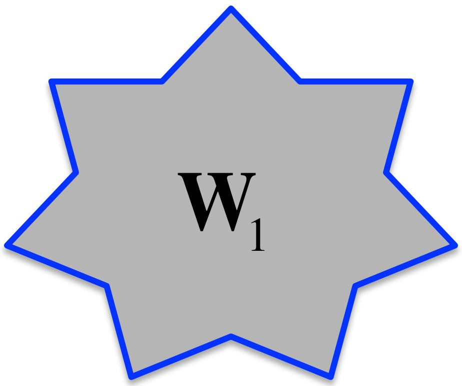
$$\operatorname{tr} Y_i \left(VV^* \right) = -s_i \qquad \text{for all } i$$

$$\Leftrightarrow \begin{cases} \operatorname{tr} Y_i W = -s_i & \text{for all } i \\ W \geq 0, \ \boxed{\operatorname{rank} W = 1} & \end{cases}$$

$$\boxed{Y_i := Y^* e_i e_i^*}$$

nonconvex
→ relaxations

BIM: solution set



$$\operatorname{tr} Y_i W = -s_i \qquad \text{for all } i$$

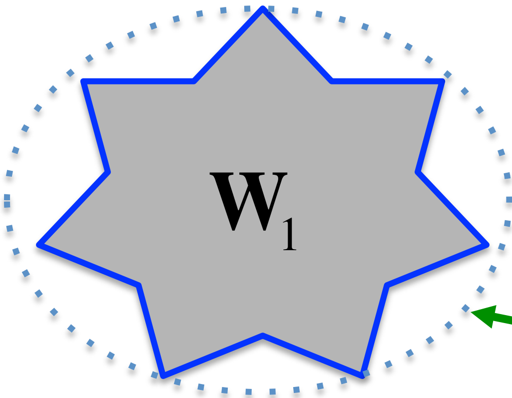
$$\mathbf{W}_1 : \quad W \geq 0$$

$$\operatorname{rank} W = 1$$

nonconvex

BIM: SDP relaxation

$$\mathrm{tr}\; Y_i W = -s_i \qquad \text{for all } i$$



$$\mathbf{W}_1 : \quad W \geq 0$$

$$\mathrm{rank}\; W = 1$$

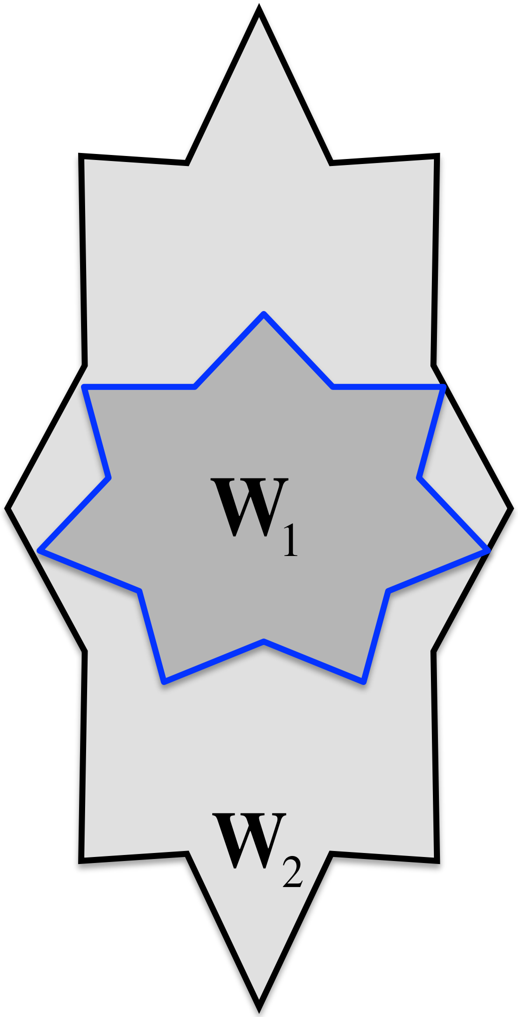
nonconvex

$$\mathbf{W}_+ : \quad W \geq 0$$

convex

→ SDP relaxation

BIM: 2x2 psd rank-1

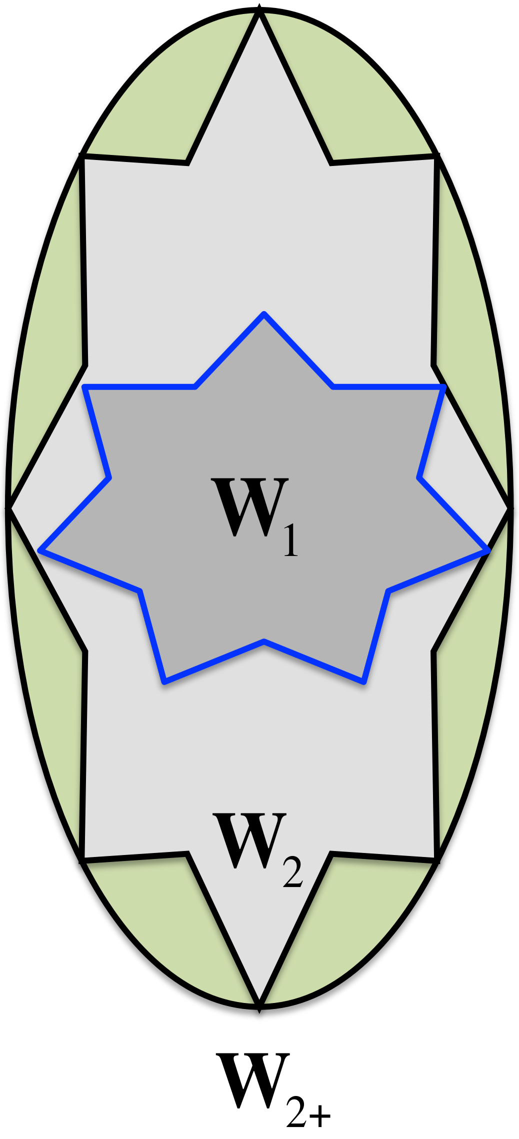


$$\operatorname{tr} Y_i W = -s_i \qquad \text{for all } i$$

$$\mathbf{W}_1: \quad W \geq 0$$
$$\operatorname{rank} W = 1$$
nonconvex

$$\mathbf{W}_2: \quad W(i,j) \geq 0$$
$$\operatorname{rank} W(i,j) = 1$$
nonconvex

BIM: 2x2 psd



$\mathrm{tr}\ Y_i W = -s_i \quad$ for all i

$\mathbf{W}_1 : \quad W \geq 0$

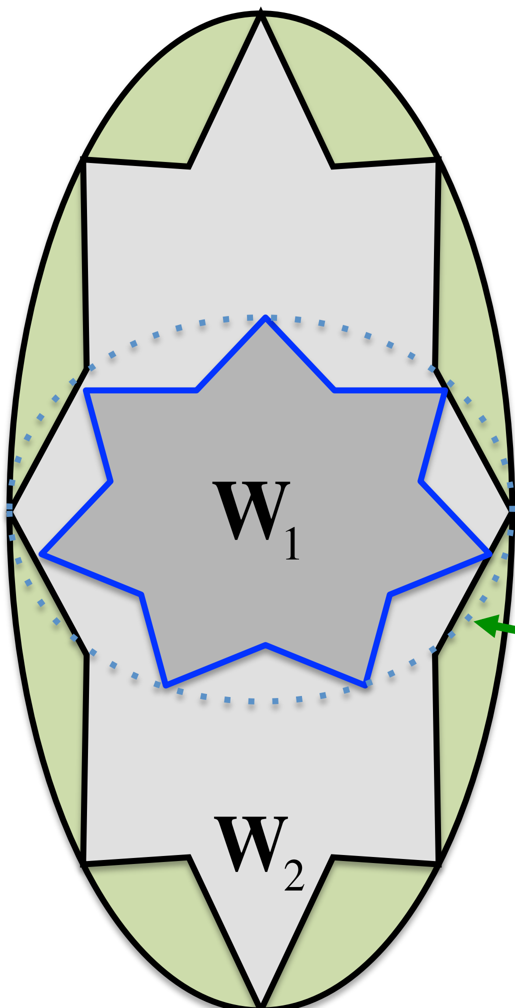
$\quad \mathrm{rank}\ W = 1$ nonconvex

$\mathbf{W}_2 : \quad W(i,j) \geq 0$

$\quad \mathrm{rank}\ W(i,j) = 1$ nonconvex

$\mathbf{W}_{2+} : \quad W(i,j) \geq 0$ convex

Relaxations: BIM



$\text{tr } Y_i W = -s_i \quad \text{for all } i$

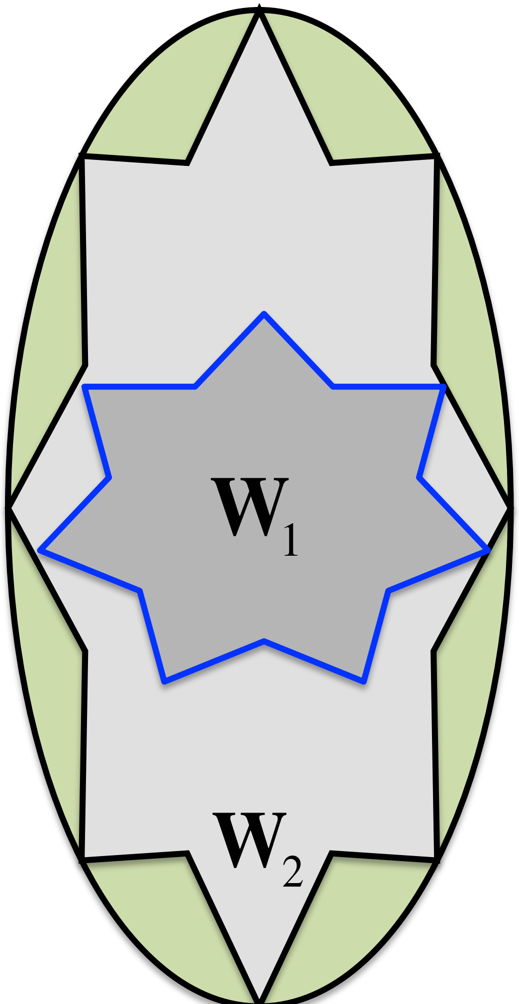
$\mathbf{W}_1 : \quad \text{rank } W = 1$ nonconvex

$\mathbf{W}_+ : \quad W \geq 0$ convex

$\mathbf{W}_2 : \quad \text{rank } W(i,j) = 1$ nonconvex

$\mathbf{W}_{2+} : \quad W(i,j) \geq 0$ convex

BIM: exactness



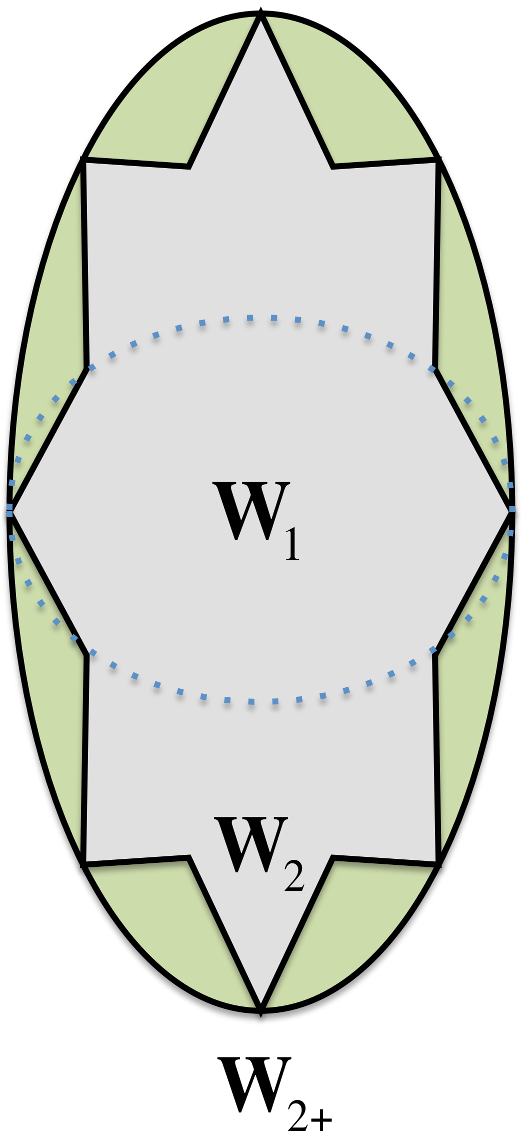
Theorem

If W psd then TFAE

- ☐ W rank-1
- ☐ $W(i,j)$ psd rank-1 and
$$\sum_{(i,j)\in cycle} \angle W_{ij} = 0$$
- ☐ $W\big(\mathrm{cl}(\mathrm{chord}(G))\big)$ psd rank-1

Moreover psd completion of $W(i,j)$ is unique

Relaxations: BIM



<u>Theorem</u>

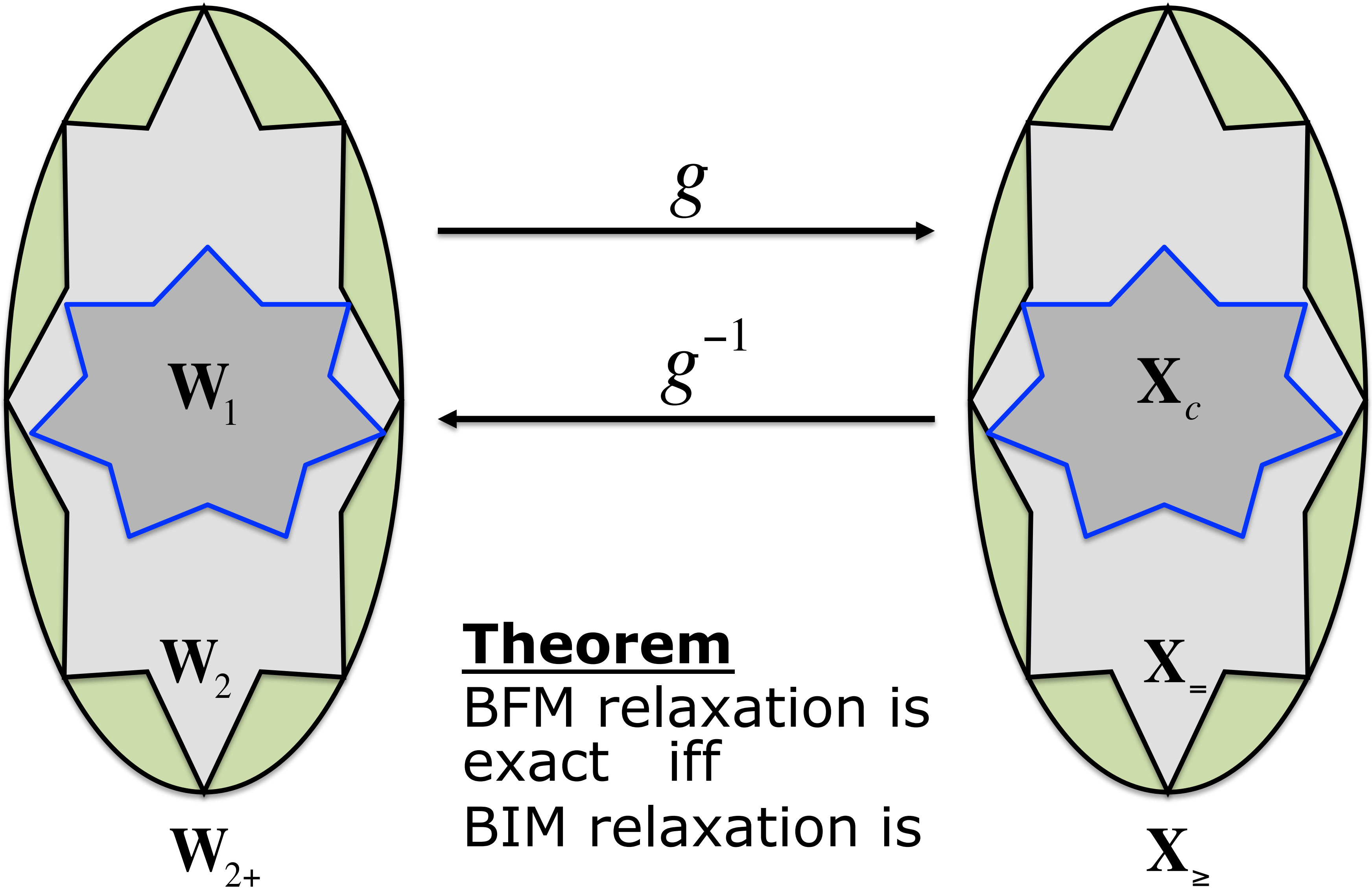
For radial networks:

W psd rank-1 iff

$W(i,j)$ psd rank-1 and

i.e. $\mathbf{W}_1 = \mathbf{W}_2 \cap \{W \geq 0\}$

Equivalence of relaxations



Theorem

BFM relaxation is exact iff

BIM relaxation is