GEOMETRIC OPTIMIZATION AND ARRANGEMENTS

Micha Sharir

Tel Aviv University
and
New York University

Main Theme

- Many geometric optimization problems reduce to problems on arrangements of curves and surfaces

- Calls for development and use of tools for arrangements

  Lower envelopes, vertical decomposition, union of regions, sandwich between two envelopes, zones, overlays of minimization diagrams, ...

Parametric Searching

Optimization: Find $\lambda^*(P)$

Decision: Determine whether $\lambda^*(P) \le \lambda_0$

Efficient solution of the decision problem $\mapsto$
Efficient solution of the optimization problem,
using clever binary search on λ^*

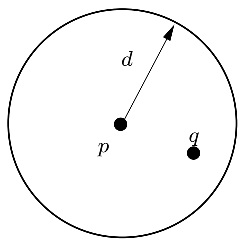
Parametric Searching

- Parallel generic simulation of the decision procedure at λ^* [Megiddo 1983]
- Random sampling of critical λ-values [Matoušek, Chan]
- Monotone matrix searching [Frederickson-Johnson]
- Expander graphs [Katz-Sharir]
- Just plain old binary search

Moral: Focus on the decision procedure

Example: Find the k-th largest inter-point distance in a set P of n points in the plane.

Decision: Given distance d, how many pairs of points at distance $\leq d$?



Draw a disk of radius d about each point of P

How many disk-point containments are there?

Standard range-searching problem; can be solved in time $O^*(n^{4/3})$ (Using machinery of partitioning of planar arrangements)

Arrangements

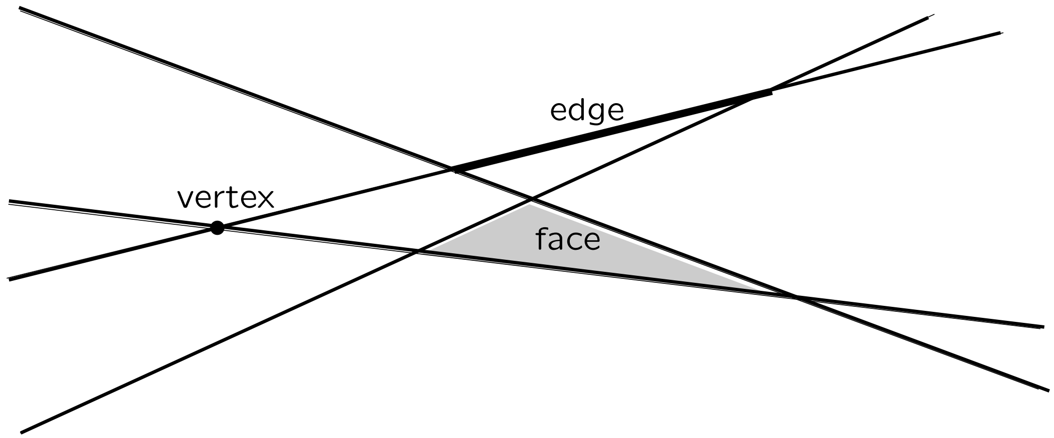
An arrangement $\mathcal{A}(S)$ of a set S of n curves (in the plane) or surfaces (in $d \geq 3$ dimensions):

Decomposition of space into maximal relatively open cells (faces) of various dimensions obtained by 'drawing' the curves / surfaces of S

The complexity of (substructures of) $\mathcal{A}(S)$: Number of faces in the substructure.

Typically, a full arrangement has complexity $\Theta(n^d)$

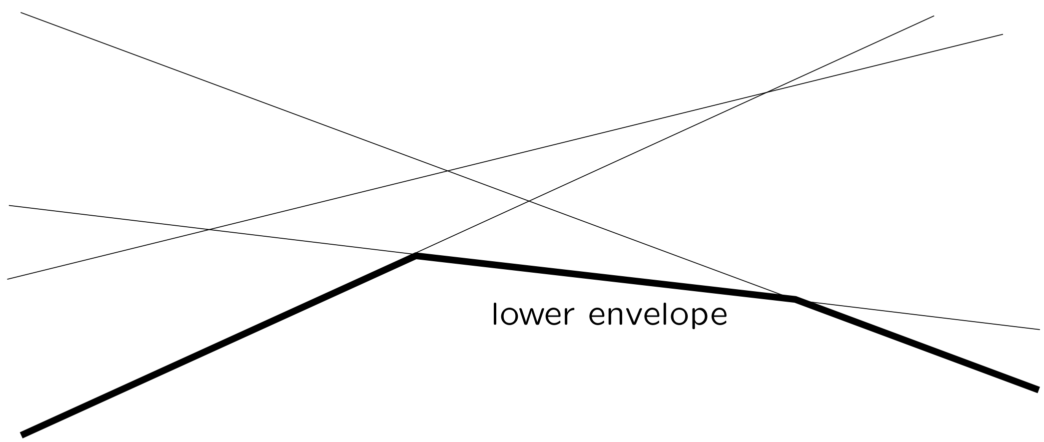
But substructures have smaller complexity



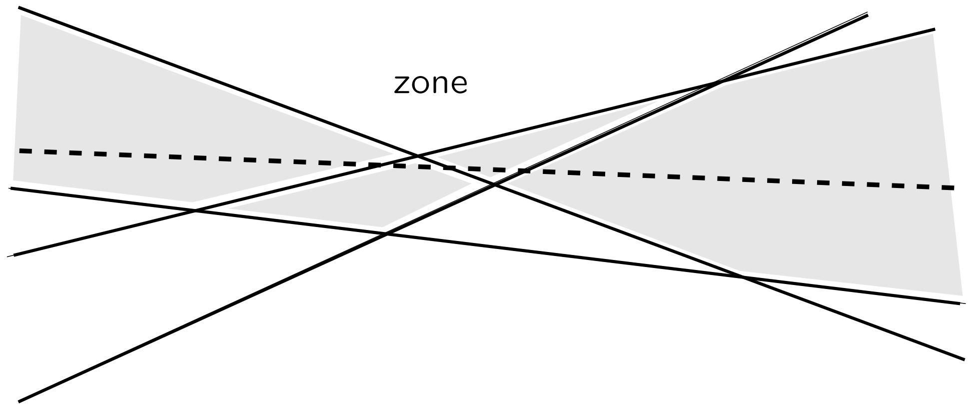
Arrangement of lines in the plane.

WHICH SUBSTRUCTURES?

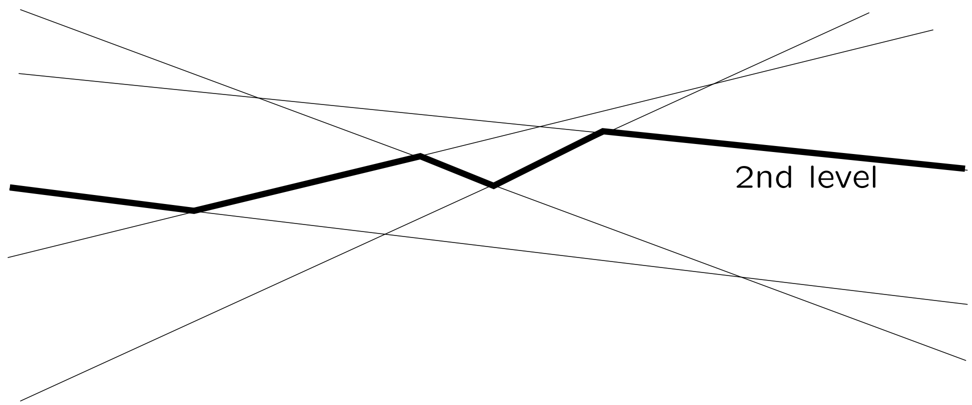
- Lower / upper envelopes



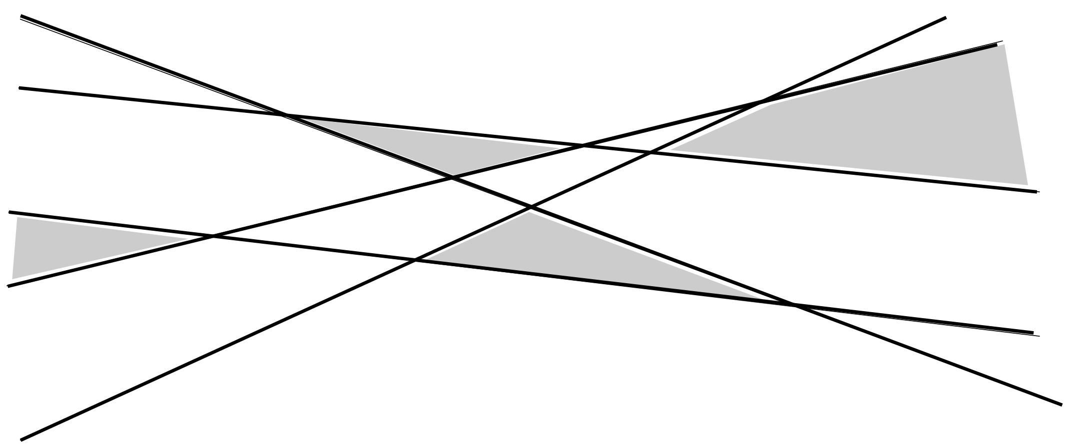
- Single cells
- Zones


zone

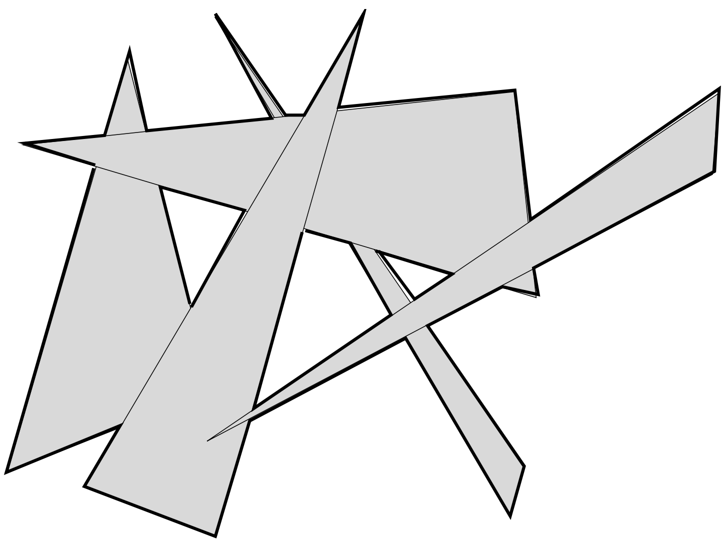
- Levels



- Many faces



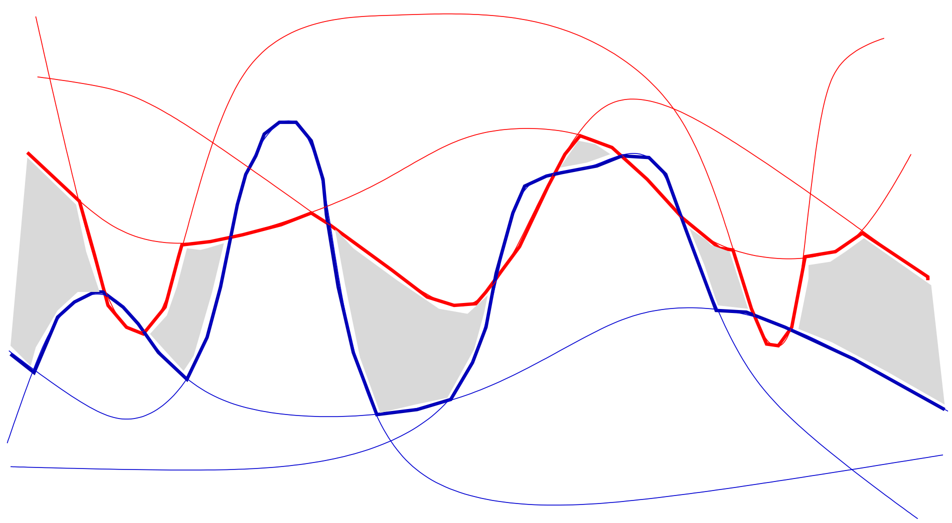
- Union of objects



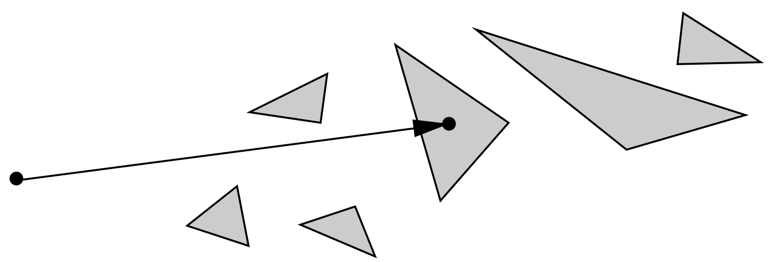
• Vertical decomposition:
Decomposing cells into cells of constant descriptive complexity

• Overlay of substructures of two arrangements

• Sandwich region between two envelopes

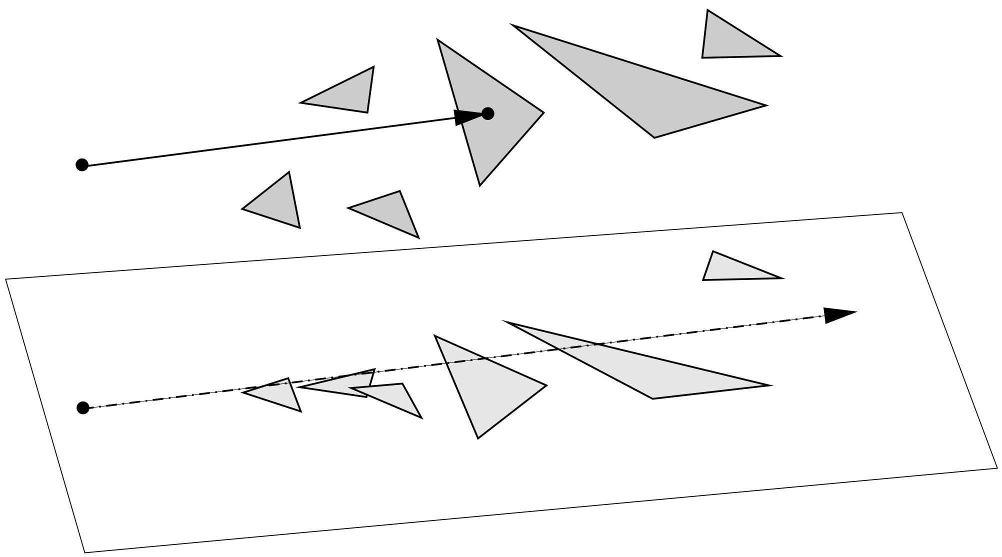


Ray Shooting amid triangles in $\mathbb{R}^3$



Parametric searching $\mapsto$ Segment Emptiness: Does a query segment s intersect any input triangle?

Ray Shooting, Cont'd



Project onto xy-plane

Find subset T_s of all triangles that s crosses in the projection

+ pair of edges of each $\Delta \in T_s$ that s crosses in the projection

Partitioning and range searching in 2D arrangements

Ray Shooting, Cont'd

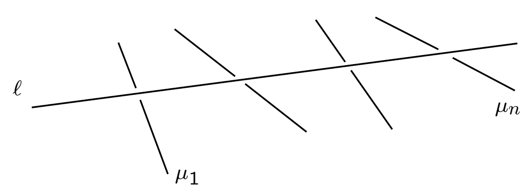
Is there a triangle $\Delta \in T_s$ for which s passes above one edge and below the other?

At this point, can think of s and these edges as lines

Set of n pairs of lines (μ_i, λ_i) and a query line ℓ

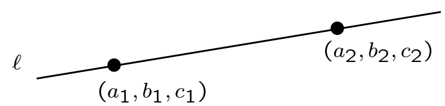
Is there a pair such that ℓ passes above μ_i and below λ_i?

Simpler variant: Does ℓ pass above all lines μ_i?



Plücker coordinates

Map a line ℓ in 3-space to a point p_ℓ and a hyperplane π_ℓ in projective 5-space



$$\begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \end{bmatrix} = (a_1b_2 - a_2b_1,\ a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3)$$

With some care:
ℓ passes above / through / below ℓ' iff
p_ℓ lies above / on / below $\pi_{\ell'}$

Lines in 3-space have only 4 degrees of freedom
All Plücker points lie on a 4-D surface
(Plücker surface), a quadric

Ray Shooting, Cont'd

Reformulation: Preprocess n hyperplanes in 5-space so as to determine whether a query point lies above their upper envelope

Upper envelope of hyperplanes $\approx$ convex hulls of points

In 5-space, convex hull of n points has $O(n^2)$ complexity

Envelope can be computed in $O(n^2)$ time and preprocessed in near-quadratic time into a data structure that supports $O(\log n)$ queries
Point location in high-dimensional arrangements

Lucky breaks:
- Hyperplanes
- Region above upper envelope

Ray Shooting, Cont'd

Set of n pairs of lines (μ_i, λ_i) and a query line ℓ

Is there a pair such that ℓ passes above μ_i and below λ_i?

Map the μ_i's into set M of n hyperplanes in 5-space

Preprocess into a data structure that supports
range searching queries:
Report in compact form all hyperplanes below a query point p_ℓ

Range Searching and Arrangement Decomposition

Random Sampling, ε-Nets

H a set of n hyperplanes in $\mathbb{R}^d$
R a random sample of r hyperplanes of H

Decompose $\mathcal{A}(R)$ into $O(r^d)$ simplices
On average, each simplex is crossed by $\frac{n}{r}$ hyperplanes of H
With high probability, crossed by at most $\frac{cn}{r} \log r$ hyperplanes
Can be improved to $\frac{n}{r}$ with some refinement

Decomposition called $(1/r)$-cutting
[Haussler-Welzl], [Clarkson], [Clarkson-Shor], [Chazelle-Friedman], [Chazelle], [Matoušek], ...

Range Searching and Arrangement Decomposition, Cont'd

Build a recursive structure: Apply same decomposition within each cell of the cutting for the hyperplanes that cross the cell

Structure requires $O^*(n^d)$ storage and preprocessing time

Querying with p:

Find all the cells τ that contain p (one in each level)

Report for each τ the set of hyperplanes passing above τ

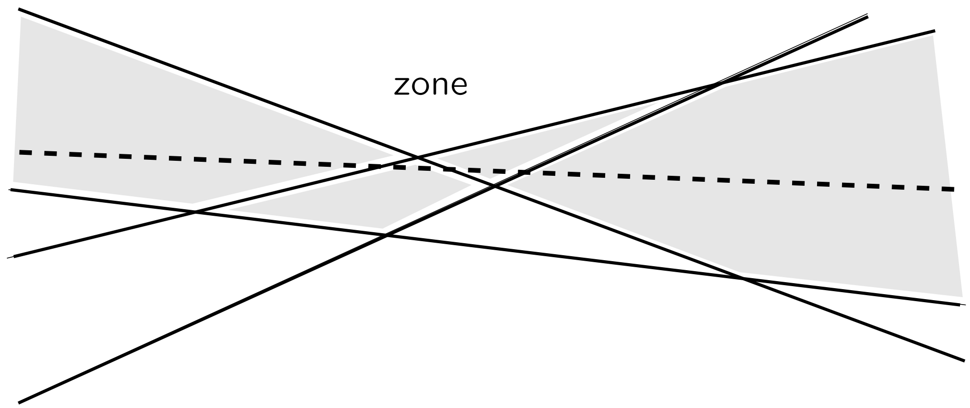
Output: Disjoint union of $O(\log n)$ canonical sets

Ray Shooting, Cont'd

For our problem, near-$O(n^5)$ storage

Too much, because the query points are restricted to lie on the Plücker surface Π!

Only the zone of Π in $\mathcal{A}(R)$ is relevant



Zone Theorem [Aronov, Pellegrini, Sharir 93]

The zone of a convex or fixed-degree algebraic surface in an arrangement of n hyperplanes in $\mathbb{R}^d$ has $O(n^{d-1})$ faces (of all dimensions)

Yields a $(1/r)$-cutting of size $O(r^4 \log r)$

Ray Shooting, End

For each canonical set of the μ_i's, take the corresponding set of the λ_i, and construct for it the preceding envelope structure for being *below* the envelope

For each output set of μ_i's for the query with ℓ
query ℓ in the data structure of the matching structure for the λ_i's

Summary:
Data structure of $O^*(n^4)$ size
Polylogarithmic query time

Son of Ray Shooting Rides Again

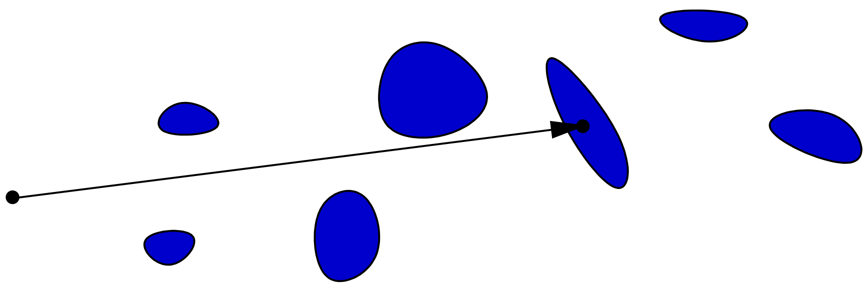
We worked very hard

Plücker linearization, Lower/upper envelopes, Cuttings, Zones

Still retaining one ace: Hyperplanes

What about general surfaces?

Ray shooting amid semi-algebraic sets

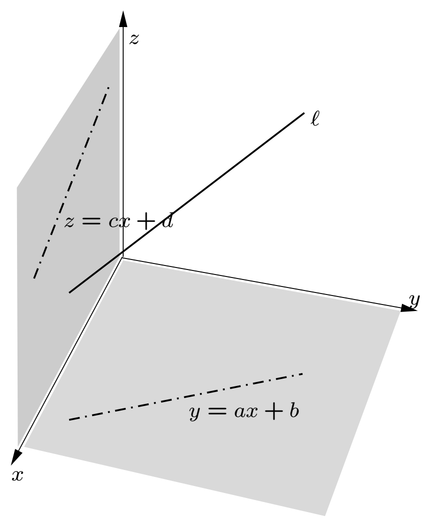


Ray Shooting II, Cont'd

Plücker transformation useless

Instead, parametrize lines as points in 4-space

E.g. map the line $\ell : y = ax + b,\ z = cx + d$
into the point $\ell^* = (a, b, c, d) \in \mathbb{R}^4$



Ray Shooting II, Cont'd

$\mathcal{C}$ – set of n bodies in $\mathbb{R}^3$

For each $C \in \mathcal{C}$
$K(C) =$ region (in $\mathbb{R}^4$) of all lines that intersect C

Need to compute $K = \bigcup_{C \in \mathcal{C}} K(C)$ and
Preprocess it for point location
$\ell^* \in K$ iff ℓ intersects some set in $\mathcal{C}$

Union of geometric objects: New substructure

Searching in K

Construct a $(1/r)$-cutting of $\mathcal{A}(\mathcal{C})$:
Arrangement of the bounding surfaces of the sets $K(C)$

Decomposition into cells (not simplices!)
each crossed by at most n/r surfaces

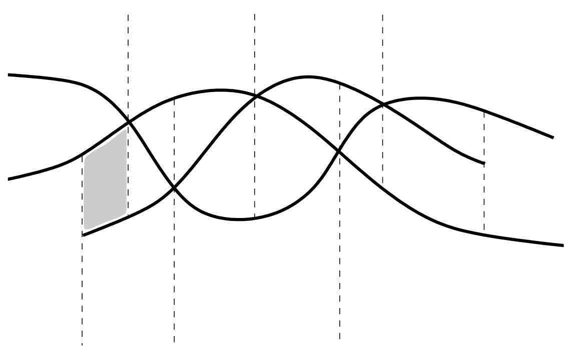
Take $r =$ large constant
Recurse in each cell τ with the $K(C)$'s whose boundaries cross τ
Terminate at cells fully contained in some $K(C)$
Search with ℓ^*: One cell in each recursive level
ℓ crosses some C iff ℓ^* reaches a terminal cell

Big questions: What is the cutting? How many cells? How to construct it efficiently?

Vertical Decomposition: A New Substructure

Decomposing cells of an arrangement $\mathcal{A}(S)$ of n “simple” surfaces into subcells of constant descriptive complexity

Needed to ensure that each subcell in a sample of r surfaces will be crossed by $\approx n/r$ surfaces of S
Random sampling / ε-net theory



Vertical Decomposition, Cont'd

Complex definition in higher dimensions

Recursive decomposition, one dimension at a time

The only general-purpose method known!

Vertical Decomposition, Cont'd

Number of cells in the V.D. of n surfaces in $\mathbb{R}^d$
(Ignore dependence on the algebraic degree)

$O(n^2)$ for $d = 2$

[Chazelle et al. 89,91]:
$O^*(n^3)$ for $d = 3$
$O^*(n^{2d-3})$ for $d > 3$

[Koltun 01]:
$O^*(n^4)$ for $d = 4$
$O^*(n^{2d-4})$ for $d > 4$

A few more studies of special cases

Optimal bounds unknown for $d \geq 5$

Unknown for substructures:

- Region above envelope
- Single cell (known for $d = 3$)
- Sandwich region between envelopes

Back to Ray Shooting

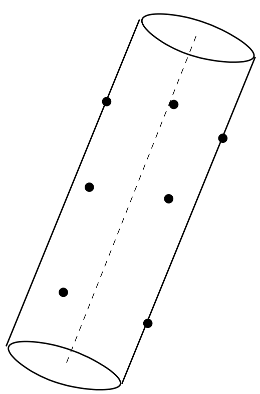
Can solve the general ray shooting with a data structure of size $O^*(n^4)$ and polylogarithmic query time

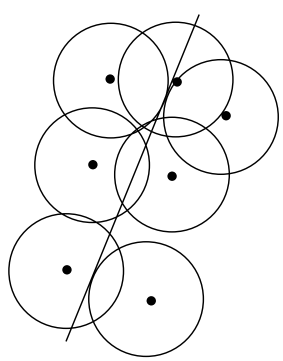
As if this wasn't bad enough... A new problem!
Smallest Enclosing Cylinder

P a set of n points in $\mathbb{R}^3$

Find cylinder of smallest radius that contains P

Paremetric searching + a simple transformation $\mapsto$
Decision procedure:
Given n unit balls in $\mathbb{R}^3$, is there a line that stabs all of them?





(A cylinder c of radius r contains a point q iff the axis of c stabs the ball of radius r centered at q)

Smallest Enclosing Cylinder, Cont'd

The "opposite" of the ray shooting problem:
There: Does a given ℓ hit at least one object?
Here: Is there any ℓ that hits all objects?

Map to 4-space. For each input ball B
$K(B)$ = region of all points ℓ^* where ℓ meets B

Map the line $\ell : y = ax + b,\ z = cx + d$
into the point $\ell^* = (a, b, c, d) \in \mathbb{R}^4$

$$K(B) = \{(a, b, c, d) \mid f_B(a, b, c) \leq d \leq g_B(a, b, c)\}$$

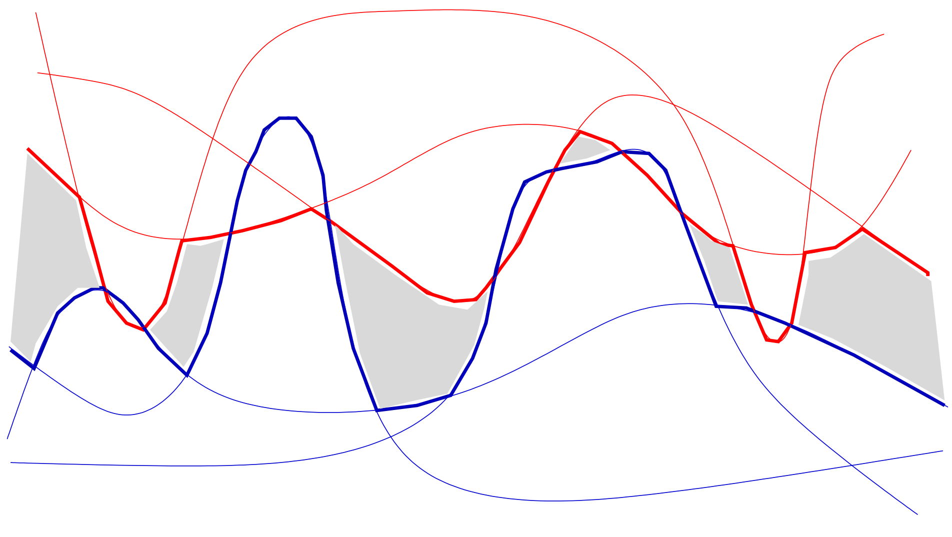
f_B, g_B: partially defined, represent lower tangent and upper tangent lines

Smallest Enclosing Cylinder, Cont'd

Line ℓ meets all balls B iff $\ell^* \in K^* = \bigcap_B K(B)$

$K^* = \{(a, b, c, d) \mid \max_B f_B(a, b, c) \le d \le \min_B g_B(a, b, c)\}$

The sandwich region between an upper envelope and a lower envelope of trivariate functions
Yet another substructure...



Smallest Enclosing Cylinder, Cont'd

[Koltun-Sharir 02]: The complexity of such a sandwich region is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$

But we don't know yet how to search in this region with near-cubic resources
(Alternative near-cubic solution: [Agarwal, Aronov, Sharir 99])

Open: Complexity of the vertical decomposition of a sandwich region in $\mathbb{R}^4$

Near-linear bound known for $d = 2$ (univariate functions)
Near-quadratic bound known for $d = 3$ (bivariate functions)
[Agarwal, Schwarzkopf, Sharir 96]

What Else?

- Width in three dimensions = Plane fitting
- Line fitting in the plane with miniumum sum of distances
- Minimum Hausdorff distance under translation
- Minimum weight bipartite Euclidean matching in the plane

and many more...

Width in three dimensions = Plane fitting

Reduces to

Given two sets L, L' of lines in space, is there a pair $(\ell, \ell') \in L \times L'$ such that $d(\ell, \ell') \leq 1$?

Given n points and m surfaces in 4-space, is there any point that lies above the lower envelope of the surfaces?

[Chazelle et al. 93], [Agarwal-Sharir 96]

Line fitting in the plane with miniumum sum of distances

Reduces to

Given n lines in the plane, compute the median level of their arrangement

Vertices of this level are duals of candidate fitting lines

Minimum Hausdorff distance under translation

Reduces to

Compute and search in the upper envelope of Voronoi surfaces in 3-space

[Huttenlocher-Kedem-Sharir 93]

Minimum weight bipartite Euclidean matching

Reduces to

Dynamic maintenance of and search in lower envelope of bivariate functions [Agarwal-Efrat-Sharir 00]

End of Agony

Thank You