Emerging Trends in Optimization

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I. Integer programming

II. Large-scale linear programming

What is (mixed-) integer programming?

$$\min c^T x$$

$$Ax + By \geq b, \quad x \text{ integral}$$

Dramatic improvements over the last ten years

- More mature methodology: branch-and-cut (theory and implementation)
- Vastly better linear programming codes
- Better computing platforms

$\longrightarrow$ Can now solve problems once thought impossible

But ...

Network design problem

- We have to build a directed network where each node has small out- and in-degree
- In the network, we have to route given multicommodity demands
- We have to carry out both tasks at the same time, so that the **maximum** total flow on any edge is **minimized**

Input data:

- A list of **traffic demands**. Traffic demand $\boldsymbol{k}$ specifies that $\boldsymbol{d^k}$ units must be routed from node $\boldsymbol{s^k}$ to node $\boldsymbol{t^k}$; $\boldsymbol{k = 1, 2, \cdots, K}$.
- The network has out-degrees and in-degrees at most p.

Mixed-integer programming formulation:

- For every pair of nodes $\boldsymbol{i}$, $\boldsymbol{j}$: $\boldsymbol{x_{ij}}$, to indicate whether the logical network contains edge $\boldsymbol{(i, j)}$
- For every demand $\boldsymbol{k}$ and every pair of nodes $\boldsymbol{i}$, $\boldsymbol{j}$: $\boldsymbol{f^k_{ij}}$, the amount of flow of commodity $\boldsymbol{k}$ that is routed on edge $\boldsymbol{(i, j)}$.

Complete formulation

$f_{ij}^k \leq d^k x_{ij}$, for all k and (i, j)

Route all traffic (Flow conservation)

$\Sigma_{j \neq s^k} f_{s^k j}^k = d^k$, for all k

$\Sigma_{j \neq i} f_{ij}^k = 0$, for all k and $i, j \neq s^k, t^k$

Degree constraint

$\Sigma_{j \neq i} x_{ij} \leq p$, for all i

$\Sigma_{j \neq i} x_{ji} \leq p$, for all i

Congestion

$\Sigma_k f_{ij}^k - \lambda \leq 0$, for all (i, j)

Objective : **Minimize** λ

$(f, x \geq 0)$

Instance **danoint** in the **MIPLIB**

1992

- Linear programming relaxation has value $\sim$ **12.0**
- Strong formulation has value (lower bound) $\sim$ **60.0** (Two minutes CPU on SUN Sparc 2)
- Branch-and-bound (Cplex 3.0) improves lower bound to $\sim$ **63.0**; best upper bound has value $\sim$ **65.6**

2002

- Cplex 8.0 solves **danoint** in one day of CPU time on 2.0 GHz P4 while enumerating approximately two million branch-and-bound nodes.

Progress?

Speedup is due to:

1. Vastly faster computers

2. Much faster linear programming solver (esp. Cplex)

3. Faster integer programming solver (Branch-and-cut)

A thought experiment:

Use the 2002 branch-and-cut module together with the 1992 LP solver + machine

$\longrightarrow$ One day CPU time (2002) becomes **one year** CPU time (1992).

Why does this matter?

$\longrightarrow$ The network in **danoint** has **8** nodes: **danoint** has 56 0-1 variables and $\sim$ 200 continuous variables.

So?

dano3mip is **danoint**'s big brother.

- 24 node network; $\sim$ 550 0-1 variables and $\sim$ 13500 continuous variables.

- **1992: strong formulation gives error bound of $\sim$ 25% which cannot be improved by branch-and-bound. Problem considered unsolvable.**

- **2003: Cplex 8.0 cannot improve 1992 bound after one week. dano3mip is now the only unsolved problem in MIPLIB.**

Starting point

Balas, Pulleyblank, Barahona, others (pre 1990).

A polyhedron $P \subseteq R^n$ can be the **projection**

of a *simpler* polyhedron $Q \subseteq R^N$ $(N > n)$

More precisely:

There exist polyhedra $P \subseteq R^n$, such that

- P has exponentially (in n) many facets, and
- P is the projection of $Q \subseteq R^N$, where
- N is polynomial in n, and Q has polynomially many facets.

$\rightarrow$ Lovász and Schrijver (1989)

Given $\mathcal{F} = \{\, x \in \{0,1\}^n \;:\; Ax \geq b \,\}$

Question: Given $x^* \in R^n_+$, is $x^* \in conv(\mathcal{F})$?

Idea: Let $N \gg n$.

Consider a function (a "lifting") that maps each

$v \in \{0,1\}^n$ into $\hat{z} = \hat{z}(v) \in \{0,1\}^N$

with $\hat{z}_i = v_i$, $1 \leq i \leq n$.

Let $\hat{\mathcal{F}}$ be the image of $\mathcal{F}$ under this operator.

Question: Can we find $y^* \in conv(\hat{\mathcal{F}})$, such that $y^*_i = x^*_i$, $1 \leq i \leq n$?

Concrete Idea

$v \in \{0,1\}^n$ mapped into $\hat{v} \in \{0,1\}^{2^n}$, where

(i) the entries of $\hat{v}$ are indexed by subsets of $\{1, 2, \cdots, n\}$, and

(ii) For $S \subseteq \{1, \ldots, n\}$, $\hat{v}_S = 1$ iff $v_j = 1$ for all $j \in S$.

Example: $v = (1, 1, 1, 0)^T$ mapped to:

$\hat{v}_\emptyset = 1$, $\hat{v}_1 = 1$, $\hat{v}_2 = 1$, $\hat{v}_3 = 1$, $\hat{v}_4 = 0$,

$\hat{v}_{\{1,2\}} = \hat{v}_{\{1,3\}} = \hat{v}_{\{2,3\}} = 1$,

$\hat{v}_{\{1,4\}} = \hat{v}_{\{2,4\}} = \hat{v}_{\{3,4\}} = 0$,

$\hat{v}_{\{1,2,3\}} = 1$, $\hat{v}_{\{1,2,4\}} = \hat{v}_{\{1,3,4\}} = \hat{v}_{\{2,3,4\}} = 0$,

$\hat{v}_{\{1,2,3,4\}} = 0$.

Take $v \in \{0,1\}^n$. The $2^n \times 2^n$ matrix $\hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^{\boldsymbol{t}}$

$\rightarrow$ Is **symmetric**, and its **main diagonal $=$ $\emptyset$-row**.

Further, suppose $x^* \in R^n$ satisfies

$$\boldsymbol{x^*} = \boldsymbol{\Sigma_i \lambda_i v_i}$$

where each $\boldsymbol{v_i} \in \boldsymbol{\{0,1\}^n}$, $\boldsymbol{0 \leq \lambda}$, and $\boldsymbol{\Sigma_i \lambda_i = 1}$.

Let $W = W(x^*) = \Sigma_i \lambda_i \hat{v}_i \hat{v}_i{}^t$ and $y = \Sigma_i \lambda_i \hat{v}_i$.

- $y_{\{j\}} = x^*_j$, for $1 \leq j \leq n$.
- W is symmetric, $W_{\emptyset,\emptyset} = 1$, diagonal $=$ $\emptyset$-column $= y$.
- $W \succeq 0$.
- $\forall\, p, q \subseteq \{1, 2, \cdots, n\}$, $W_{p,q} = y_{p \cup q}$

So we can write $W = W^y$.

$x^* = \Sigma_i \lambda_i v_i, \quad 0 \leq \lambda$ and $\Sigma_i \lambda_i = 1.$

$y = \Sigma_i \lambda_i \hat{v}_i, \qquad W^y \; = \; \Sigma_i \lambda_i \hat{v}_i {\hat{v}_i}^t,$ (cont'd)

Assume each $v_i \in \mathcal{F}$.

Theorem

Suppose $\Sigma_{j=1}^n \alpha_j x_j \geq \alpha_0 \;\; \forall x \in \mathcal{F}$.

Let $p \subseteq \{1, 2, \cdots, n\}$.

Then:

$$\boldsymbol{\Sigma_{j=1}^n \alpha_j W^y_{\{j\},p} \; - \; \alpha_0 W^y_{\emptyset,p} \; \geq \; 0}$$

e.g. the p-column of W satisfies
every constraint valid for $\mathcal{F}$, homogenized.

$\longrightarrow$ just show that for every i,

$$\Sigma_{j=1}^n \alpha_j [\hat{v}_i {\hat{v}_i}^t]_{\{j\},p} \; - \; \alpha_0 [\hat{v}_i {\hat{v}_i}^t]_{\emptyset,p} \; \geq \; 0$$

Also holds for the $\emptyset$-column minus the p^{th}-column.

Lovász-Schrijver Operator, for $\mathcal{F} = \{ x \in \{0,1\}^n \ : \ Ax \geq b \}$

`*************************************************`

1. Form an $(n+1) \times (n+1)$-matrix W of **variables**
2. **Constraint:** $W_{0,0} = 1$, W symmetric, $W \succeq 0$.
3. **Constraint:** $0 \leq W_{i,j} \leq W_{0,j}$, for all i, j.
4. **Constraint:** The main diagonal of W equals its 0-row.
5. **Constraint:** For every column u of W,

$$\Sigma_{h=1}^{n} a_{i,h} u_h \ - \ b_i u_0 \ \geq \ 0, \quad \forall \text{ row } \ i \ \text{ of } \ A$$

   and

$$\Sigma_{h=1}^{n} a_{i,h} (W_{h,0} - u_h) \ - \ b_i (1 - u_0) \ \geq \ 0, \quad \forall \text{ row } \ i \ \text{ of } \ A$$

`*************************************************`

Let $C = \{ x \in R^n : \ 0 \leq x \leq 1, \ Ax \geq b \}$
and $N_+(C)$ = set of $x \in R^n$, such that
there exists W satisfying 1-5, with $W_{j,0} = x_j$, $1 \leq j \leq n$.

Theorem. $\boldsymbol{C \supseteq N_+(C) \supseteq N_+^2(C) \supseteq \cdots \supseteq N_+^n(C) = conv(\mathcal{F})}$.

Lovász-Schrijver revisited

$v \in \{0,1\}^n$ lifted to $\hat{v} \in \{0,1\}^{2^n}$, where

(i) the entries of $\hat{v}$ are indexed by subsets of $\{1, 2, \cdots, n\}$, and

(ii) For $S \subseteq \{1, \ldots, n\}$, $\hat{v}_S = 1$ iff $v_j = 1$ for all $j \in S$.

$\longrightarrow$ this approach makes statements about *sets* of variables that simultaneously equal 1

How about more complex logical statements?

Subset algebra lifting

For $1 \le j \le n$, let

$$Y_j = \{z \in \{0,1\}^n : z_j = 1\}, \quad N_j = \{z \in \{0,1\}^n : z_j = 0\}$$

Let $\mathcal{A}$ denote the set of all set-theoretic expressions

involving the Y_j, the N_j, and $\emptyset$.

Note:

(i) $\mathcal{A}$ is isomorphic to the set of subsets of $\{0,1\}^n$.

(ii) $|\mathcal{A}| = 2^{2^n}$

(iii) $\mathcal{A}$ is a lattice under $\supseteq$; containing an isomorphic copy of the lattice of subsets of $\{1, 2, \cdots, n\}$.

Lift $\boldsymbol{v \in \{0,1\}^n}$ **to** $\boldsymbol{\breve{v} \in \{0,1\}^{\mathcal{A}}}$

where for each $\boldsymbol{S \subseteq \{0,1\}^n}$, $\boldsymbol{\breve{v}_S = 1}$ **iff** $\boldsymbol{v \in S}$.

Example

$v = (1, 1, 1, 0, 0) \in \{0, 1\}^5$ is lifted to

$\breve{v} \in \{0, 1\}^{2^{32}}$ which satisfies

$\breve{v}[(Y_1 \cap Y_2) \cup Y_5] = 1$

$\breve{v}[Y_3 \cap Y_4] = 0$

$\breve{v}[Y_3 \cap (Y_4 \cup N_5)] = 1$

$\ldots$

$\breve{v}[S] = 1$ iff $(1, 1, 1, 0, 0) \in S$

Note: if $v \in \mathcal{F}$ then $\breve{v}[\mathcal{F}] = 1$.

$\longrightarrow$ Family of algorithms that generalize Lovász-Schrijver, Sherali-Adams, Lasserre

Generic Algorithm

1. Form a family of set-theoretic indices, $\boldsymbol{\mathcal{V}}$.

These include, for $1 \leq j \leq n$:

Y_j, to represent $\{x \in \{0,1\}^n \; : \; x_j = 1\}$
N_j, to represent $\{x \in \{0,1\}^n \; : \; x_j = 0\}$

Also $\emptyset$, $\mathcal{F}$.

2. Impose all constraints known to be valid for $\boldsymbol{\mathcal{F}}$:

e.g.

$$x_1 + 4x_2 \geq 3 \text{ valid } \rightarrow \; X[\mathrm{Y}_1] + 4X[\mathrm{Y}_2] - 3 \geq 0$$

Also set theoretic constraints, e.g. $X[\mathrm{N}_5] \geq X[\mathrm{Y}_2 \cap \mathrm{N}_5]$.

3. Form a matrix $\boldsymbol{U \in R^{\mathcal{V} \times \mathcal{V}}}$ of variables, with

- U symmetric, main diagonal $= \mathcal{F}$-row $= X$
- For $p, q \in \mathcal{V}$,

  $U_{p,q} \; = \; X[p \cap q]$ if $p \cap q \in \mathcal{V}$
  $U_{p,q} \; = \;$ a new variable, otherwise
- All columns of U satisfies every constraint; optionally $U \succeq 0$.

How do we algorithmically choose small (polynomial-size) $\mathcal{V}$?

Example (level 2):

$$\begin{array}{rcl} x_1 + 5x_2 + x_3 + x_4 + x_5 - 2x_6 & \geq 2 & \rightarrow \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_6 \\ -x_2 + 2x_3 + x_4 \qquad\qquad + x_6 & \leq 3 & \rightarrow \mathrm{N}_2 \cap \mathrm{Y}_3 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6 \\ x_1 \; + x_2 + x_3 - x_4 \qquad\qquad\qquad & \geq 1 & \rightarrow \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_3 \cap \mathrm{Y}_4 \end{array}$$

$\longrightarrow$ construct $\boldsymbol{\omega} = \mathbf{N_1} \cap \mathbf{N_2} \cap \mathbf{Y_4} \cap \mathbf{Y_6}$

also $\quad \mathbf{Y_1} \cap \mathbf{Y_2} \cap \mathbf{Y_4} \cap \mathbf{Y_6}$ (a *negation* of **order 2** of ω)

and all other negations of order 2 (e.g. $\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_4 \cap \mathrm{N}_6$).

also:

$$\omega^{>2} = \bigcup_{t>2} \{\omega' : \omega' \text{ is a negation of order } t \text{ of } \omega\}$$

In general, the $w^{>r}$ expressions are unions (disjuntions) of exponentially many intersections.

Constraints

"Box" constraints:

$$0 \leq X, \quad X[\mathcal{F}] = 1, \quad X[p] - X[\mathcal{F}] \leq 0$$

Also, say:

$$\omega = \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6.$$

Then e.g.

$$X[\mathrm{N}_1] - X[\omega] \geq 0.$$

Also,

$$X[\mathrm{Y}_1] + X[\mathrm{Y}_2] + X[\mathrm{N}_4] + X[\mathrm{N}_6] - 2X[\omega^{>1}] \geq 0.$$

Finally,

$$X[\omega] + X[\mathrm{Y}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6] + X[\mathrm{N}_1 \cap \mathrm{Y}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6] +$$

$$+ X[\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_4 \cap \mathrm{Y}_6] + X[\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{N}_6] +$$

$$+ X[\omega^{>1}] \; = \; 1$$

$\longrightarrow$ Implications for "matrix of variables"

Set covering problems

$$\begin{aligned} \min \quad & c^T x \\ s.t. \ & x \in \mathcal{F} \end{aligned}$$

$\mathcal{F} = \{\, x \in \{0,1\}^n \ : \ Ax \geq 1 \,\}, \quad A$ a $0-1$-matrix.

$\longrightarrow$ All nontrivial valid inequalities $\alpha^T x \geq \alpha_0$ satisfy $\alpha \geq 0$ and integral

Theorem

For any integer $k \geq 1$, there is a polynomial-size relaxation guaranteed to satisfy all valid inequalities with coefficients in $\boldsymbol{\{0, 1, \cdots, k\}}$. ∎

$\longrightarrow$ $k = 2$ requires **exponential** time for Lovász-Schrijver

Chvátal-Gomory cuts

Given constraints:

$$\Sigma_j a_{ij} x_j \ \geq \ b_i, \ \ 1 \leq i \leq m$$

and multipliers $\pi_i \geq 0, \ \ 1 \leq i \leq m$

$x \in Z_+$ implies that

$$\Sigma_j \left\lceil \Sigma_i \pi_i a_{ij} \right\rceil x_j \ \geq \ \left\lceil \Sigma_i \pi_i b_i \right\rceil$$

is valid

$\rightarrow$ a C-G **rank 1** cut (ca. 1960)

Example:

$$\begin{array}{lllll} 3x_1 & -2x_2 & & \geq & 2 & \times\frac{1}{2} \\ 2x_1 & & +4x_3 & \geq & 5 & \times\frac{1}{3} \\ & x_2 & +x_3 & \geq & 1 & \times 1 \end{array}$$

get:

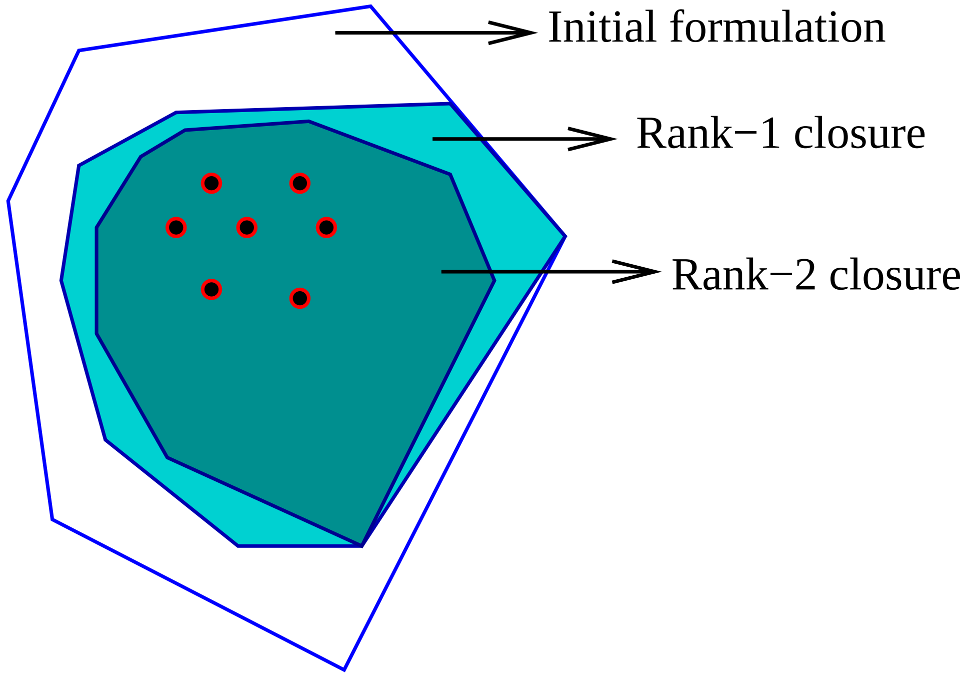
$$\begin{array}{lllll} \frac{13}{6}x_1 & & +\frac{7}{3}x_3 & \geq & \frac{11}{3} \end{array}$$

round-up:

$$\begin{array}{lllll} 3x_1 & & +3x_3 & \geq & 4 \end{array}$$

→ the polyhedron defined by all rank-1 cuts is called the rank-1 **closure** (Cook, Kannan, Schrijver 1990)

Similarly, one can define the rank-2, rank-3, etc. closures ... the convex hull of 0-1 solutions always has finite rank



Recent results: the separation problem over the rank-1 closure of a problem is NP-hard (Eisenbrand, Fischetti-Caprara, Cornuejols-Li)

Chvátal-Gomory cuts and Set Covering Problems

$$\begin{aligned} \min \quad & c^T x \\ s.t. \; & x \in \mathcal{F} \end{aligned}$$

$\mathcal{F} = \{\, x \in \{0,1\}^n \ : \ Ax \geq 1 \,\}, \quad A$ a $0-1$-matrix.

Suppose $r > 0$ integral. Let:

- $\mathcal{F}_r \ = \$ rank-r closure of $\mathcal{F}$,
- $\tau_r \ = \ \min\{c^T x \ : x \in \mathcal{F}_r\}$

Theorem

For each $\boldsymbol{r > 0}$ and $\boldsymbol{0 < \epsilon < 1}$, there is a polynomial-time relaxation with value at least

$$\boldsymbol{(1 - \epsilon)\, \tau_r}$$

PROGRESS IN LARGE-SCALE LINEAR PROGRAMMING
LAST 10-20 YEARS

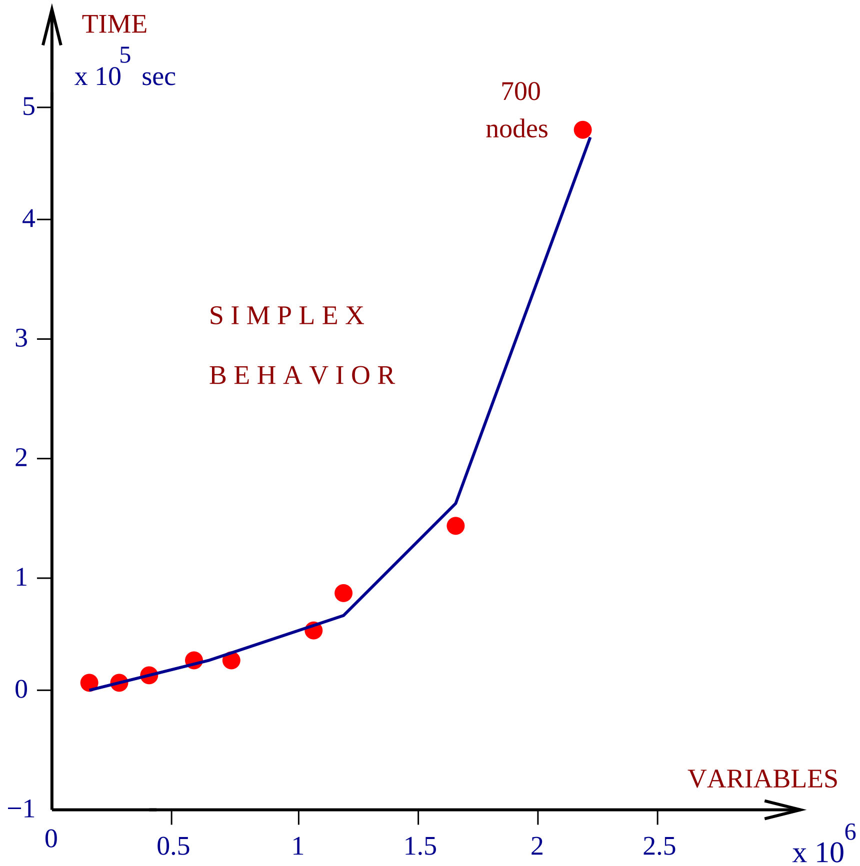
- **THE ELLIPSOID METHOD** - first provably good algorithm
- **PROBABILISTIC ANALYSIS OF THE SIMPLEX METHOD** - is it fast on "average" problems?
- **KARMARKAR'S ALGORITHM** - provably good interior point methods = logarithmic barrier methods
- **STEEPEST-EDGE PIVOTING SCHEMES FOR SIMPLEX METHODS** - fewer iterations
- **BETTER USE OF SPARSITY IN SIMPLEX METHODS** - faster pivots
- **MORE EFFECTIVE CHOLESKY FACTORIZATION METHODS** - faster, sparser
- **IMPROVEMENTS IN NUMERICAL LINEAR ALGEBRA** - improved numerical stability

- **GREATLY EXPANDED TEST-BEDS WITH LARGER PROBLEMS OF DIFFERENT TYPES**
- **MUCH FASTER COMPUTERS WITH VASTLY LARGER STORAGE**
- **MUCH BETTER SOFTWARE DEVELOPMENT ENVIRONMENTS**

→ LPs CAN BE SOLVED THOUSANDS OF TIMES FASTER THAN 10 YEARS AGO

ARE THE ALGORITHMS SCALABLE?

EXPERIMENTS USING CONCURRENT FLOW PROBLEMS



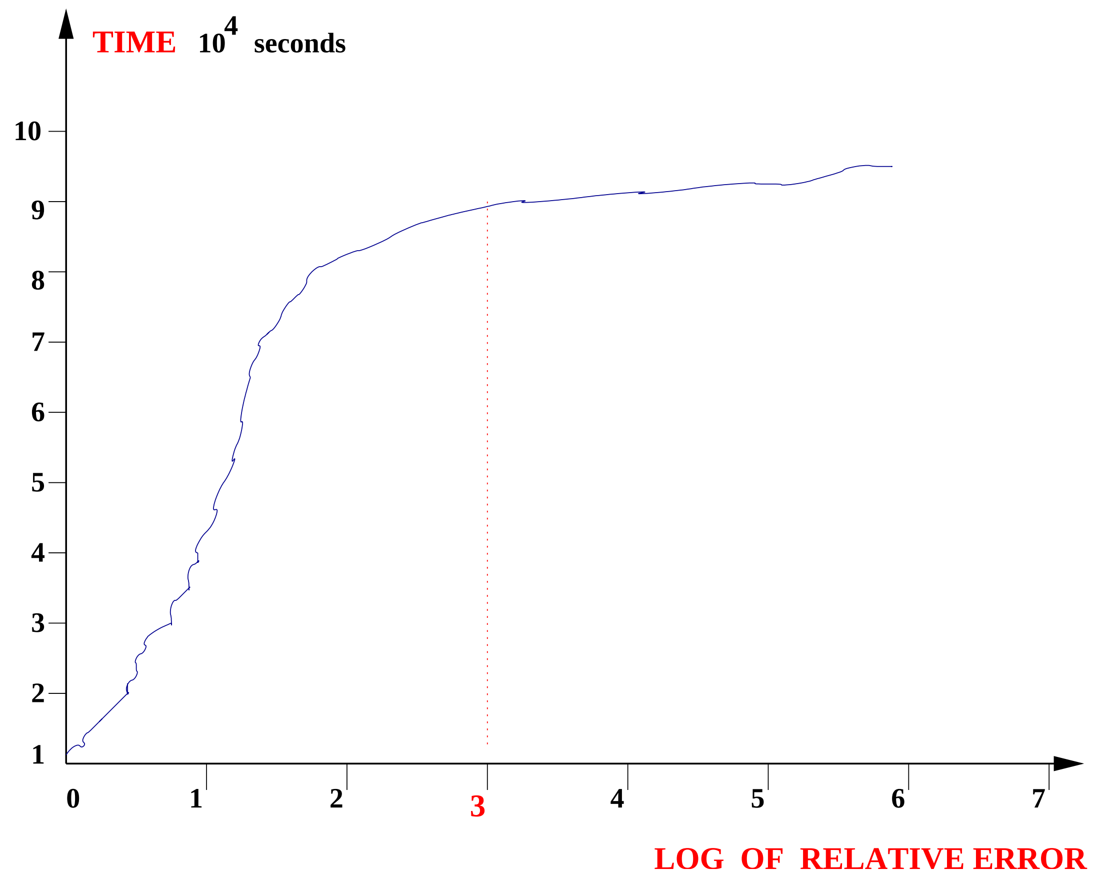
polygonal line is best-fit cubic

WHAT IS THE PROBLEM SIZE WE NEED TO HANLDE?

- **TEN YEARS AGO: THOUSANDS TO (SOMETIMES) MILLIONS OF VARIABLES AND CONSTRAINTS**
- **NOW: TENS OR HUNDREDS OF MILLIONS**

TYPICAL CONVERGENCE BEHAVIOR

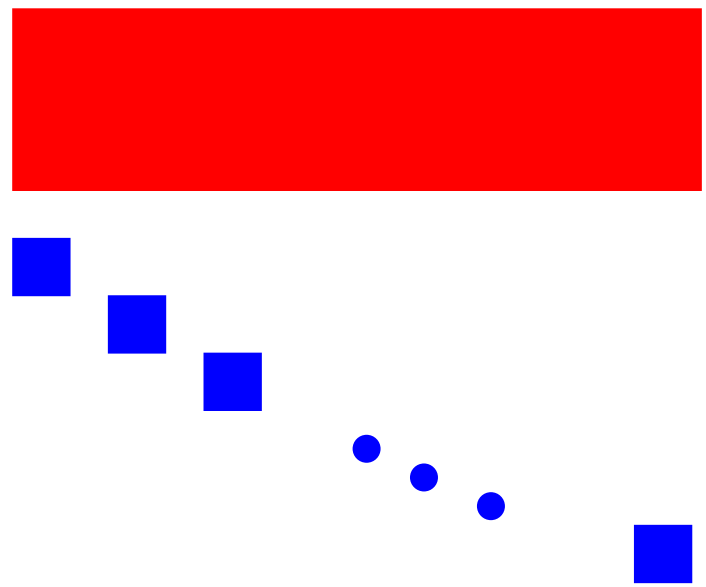
(interior point method)



On large, difficult problems:

- **Interior Point Methods** – based on polynomial-time algorithms. In practice: “few” iterations, punishingly slow per iteration, could require gigantic amounts of memory
- **Simplex-like methods** – no really satisfactory theoretical basis. In practice: astronomically many iterations, could require a lot of memory
- at least **ten years** have elapsed since last major speedup

Block-angular linear programs



Prototypical example: **routing problems**

A special case: **the maximum concurrent flow problem**:

Given a network with multicommodity demands and with capacities, route a maximum common percentage of all the demands without exceeding capacities.

Approximation algorithms

$\longrightarrow$ Given a tolerance $\mathbf{0 < \epsilon < 1}$, find a throughput of value at least $\mathbf{(1-\epsilon)}$ times the optimal

$\longrightarrow$ Do so with a ***fast*** algorithm with low memory requirements

- Shahrokhi and Matula (1989): running time at most $\boldsymbol{O(\epsilon^{-7})}$ (times polynomial)
- Plotkin, Shmoys, Tardos (1990): $\boldsymbol{\epsilon^{-3}}$
- Plotkin, Shmoys, Tardos; Grigoriadis and Khachiyan (1991): $\boldsymbol{\epsilon^{-2}}$
- $\cdots$
- Young; Garg and Könemann; Fleischer (1999); better methods, still $\boldsymbol{\epsilon^{-2}}$

$\star$ Old heuristic popular among EE crowd: the **"Flow Deviation Method"**
Fratta, Gerla, Kleinrock **(1971)**

Theorem: The flow deviation method requires $\boldsymbol{O(\epsilon^{-2})}$ iterations.
[Bienstock and Raskina, 2000]

But is it practical?

Experiments

Nodes	Vars	FD Time (sec.)	Cplex Dual Time (sec.)
484	1095606	746	20184
500	1019054	1024	33056
600	1569446	1303	68842
700	2131038	2103	227770
1000	4396604	5487	1734618

[**FD using** $\boldsymbol{\epsilon = 10^{-4}}$]

⋆ On-going work: parallel implementation to handle massively large cases (hundreds of millions of variables)

Basic Idea

- **Penalize** flows according to the degree that they utilize links
- **Reroute** flows according to penalties
- **Scale** flows (and throughput) whenever very slack utilization

Given a flow $\boldsymbol{f}$, $\boldsymbol{\lambda(f)}$ is the **highest link ufilization** by $\boldsymbol{f}$.

Algorith outline

A1. Let $\boldsymbol{f}$ be a strictly feasible flow, with throughput $\boldsymbol{\theta}$.

B1. Let $\boldsymbol{g} = \frac{1}{\lambda(f)} \boldsymbol{f}$.

C1. Find a feasible flow $\boldsymbol{h}$ with throughput $\frac{1}{\lambda(f)} \boldsymbol{\theta}$ and $\boldsymbol{\lambda(h)}$ substantially smaller than $\boldsymbol{\lambda(g)}$.

D1. Reset $\boldsymbol{f} \leftarrow \boldsymbol{h}$ and $\boldsymbol{\theta} \leftarrow \frac{1}{\lambda(f)} \boldsymbol{\theta}$, and go to **B1**.

$\longrightarrow$ To accomplish **C1**, reduce $\Sigma_e \frac{f_e}{u_e - f_e}$

$\longrightarrow$ This is done with a "gradient" method