On the Inefficiency of Equilibria in Congestion Games
Extended Abstract

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Abstract We present a short geometric proof for the price of anarchy results that have recently been established in a series of papers on selfish routing in multicommodity flow networks. This novel proof also facilitates two new types of results: On the one hand, we give pseudo-approximation results that depend on the class of allowable cost functions. On the other hand, we derive improved bounds on the inefficiency of Nash equilibria for situations in which the equilibrium travel times are within reasonable limits of the free-flow travel times. These tighter bounds help to explain empirical observations in vehicular traffic networks. Our analysis holds in the more general context of congestion games, which provides the framework in which we describe this work.

1 Introduction

Congestion games (Rosenthal 1973) are noncooperative games in which a player's strategy is to choose a subset of resources, and the utility of each player only depends on the number of players choosing the same or some overlapping strategy. In this paper, we consider a particular variant of *atomic* congestion games, Rosenthal's original setting where the number of players is finite, as well as *nonatomic* congestion games. Nonatomic games (Schmeidler 1973) model interactions involving a large number of players, each with a negligible ability to affect the others. Nonatomic congestion games have been studied, among others, by Milchtaich (2000, 2004), Chau and Sim (2003), and Roughgarden and Tardos (2004).

The most prominent example of a nonatomic congestion game is the traffic routing model of Wardrop (1952). The arcs in a given network represent the resources, the different origin-destination pairs correspond to the player types, and the strategies available to a particular player type are the paths in the network between its origin-destination pair. The cost of an arc describes the delay experienced by traffic traversing that arc as a function of the flow on that arc. A social optimum corresponds to a multicommodity flow of minimum total delay, while a Nash equilibrium equals a user equilibrium flow, where every player is traveling on a shortest path under the prevailing conditions.

Nash equilibria in general and user equilibria in particular are typically inefficient: They generally do not minimize the social cost. Koutsoupias and Papadimitriou (1999) proposed to analyze the inefficiency of equilibria from a worst-case perspective; this led to the notion of "price of anarchy" (Papadimitriou 2001), which is the ratio of the worst social cost of a Nash equilibrium to the cost of an optimal solution. In the context of selfish routing (i.e., the traffic model described in the previous paragraph), the price of anarchy was analyzed in a series of papers for increasingly more general classes of cost functions and model features; see, among others, Roughgarden and Tardos (2002), Roughgarden (2003), Schulz and Stier-Moses (2003), Chau and Sim (2003), Correa, Schulz, and Stier-Moses (2004), Roughgarden and Tardos (2004), Perakis (2004), and Roughgarden (2005).

In this paper, we give alternative proofs for most results in the above-mentioned papers. Our proofs simplify and unify previous arguments, and they enable us to extend these insights to more general settings. In Section 2, we study nonatomic congestion games with separable cost functions. For two known bounds on the inefficiency of Nash equilibria, we provide proofs that rely on a new interpretation of the parameter β introduced by Correa, Schulz, and Stier-Moses (2004) in the context of traffic routing. This interpretation sets the stage for various generalizations. In particular, we obtain new tight pseudo-approximation (sometimes also called bicriteria) results that depend on the class of allowable cost functions (Sections 2.1 and 2.2), and a more realistic bound on the price of anarchy for situations where the variable cost of a resource does not exceed its fixed cost by too much (Section 2.3). Section 2.4 considers costs functions that do not include fixed costs. We show that for polynomials of small degree the price of anarchy is significantly smaller than with arbitrary fixed costs. In Sections 3 and 4 we extend these results to atomic congestion games with divisible demands and to nonseparable cost functions, respectively.

2 Nonatomic Congestion Games with Separable Costs

In nonatomic congestion games, we are given a finite set A of resources, and k different types of players. Players are infinitesimal agents, and the continuum of players of type i is represented by the interval $[0, n_i]$. Each player type i possesses a set $\mathcal{S}_i$ of strategies, where each strategy consists of a subset of the resources. For notational convenience, we assume that the sets $\mathcal{S}_i$, $i = 1, 2, \ldots, k$, are disjoint, and we denote their union by $\mathcal{S}$. The rate of consumption of a resource $a \in S$ by a strategy $S \in \mathcal{S}_i$ is given by $r_{a,S} \geq 0$. Each player selects a strategy, which leads to a strategy distribution $x = (x_S)_{S \in \mathcal{S}}$ with $\sum_{S \in \mathcal{S}_i} x_S = n_i$ for each player type i, and $x_S \geq 0$ for all $S \in \mathcal{S}$. A strategy distribution generates a utilization rate $x_a := \sum_{i=1}^{k} \sum_{S\in\mathcal{S}_i : a\in S} r_{a,S}\, x_S$ for each resource $a \in A$. The cost $c_S(x)$ of using a strategy S is the sum of the costs of the resources associated with that strategy, amplified with the corresponding rate of consumption. The cost of resource $a \in A$ is given by a a nondecreasing and continuous function $c_a : \mathbb{R}_+^A \to \mathbb{R}_+$. Hence, $c_S(x) := \sum_{a\in S} r_{a,S}\, c_a(x)$. The social cost $C(x)$ of a strategy distribution x is defined as the total disutility experienced by the players: $C(x) := \sum_{i=1}^{k} \sum_{S\in\mathcal{S}_i} c_S(x)x_S = \sum_{a\in A} c_a(x)x_a$.[4]

A social optimum x^{OPT} is a strategy distribution of minimum social cost; i.e., $C(x^{\text{OPT}}) \le C(x)$ for all strategy distributions x. A strategy distribution x^{NE} is a Nash equilibrium when no player has an incentive to unilaterally change his or her strategy; i.e., $c_S(x^{\text{NE}}) \le c_{S'}(x^{\text{NE}})$ for any two strategies $S, S' \in \mathcal{S}_i$ with $x_S^{\text{NE}} > 0$, for each $i = 1, 2, \ldots, k$. It is well known (e.g., Dafermos and Sparrow (1969), Smith (1979)) that a strategy distribution x^{NE} is a Nash equilibrium if and only if it satisfies

$$\sum_{a\in A} c_a(x^{\text{NE}})(x_a^{\text{NE}} - x_a) \le 0 \text{ for all strategy distributions } x. \tag{1}$$

In this section we present different bounds on the inefficiency of equilibria in nonatomic congestion games with separable cost functions. The cost functions c_a are separable if $c_a(x) = c_a(x_a)$ for all $a \in A$. Beckmann, McGuire, and Winsten (1956) showed that in this case a Nash equilibrium x^{NE} exists, and any two different equilibria have the same social cost. While two of the bounds are known, at least in the context of selfish routing, we propose a different way to look at their proofs, which allows us to derive additional types of bounds.

2.1 Affine Cost Functions

Let us begin by studying the simplest case of nonatomic congestion games, namely that of separable and affine cost functions. In other words, the cost of resource $a \in A$ under the utilization rate vector x is $c_a(x) = c_a x_a + b_a$ for some nonnegative coefficients c_a, b_a. The most compact proof of the following result was presented in the context of selfish routing by Correa, Schulz, and Stier-Moses (2004), and our proof below can be viewed as a geometric variant of it. The result itself is due to Roughgarden and Tardos (2002, 2004).

Theorem 1 (Roughgarden and Tardos 2004). *Let x^{NE} be a Nash equilibrium of a nonatomic congestion game with separable, affine cost functions, and let x^{OPT} be a social optimum. Then, $C(x^{\text{NE}}) \le 4/3\, C(x^{\text{OPT}})$.*

Proof. Let x be an arbitrary strategy distribution. Because of (1), we have

$$C(x^{\text{NE}}) \le \sum_{a\in A} c_a(x_a^{\text{NE}})x_a = \sum_{a\in A} c_a(x_a)x_a + \sum_{a\in A} (c_a(x_a^{\text{NE}}) - c_a(x_a))x_a\,. \tag{2}$$

[4] Because of the equivalence of the strategy-by-strategy view and the resource-by-resource view of the social cost, we interchangeably use x to denote a strategy distribution and its associated vector of utilization rates.

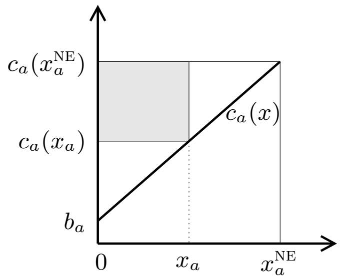

Figure 1. Illustration of the proof of Theorem 1

Since each function $c_a(\cdot)$ is nondecreasing, we only need to focus on the expressions $(c_a(x_a^{\mathrm{NE}})-c_a(x_a))x_a$ for which $x_a < x_a^{\mathrm{NE}}$ to bound the last term from above. In this case, $(c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a$ is equal to the area of the shaded rectangle in Figure 1. Note that the area of any rectangle whose upper-left corner point is $(0, c_a(x_a^{\mathrm{NE}}))$ and whose lower-right corner point lies on the line representing $c_a(x_a) = c_a x_a + b_a$, is at most half that of the triangle defined by the three points $(0, c_a(x_a^{\mathrm{NE}}))$, $(0, b_a)$, and $(x_a^{\mathrm{NE}}, c_a(x_a^{\mathrm{NE}}))$. In particular,

$$(c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a \leq \frac{1}{4}\, c_a(x_a^{\mathrm{NE}})x_a^{\mathrm{NE}} \;\; ,$$

which completes the proof. □

An immediate consequence of this proof is a pseudo-approximation result, which upper bounds the social cost of a Nash equilibrium by that of an optimal strategy distribution for the same game with more players of each type. More precisely, one only needs the following inequality derived in the preceding proof,

$$\sum_{a\in A} c_a(x_a^{\mathrm{NE}})x_a \leq C(x) + \frac{1}{4}C(x^{\mathrm{NE}}) \;\; , \tag{3}$$

which holds for *any* nonnegative vector x (i.e., x need not be a strategy distribution), to obtain this result:

Corollary 2. *If x^{NE} is a Nash equilibrium of a nonatomic congestion game with separable, affine cost functions and x^{OPT} is a social optimum for the same game with 5/4 times as many players of each type*[5]*, then $C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$.*

Proof. Let x be an arbitrary strategy distribution of the nonatomic congestion game with 5/4 times as many players as in the original game. Then,

$$C(x^{\mathrm{NE}}) = \frac{5}{4}\sum_{a\in A} c_a(x_a^{\mathrm{NE}})x_a^{\mathrm{NE}} - \frac{1}{4}\, C(x^{\mathrm{NE}}) \leq \sum_{a\in A} c_a(x_a^{\mathrm{NE}})x_a - \frac{1}{4}C(x^{\mathrm{NE}}) \leq C(x) \;\; .$$

[5] Formally, the continuum of players of type i in the new game is represented by the interval $[0, \frac{5}{4}n_i]$, $i = 1, 2, \ldots, k$.

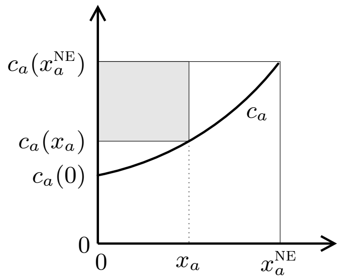

Figure 2. Illustration of the definition of β

The first inequality is implied by (1) and the feasibility of the vector $4\,x/5$ for the original game, while the second one follows from (3). $\square$

The first result of this kind was given by Roughgarden and Tardos (2002), who showed that an equilibrium traffic assignment causes a total travel time of at most that of a social optimum routing twice as much traffic. This result and its subsequent extension to general nonatomic congestion games (Roughgarden and Tardos 2004) hold for arbitrary (separable) cost functions. The selfish routing version of Corollary 2 is due to Chakrabarty (2004) and inspired us to qualify the pseudo-approximation bounds according to the class of cost functions considered; see the next section for details.

2.2 General Cost Functions

Note that we used the linearity of the cost functions $c_a(\cdot)$ in only one place, namely when we proved (3). Hence, a suited generalization of (3) is the key for extending the results in Theorem 1 and Corollary 2 to more general classes of cost functions. For arbitrary (but still separable) cost functions, we are confronted with the situation depicted in Figure 2. To upper bound the area of the shaded rectangle, i.e., $(c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a$, in terms of the area of the large rectangle, which is of size $c_a(x_a^{\mathrm{NE}})x_a^{\mathrm{NE}}$, we simply define for each cost function $c_a(\cdot)$ and nonnegative scalar $v_a \geq 0$,

$$\beta(c_a, v_a) := \max_{x_a \geq 0} \frac{(c_a(v_a) - c_a(x_a))x_a}{c_a(v_a)v_a} \ .$$

Here, $0/0 = 0$ by convention. Note that $0 \leq \beta(c_a, v_a) \leq 1$. For a given class $\mathcal{C}$ of cost functions (e.g., polynomials of a certain degree), we let

$$\beta(\mathcal{C}) := \sup_{c_a \in \mathcal{C}, v_a \geq 0} \beta(c_a, v_a) \ .$$

This definition leads directly to the following lemma.

Lemma 3. *Let x^{NE} be a Nash equilibrium of a nonatomic congestion game with separable cost functions drawn from a given class $\mathcal{C}$, and let x be an arbitrary nonnegative vector. Then,*

$$\sum_{a \in A} c_a(x_a^{\mathrm{NE}}) x_a \leq C(x) + \beta(\mathcal{C}) C(x^{\mathrm{NE}}) \ .$$

In turn, Lemma 3 yields the following generalizations of Theorem 1 and Corollary 2, with virtually no change in proof except for the replacement of (3) with Lemma 3.

Theorem 4. *Let x^{NE} be a Nash equilibrium of a nonatomic congestion game with separable cost functions drawn from a given class $\mathcal{C}$.*

(a) *If x^{OPT} is a social optimum of this game, then $C(x^{\mathrm{NE}}) \leq (1-\beta(\mathcal{C}))^{-1} C(x^{\mathrm{OPT}})$.*
(b) *If x^{OPT} is a social optimum of the same game with $1 + \beta(\mathcal{C})$ times as many players of each type, then $C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$.*

The parameter $\beta(\mathcal{C})$ was first defined in the context of selfish routing by Correa, Schulz, and Stier-Moses (2004), but without mentioning the motivation offered by Figure 2. It is related to the anarchy value $\alpha(\mathcal{C})$ of Roughgarden (2003) (in the context of selfish routing) and Roughgarden and Tardos (2004), but $\alpha(\mathcal{C})$ is only defined for cost functions $c_a(\cdot)$ that are differentiable and for which $c_a(x_a)x_a$ is convex. If $\alpha(\mathcal{C})$ exists, then $\alpha(\mathcal{C}) = (1 - \beta(\mathcal{C}))^{-1}$. Concrete values of $\beta(\mathcal{C})$ can readily be calculated for several classes of cost functions; see Roughgarden (2003) and Correa, Schulz, and Stier-Moses (2004) as well as Figure 4 (a) and Table 1 below.

2.3 Cost Functions with Limited Congestion Effects

To motivate our next set of results, it is helpful to consider selfish routing in traffic networks. The empirically observed ratio of the total travel time of a user equilibrium to that of a system optimum is typically significantly smaller than predicted by the price-of-anarchy results of Theorem 4. For instance, in the computational studies of Jahn, Möhring, Schulz, and Stier-Moses (2005) the largest ratio of user equilibrium cost to system optimum cost over several realistic instances is 1.15 (instead of the theoretical worst case of 2.151). Qiu et al. (2003) made a similar observation in the context of telecommunication networks. For a given class of latency functions, the corresponding price of anarchy is a worst-case measure, taken over all possible instances. However, if one compares the time needed to drive to work during rush hour with the duration of the same trip at night, then the "free-flow travel time" is usually not a negligible fraction of the rush-hour experience. Formally, we make the following assumption: The cost of any given resource at utilization rate zero is at least a constant fraction of that of the same resource at the utilization rate in equilibrium. A different way of illustrating this assumption is by looking at a production process, where the fixed cost of any resource accounts for a substantial fraction of the total cost

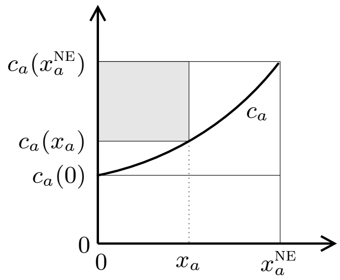

Figure 3. Illustration of the proof of Lemma 6

(fixed plus variable costs) in equilibrium. Depending on the ratio of fixed to total cost, we obtain a parametrized sequence of improved bounds on the inefficiency of equilibria.

Theorem 5. *Let x^{NE} be a Nash equilibrium of a nonatomic congestion game with separable cost functions drawn from a given class $\mathcal{C}$ such that $c_a(0) \geq \eta\, c_a(x_a^{\mathrm{NE}})$ for all $a \in A$, for some constant $0 \leq \eta \leq 1$.*[6]

(a) *If x^{OPT} is a social optimum, then $C(x^{\mathrm{NE}}) \leq (1-(1-\eta)\beta(\mathcal{C}))^{-1}\, C(x^{\mathrm{OPT}})$.*
(b) *If x^{OPT} is a social optimum of the same game with $1+(1-\eta)\beta(\mathcal{C})$ times as many players of each type, then $C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$.*

The proof of Theorem 5 rests upon the following generalization of Lemma 3.

Lemma 6. *Let x^{NE} be a Nash equilibrium of a nonatomic congestion game with separable cost functions drawn from a given class $\mathcal{C}$ such that $c_a(0) \geq \eta\, c_a(x_a^{\mathrm{NE}})$ for all $a \in A$, for some constant $0 \leq \eta \leq 1$. Moreover, let x be an arbitrary nonnegative vector. Then,*

$$\sum_{a \in A} c_a(x_a^{\mathrm{NE}}) x_a \leq C(x) + (1-\eta)\beta(\mathcal{C}) C(x^{\mathrm{NE}}) \ .$$

Proof. We again rewrite $c_a(x_a^{\mathrm{NE}})x_a$ as $c_a(x_a)x_a + (c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a$. If $x_a \geq x_a^{\mathrm{NE}}$, there is nothing left to prove. So assume $x_a < x_a^{\mathrm{NE}}$, and consider Figure 3. Because $c_a(0) \geq \eta\, c_a(x_a^{\mathrm{NE}})$, the area $(c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a$ of the small shaded rectangle is at most $\beta(\mathcal{C})$ times that of the rectangle with upper-left corner point $(0, c_a(x_a^{\mathrm{NE}}))$ and lower-right corner point $(x_a^{\mathrm{NE}}, c_a(0))$, which is of size at most $(1-\eta)c_a(x_a^{\mathrm{NE}})x_a^{\mathrm{NE}}$. The result follows. □

Figure 4 (a) displays the relationship between η and the bound on the price of anarchy given by Theorem 5 (a) for polynomials of different degrees. Note that

[6] We also assume that $\mathcal{C}$ is closed under adding constants $b \in \mathbb{R}$ for which $c(0)+b \geq 0$; otherwise, the resulting improvement would not be directly visible but would remain hidden in the value of $\beta(\mathcal{C})$.

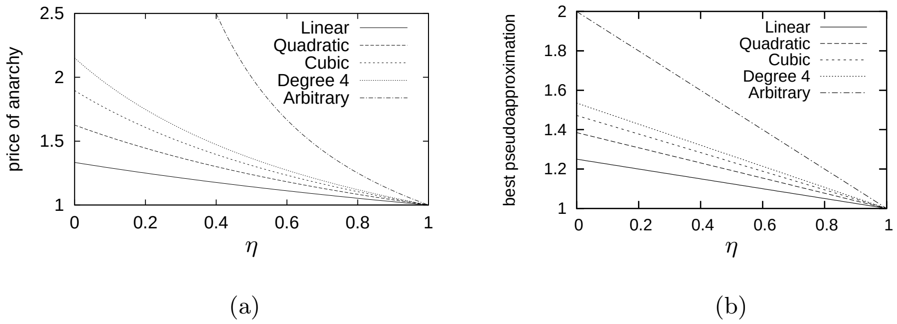

Figure 4. (a) Price of anarchy as a function of fixed to total costs. (b) Minimum value for which the pseudo-approximation result holds. Each curve refers to a set $\mathcal{C}$ that contains nonnegative polynomials of fixed degree.

the price of anarchy is at most $1/\eta$, even if we do not place any restriction on $\mathcal{C}$. This observation qualifies the unboundedness of the price of anarchy for instances with general cost functions described by Roughgarden and Tardos (2002, 2004). As another example, consider a vehicular network in which users travel at most twice as long when the network is congested compared to the situation when it is not. If arc latencies are modeled as polynomials of degree 4, as it is the case for the widely used Bureau of Public Roads' functions (1964), Theorem 4 (a) gives an upper bound of 2.151 on the price of anarchy. However, Theorem 5 (a) gives a more accurate bound of 1.365. More generally, we believe that the bounds presented in Theorem 5 offer a good explanation of the satisfactory performance of Nash equilibria in many practical situations. Figure 4 (b) illustrates part (b) of Theorem 5.

Let us also point out that the bounds given in Theorem 5 are tight. Consider the traffic assignment instance in Figure 5, where v units of flow must be routed from one node to the other over two parallel arcs a and a'. The arc latencies are $c_a(x_a) = c(v)$ (a constant) and $c_{a'}(x_{a'}) = \eta\, c(v) + (1-\eta)c(x_{a'})$, respectively. Here, the function c and the scalar v are chosen such that $\beta(\mathcal{C}) = \beta(c, v)$. Both bounds given in Theorem 5 are simultaneously tight for this instance.

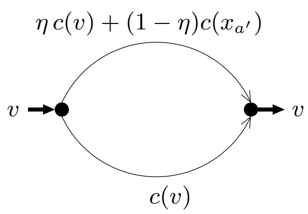

Figure 5. Instance for which the bounds in Theorem 5 are tight

2.4 Cost Functions without Fixed Costs

In contrast to the preceding section, we now consider instances where the cost of each resource at utilization rate zero is equal to zero. This model helps to capture situations in which variable costs dominate fixed costs or where fixed costs can be neglected altogether. Examples include telecommunication networks where the propagation delays are usually negligible compared to the queueing delays in the routers, and production systems where the equipment has been installed already. We give the first results of this kind; in particular, the price of anarchy for small-degree polynomials in this setting is smaller than the bounds given in Theorem 4. Polynomials are of particular interest in several application contexts. For instance, the Bureau of Public Roads' cost functions used in traffic networks are polynomials of degree 4.

Dafermos and Sparrow (1969) showed that when all cost functions are monomials of the same degree, Nash equilibria and system optima coincide. In particular, the price of anarchy is 1 if we only allow linear functions (affine functions without constant term) instead of 4/3, the value for the set of all affine functions. Before we show that a similar effect can be observed for polynomials of (somewhat) higher degree, let us introduce some notation to facilitate the proofs. We represent the cost functions as $c_a(x_a) := \sum_{1 \leq i \leq d} c_{a,i}\, x_a^i$ whenever we consider polynomials of degree d. Moreover, we denote the total cost of a strategy distribution x when resources are utilized according to another vector x' by $C^{x'}(x) := \sum_{a \in A} c_a(x'_a) x_a$. Also, we denote the total cost restricted to the degree i monomials by $C_i(x) := \sum_{a \in A} c_{a,i}\, x_a^{i+1}$ and $C_i^{x'}(x) := \sum_{a \in A} c_{a,i}\, x_a\, (x'_a)^i$, respectively. Using this notation, we can rewrite (1) as $C(x^{\text{NE}}) \leq C^{x^{\text{NE}}}(x)$, which is valid for any strategy distribution x and Nash equilibrium x^{NE}. The following lemma is well known in the context of variational inequalities; see, e.g., Smith (1979).

Lemma 7. *If x^{NE} is a Nash equilibrium and x is an arbitrary strategy distribution of a nonatomic congestion game, then $C^x(x^{\text{NE}}) \leq C(x)$.*

With the notation and preliminary results in place, we are prepared to bound the price of anarchy in networks without fixed costs.

Theorem 8. *Consider the family of nonatomic congestion games with separable cost functions that are polynomials of degree at most d, for $1 \leq d \leq 4$. Then, the price of anarchy for this class of games is bounded from above and below by the values displayed in Table 1.*

Due to space limitations, we restrict ourselves to prove the upper bound of 5/4 for cubic functions.

Proof. We use two simple bounds: $(\delta x_a - x_a^{\text{NE}})^2 \geq 0$ with $\delta \in \mathbb{R}$, which implies that $C_1^x(x^{\text{NE}}) \leq C_1(x^{\text{NE}}) + \frac{1}{4} C_1(x)$ and $C_2^{x^{\text{NE}}}(x) \leq \frac{1}{2} C_2^x(x^{\text{NE}}) + \frac{1}{2} C_2(x^{\text{NE}})$, and $(\frac{1}{2} x_a^2 - (x_a^{\text{NE}})^2 + x_a\, x_a^{\text{NE}})^2 \geq 0$, which implies that $C_3^{x^{\text{NE}}}(x) \leq \frac{1}{2} C_3^x(x^{\text{NE}}) + \frac{1}{2} C_3(x^{\text{NE}}) + \frac{1}{8} C_3(x)$. Now,

$$C(x^{\text{NE}}) \leq C^{x^{\text{NE}}}(x) = C_3^{x^{\text{NE}}}(x) + C_2^{x^{\text{NE}}}(x) + C_1^{x^{\text{NE}}}(x)$$

$$\leq \frac{C_3^x(x^{\text{NE}})}{2} + \frac{C_3(x^{\text{NE}})}{2} + \frac{C_3(x)}{8} \quad + \quad \frac{C_2^x(x^{\text{NE}})}{2} + \frac{C_2(x^{\text{NE}})}{2} \quad + \frac{C_1^x(x^{\text{NE}})}{2} + \frac{C_1(x^{\text{NE}})}{2} + \frac{C_1(x)}{8}$$

$$\leq \frac{C(x^{\text{NE}})}{2} + \frac{C^x(x^{\text{NE}})}{2} + \frac{C(x)}{8} \leq \frac{1}{2}C(x^{\text{NE}}) + \frac{5}{8}C(x) \ . \qquad \square$$

Table 1. Comparison of guarantees for the efficiency of equilibria with and without fixed costs. All coefficients are assumed to be nonnegative. The column entitled '$c_a(0) = 0$' shows lower and upper bounds on the price of anarchy without fixed costs. The lower bounds represent the worst-case coordination ratio of a Pigou-type instance with two parallel arcs, cost functions in $\mathcal{C}$, and no fixed costs. The column entitled '$c_a(0)$ arbitrary' shows the exact price of anarchy with arbitrary fixed costs.

Set $\mathcal{C}$ of allowable cost functions	Example	Price of Anarchy $\alpha(\mathcal{C})$		
		$c_a(0) = 0$		$c_a(0)$ arbitrary
		LB	UB	
linear functions	$a_1x + a_0$	1	1	1.334
quadratic functions	$a_2x^2 + a_1x + a_0$	1.035	1.185	1.626
cubic functions	$a_3x^3 + a_2x^2 + a_1x + a_0$	1.098	1.25	1.896
polynomials of degree 4	$\sum_{i=0}^4 a_i x^i$	1.167	1.999	2.151

3 Weighted Congestion Games with Divisible Demands

Rosenthal's (1973) original version of congestion games has a finite number of players, and each player controls one unit of demand that cannot be split across strategies. While he showed that a pure-strategy Nash equilibrium always exists, this is not necessarily the case for weighted congestion games, where players can have arbitrary demands (Fotakis et al. 2004). We consider a hybrid between weighted congestion games and nonatomic congestion games, which is guaranteed to possess a pure-strategy Nash equilibrium. Moreover, the price of anarchy can be bounded in similar terms to the results discussed above.

An *atomic weighted congestion game with divisible demands* is similar to a nonatomic congestion game, except that there is a finite number of players $1, 2, \ldots, k$. Each player i has a divisible demand $n_i > 0$ and assigns portions $x_S \geq 0$ of it to various strategies S in his or her strategy space $\mathcal{S}_i$ such that $\sum_{S \in \mathcal{S}_i} x_S = n_i$. For a given strategy distribution $x = (x_S)_{S \in \mathcal{S}}$, the cost to player i is $C_i(x) := \sum_{S \in \mathcal{S}_i} c_S(x) x_S$. The social cost $C(x)$ is equal to the total cost; i.e., $C(x) := \sum_{i=1}^k C_i(x) = \sum_{a \in A} c_a(x_a) x_a$, where $x_a = \sum_{i=1}^k \sum_{a \in S \in \mathcal{S}_i} r_{a,S} x_S$ is the utilization rate of resource a. A strategy distribution x constitutes a Nash equilibrium if, for each player i, $(x_S)_{S \in \mathcal{S}_i}$ minimizes

$C_i(x)$, keeping the actions of the other players fixed. This model provides an abstract framework for the "atomic splittable selfish routing problem" considered in previous papers. Catoni and Pallotino (1991) presented instances for which the ratio of the equilibrium cost to the optimal cost is smaller for the nonatomic version than its atomic splittable counterpart, while Roughgarden (2005) proved that the price of anarchy for any given class of allowable cost functions is always dominated by that of nonatomic games. Harker (1988) and Orda, Rom, and Shimkin (1993) established the existence of pure-strategy Nash equilibria, and Roughgarden and Tardos (2002) showed that their pseudo-approximation results for nonatomic selfish routing games also holds for the atomic splittable case.

Although the variational inequality (1) does in general not hold for Nash equilibria of atomic games with divisible demands, it actually suffices to prove that (2) is still true in order to derive results similar to Theorems 4 and 5. Consider a Nash equilibrium x^{NE} and define, for $a \in A$, $\bar{c}_a(x_a) := c_a(x_a^{\mathrm{NE}})$ if $x_a \leq x_a^{\mathrm{NE}}$ and $\bar{c}_a(x_a) := c_a(x_a)$, otherwise. It is straightforward to extend a result in the proof of Roughgarden and Tardos (2002, Theorem 5.4) to characterize Nash equilibria in atomic congestion games with divisible demands:

Lemma 9 (Roughgarden and Tardos 2002). *Let x^{NE} be a Nash equilibrium of an atomic congestion game with divisible demands. Furthermore, assume that the functions $c_a(x_a)x_a$ are convex, for all $a \in A$. Then, $\sum_{a\in A} \bar{c}_a(x_a^{\mathrm{NE}})x_a^{\mathrm{NE}} \leq \sum_{a\in A} \bar{c}_a(x_a)x_a$ for all strategy distributions x.*

Using slightly weaker versions of (2) and Lemma 3, we can bound the price of anarchy of atomic congestion games with divisible demands.

Theorem 10. *Let x^{NE} be a Nash equilibrium of an atomic congestion game with divisible demand and separable cost functions drawn from a given class $\mathcal{C}$. Assume that $c_a(x_a)x_a$ is a convex function, for all $c_a \in \mathcal{C}$.*

(a) *If x^{OPT} is a social optimum of this game, then $C(x^{\mathrm{NE}}) \leq (1-\beta(\mathcal{C}))^{-1}\, C(x^{\mathrm{OPT}})$.*
(b) *If x^{OPT} is a social optimum of the same game with $1+\beta(\mathcal{C})$ times as many players of each type, then $C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$.*

Proof. Lemma 9 implies that the cost $C(x^{\mathrm{NE}})$ of a Nash equilibrium is bounded from above by $\sum_{a\in A} \bar{c}_a(x_a)x_a$ for any strategy distribution x. Let us consider the terms $\bar{c}_a(x_a)x_a$ individually. If $x_a \geq x_a^{\mathrm{NE}}$, then $\bar{c}_a(x_a)x_a = c_a(x_a)x_a$, and we are done. Otherwise, $\bar{c}_a(x_a)x_a = c_a(x_a)x_a + (c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a$. Hence, $C(x^{\mathrm{NE}}) \leq \sum_{a\in A} \bar{c}_a(x_a)x_a \leq C(x) + \beta(\mathcal{C})C(x^{\mathrm{NE}})$, which shows part (a). For part (b), notice that $(1+\beta(\mathcal{C}))^{-1}x^{\mathrm{OPT}}$ is feasible for the original problem. Therefore,

$$C(x^{\mathrm{NE}}) \leq \sum_{a\in A} \bar{c}_a\left(\frac{x_a^{\mathrm{OPT}}}{1+\beta(\mathcal{C})}\right)\frac{x_a^{\mathrm{OPT}}}{1+\beta(\mathcal{C})} \leq \frac{1}{1+\beta(\mathcal{C})}\sum_{a\in A} \bar{c}_a(x_a^{\mathrm{OPT}})x_a^{\mathrm{OPT}},$$

where the second inequality follows from the monotonicity of the cost functions. Now, we can proceed as in Corollary 2:

$$C(x^{\mathrm{NE}}) = (1+\beta(\mathcal{C}))C(x^{\mathrm{NE}}) - \beta(\mathcal{C})\,C(x^{\mathrm{NE}}) \leq \sum_{a\in A} \bar{c}_a(x_a^{\mathrm{OPT}})x_a^{\mathrm{OPT}} - \beta(\mathcal{C})\,C(x^{\mathrm{NE}}) \leq C(x)\,.$$

□

Part (a) of Theorem 10 extends and simplifies a result of Roughgarden (2005) who proved a similar bound for network games. Part (b) is new. Using the arguments in the proof of Theorem 10, it is straightforward to extend Theorem 5 to atomic games.

4 Nonatomic Games with Nonseparable Costs

In some practical situations, the cost of using one resource may depend on the rate of consumption of others. For instance, the time a vehicle needs to cross through a stop sign clearly depends on the amount of flow traversing the perpendicular street; the waiting time of passengers at a given bus stop depends on the number of passengers boarding the bus at previous stops; or, to give an example in the context of wireless communication networks, transmission delays depend on the load of neighboring cells, because of interference.

The purpose of this section is to extend our results on nonatomic congestion games to the case where cost functions are not necessarily separable. In this context it is convenient to write the social cost of a strategy distribution x as an inner product, $C(x) = \langle c(x), x\rangle$, with $c : X \to \mathbb{R}_+^A$ continuous. Here, X denotes the convex and compact space of feasible utilization vectors. (In particular, X can be used to model side constraints such as resource capacities; see Correa, Schulz, and Stier-Moses (2004) for a discussion.) In the spirit of Equation (1), a strategy distribution x^{NE} is an equilibrium if it satisfies the variational inequality

$$\langle c(x^{\text{NE}}), x^{\text{NE}} - x\rangle \leq 0 \quad \text{for all } x \in X. \tag{4}$$

Moreover, a social optimum x^{OPT} is an optimal solution to $\min_{x\in X}\langle c(x), x\rangle$. Under the continuity of c as well as the compactness and convexity of X, an equilibrium exists by the classic result of Hartman and Stampacchia (1966) (see also Smith 1979). Of course, a system optimum also exists as c is continuous and X is compact.

For a cost function c and a utilization vector $v \in \mathbb{R}_+^A$, a natural extension of the parameter β is

$$\beta(c, v) := \max_{x\geq 0} \frac{\langle c(v) - c(x), x\rangle}{\langle c(v), v\rangle} \ .$$

With the definition of $\beta(\mathcal{C}) := \sup_{c\in\mathcal{C}, v\in X} \beta(c, v)$, we can extend Lemma 3 to nonseparable cost functions, thereby simplifying and extending earlier work. Chau and Sim (2003) proved that the price of anarchy for nonseparable and symmetric cost functions is bounded by a natural extension of the parameter $\alpha(\mathcal{C})$ of Roughgarden and Tardos (see Section 2.2). Perakis (2004) considered general nonseparable cost functions. Let us mention that the known bounds require stronger assumptions on the cost functions, such as convexity, differentiability, and monotonicity.

Lemma 11. *Let x^{NE} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}$ of nonseparable cost functions, and let x be a nonnegative vector. Then, $\langle c(x^{\text{NE}}), x\rangle \leq C(x) + \beta(\mathcal{C})C(x^{\text{NE}})$.*

This lemma yields the following price-of-anarchy and pseudo-approximation results for nonatomic congestion games with nonseparable cost functions.

Theorem 12. *Let x^{NE} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}$ of nonseparable cost functions.*

(a) *If x^{OPT} is a social optimum for this game, then $C(x^{\mathrm{NE}}) \leq (1-\beta(\mathcal{C}))^{-1} C(x^{\mathrm{OPT}})$.*
(b) *If x^{OPT} is a social optimum for the same game with $1+\beta(\mathcal{C})$ times as many players of each type, then $C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$.*

Proof. For (a), it suffices to use (4) and Lemma 11:

$$C(x^{\mathrm{NE}}) = \langle c(x^{\mathrm{NE}}), x^{\mathrm{NE}} \rangle \leq \langle c(x^{\mathrm{NE}}), x^{\mathrm{OPT}} \rangle \leq C(x^{\mathrm{OPT}}) + \beta(\mathcal{C}) C(x^{\mathrm{NE}}) \ .$$

Let us now prove part (b). Because of the feasibility of $(1+\beta(\mathcal{C}))^{-1}\, x^{\mathrm{OPT}}$ for the original game, we have that $\langle c(x^{\mathrm{NE}}), x^{\mathrm{NE}} \rangle \leq \langle c(x^{\mathrm{NE}}), (1+\beta(\mathcal{C}))^{-1}\, x^{\mathrm{OPT}} \rangle$. Therefore,

$$\begin{aligned}
\langle c(x^{\mathrm{NE}}), x^{\mathrm{NE}} \rangle &= (1+\beta(\mathcal{C}))\langle c(x^{\mathrm{NE}}), x^{\mathrm{NE}} \rangle - \beta(\mathcal{C})\langle c(x^{\mathrm{NE}}), x^{\mathrm{NE}} \rangle \\
&\leq (1+\beta(\mathcal{C}))\langle c(x^{\mathrm{NE}}), (1+\beta(\mathcal{C}))^{-1} x^{\mathrm{OPT}} \rangle - \beta(\mathcal{C})\langle c(x^{\mathrm{NE}}), x^{\mathrm{NE}} \rangle \\
&\leq C(x^{\mathrm{OPT}}) + \beta(\mathcal{C})C(x^{\mathrm{NE}}) - \beta(\mathcal{C})C(x^{\mathrm{NE}}) \\
&= C(x^{\mathrm{OPT}}) \ .
\end{aligned}$$

□

A particular class of nonseparable cost functions that has been studied before are affine functions; i.e., $c(x) = Ax + b$, with $b \geq 0$, and A symmetric and positive semidefinite. Theorem 12 provides a simple proof of a result by Chau and Sim (2003), which established that the price of anarchy for this kind of cost functions is at most 4/3. Indeed, in this case

$$\beta(c,v) = \max_{x \geq 0} \frac{\langle c(v) - c(x), x \rangle}{\langle c(v), v \rangle} = \frac{\max_{x \geq 0} \langle A(v-x), x \rangle}{\langle Av, v \rangle + \langle b, v \rangle} \ .$$

As A is symmetric and positive semidefinite, the numerator amounts to a convex minimization problem, and the optimum is attained at $x = v/2$, leading to $\beta(\text{affine costs}) = 1/4$. Theorem 12 yields $C(x^{\mathrm{NE}}) \leq 4/3\, C(x^{\mathrm{OPT}})$, where x^{OPT} is a social optimum for this game. Moreover, $C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$, for a social optimum x^{OPT} of the same game with 5/4 times as many players of each type.

Let us finally note that the improved results for games with limited congestion (i.e., Theorem 5) also hold in this setting. Indeed, we only need to generalize Lemma 6. Similarly to Section 2.3, we assume that $\mathcal{C}$ satisfies that for $c \in \mathcal{C}$ and a constant vector $b \in \mathbb{R}^A$ for which $c(x) + b \in \mathbb{R}^A_+$ for all $x \in \mathbb{R}^A_+$, $c(x) + b \in \mathcal{C}$.

Lemma 13. *Let x^{NE} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}$ such that $c(0) \geq \eta\, c(x^{\mathrm{NE}})$ for $c \in \mathcal{C}$ and $0 \leq \eta \leq 1$. If x is a nonnegative vector, then $\langle c(x^{\mathrm{NE}}), x \rangle \leq C(x) + (1-\eta)\beta(\mathcal{C})C(x^{\mathrm{NE}})$.*

Proof. Let us write $c \in \mathcal{C}$ as $c(x) = M(x) + b$, where $b = c(0) \geq \eta\, c(x^{\text{NE}}) \geq 0$. Thus,

$$\beta(c, x^{\text{NE}}) = \max_{x \geq 0} \frac{\langle c(x^{\text{NE}}) - c(x), x\rangle}{\langle c(x^{\text{NE}}), x^{\text{NE}}\rangle} \leq \max_{x \geq 0} \frac{\langle M(x^{\text{NE}}) - M(x), x\rangle}{\langle M(x^{\text{NE}}), x^{\text{NE}}\rangle + \frac{\eta}{1-\eta}\langle M(x^{\text{NE}}), x^{\text{NE}}\rangle}$$
$$= (1-\eta) \max_{x \geq 0} \frac{\langle M(x^{\text{NE}}) - M(x), x\rangle}{\langle M(x^{\text{NE}}), x^{\text{NE}}\rangle} \leq (1-\eta)\beta(\mathcal{C}) \ .$$

So, $\langle c(x^{\text{NE}}), x\rangle \leq \langle c(x), x\rangle + \beta(c, x^{\text{NE}})\langle c(x^{\text{NE}}), x^{\text{NE}}\rangle \leq C(x) + (1-\eta)\beta(\mathcal{C})C(x^{\text{NE}})$. □

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