# Ranking sets of objects using the Shapley value and other regular semivalues

Stefano Moretti, Alexis Tsoukiàs

Laboratoire d'Analyse et Modélisation de Systémes pour l'Aide à la DEcision (Lamsade)
CNRS UMR7243, Paris Dauphine University

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# Summary

Preferences over sets

Properties that prevent the interaction

Alignment with regular semivalues

Interaction among objects

# Two recent papers

- Moretti S., Tsoukias A. (2012). Ranking Sets of Possibly Interacting Objects Using Shapley Extensions. In *Thirteenth International Conference on the Principles of Knowledge Representation and Reasoning* (KR2012).

- Lucchetti R., Moretti S., Patrone F. (2012) A probabilistic approach to ranking sets of interacting objects, in progress.

# Central question

How to derive a ranking over the set of all subsets of $N$ in a way that is "compatible" with a primitive ranking over the single elements of $N$?

- Relevant number of papers focused on the problem of deriving a preference relation on the power set of $N$ from a preference relation over single objects in $N$. Most of them provide an axiomatic approach (Kannai and Peleg (1984), Barbera et al (2004), Bossert (1995), Fishburn (1992), Roth (1985) etc.)

- **Extension axiom**: Given a total preorder $\succcurlyeq$ on $N$, we say that a total preorder $\sqsupseteq$ on $2^N$ is an *extension* of $\succcurlyeq$ if and only if for each $x, y \in N$,

$$\{x\} \sqsupseteq \{y\} \Leftrightarrow x \succcurlyeq y$$

# Well-known properties prevent interaction

**Axiom** [Responsiveness, RESP] A total preorder $\sqsupseteq$ on $2^N$ satisfies the *responsiveness* property iff for all $A \in 2^N \setminus \{N, \emptyset\}$, for all $x \in A$ and for all $y \in N \setminus A$ the following conditions holds

$$A \sqsupseteq (A \setminus \{x\}) \cup \{y\} \Leftrightarrow \{x\} \sqsupseteq \{y\}$$

■

- This axiom was introduced by Roth (1985) studying colleges' preferences for the "college admission problem" (see also Gale and Shapley (1962)).

- Bossert (1995) used the same property for ranking sets of alternatives with a fixed cardinality and to characterize the class of *rank-ordered lexicographic* extensions.

# Well-known extensions prevent interaction

Most of the axiomatic approaches from the literature make use of the RESP axiom to prevent any kind of interaction among the objects in $N$.:

- max and min extensions (Kreps 1979, Barberà, Bossert, and Pattanaik 2004)

- lexi-min and lexi-max extensions (Holzman 1984, Pattanaik and Peleg 1984)

- median-based extensions (Nitzan and Pattanaik 1984)

- rank-ordered lexicographic extensions (Bossert 1995)

- many others...

# Basic-Basic on coalitional games

A *coalitional game* (many names...) is a pair $(N, v)$, where $N$ denotes the finite set of *players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$.

Given a game, a regular semivalue (see Dubey et al. 1981, Carreras and Freixas 1999; 2000) may be computed to convert information about the worth that coalitions can achieve into a personal attribution (of payoff) to each of the players:

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subset N : i \notin S} p_s\big( v(S \cup \{i\}) - v(S)\big)$$

for each $i \in N$, where $p_s$ represents the probability that a coalition $S \in 2^N$ (of cardinality $s$) with $i \notin S$ forms. So coalitions of the same size have the same probability to form!

(of course $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$, but we also assume $p_s > 0$.)

# Shapley and Banzhaf regular semivalues

- The Shapley value (Shapley 1953) is a regular semivalue $\pi^{\hat{\mathbf{p}}}(v)$, where

$$\hat{p}_s = \frac{1}{n\binom{n-1}{s}} = \frac{s!(n-s-1)!}{n!}$$

for each $s = 0, 1, \ldots, n-1$ (i.e., the cardinality is selected with the same probability).

- Another very well studied probabilistic value is the *Banzhaf value* (Banzhaf III 1964), which is defined as the regular semivalue $\pi^{\tilde{\mathbf{p}}}(v)$, where

$$\tilde{p}_s = \frac{1}{2^{n-1}}$$

for each $s = 0, 1, \ldots, n-1$, (i.e., each coalition has an equal probability to be chosen)

# $\pi^{\mathbf{p}}$-aligned total preorders

Given a total preorder $\sqsupseteq$ on $2^N$, we denote by $V(\sqsupseteq)$ the class of coalitional games that numerically represent $\sqsupseteq$ (for each $S, V \in 2^N$, $S \sqsupseteq V \Leftrightarrow u(S) \geq u(V)$ for each $u \in V(\sqsupseteq)$).

**DEF.** Let $\pi^{\mathbf{p}}$ be a regular semivalue. A total prorder $\sqsupseteq$ on $2^N$ is *$\pi^{\mathbf{p}}$-aligned* iff for each numerical representation $v \in V(\sqsupseteq)$ we have that

$$\{i\} \sqsupseteq \{j\} \Leftrightarrow \pi^{\mathbf{p}}_i(v) \geq \pi^{\mathbf{p}}_j(v)$$

for all $i, j \in N$. ■

Here we use regular semivalues to impose a constraint to the possibilities of interaction among objects: complementarities or redundancy are possible but, globally, their effects cannot overwhelm the limitation imposed by the original ranking.

# Example: Shapley-aligned total preorder...

For each coalitional game $v$, the Shapley value is denoted by $\phi(v) = \pi^{\hat{\mathbf{p}}}(v)$.
Let $N = \{1, 2, 3\}$ and let $\sqsupseteq^a$ be a total preorder on $N$ such that $\{1, 2, 3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1, 3\} \sqsupset^a \{2, 3\} \sqsupset^a \{1\} \sqsupset^a \{1, 2\} \sqsupset^a \emptyset$.

For every $v \in V(\sqsupseteq^a)$

$$\phi_2(v) - \phi_1(v) = \frac{1}{2}\big(v(2) - v(1)\big) + \frac{1}{2}\big(v(2, 3) - v(1, 3)\big) > 0$$

On the other hand

$$\phi_3(v) - \phi_2(v) = \frac{1}{2}\big(v(3) - v(2)\big) + \frac{1}{2}\big(v(1, 3) - v(1, 2)\big) > 0.$$

# ... $\pi^{\mathbf{p}}$-aligned for other regular semivalues

Note that $\sqsupseteq^a$ is $\pi^{\mathbf{p}}$-aligned for every regular semivalue such that $p_0 \geq p_2$:

$$\pi_2^{\mathbf{p}}(v)-\pi_1^{\mathbf{p}}(v) = (p_0+p_1)\big(v(2)-v(1)\big)+(p_1+p_2)\big(v(2,3)-v(1,3)\big) > 0$$

On the other hand

$$\pi_3^{\mathbf{p}}(v)-\pi_2^{\mathbf{p}}(v) = (p_0+p_1)\big(v(3)-v(2)\big)+(p_1+p_2)\big(v(1,3)-v(1,2)\big) > 0$$

for every $v \in V(\sqsupseteq^a)$.

# Total preorder $\pi^{\mathbf{p}}$-aligned for no regular semivalues

It is quite possible that for a given preorder there is no $\pi^{\mathbf{p}}$-ordinal semivalue associated to it. It is enough, for instance, to consider the case $N = \{1, 2, 3\}$ and the following total preorder:

$$N \sqsupset \{1, 2\} \sqsupset \{2, 3\} \sqsupset \{1\} \sqsupset \{1, 3\} \sqsupset \{2\} \sqsupset \{3\} \sqsupset \emptyset.$$

Then it is easy to see that 1 and 2 cannot be ordered since, fixed a semivalue $\mathbf{p}$ the quantity

$$\pi_2^{\mathbf{p}}(v) - \pi_1^{\mathbf{p}}(v) = (p_0 + p_1)(v(\{1\}) - v(\{2\})) + (p_1 + p_2)(v(\{1, 3\}) - v(\{2, 3\})$$

can be made both positive and negative by suitable choices of $v$.

**Proposition** Let $\sqsupseteq$ be a total preorder on $2^N$. If $\sqsupseteq$ satisfies the RESP property, then it is $\pi^{\mathbf{p}}$-aligned with every regular semivalue $\pi^{\mathbf{p}}$. ■

- All the extensions from the literature listed in the previous slide are $\pi^{\mathbf{p}}$-aligned with all regular semivalues...

$\{1,2,3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1,3\} \sqsupset^a \{2,3\} \sqsupset^a \{1\} \sqsupset^a \{1,2\} \sqsupset^a \emptyset$
is not RESP but is $\pi^{\mathbf{p}}$-aligned with all $\pi^{\mathbf{p}}$ such that $p_0 \geq p_2$.

- We can say something more....

# Monotonic total preorders

**Axiom** [Monotonicity, MON] A total preorder $\sqsupseteq$ on $2^N$ satisfies the *monotonicity* property iff for each $S, T \in 2^N$ we have that

$$S \subseteq T \Rightarrow T \sqsupseteq S.$$

■

$\sqsupseteq^a$ introduced in the previous example does not satisfy MON:
$\{1,2,3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1,3\} \sqsupset^a \{2,3\} \sqsupset^a \{1\} \sqsupset^a \{1,2\} \sqsupset^a \emptyset$.

- Min extension is a $\pi^{\mathbf{p}}$-aligned for all regular semivalues, it satisfies RESP, but it does not satisfy MON.

# An axiomatic characterization (with no interaction)

Let $\sqsupseteq$ be a total preorder on $2^N$. For each $S \in 2^N \setminus \{\emptyset\}$, denote by $\sqsupseteq_S$ the restriction of $\sqsupseteq$ on $2^S$ such that for each $U, V \in 2^S$,

$$U \sqsupseteq V \Leftrightarrow U \sqsupseteq_S V.$$

**Theorem** Let $\pi^{\mathbf{p}}$ be a regular semivalue. Let $\sqsupseteq$ be a total preorder on $2^N$ which satisfies the MON property. The following two statements are equivalent:

(i) $\sqsupseteq$ satisfies the RESP property.

(ii) $\sqsupseteq_S$ is $\pi^{\mathbf{p}}$-aligned for every $S \in 2^N \setminus \{\emptyset\}$. ∎

- side-product: for a large family of coalitional games all regular semivalues are ordinal equivalent (e.g. *airport games* (Littlechild and Owen (1973), Littlechild and Thompson (1977))

# A generalization of RESP which admits the interaction

We denote by $\Sigma_{ij}^s$ the set of all subsets of $N$ of cardinality $s$ which do not contain neither $i$ nor $j$, i.e.
$\Sigma_{ij}^s = \{S \in 2^N : i, j \notin S, |S| = s\}$.

Order the sets $S_1, S_2, \ldots, S_{n_s}$ in $\Sigma_{ij}^s$ when you add $i$ and $j$, respectively:

$$
\begin{array}{ccc}
S_1 \cup \{i\} & & S_{l(1)} \cup \{j\} \\
| \sqcup & & | \sqcup \\
S_2 \cup \{i\} & & S_{l(2)} \cup \{j\} \\
| \sqcup & & | \sqcup \\
\cdots & & \cdots \\
| \sqcup & & | \sqcup \\
S_{n_s} \cup \{i\} & & S_{l(n_s)} \cup \{j\}
\end{array}
$$

# **Axiom**[Permutational Responsiveness, PR]

We denote by $\Sigma_{ij}^s$ the set of all subsets of $N$ of cardinality $s$ which do not contain neither $i$ nor $j$, i.e.
$\Sigma_{ij}^s = \{S \in 2^N : i, j \notin S, |S| = s\}$.

Order the sets $S_1, S_2, \ldots, S_{n_s}$ in $\Sigma_{ij}^s$ when you add $i$ and $j$, respectively:

$$\begin{array}{ccc} S_1 \cup \{i\} & \sqsupseteq & S_{l(1)} \cup \{j\} \\ |\sqcup & & |\sqcup \\ S_2 \cup \{i\} & \sqsupseteq & S_{l(2)} \cup \{j\} \\ |\sqcup & & |\sqcup \\ \ldots & \sqsupseteq & \ldots \\ |\sqcup & & |\sqcup \\ S_{n_s} \cup \{i\} & \sqsupseteq & S_{l(n_s)} \cup \{j\} \end{array}$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

# Again a sufficient condition...

**Proposition** Let $\sqsupseteq$ be a total preorder on $2^N$. If $\sqsupseteq$ satisfies the PR property, then $\sqsupseteq$ is $\pi^{\mathbf{p}}$-aligned with every regular semivalue. ■

- Consider the (Shapley-aligned) total prorder $\sqsupseteq^a$ of previous $\{1,2,3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1,3\} \sqsupset^a \{2,3\} \sqsupset^a \{1\} \sqsupset^a \{1,2\} \sqsupset^a \emptyset$. Note that $\{2\} \sqsupset \{1\}$, but $\{1,3\} \sqsupset \{2,3\}$.

- $\{1,2,3,4\} \sqsupset^b \{2,3,4\} \sqsupset^b \{3,4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{2\} \sqsupset^b \{2,4\} \sqsupset^b \{1,4\} \sqsupset^b \{1,3\} \sqsupset^b \{2,3\} \sqsupset^b \{1,3,4\} \sqsupset^b \{1,2,4\} \sqsupset^b \{1,2,3\} \sqsupset^b \{1,2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$ is $\pi^{\mathbf{p}}$-aligned for all **p** but does not satisfy the PR property.

# Work in progress: Lucchetti, Moretti, Patrone (2012) A probabilistic approach to ranking sets of interacting objects

- A new interpretation of $\pi^{\mathbf{p}}$-aligned total preorders in terms of "ranking sets of objects" under uncertainty.

- Characterizations of total preorders which are $\pi^{\mathbf{p}}$-aligned with all semivalues.

- Characterizations of specific $\pi^{\mathbf{p}}$-aligned total preorders (with or without the comparison of ordered lists of sets)

# Why not to consider probabilistic values?

A probabilistic value $\pi^p$ (or probabilistic power index ) $\pi$ for the game $v$ is an $n$-vector $\pi^p(v) = (\pi_1^p(v), \pi_2^p(v), \ldots, \pi_n^p(v))$, such that

$$\pi_i^p(v) = \sum_{S \in 2^{N \setminus \{i\}}} p^i(S)\big(v(S \cup \{i\}) - v(S)\big) \qquad (1)$$

for each $i \in N$ and $S \in 2^{N \setminus \{i\}}$, and $p = (p^i : 2^{N \setminus \{i\}} \to \mathbb{R}^+)_{i_i n N}$, is a collection of non negative real functions fulfilling the condition $\sum_{S \in 2^{N \setminus \{i\}}} p^i(S) = 1$.

# Again RESP...

Theorem (R. Lucchetti, S. Moretti, F. Patrone 2012)

*Let $N$ be a finite set and let $\sqsupseteq$ be a total preorder on $2^N$. Then the following are equivalent:*

1. $\sqsupseteq$ *is aligned w.r.t. all the probabilistic values;*
2. $\sqsupseteq$ *satisfies the RESP property.*

# **Axiom**[Double Permutational Responsiveness, DPR]

Order the sets $S_1, S_2, \ldots, S_{n_s+n_{s-1}}$ in $\Sigma_{ij}^{s} \cup \Sigma_{ij}^{s-1}$ when you add $i$ and $j$, respectively:

$$\begin{array}{ccc}
S_1 \cup \{i\} & \sqsupseteq & S_{l(1)} \cup \{j\} \\
|\bigsqcup & \sqsupseteq & |\bigsqcup \\
S_2 \cup \{i\} & & S_{l(2)} \cup \{j\} \\
|\bigsqcup & & |\bigsqcup \\
\ldots & \sqsupseteq & \ldots \\
|\bigsqcup & & |\bigsqcup \\
S_{n_s+n_{s-1}} \cup \{i\} & \sqsupseteq & S_{l(n_s+n_{s-1})} \cup \{j\}
\end{array}$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

# A characterization with possibility of interaction

Theorem (R. Lucchetti, S. Moretti, F. Patrone 2012)

*Let $N$ be a finite set and let $\sqsupseteq$ be a total preorder on $2^N$. The following statements are equivalent:*

1) $\sqsupseteq$ *fulfills the DPR property;*
2) $\sqsupseteq$ *is $\pi^{\mathbf{p}}$-aligned w.r.t. all the semivalues.*

- $\{1,2,3,4\} \sqsupset^b \{2,3,4\} \sqsupset^b \{3,4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{2\} \sqsupset^b \{2,4\} \sqsupset^b \{1,4\} \sqsupset^b \{1,3\} \sqsupset^b \{2,3\} \sqsupset^b \{1,3,4\} \sqsupset^b \{1,2,4\} \sqsupset^b \{1,2,3\} \sqsupset^b \{1,2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$ is $\pi^{\mathbf{p}}$-aligned for all $\mathbf{p}$, is not PR, but it is DPR.

# Next steps

- generalizing: partial orders...

- particularizing: how to represent interaction on specific applications?

- thinking of the possibility to do a kind a inverse process, not necessarily respecting the ranking restricted to the singletons.

# Thanks!