Ramsey Number for Hypergraph Cycles

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Hypergraphs

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The *tight cycle* $\mathcal{T}_n$ has vertex set $v_1, \ldots, v_n$ and edges $v_1v_2v_3$, $v_2v_3v_4$, $v_3v_4v_5, \ldots, v_nv_1v_2$.

Loose and tight cycles for $n = 12$.

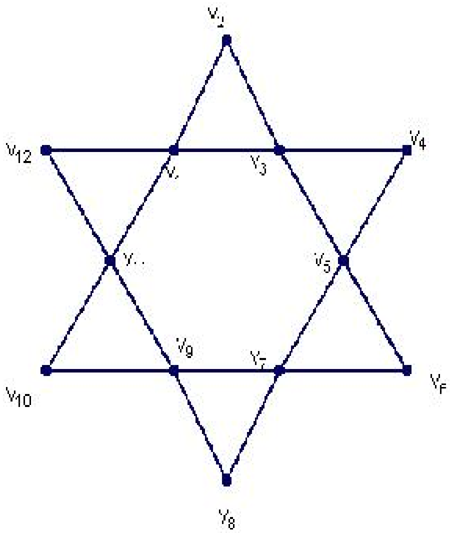

Figure 1: loose cycle $\mathcal{C}_{12}$

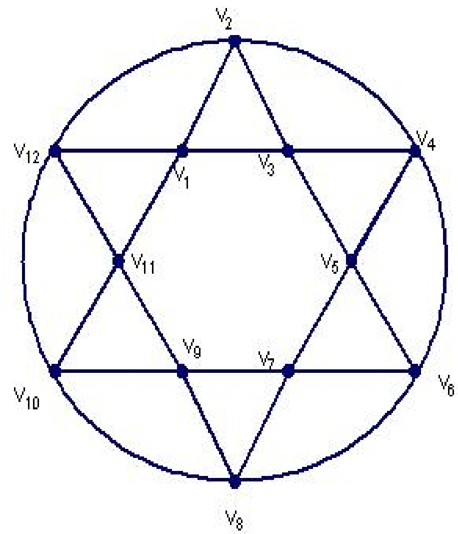

Figure 2: tight cycle $\mathcal{T}_{12}$

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$$R(\mathcal{C}_{4k}, \mathcal{C}_{4k}) > 5k - 2 \text{ and } R(\mathcal{C}_{4k+2}, \mathcal{C}_{4k+2}) > 5k - 1$$

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$$R(\mathcal{C}_n,\mathcal{C}_n) < (5+o(1))n/4.$$

Lower bound: $R(\mathcal{C}_{4k}, \mathcal{C}_{4k}) > 5k - 2$

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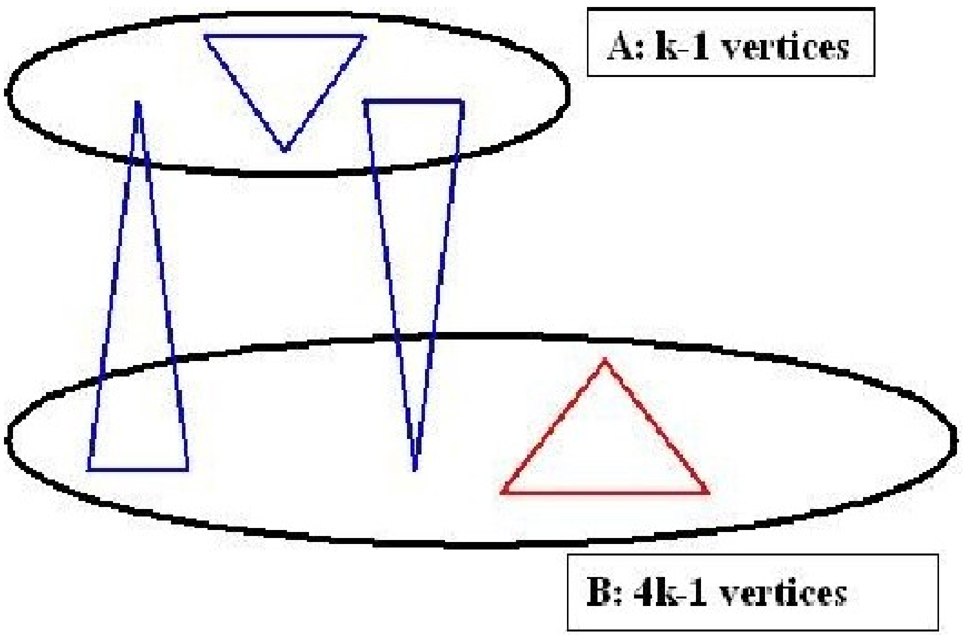
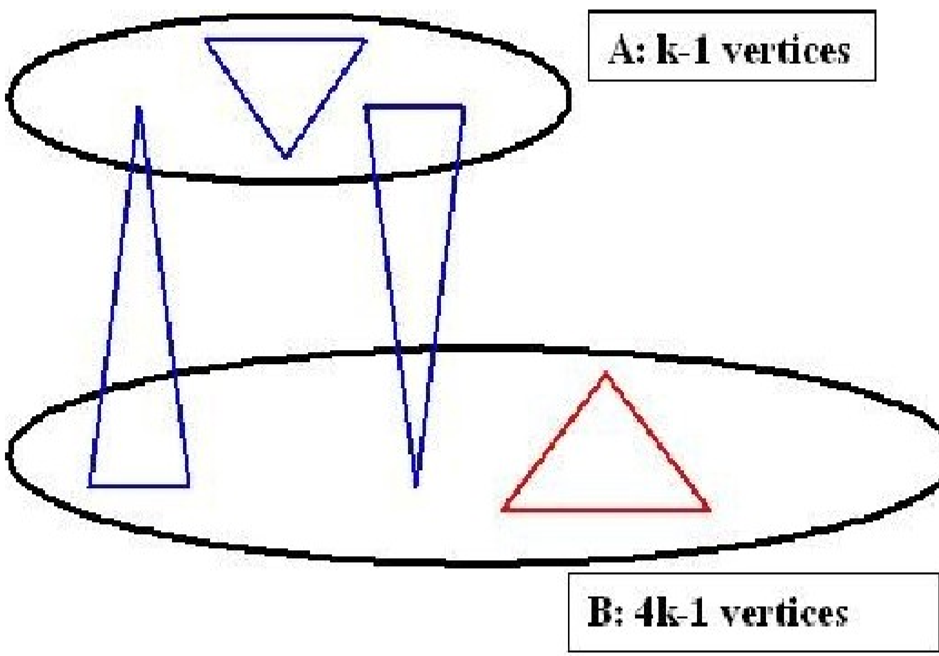
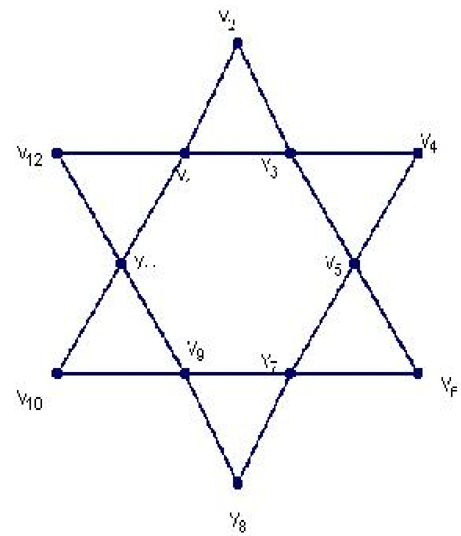


There is no RED $\mathcal{C}_{4k}$ because all RED triples are inside B and B contains only $4k - 1$ vertices.

There is no BLUE $\mathcal{C}_{4k}$: each of $2k$ edges must contain a vertex from A. Since each vertex in $\mathcal{C}_{4k}$ can be in at most 2 edges, A must contain at least $2k/2 = k$ vertices - a contradiction!





Upper bound

Theorem 2. *For every* $\eta > 0$ *there is* $n_0 = n_0(\eta)$ *such that for every* $n > n_0$*, every 2-coloring of* $K^{(3)}_{5(1+\eta)n/4}$ *contains a monochromatic copy of* $\mathcal{C}_n$*.*

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We must find a RED or BLUE copy of $\mathcal{C}_n$.

Step 1: apply the Regularity Lemma on $\mathcal{F}_R$ with a small ϵ

The vertex set $V(\mathcal{F}_R)$ is partitioned into $t+1$ classes $V_0 \cup V_1 \cup \ldots \cup V_t$ such that

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  $\mathcal{F}_R[V_i', V_j', V_k']$ contains $(d_{ijk} \pm \epsilon)|V_i'||V_j'||V_k'|$ RED edges for each sample $V_i' \subset V_i, V_j' \subset V_j, V_k' \subset V_k$, $|V_i'| \geq \epsilon|V_i|, |V_j'| \geq \epsilon|V_j|, |V_k'| \geq \epsilon|V_k|$.

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Note that $\mathcal{F}_B[V_i', V_j', V_k']$ then contains $\big((1-d_{ijk}) \pm \epsilon\big)|V_i'||V_j'||V_k'|$ BLUE edges and is also ϵ-regular.

Step 2: define the cluster hypergraph $\mathcal{J}$ and its 2-coloring

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We color the edge $ijk \in \mathcal{J}$ RED if the density of $\mathcal{F}_R[V_i, V_j, V_k]$ is at least 1/2, otherwise we color it BLUE ; i.e. $\mathcal{J} = \mathcal{J}_R \cup \mathcal{J}_B$.

Step 3: shadow graph monochromatic components

The *shadow graph* $\Gamma(\mathcal{J}_*)$, $* \in \{R, B\}$, has vertices $1, \ldots, t$ and edge set

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Each vertex is in exactly one RED and one BLUE component.

Step 4: looking for (many) diamonds

A *diamond* $\mathcal{D}$ is a hypergraph with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2v_3, v_2v_3v_4\}$. We will call v_2v_3 the *central edge* of $\mathcal{D}$.

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Lemma 3. *The cluster graph $\mathcal{J}$ contains a monochromatic component $\mathcal{L}$ that contains (i.e. $\mathcal{J}[\mathcal{L}]$ does) $s \sim t/5$ vertex-disjoint diamonds $\mathcal{D}_1, \ldots, \mathcal{D}_s$.*

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Fix a (RED) component $\mathcal{L}$ and (RED) vertex-disjoint diamonds $\mathcal{D}_1, \ldots, \mathcal{D}_s$. Let $M = \{e_1, \ldots, e_s\}$ be the set of all central edges of these diamonds.

Step 5: finding a trail with all diamonds

Lemma 4. *In $\Gamma(\mathcal{J}_R)$, there exists a closed (oriented) trail $x_1x_2x_3\dots x_r = x_1$*

- *containing M;*
- *with $r \leq 2t$ (remember $\Gamma(\mathcal{J}_R)$ has t vertices);*
- *each edge of $\Gamma(\mathcal{J}_R)$ is used at most once in each direction.*

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Construct the trail: walk from x_1 around the boundary of the (single) face of the embedding, ending back at x_1.

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A vertex $x \in V_i$ is *good* for the triple $\mathcal{J}_R[V_i, V_j, V_k]$ if

(i) for at least $d|V_j|/2$ vertices $y \in V_j$, there are at least $d|V_k|/2$ vertices $z \in V_k$ such that $xyz \in \mathcal{J}_R$, and

(ii) for at least $d|V_k|/2$ vertices $z \in V_k$, there are at least $d|V_j|/2$ vertices $y \in V_j$ such that $xyz \in \mathcal{J}_R$.

We define vertices in V_j and V_k to be good for $\mathcal{J}_R[V_i, V_j, V_k]$ in a similar way.

Observation: ϵ-regularity $\Rightarrow$ almost all vertices in V_i, V_j, V_k are good.

Step 5 and 3/4: properties of good vertices

Let $\mathcal{J}_R[V_i, V_j, V_k]$ be ϵ-regular with density $d > 2\epsilon^{1/3}$.

Let $u \in V_i$ and $w \in V_j$ be any two good vertices and B be a (small) set of vertices.

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Let $\mathcal{J}_1 = \mathcal{J}_R[V_i, V_j, V_k]$, $\mathcal{J}_2 = \mathcal{J}_R[V_i, V_j, V_\ell]$ be ϵ-regular with density $d > 2\epsilon^{1/6}$.

Let $u \in V_i$ and $w \in V_j$ be any two good vertices (for $\mathcal{J}_1$ and $\mathcal{J}_2$) and B be a (small) set of vertices.

Then there is a u, v-path of length up to $\sim 2m$ in $\mathcal{J}_1 \cup \mathcal{J}_2$ avoiding B.

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For each arc (x_{i-1}, x_i), choose an ϵ-regular triple $\mathcal{J}_i = \mathcal{J}_R[V_{x_{i-1}}, V_{x_i}, V_{k_i}]$ of density at least $1/2$ for some k_i.

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For $i = 2, 3, \dots, r, 1$, we join v_{i-1} and v_i by a path of length 3, disjoint from all previously defined paths.

These short paths link to form a loose cycle of length $3(r-1)$.

If the length $n/2$ of $\mathcal{C}_n$ has different parity from $3(r-1)$, replace one path of length 3 by a path of length 6 to make the parities agree. This gives a loose cycle $\mathcal{C}$ of length c, where $3(r-1) \leq c \leq 3r$ and $c \equiv n/2 \pmod 2$.

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Then we find a long (odd) path (od length $\sim n/2s = 5n/2t \leq 2N/t = 2m$) in $\mathcal{J}_R[V_{x_{i-1}}, V_{x_i}, V_k] \cup \mathcal{J}_R[V_{x_{i-1}}, V_{x_i}, V_p]$ avoiding $\mathcal{C}$.

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This way we obtain a cycle of length $\sim s \times n/2s = n/2$. With a bit of extra work (and details), we get length exactly $n/2$.

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- one of few applications of this version of the regularity lemma for 3-uniform hypergraphs;

- we need to look for $s \sim t/5$ connected diamonds. If one would like to use a matching to find $\mathcal{C}_n$, its size s would have to be $s \sim 4t/15$ (so that $s \times 3\frac{5n/4}{t} \sim n$ and this cannot be guaranteed.

Tight cycle $\mathcal{T}_n$

We can prove

$$R(\mathcal{C}_{3k}, \mathcal{C}_{3k}) > 4k - 2$$

and

$$R(\mathcal{C}_{3k+1}, \mathcal{C}_{3k+1}), R(\mathcal{C}_{3k+2}, \mathcal{C}_{3k+2}) > 6k$$

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$$R(\mathcal{C}_{3n}, \mathcal{C}_{3n}) < (4 + o(1))n$$

and

$$R(\mathcal{C}_{3n+1}, \mathcal{C}_{3n+1}), R(\mathcal{C}_{3n+2}, \mathcal{C}_{3n+2}) < (6 + o(1))n.$$

Construction for the lower bound

Coloring of $K^{(3)}_{4k-2}$ without monochromatic $\mathcal{C}_{3k}$: take X and Y $|X| = |Y| = 2k-1$, color all triples inside X and those with exactly two vertices in Y RED , all others by BLUE .

In this coloring, there is no monochromatic matching saturating $3k$ vertices contained in one strong component, and thus also no monochromatic copy of $\mathcal{C}_{3k}$.

Differences

- use of the Regularity Lemma (RL) of Frankl-Rödl;
- we look for a strongly connected matching of size $s \sim t/4$;
- we need to find long path in a regular triad produced by RL (done by Polcyn, Rödl, Ruciński, Szemerédi).