On the Regularity Method for Hypergraphs

Mathias Schacht

October 2004

Outline

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Arithmetic Progressions

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- polynomial extensions
  $\longrightarrow$ Bergelson & Leibman 1996, Bergelson & McCutcheon 2000

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An Extremal Hypergraph Problem

Theorem (Gowers 2003+, Nagle, Rödl, S. & Skokan 2003+). If $\mathcal{H}^{(k)}$ is a k-uniform hypergraph on n vertices such that every edge of $\mathcal{H}^{(k)}$ is contained in **precisely one** $K_{k+1}^{(k)}$, then

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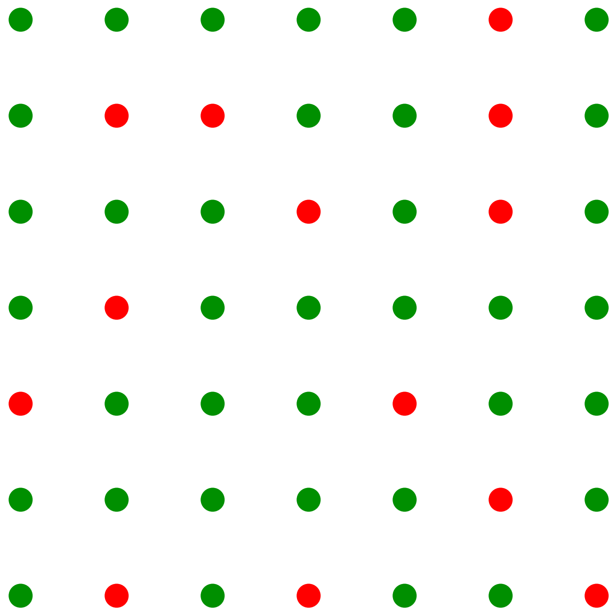
- Let $Z \subseteq [n]^2$ not containing a homothetic copy of an isosceles triangle



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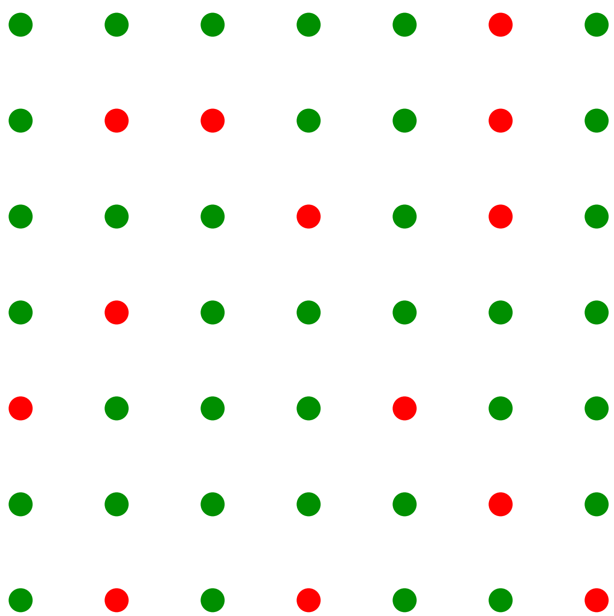

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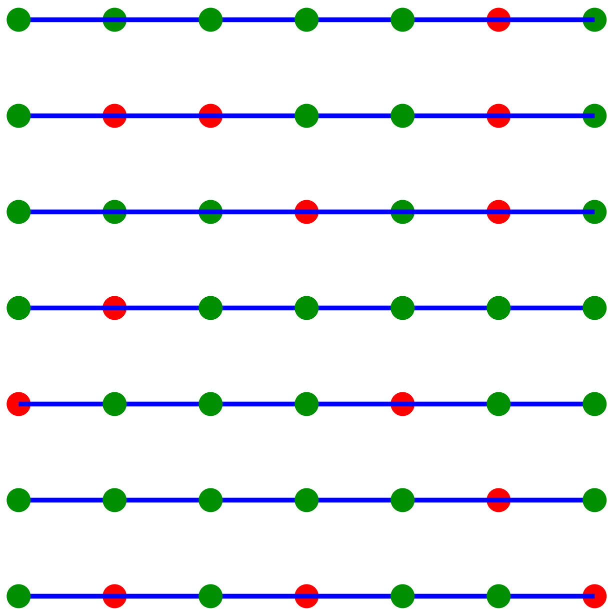

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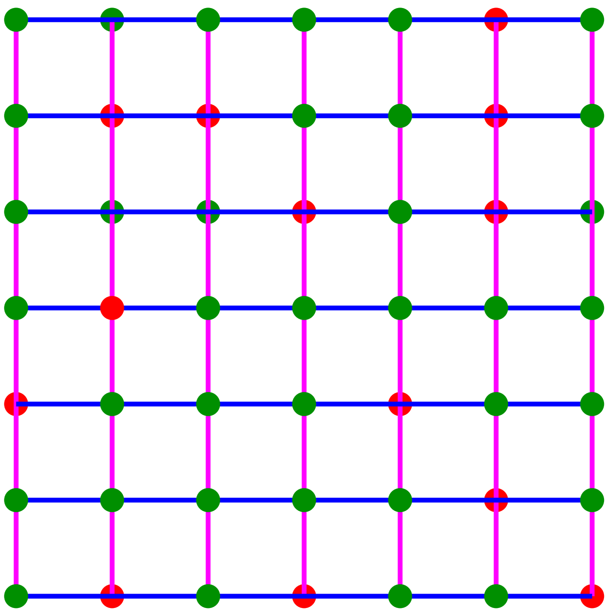

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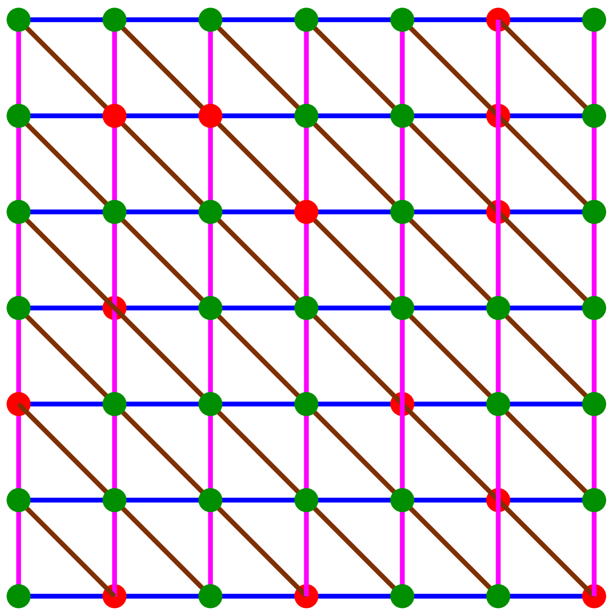

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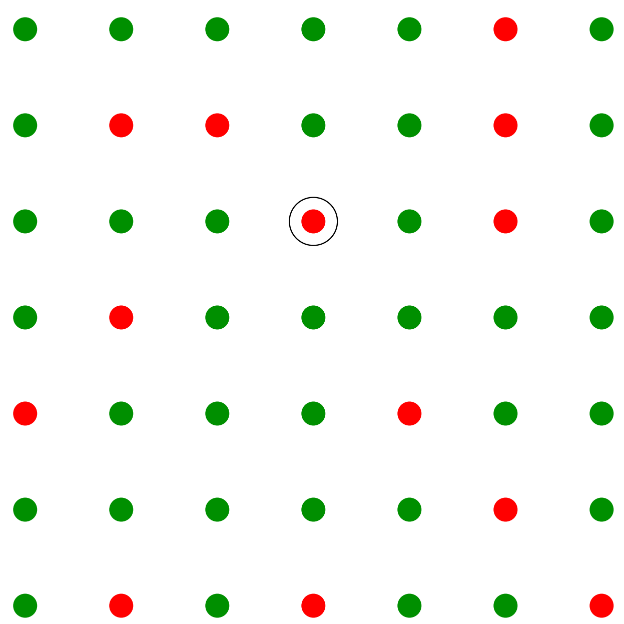

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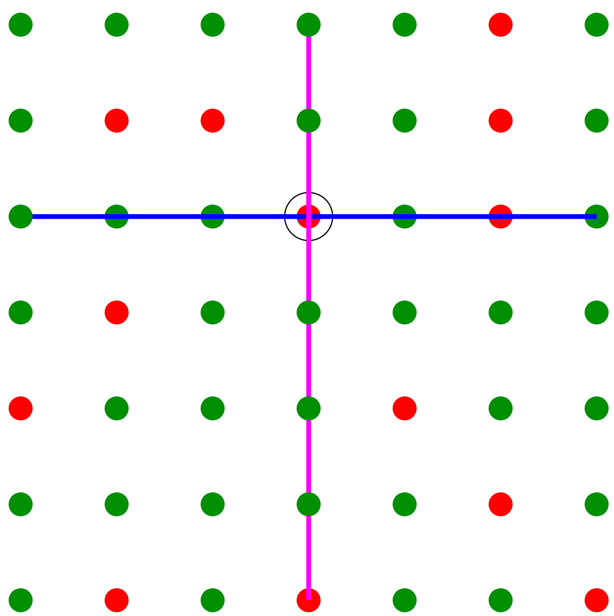

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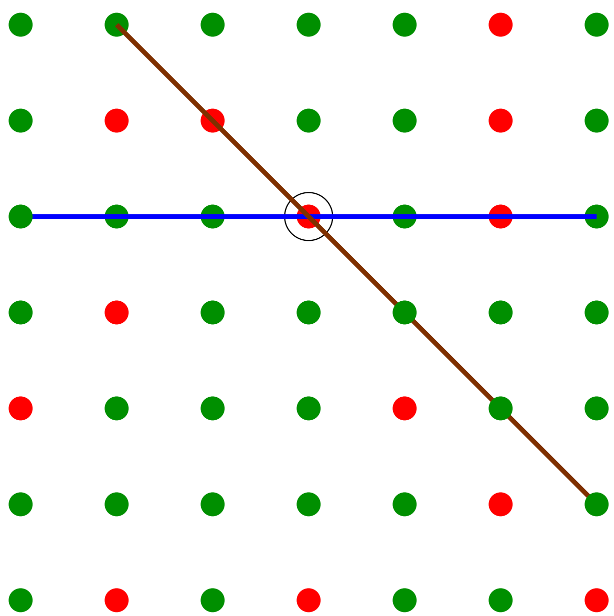

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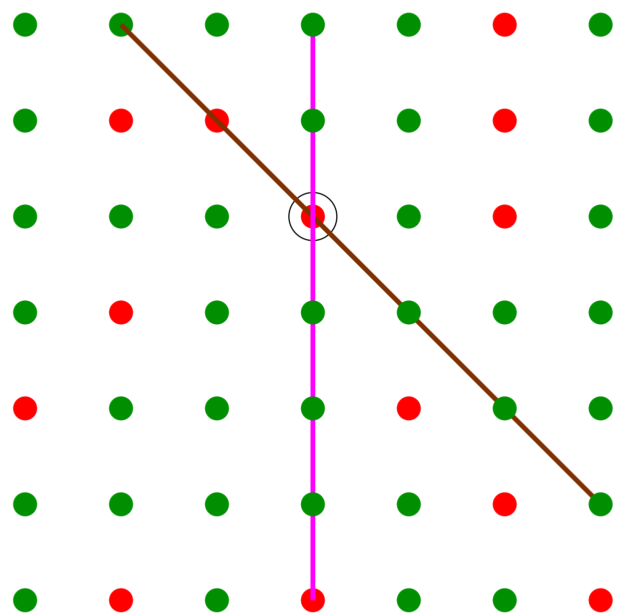

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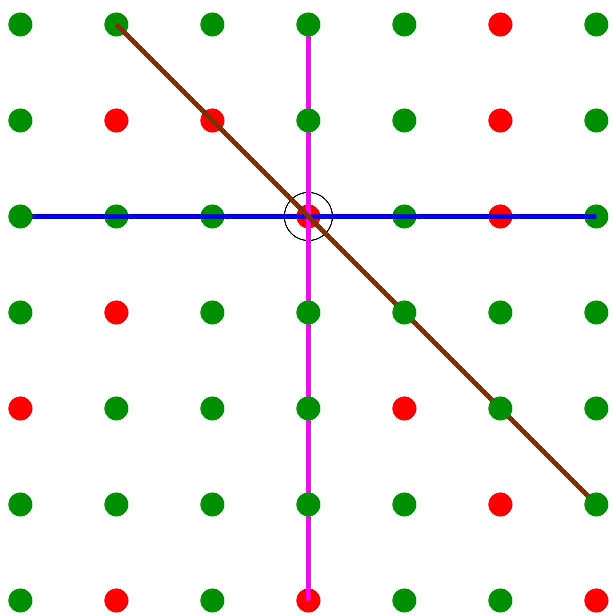

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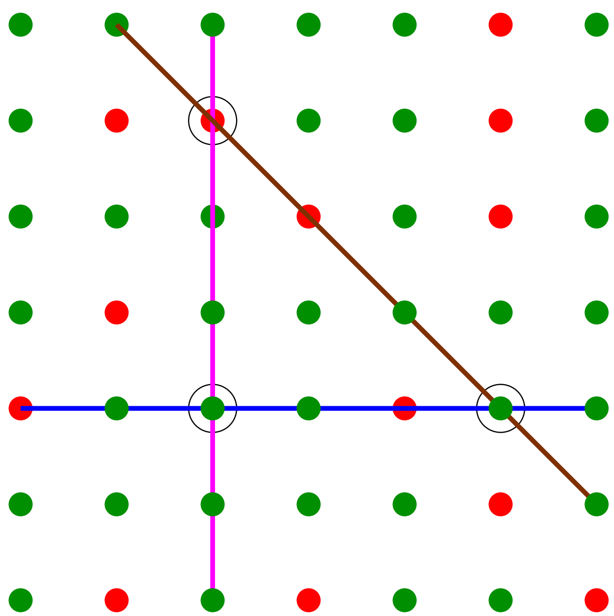

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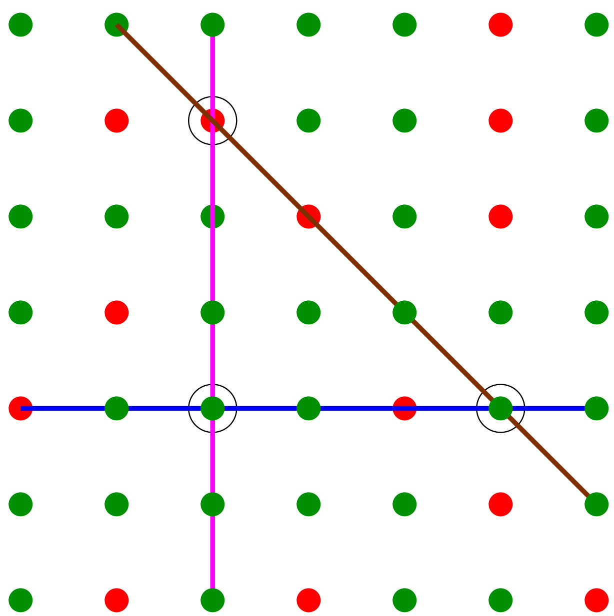

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⇒ $3|Z| = |E| = o(|V|^2) = o(n^2)$ □

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Short version. tripartite, ε-regular, and d-dense

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then

Short version. tripartite, ε-regular, and d-dense $\Rightarrow$

Counting Lemma for Graphs

Fact. Let $1 \geq d \geq 2\varepsilon > 0$. If

- $G = (V_1 \cup V_2 \cup V_3, E)$ is a tripartite graph with $|V_1| = |V_2| = |V_3| = m$

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then

G contains at least $(1 - 2\varepsilon)(d - \varepsilon)^3 m^3$ triangles.

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Many more applications of the Regularity Lemma, the Counting Lemma and its extensions in:

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Hope. Extension to hypergraphs is useful in the same areas.

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Theorem (Frankl & Rödl 2002). For all reals $\mu > 0$ and $\delta_3 > 0$ and all functions

$$\delta_2\colon \mathbb{N} \to (0, 1] \text{ and } r\colon \mathbb{N}^2 \to \mathbb{N}$$

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(iii) $\mathcal{H}^{(3)}$ is $(\delta_3, r(t, \ell))$-regular w.r.t. $\mathscr{P}$.

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with $m \geq m_0$, contains

$$(1 \pm \gamma) d_2^6 d_3^4 m^4$$

copies of $K_4^{(3)}$.

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- What about “Blow-up type” extensions of the Counting Lemma?