$[n] = \{1, 2, \ldots, n\}$, $\mathcal{F} \subseteq 2^{[n]}$

$f_i(\mathcal{F}) = f_i = |\{F \in \mathcal{F} : |F| = i\}|$

$(0 \leq i \leq n)$

profile vector

$$f(\mathcal{F}) = f = (f_0, f_1, \ldots, f_n)$$

Theorem (YBLM-inequality)

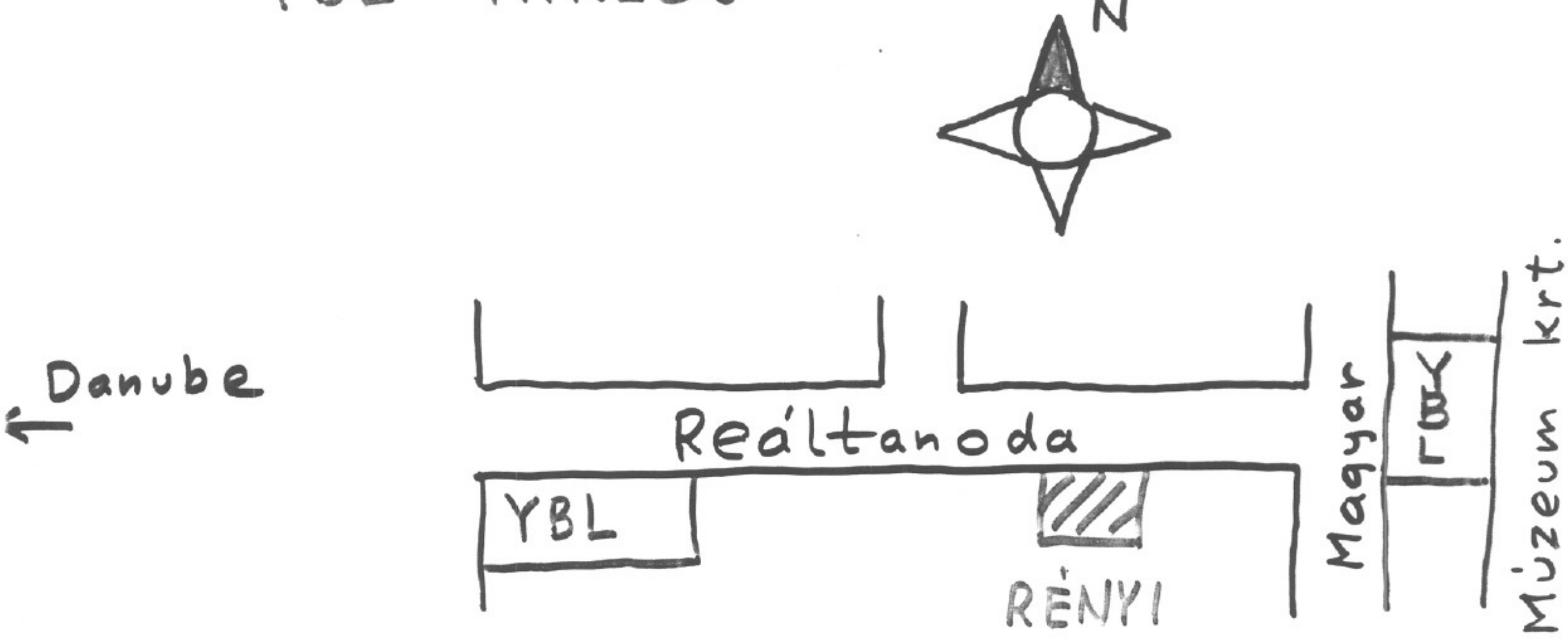
If $\mathcal{F}$ is Sperner $(F, G \in \mathcal{F} \Rightarrow F \not\subseteq G)$ then

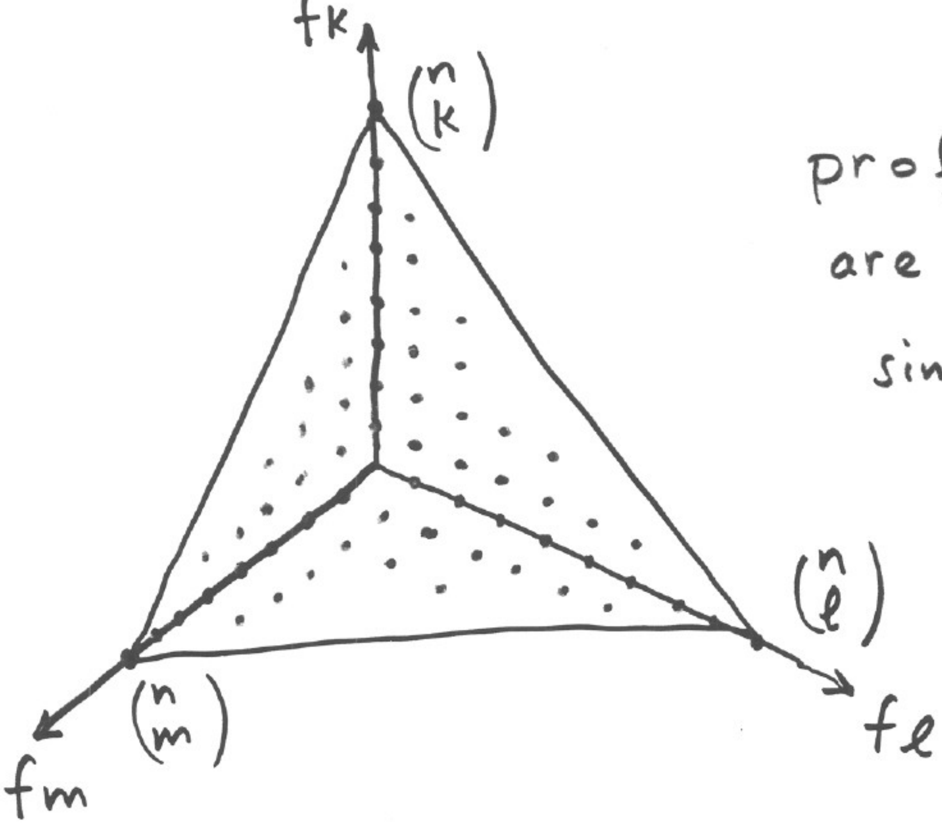
$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leq 1.$$

Y B L M 1954, 1963, 1966, 1963

Yamamoto, Bollobás, Lubell, Meshalkin

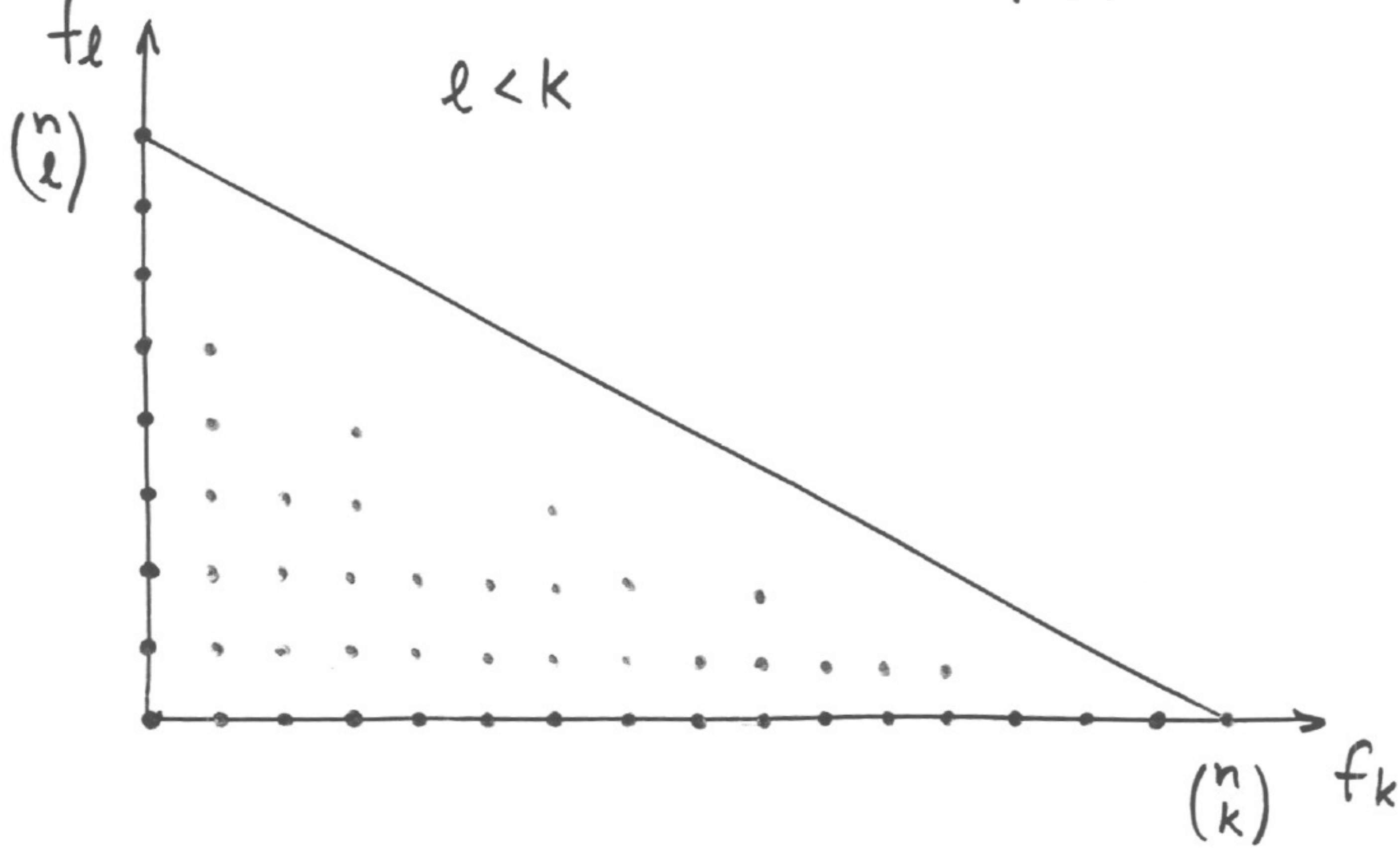
YBL MIKLÓS

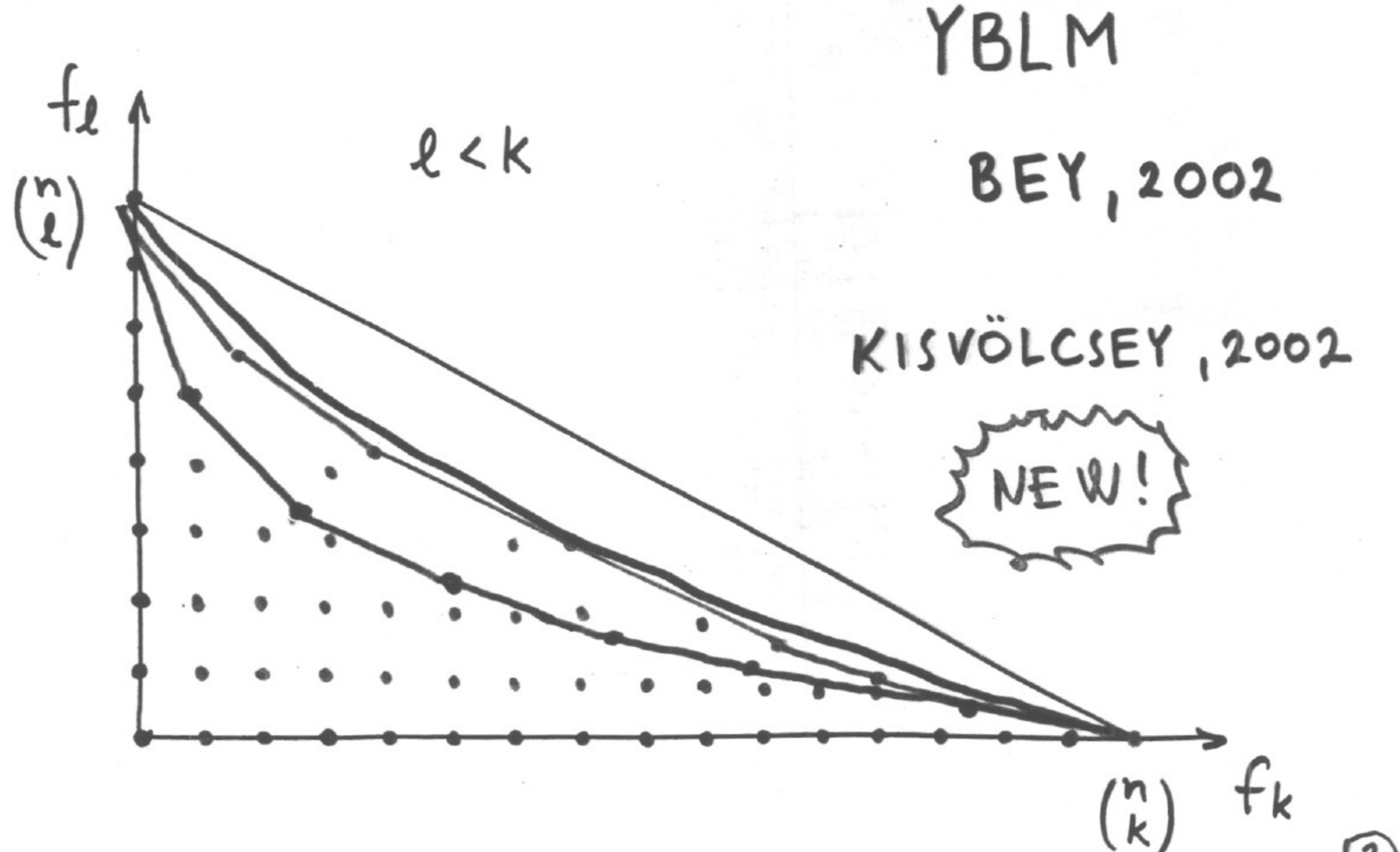




profile vectors are in this simplex

YBLM

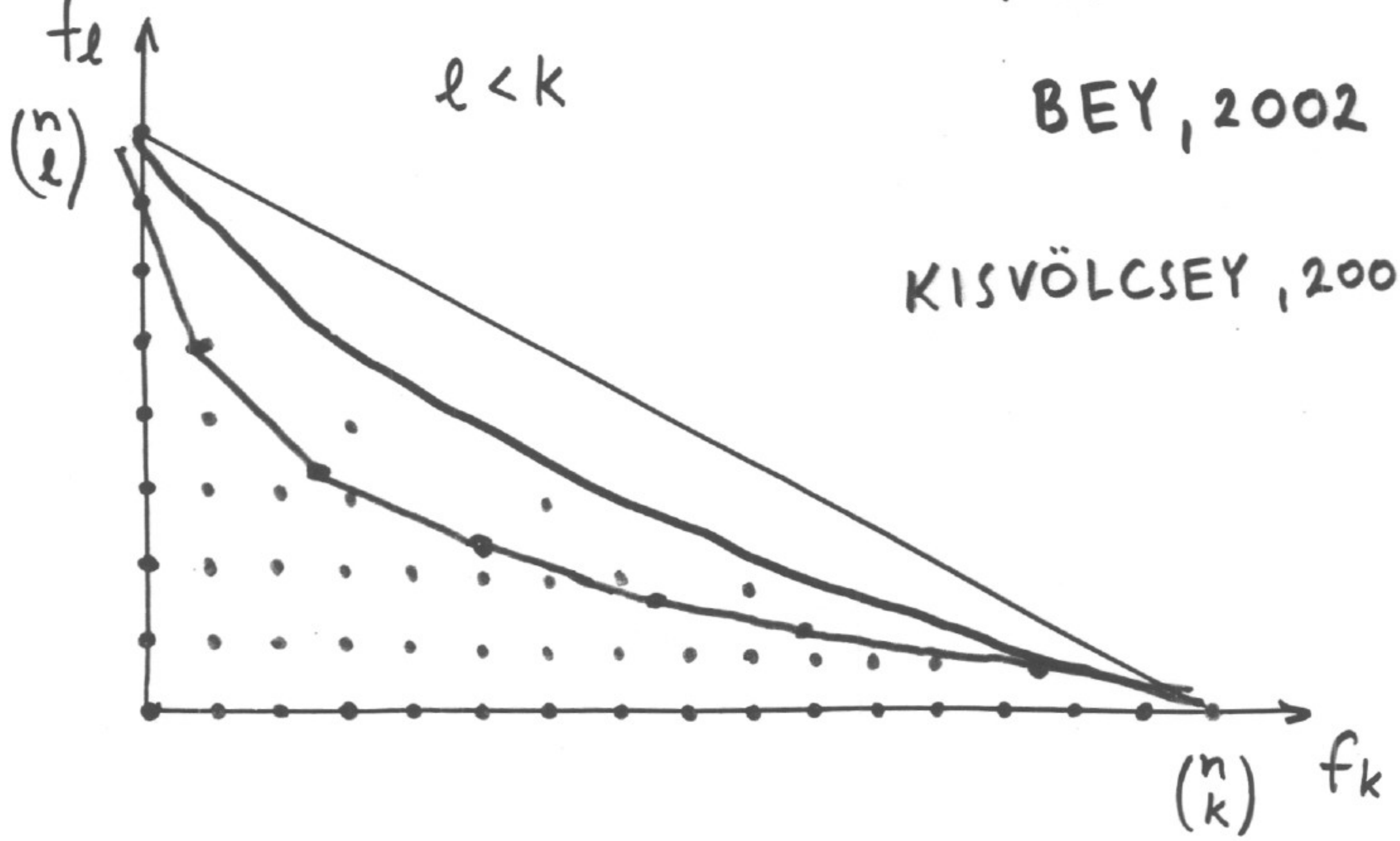



f_ℓ
$\ell < k$
$\binom{n}{\ell}$
$\binom{n}{k}$
f_k
YBLM
BEY, 2002
KISVÖLCSEY, 2002
NEW!

YBLM

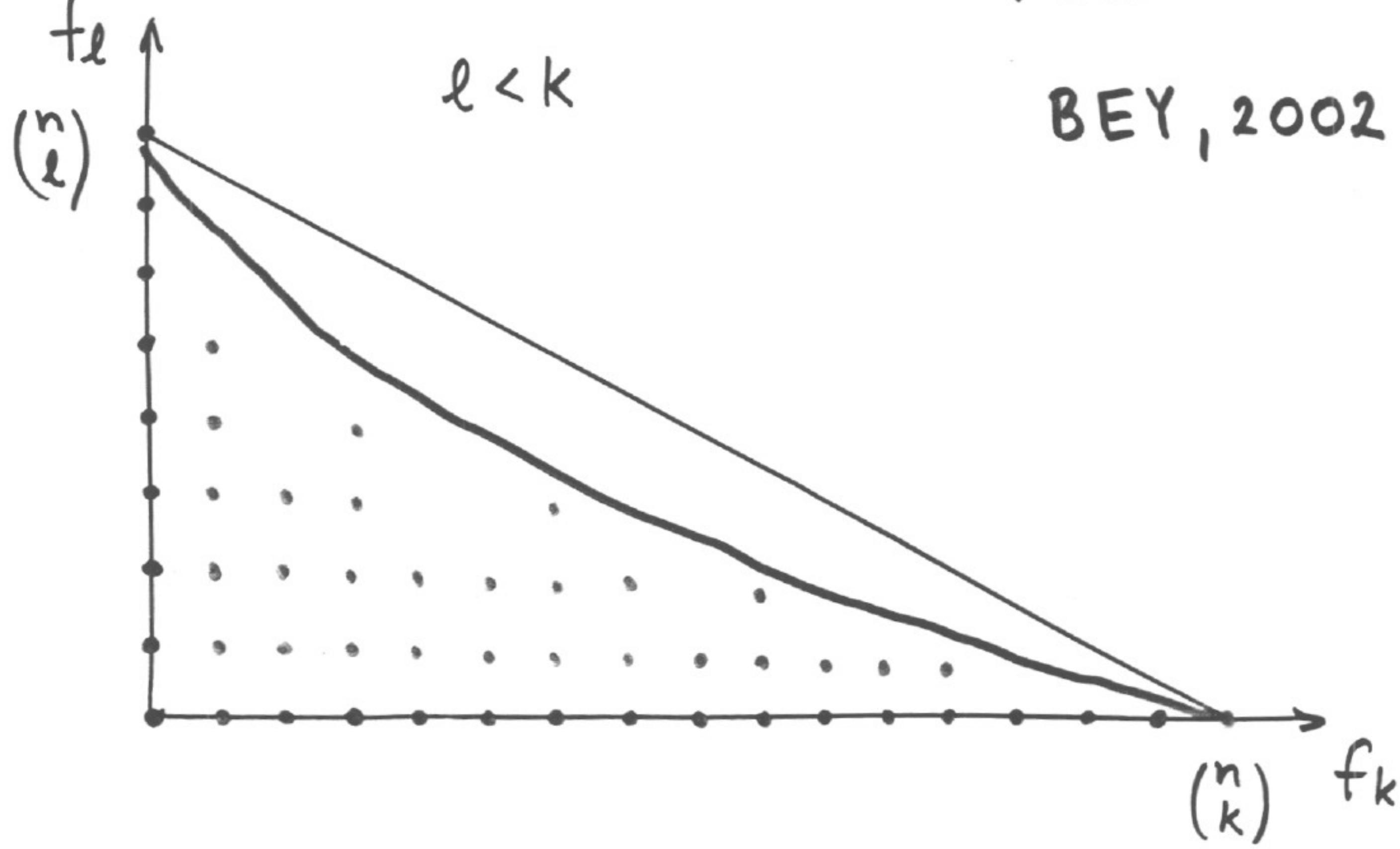
BEY, 2002

KISVÖLCSEY, 2002



YBLM

BEY, 2002

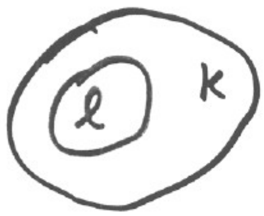


$\binom{[n]}{k}$ = all k-element subsets

$$\mathcal{F}_k \subseteq \binom{[n]}{k}$$

ℓ-shadow of $\mathcal{F}_k$: $(\ell < k)$

$$\sigma_\ell(\mathcal{F}_k) = \{ G : |G| = \ell,\ \exists\, F \in \mathcal{F}_k \text{ such that } G \subset F \}$$



k-canonical representation of m:

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_t}{t} \qquad a_k > a_{k-1} > \dots > a_t$$

$$a_t \geq t \geq 1$$

exists and unique

$$A_{k \to \ell}(m) = \binom{a_k}{\ell} + \binom{a_{k-1}}{\ell - 1} + \dots + \binom{a_t}{t-k+\ell}$$

Shadow theorem

$\mathcal{F} \subseteq \binom{[n]}{k}$

$$A_{k \to \ell}(|\mathcal{F}|) \leq |\sigma_\ell(\mathcal{F})|.$$

One can deduce a necessary and sufficient condition, using this therem, for a vector $(f_0, f_1, \dots, f_n)$ to be the profile vector of a Sperner family.

$A_{k \to \ell}(m)$ is a "fractal".

(Frankl – Tokushige)

$$m = \frac{x(x-1)\dots(x-k+1)}{k!} \qquad (x \text{ real})$$

$$B_{k\to\ell}(m) = \frac{x(x-1)\dots(x-\ell+1)}{\ell!}$$

Obvious:

$$B_{k\to\ell}(m) \leq A_{k\to\ell}(m)$$

with equality if $m = \binom{a}{k}$ for some integer a.

Lovász' variant of the shadow theorem

$$\mathcal{F} \subseteq \binom{[n]}{k}$$

$$B_{k\to\ell}(|\mathcal{F}|) \leq |\sigma_\ell(\mathcal{F})|.$$

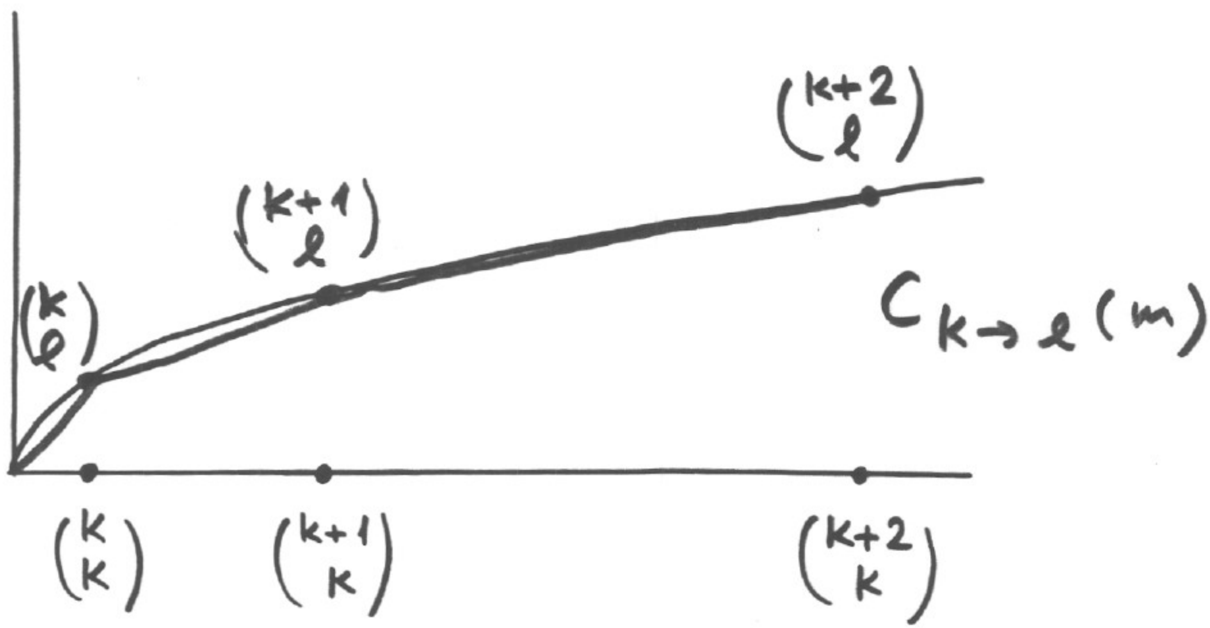
$B_{k\to\ell}$ roughly: $m \approx \frac{x^k}{k!}$

$$x \approx \sqrt[k]{m\,k!}$$

$$B_{k\to\ell}(m) \approx \frac{\left(\sqrt[k]{m\,k!}\right)^\ell}{\ell!} = \frac{(k!)^{\frac{\ell}{k}}}{\ell!} \times^{\frac{\ell}{k}}$$

Lemma (Griggs – K)

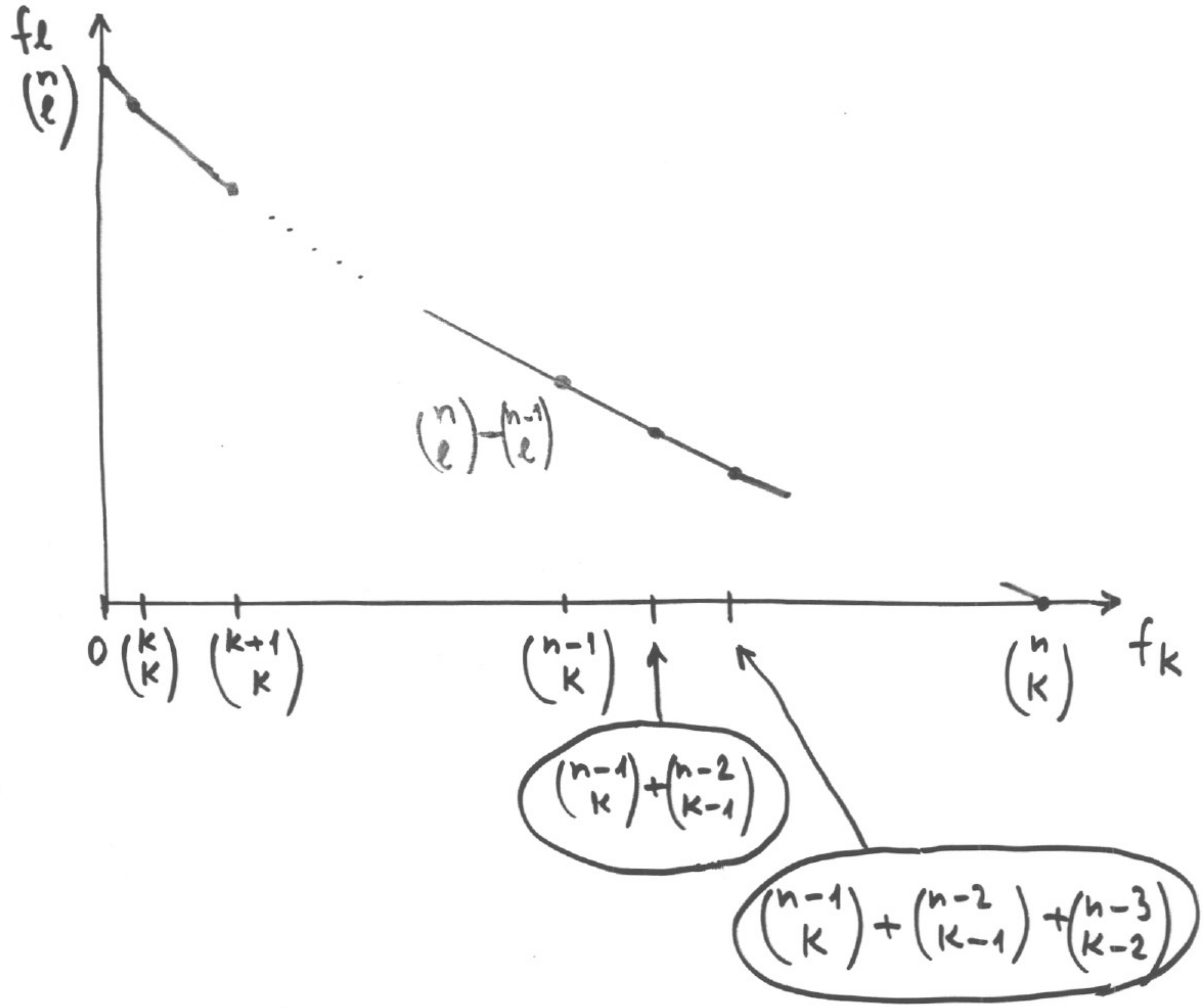
$B_{k\to\ell}(m)$ is concave from below.

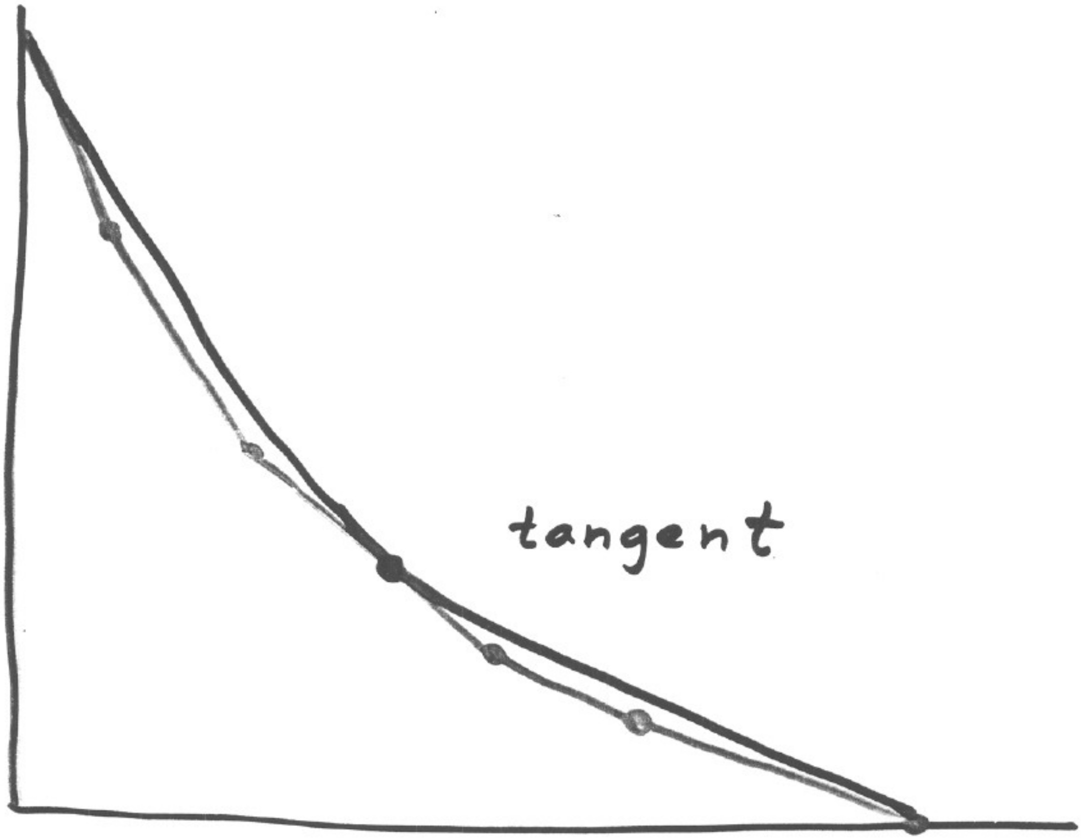


Theorem (Lazy Lovász Shadow)

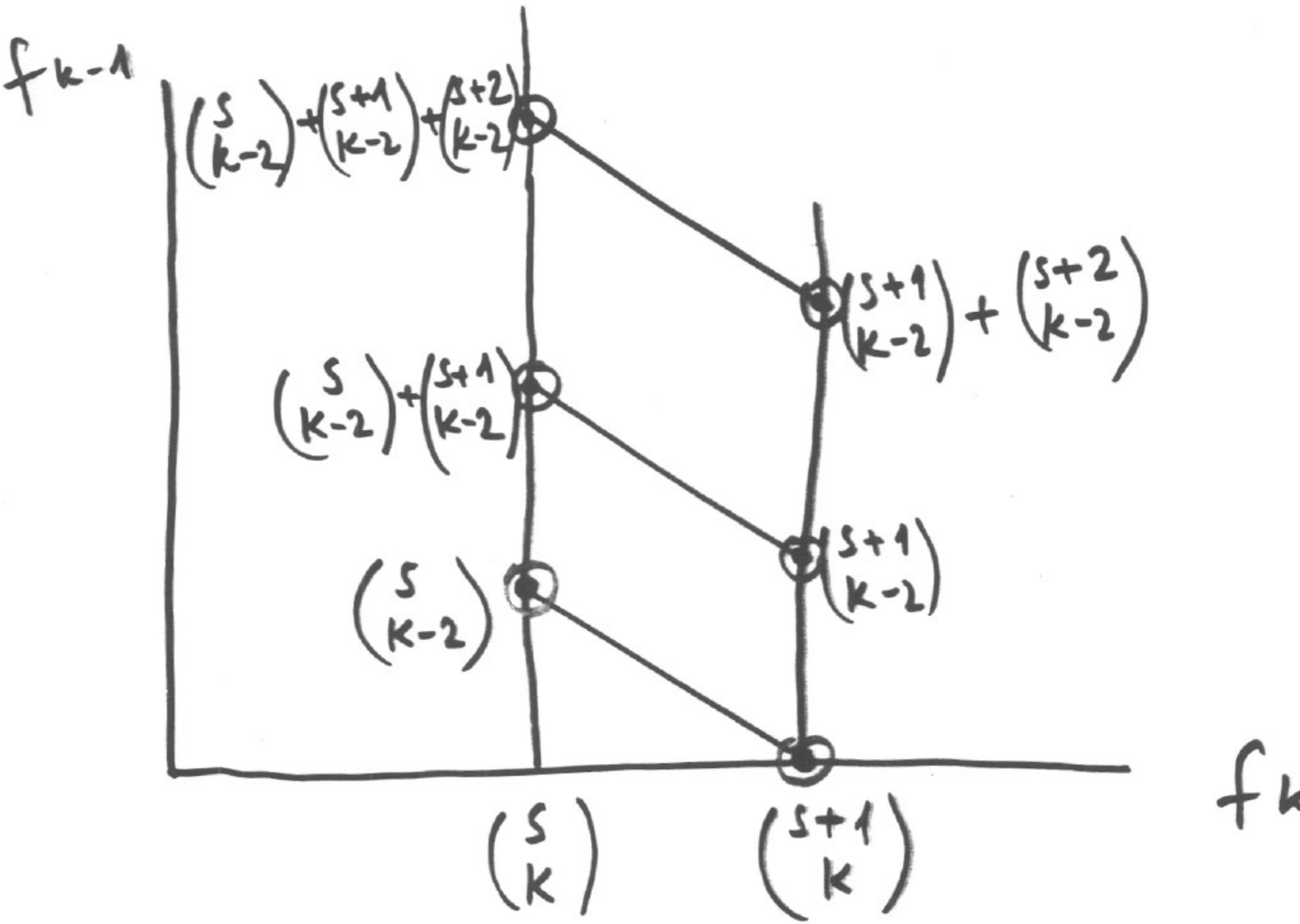
$$C_{k\to\ell}(|\mathcal{F}|) \le |\sigma_\ell(\mathcal{F})|$$

Theorem (Griggs-K) The following convex broken line is an upper bound for the two-level profile vectors (f_k, f_ℓ):





BEY is not BAD!


f_{k-1}
$\binom{s}{k-2}+\binom{s+1}{k-2}+\binom{s+2}{k-2}$
$\binom{s+1}{k-2}+\binom{s+2}{k-2}$
$\binom{s}{k-2}+\binom{s+1}{k-2}$
$\binom{s+1}{k-2}$
$\binom{s}{k-2}$
f_k
$\binom{s}{k}$
$\binom{s+1}{k}$