Overlapping Expert Information: Learning about Dependencies in Expert Judgment

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Extending Winkler's Multivariate Normal Aggregation

$P(\text{Event})$

$P(\text{Event} \mid X_1,\ldots,X_n)$

$P(\text{Event} \mid X_1,\ldots,X_n)$

Event = Collision between two vessels

X_1 = Type of other vessel

X_2 = Proximity of other vessel

X_3 = Wind speed

X_4 = Wind direction

X_5 = Current speed

X_6 = Current direction

X_7 = Visibility

$P(\text{Event} \mid X_1,\ldots,X_n)$

Event = Incoming vessel contains RDD

X_1 = Last country docked

X_2 = 2nd to last country docked

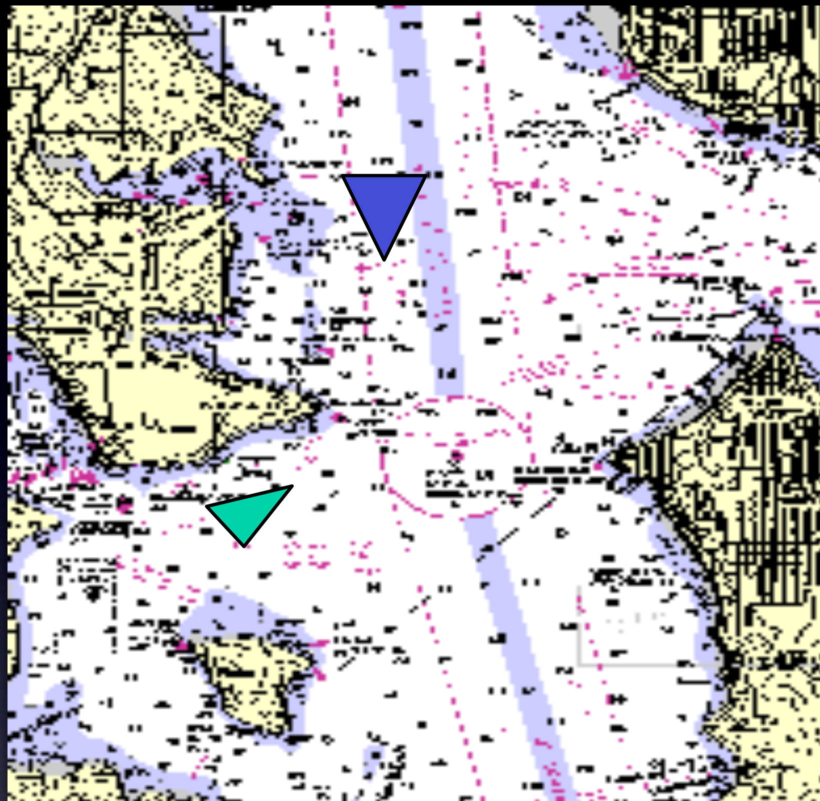
X_3 = 3rd to last country docked

X_4 = Frequency of US calls

X_5 = Vessel ownership

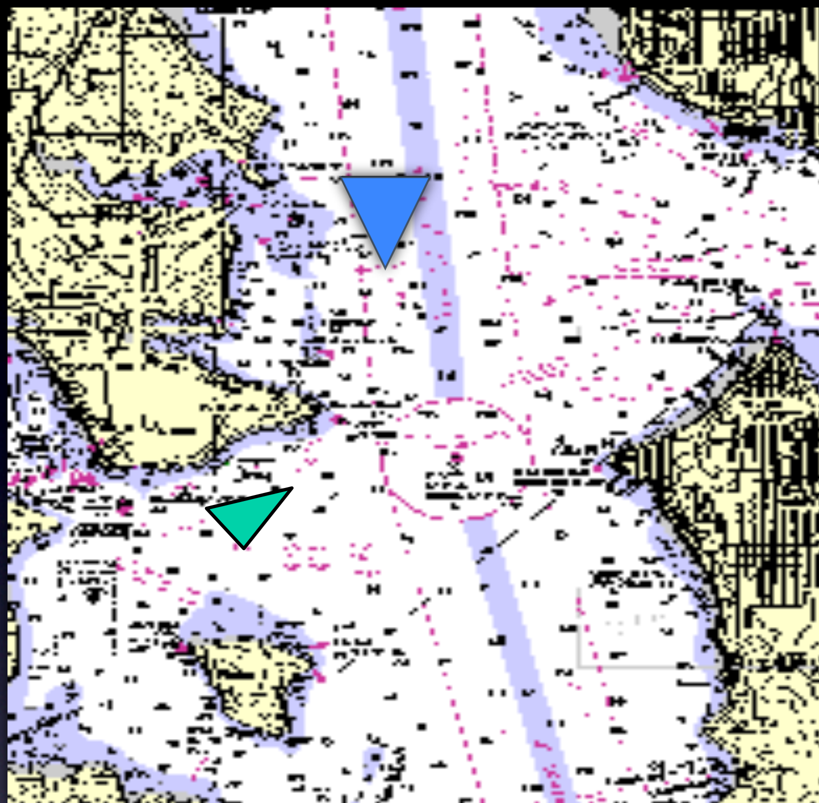
X_6 = Type of vessel

X_7 = Type of crew

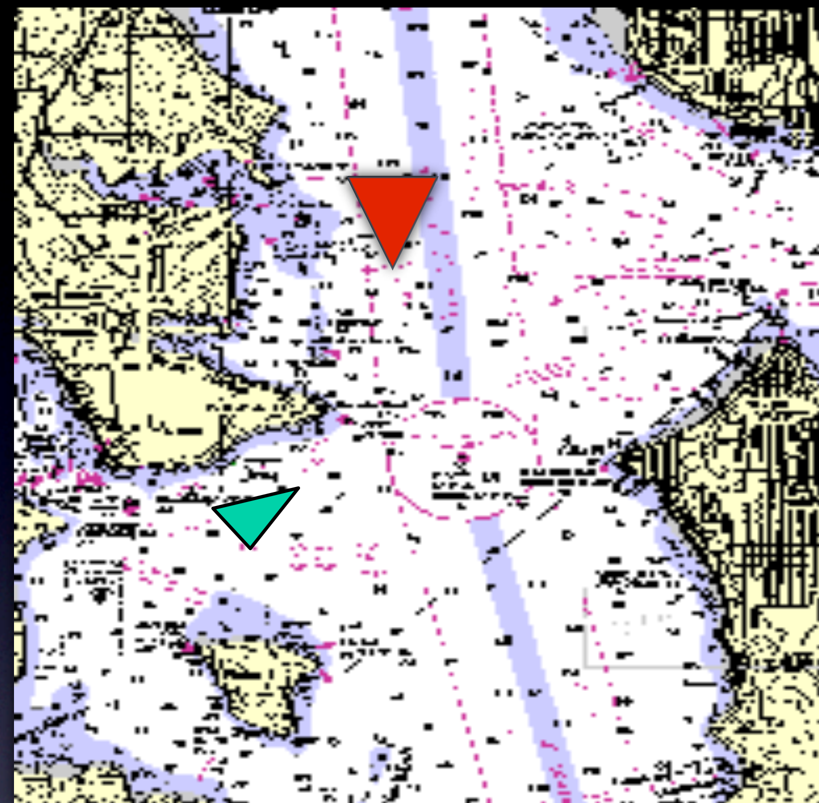


Issaquah class ferry
On the Bremerton to Seattle route
Crossing situation within 15 minutes
Other vessel is a navy vessel
No other vessels around
Good visibility
Negligible wind

What is the probability of a collision?



Issaquah class ferry
On the Bremerton to Seattle route
Crossing situation within 15 minutes
Other vessel is a navy vessel
No other vessels around
Good visibility
Negligible wind



Issaquah class ferry
On the Bremerton to Seattle route
Crossing situation within 15 minutes
Other vessel is a product tanker
No other vessels around
Good visibility
Negligible wind

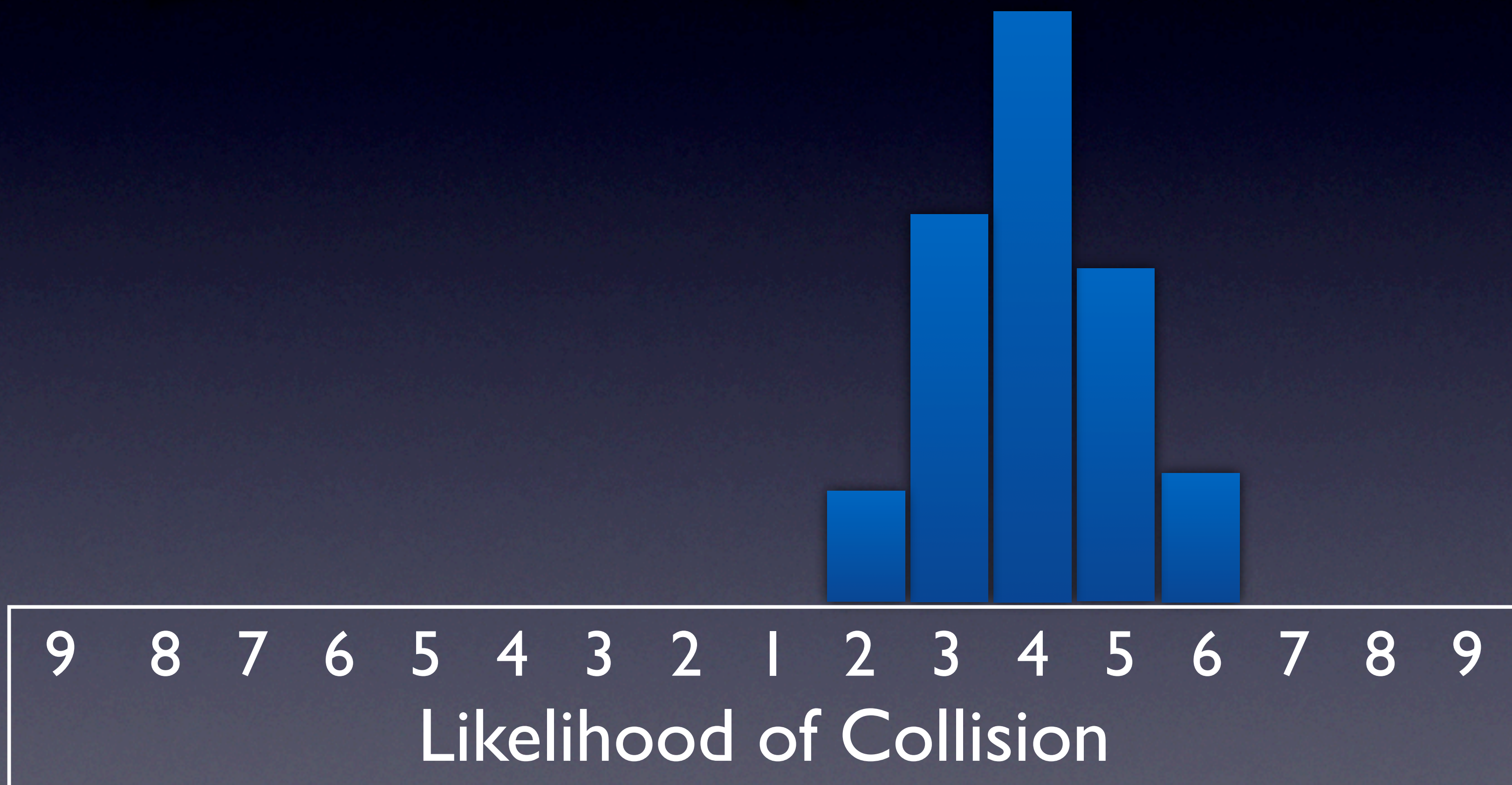
Issaquah	Ferry Class	-
SEA-BRE(A)	Ferry Route	-
Navy	1st Interacting Vessel	Product
Crossing	Traffic Scenario 1st Vessel	-
< 1 mile	Traffic Proximity 1st Vessel	-
No Vessel	2nd Interacting Vessel	-
No Vessel	Traffic Scenario 2nd Vessel	-
No Vessel	Traffic Proximity 2nd Vessel	-
> 0.5 Miles	Visibility	-
Along Ferry	Wind Direction	-
0	Wind Speed	-
	Likelihood of Collision	-
9 8 7 6 5 4 3 2 1 2 3 4 5 6 7 8 9		

$$P(Event \mid X, p_0, \beta) = p_0 \exp\left(X^T \beta\right)$$

$$\frac{P(Event \mid R, \beta)}{P(Event \mid L, \beta)} = \frac{p_0 \exp(R^T \beta)}{p_0 \exp(L^T \beta)} = \exp\left((R - L)^T \beta\right)$$

$$y_{i,j} = \ln(z_{i,j}) = X_i^T \beta + u_{i,j}$$

$$y_{i,j} = \ln(z_{i,j}) = X_i^T \beta + u_{i,j}$$



$$\theta$$

$$u_i = \mu_i - \theta$$

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \sim MVNormal\left(\underline{0}, \Sigma\right)$$

$$\pi(\theta;\mu,\Sigma) \propto \exp\left(-\left(\theta-\mu^{*}\right)^{2}/2\sigma^{*2}\right)$$

$$\mu^{*} = \frac{\underline{1}^{T}\Sigma^{-1}\mu}{\underline{1}^{T}\Sigma^{-1}\underline{1}}$$

$$\sigma^{*2} = \frac{1}{\underline{1}^{T}\Sigma^{-1}\underline{1}}$$

$$\theta$$

$$X_i^T \beta$$

$$u_i = \mu_i - \theta$$

$$u_{i,j} = y_{i,j} - X_i^T \beta$$

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \sim MVNormal(\underline{0}, \Sigma)$$

$$\underline{u}_i = \begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,p} \end{pmatrix} \sim MVNormal(\underline{0}, \Sigma)$$

$$\begin{pmatrix} y_{1,1} & \cdots & y_{1,p} \\ \vdots & \ddots & \vdots \\ y_{N,1} & \cdots & y_{N,p} \end{pmatrix} = \begin{pmatrix} x_{1,1} & \cdots & x_{1,q} \\ \vdots & \ddots & \vdots \\ x_{N,1} & \cdots & x_{N,q} \end{pmatrix} \begin{pmatrix} \beta_1 & \cdots & \beta_1 \\ \vdots & \ddots & \vdots \\ \beta_q & \cdots & \beta_q \end{pmatrix} + \begin{pmatrix} u_{1,1} & \cdots & u_{1,p} \\ \vdots & \ddots & \vdots \\ u_{N,1} & \cdots & u_{N,p} \end{pmatrix}$$

$$\mathbf{Y} = \mathbf{X}\beta\underline{1}^T + \mathbf{U}$$

$$p\left(\mathbf{Y} \mid \mathbf{X}, \beta, \Sigma\right) \propto |\Sigma|^{-\frac{N}{2}} \exp\left\{-\tfrac{1}{2} tr\left(\mathbf{V}\Sigma^{-1}\right)\right\} \quad \exp\left\{-\tfrac{1}{2} tr\left(\left(\hat{\mathbf{B}} - \beta \underline{1}^T\right)^T \mathbf{X}^T \mathbf{X} \left(\hat{\mathbf{B}} - \beta \underline{1}^T\right) \Sigma^{-1}\right)\right\}$$

$$\mu_*^\beta = \frac{\hat{\mathbf{B}} \Sigma^{-1} \underline{1}}{\underline{1}^T \Sigma^{-1} \underline{1}}$$

$$\Sigma_*^\beta = \frac{\left(X^T X\right)^{-1}}{\underline{1}^T \Sigma^{-1} \underline{1}}$$

$$(\Sigma) \sim Inv-Wishart(\mathbf{G}, m)$$

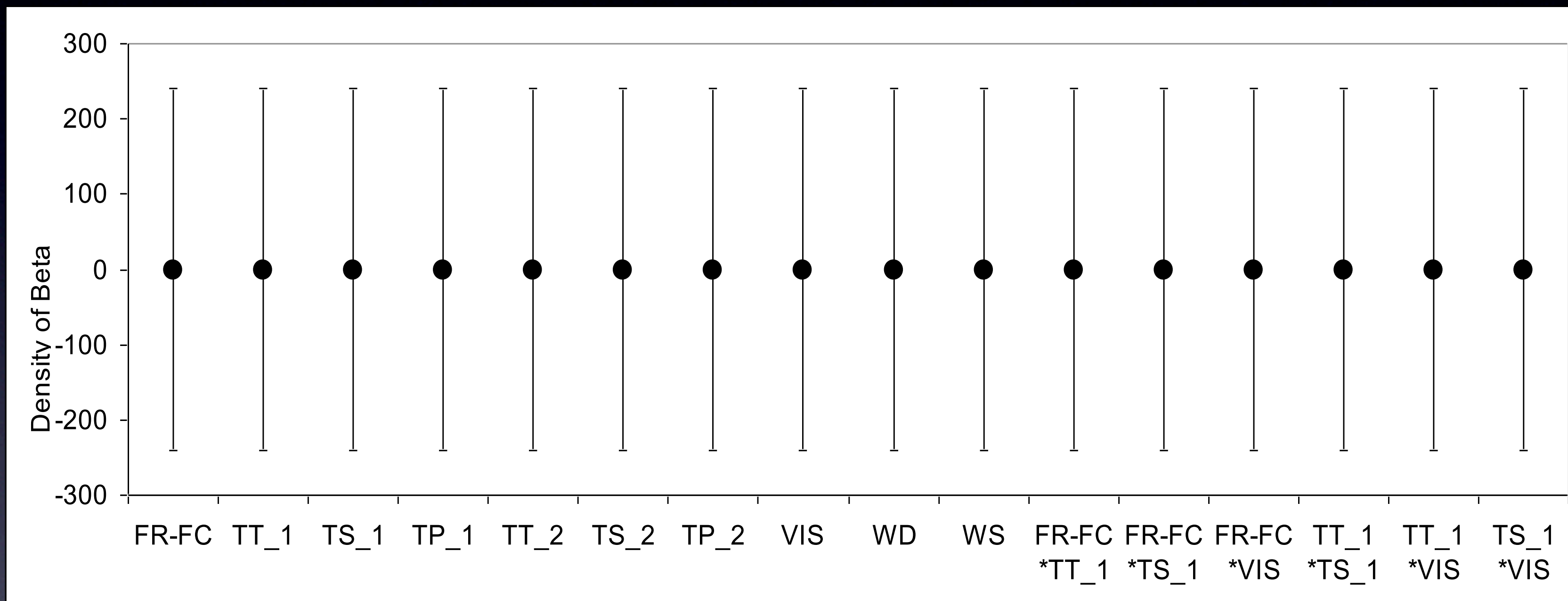
$$(\beta \mid \mathbf{Y}, \mathbf{X}, \Sigma) \sim MVNormal\left(\varphi, \frac{\mathbf{A}}{\underline{1}^T \Sigma^{-1} \underline{1}}\right)$$

$$\mathbf{V} = \left(\mathbf{Y} - \mathbf{X}\hat{\mathbf{B}}\right)^T \left(\mathbf{Y} - \mathbf{X}\hat{\mathbf{B}}\right)$$

$$(\Sigma \mid \mathbf{Y}, \mathbf{X}) \sim Inv-Wishart(\mathbf{G} + \mathbf{V}, m + N)$$

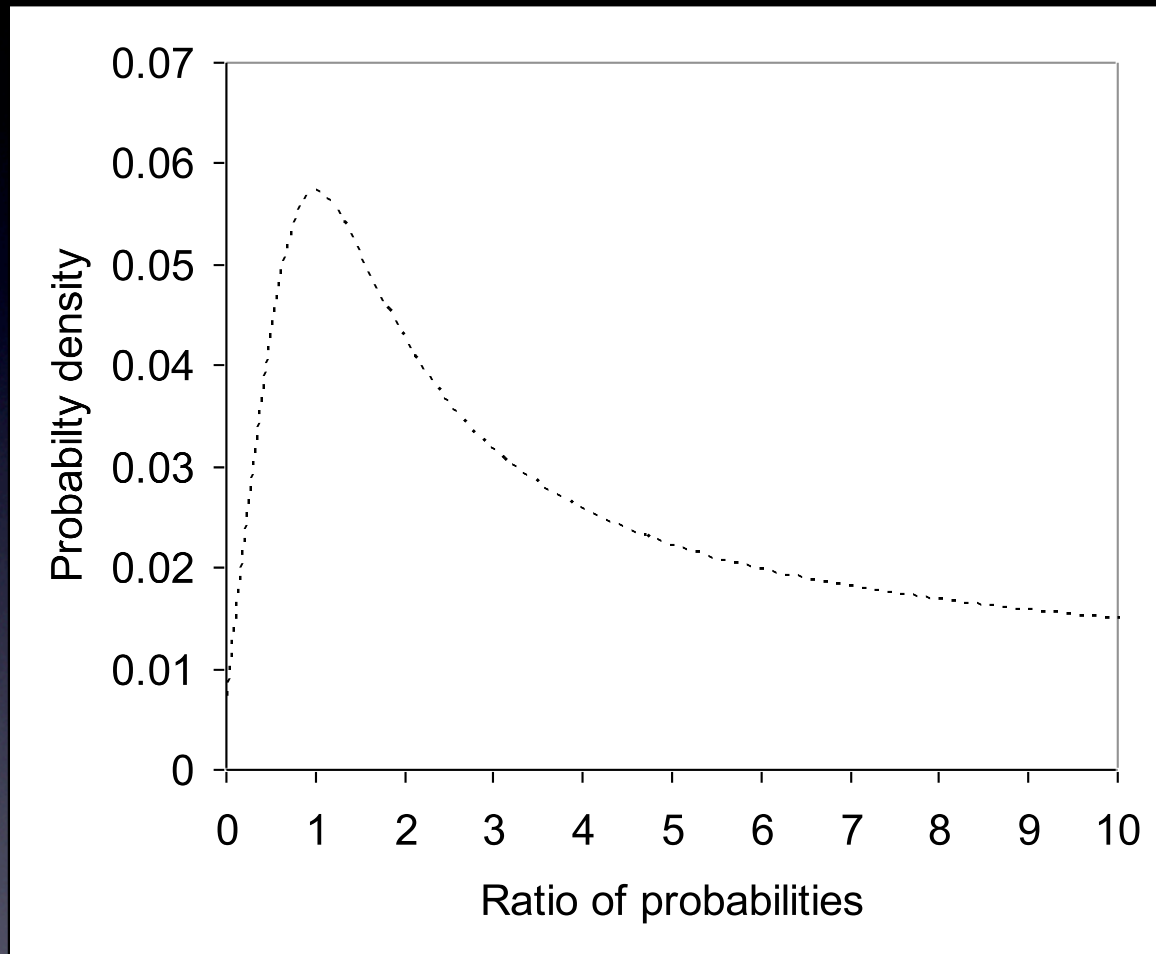
$$(\beta \mid \mathbf{Y}, \mathbf{X}, \Sigma) \sim MVNormal\left(\left(\mathbf{A}^{-1} + \mathbf{X}^T\mathbf{X}\right)^{-1}\left(\mathbf{X}^T\mathbf{X}\frac{\hat{\mathbf{B}}\Sigma^{-1}\underline{1}}{\underline{1}^T\Sigma^{-1}\underline{1}} + \mathbf{A}^{-1}\varphi\right), \frac{\left(\mathbf{A}^{-1} + \mathbf{X}^T\mathbf{X}\right)^{-1}}{\underline{1}^T\Sigma^{-1}\underline{1}}\right)$$

Description	Notation	Values
Ferry route and class	FR_FC	26
Type of 1st interacting vessel	TT_1	13
Scenario of 1st interacting vessel	TS_1	4
Proximity of 1st interacting vessel	TP_1	Binary
Type of 2nd interacting vessel	TT_2	5
Scenario of 2nd interacting vessel	TS_2	4
Proximity of 2nd interacting vessel	TP_2	Binary
Visibility	VIS	Binary
Wind direction	WD	Binary
Wind speed	WS	Continuous

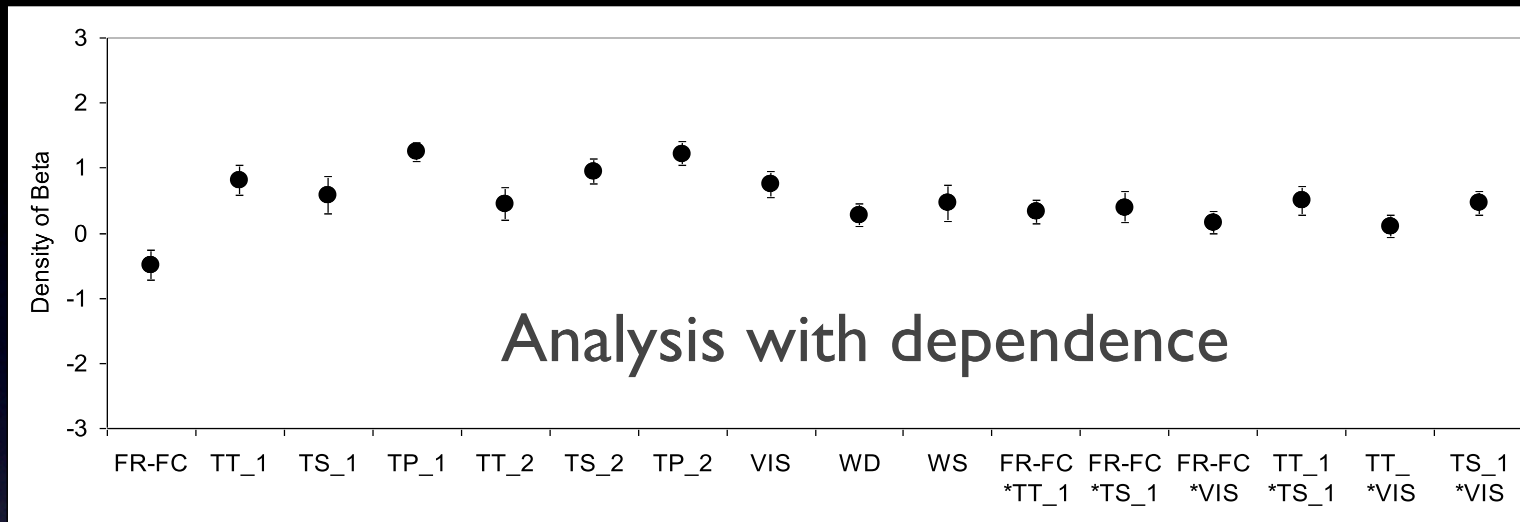


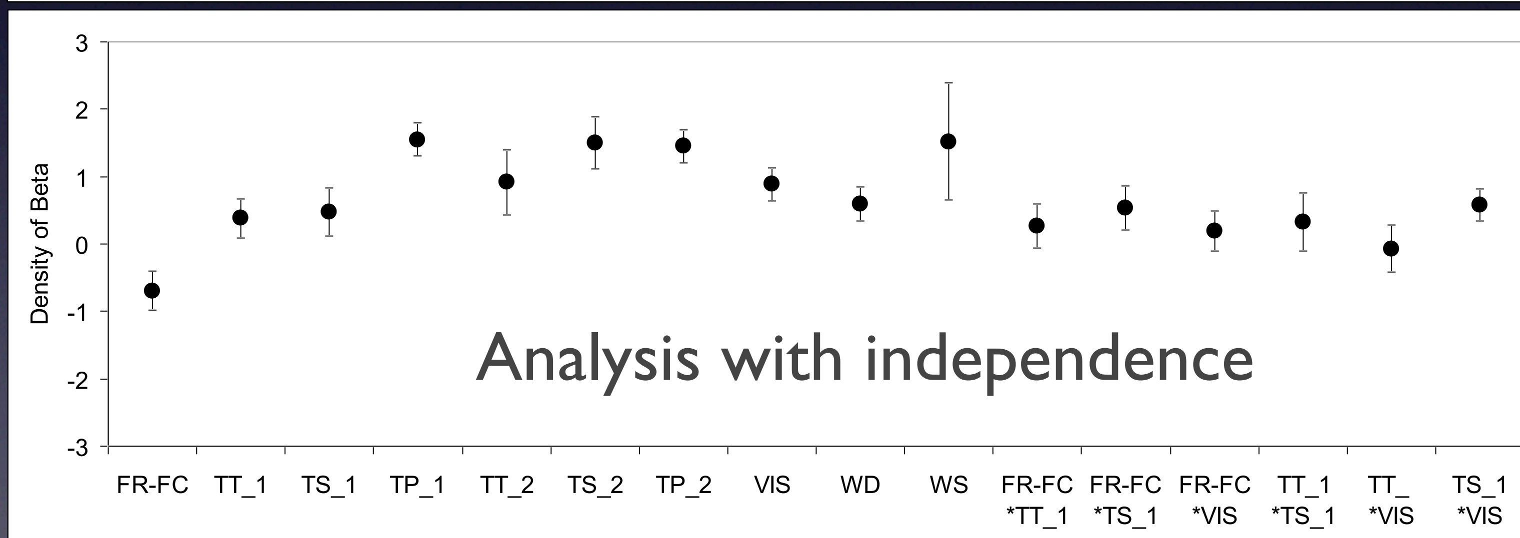
Assume independence between the experts a priori

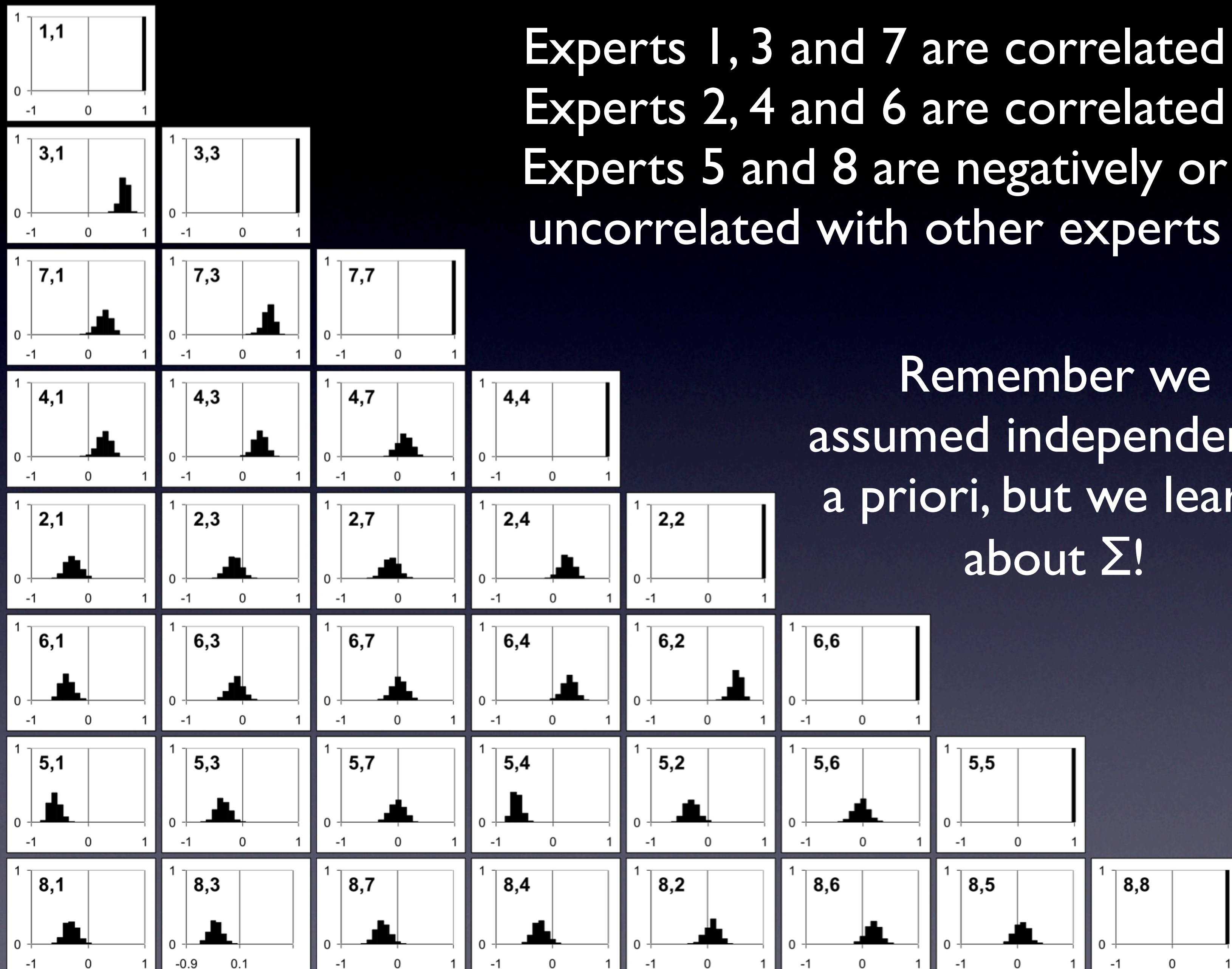
Comparing the two scenarios we pictured earlier a priori



Doesn't dependence between experts increase posterior variance?



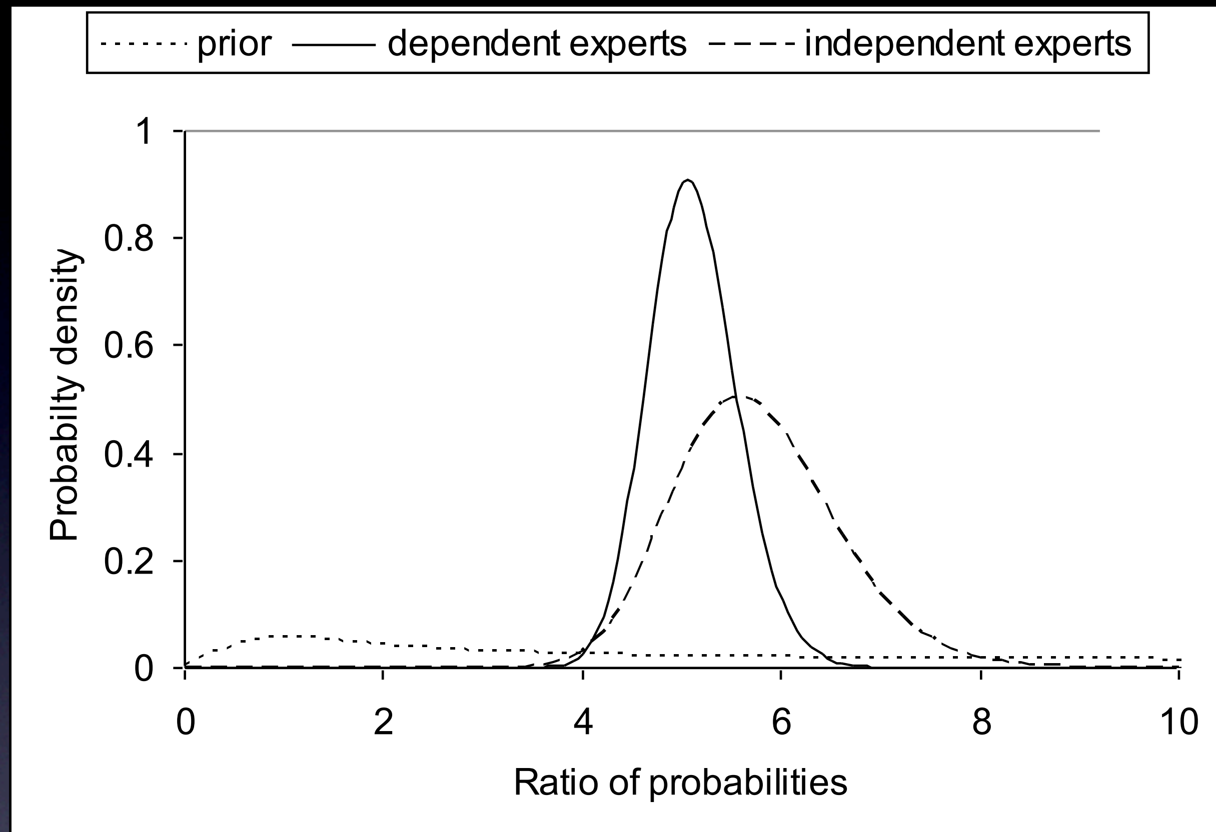




Experts 1, 3 and 7 are correlated
Experts 2, 4 and 6 are correlated
Experts 5 and 8 are negatively or uncorrelated with other experts

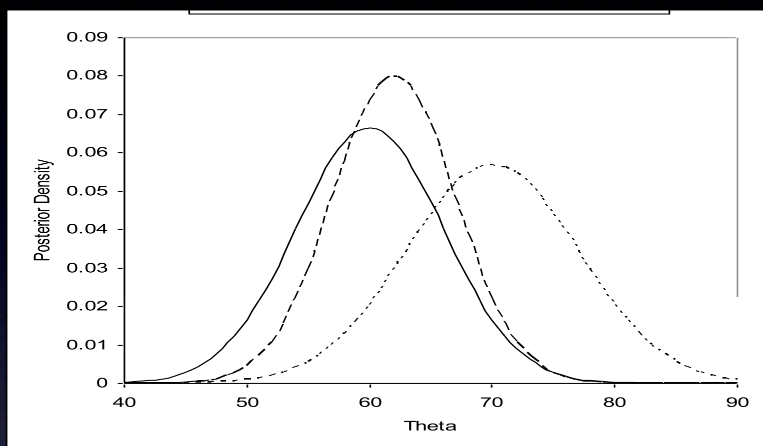
Remember we assumed independence a priori, but we learnt about Σ!

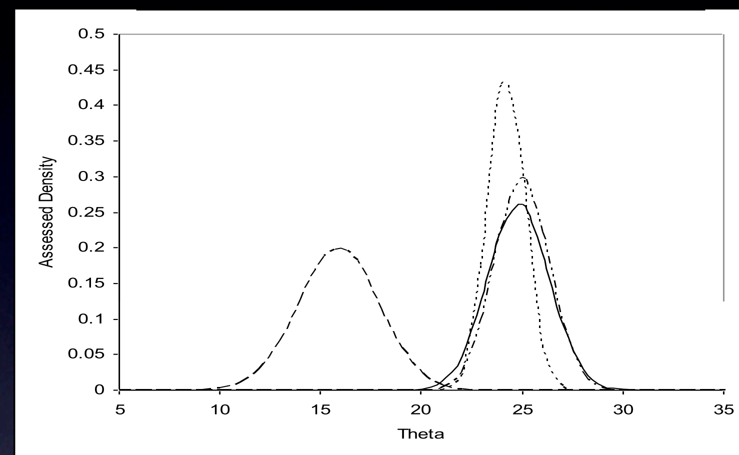
Comparing the two scenarios we pictured earlier



90% Credibility Interval	Prior	$[1.88*10^{-35}, 5.32*10^{34}]$	
	Dependent	[4.38,5.84]	½ width = 0.73
	Independent	[4.43,7.04]	½ width = 1.3

Getting the Right Mix of Experts

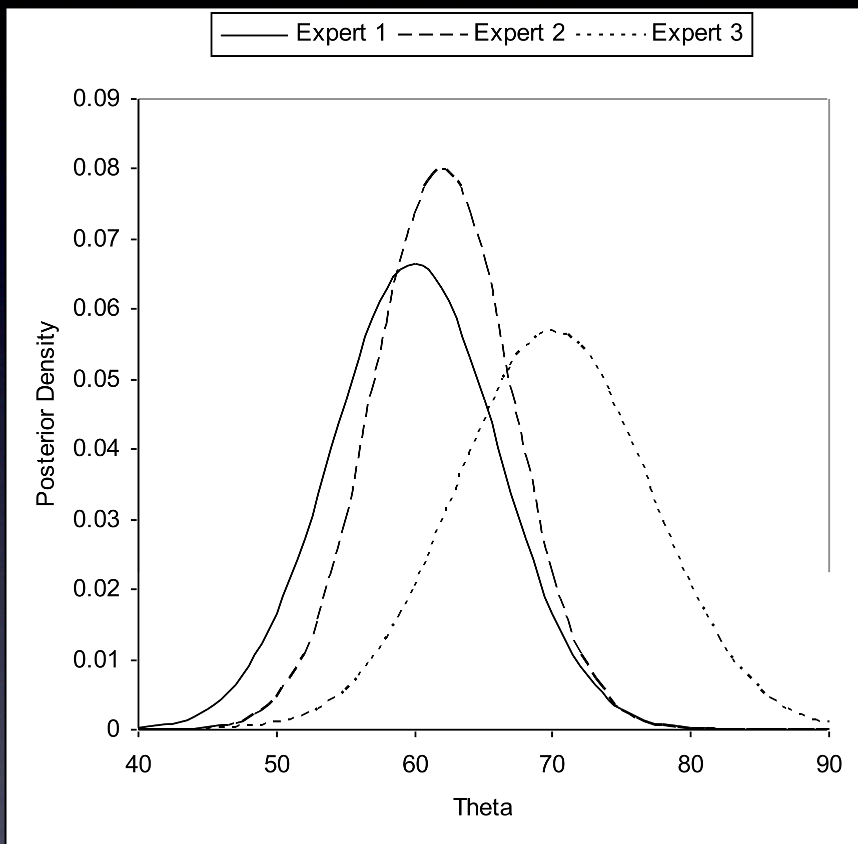




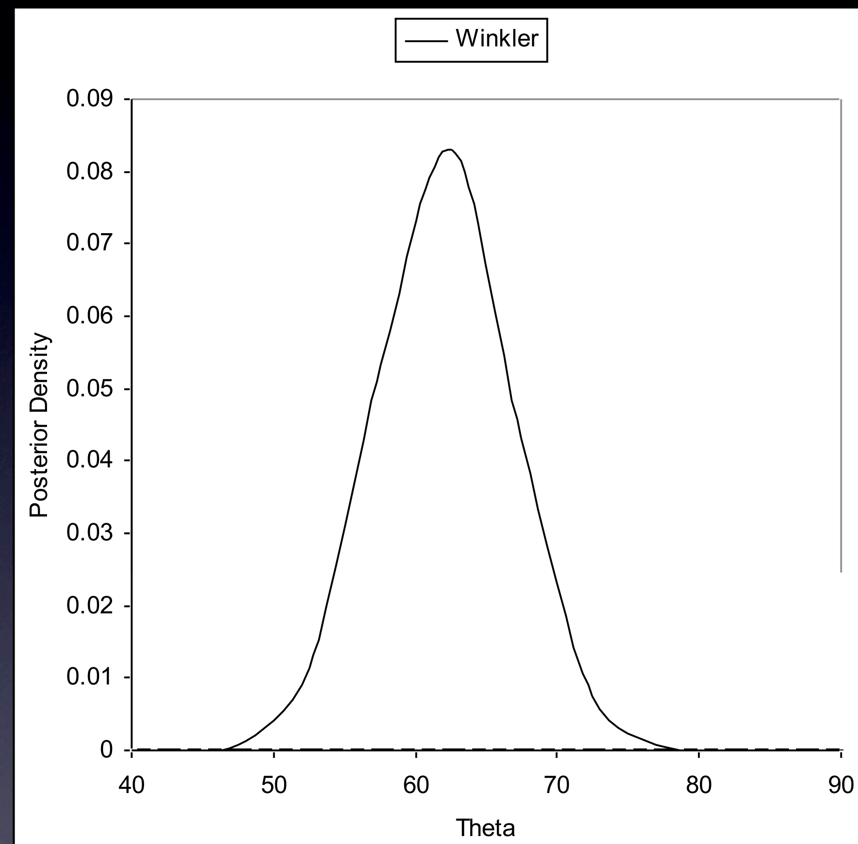
$$\left(z_1,\ldots,z_p\right)$$

$$\left(r_1,\ldots,r_p\right)$$

$$\left(z_1, \ldots, z_n \mid \theta, \Sigma\right) \sim MVNormal(\underline{\theta}, \Sigma)$$



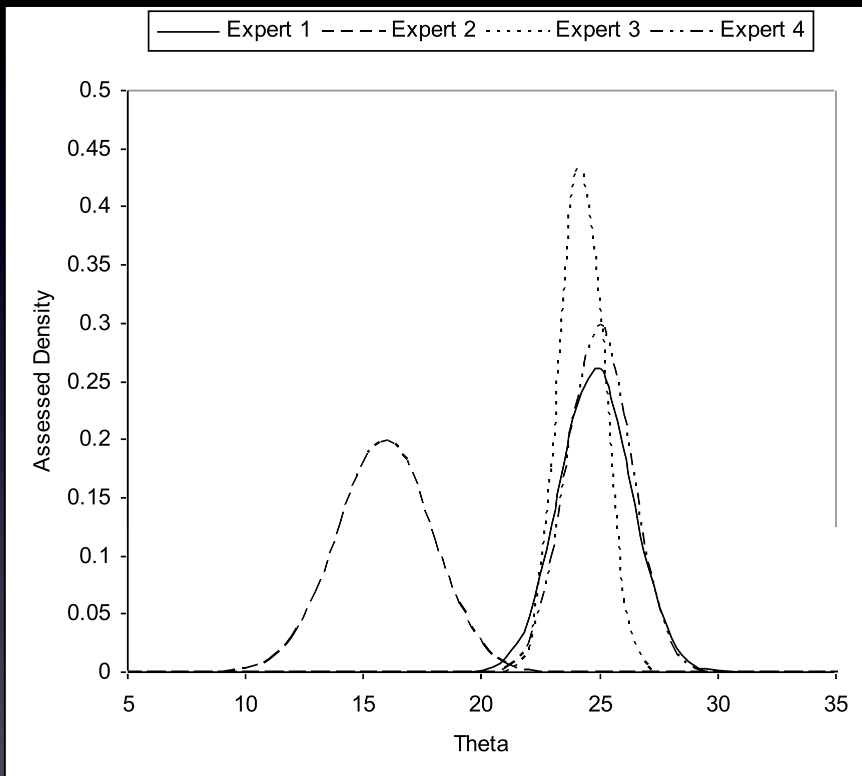
Expert 1
Expert 2
Expert 3
Posterior Density
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
40
50
60
70
80
90
Theta



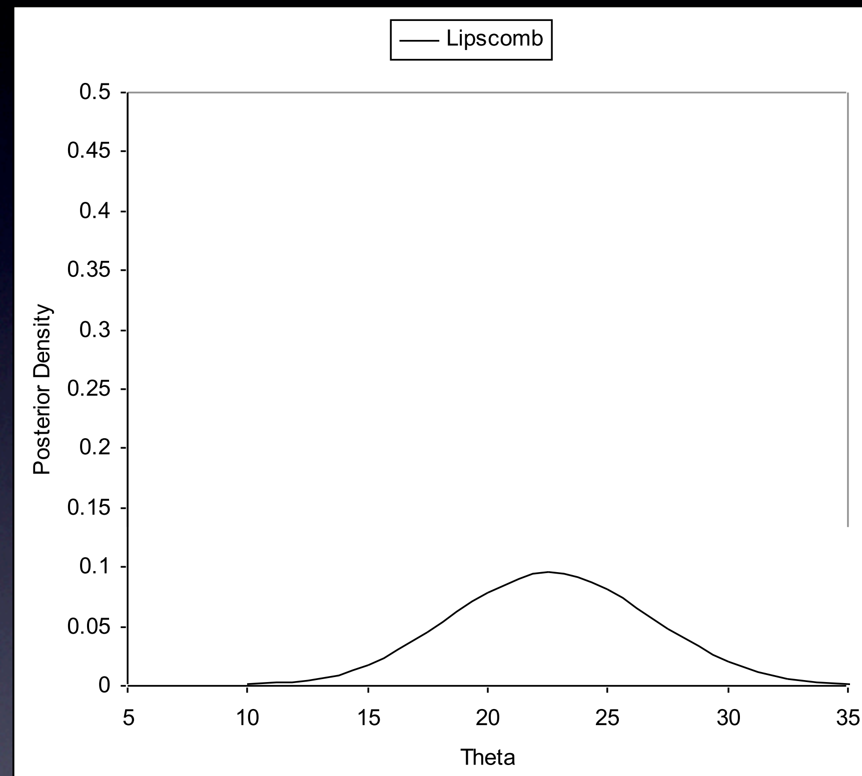
Winkler
Posterior Density
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
40
50
60
70
80
90
Theta

$$\left(z_i \mid \mu_i, \alpha_i, \gamma_i\right) \sim N(\mu_i, r_i)$$

$$\left(\mu_i \mid \theta, \lambda\right) \sim N(\theta, \lambda)$$



Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
35
Theta



Lipscomb
Posterior Density
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
35
Theta

$$\left(z_i \mid \theta, r_i, \alpha_i, \gamma_i\right) \sim N(\theta + \alpha_i, \gamma_i r_i)$$

$$\left(r_i \mid \theta, z_i, a_\gamma, b_\gamma\right) \sim Ga(a_\gamma, b_\gamma)$$

$$\alpha_1,,\ldots,\alpha_p \sim N(0,\lambda)$$

$$\gamma_1,,\ldots,\gamma_p \sim Gamma(a,b)$$

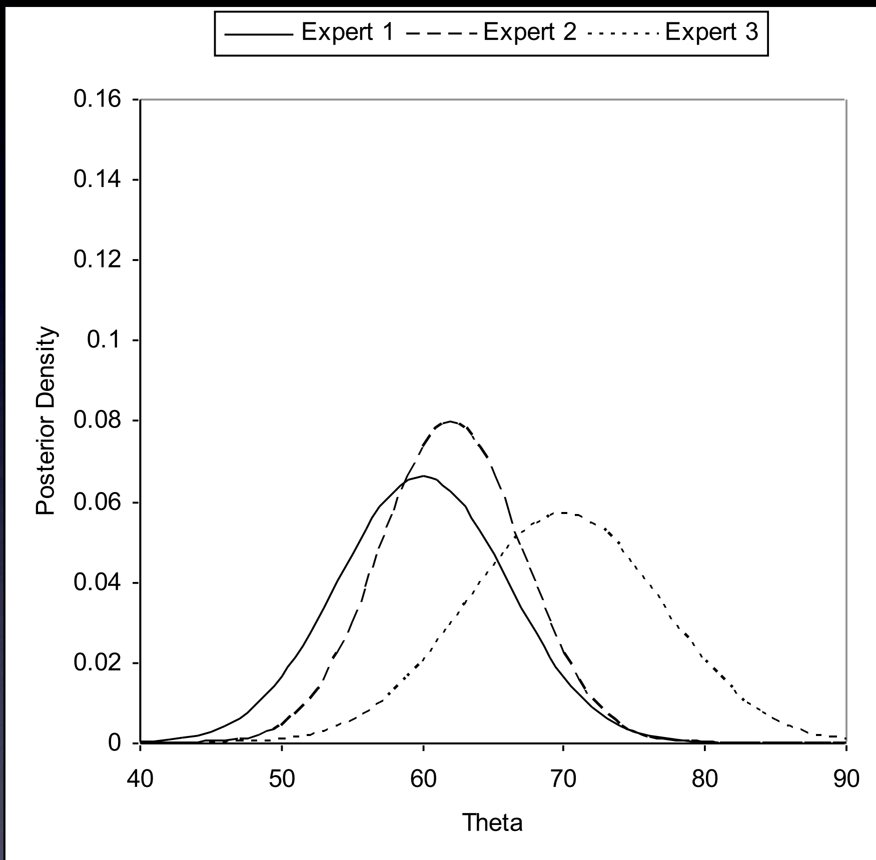
$$\left(z_i \mid \theta, r_i, \alpha_i, \gamma_i\right) \sim N(\theta + \alpha_i, \gamma_i r_i)$$

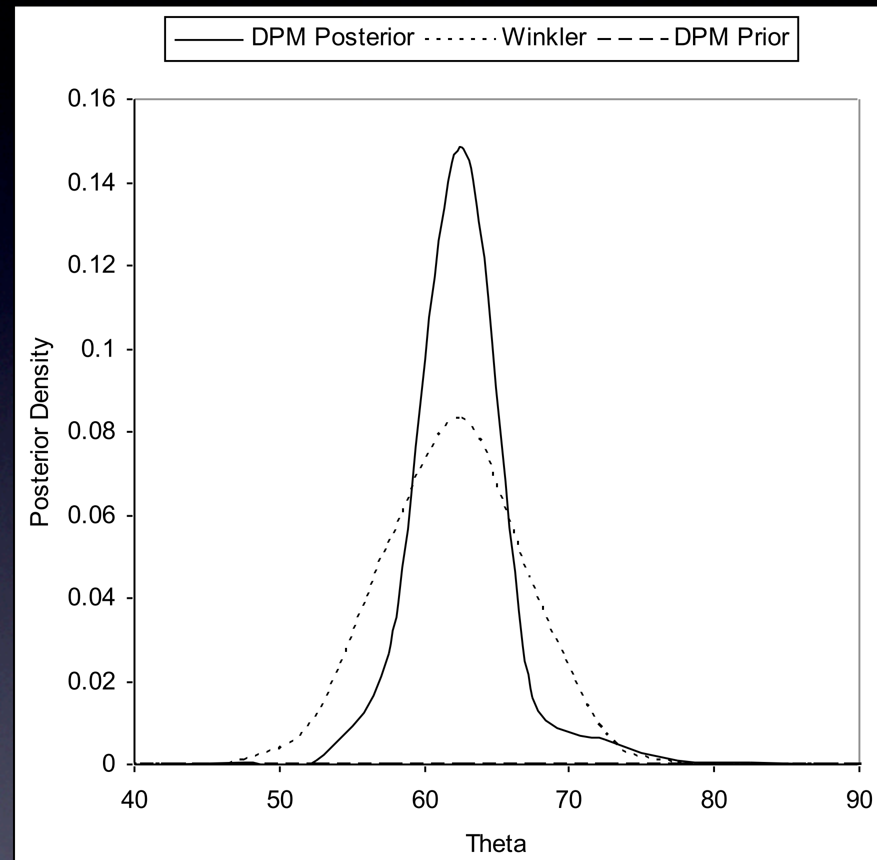
$$\left(r_i \mid \theta, z_i, a_\gamma, b_\gamma\right) \sim Ga(a_\gamma, b_\gamma)$$

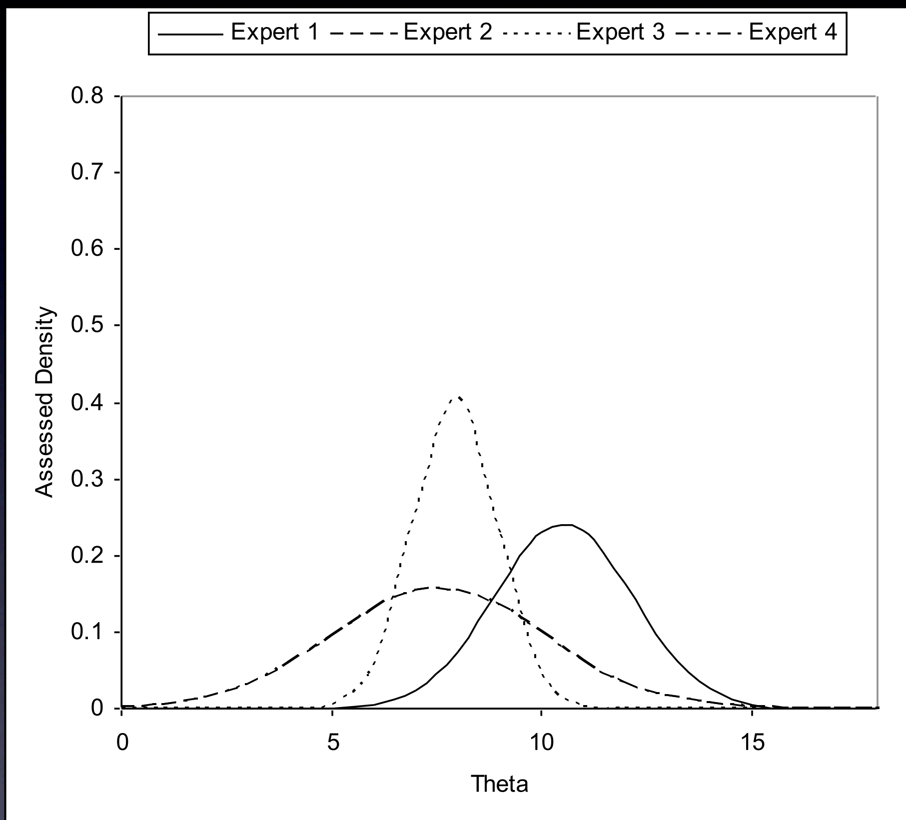
$$(\alpha_1, \gamma_1), \ldots, (\alpha_p, \gamma_p) \sim G$$

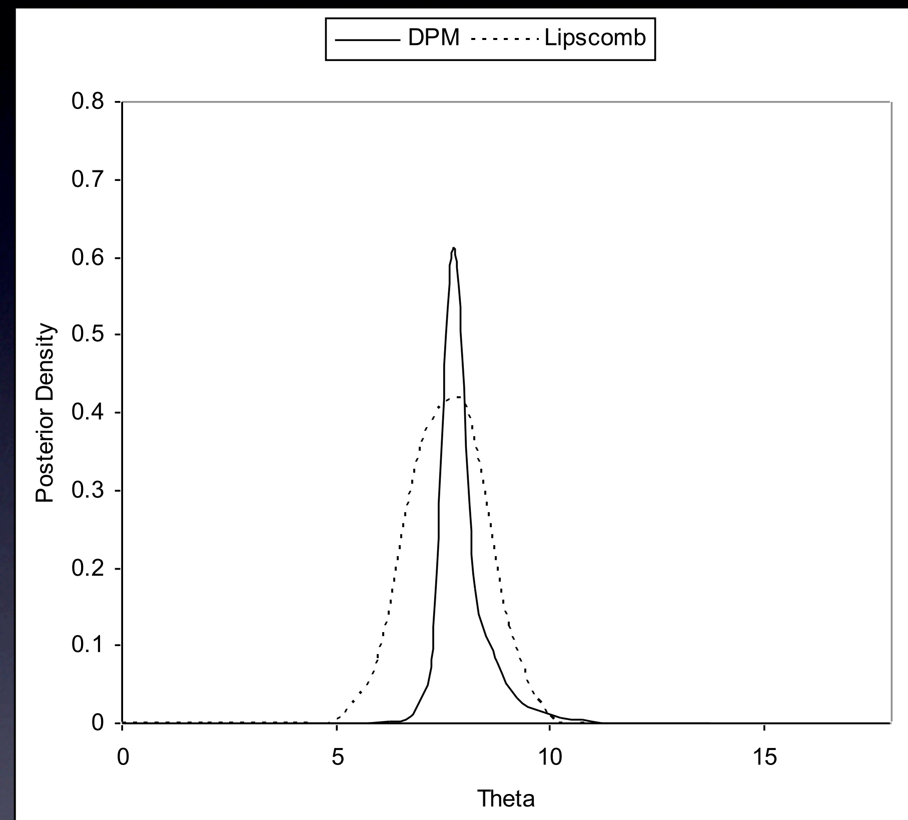
$$G \sim DP(G_0, M)$$

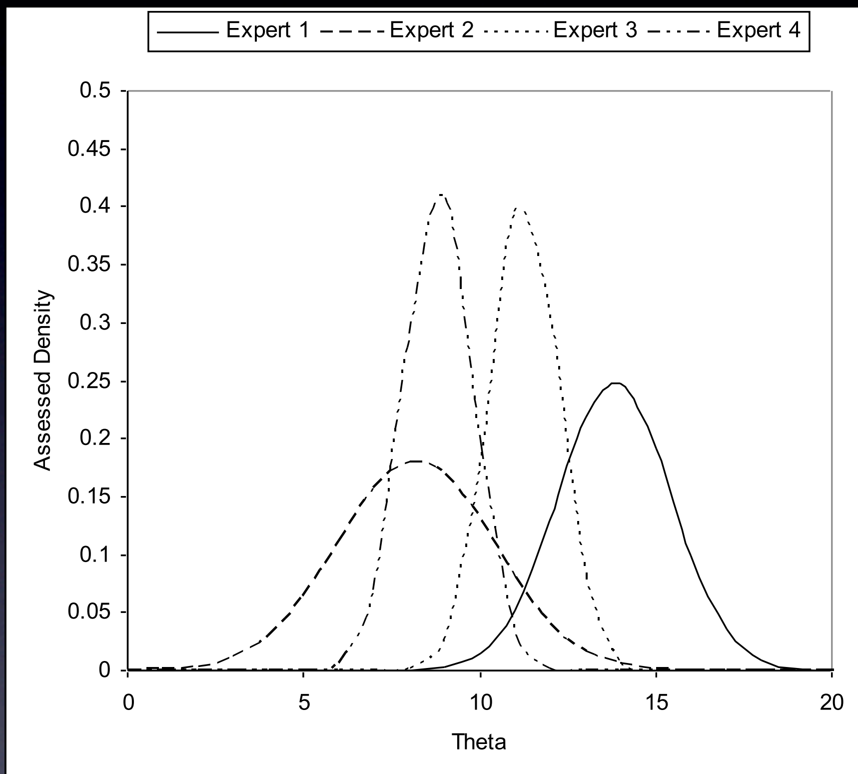
$$G_0 = gamma(a, b)$$

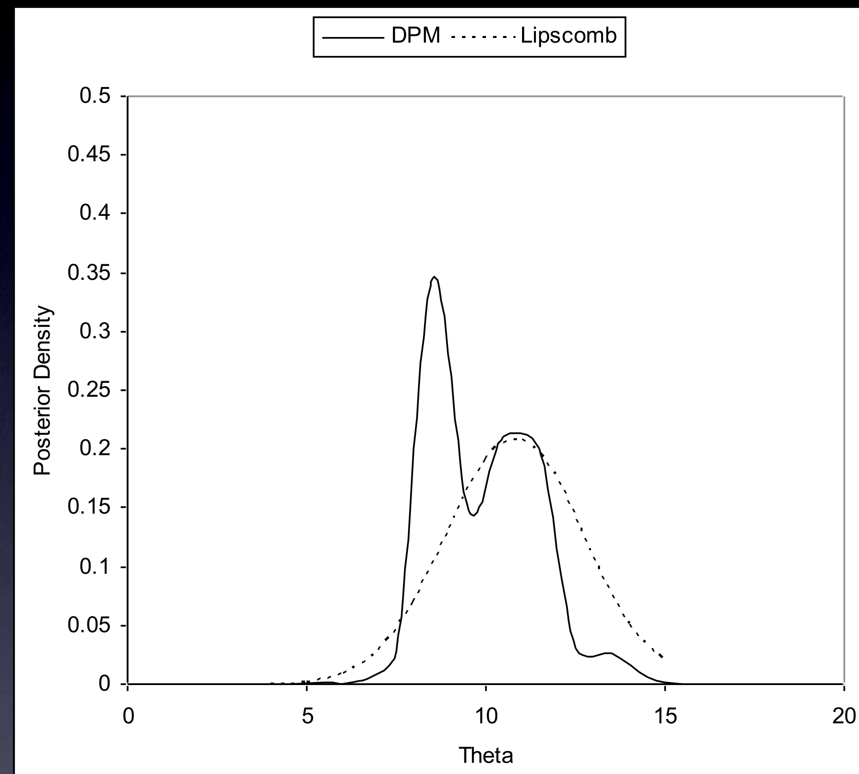

Expert 1
Expert 2
Expert 3
Posterior Density
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
40
50
60
70
80
90
Theta

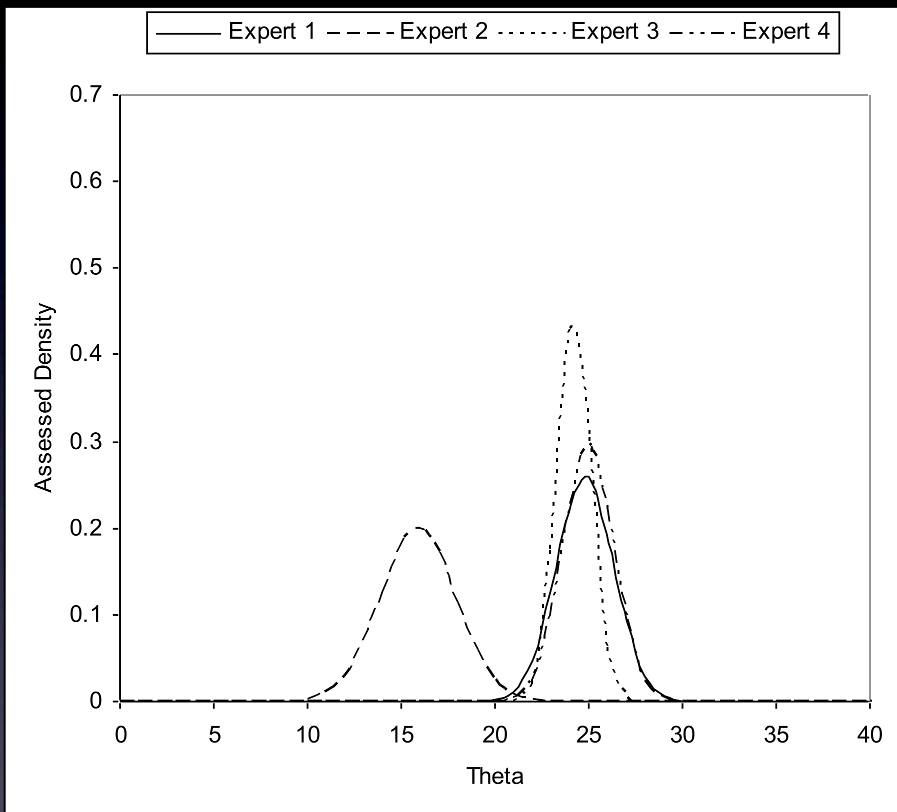

DPM Posterior
Winkler
DPM Prior
Posterior Density
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
40
50
60
70
80
90
Theta


Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
Theta

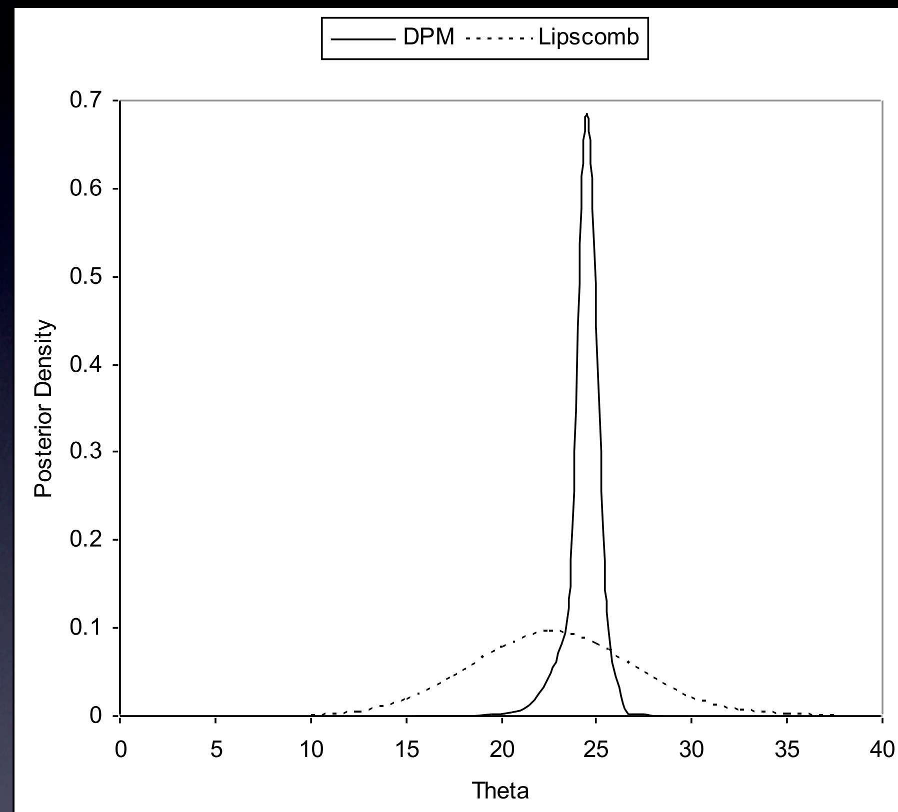

DPM
Lipscomb
Posterior Density
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
Theta


Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
Theta
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
5
10
15
20

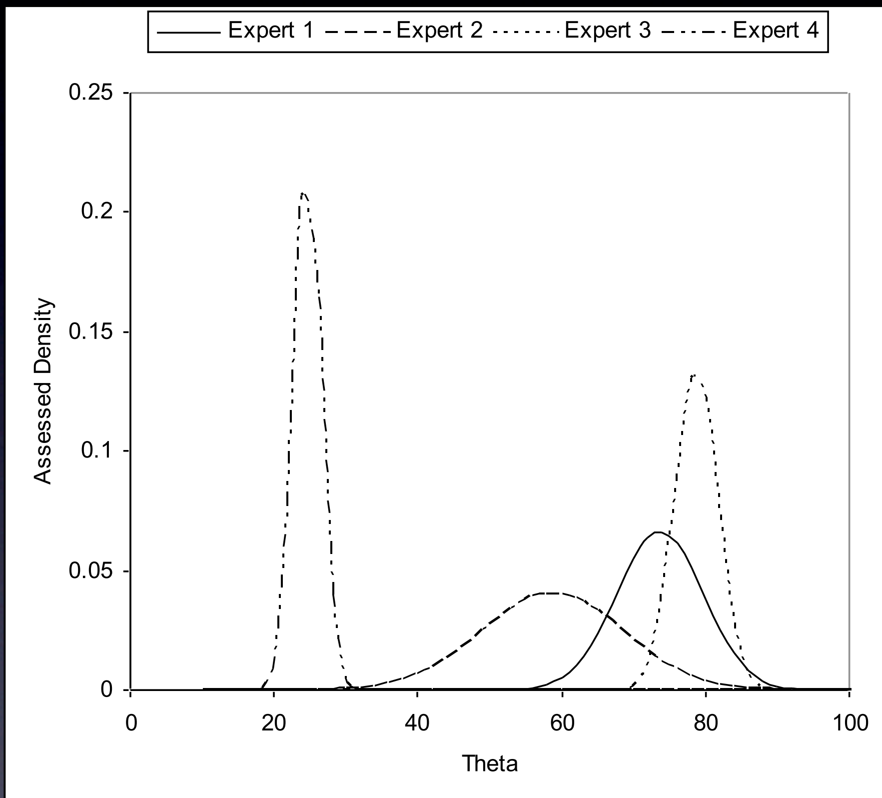

DPM
Lipscomb
Posterior Density
Theta
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
5
10
15
20

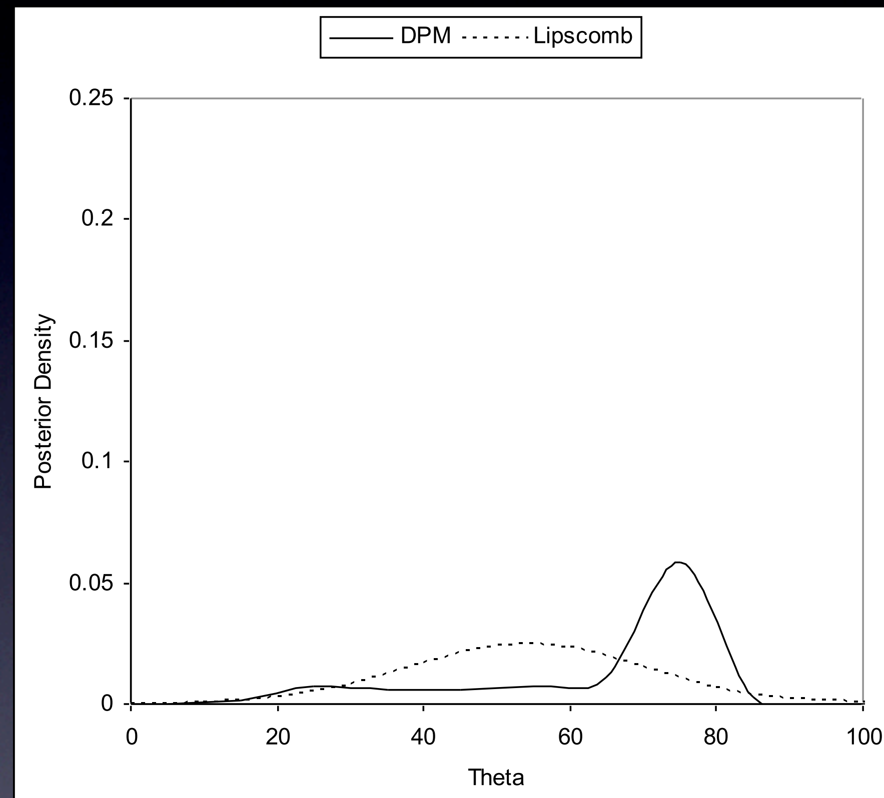


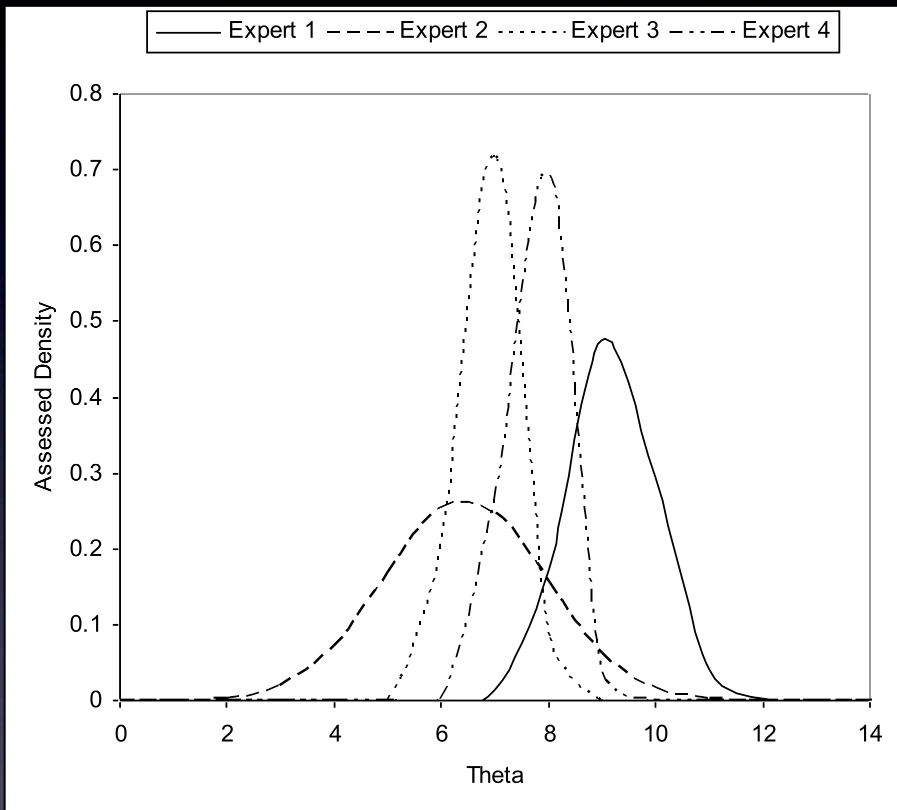
Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
Theta
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
20
25
30
35
40

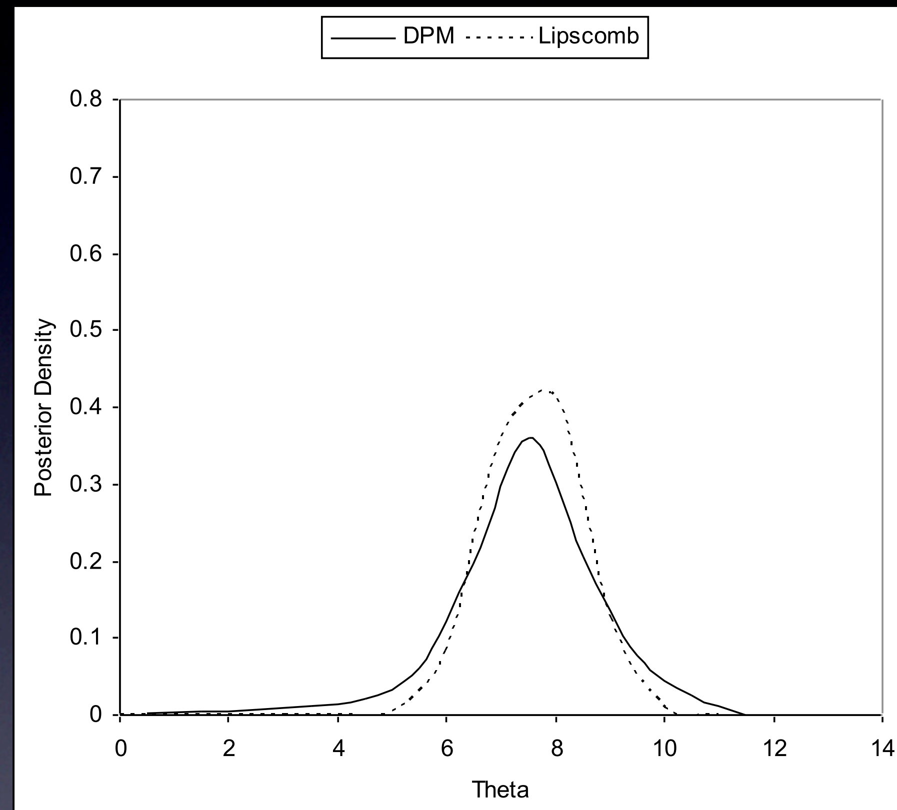


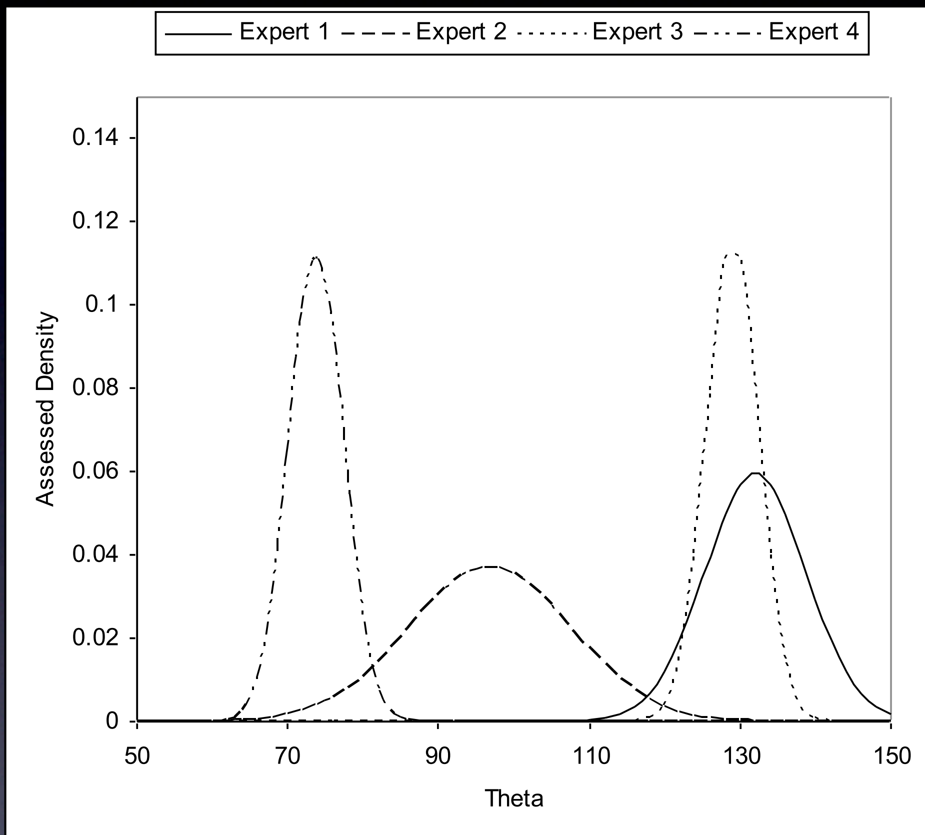
DPM
Lipscomb
Posterior Density
Theta
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
20
25
30
35
40


Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100
Theta

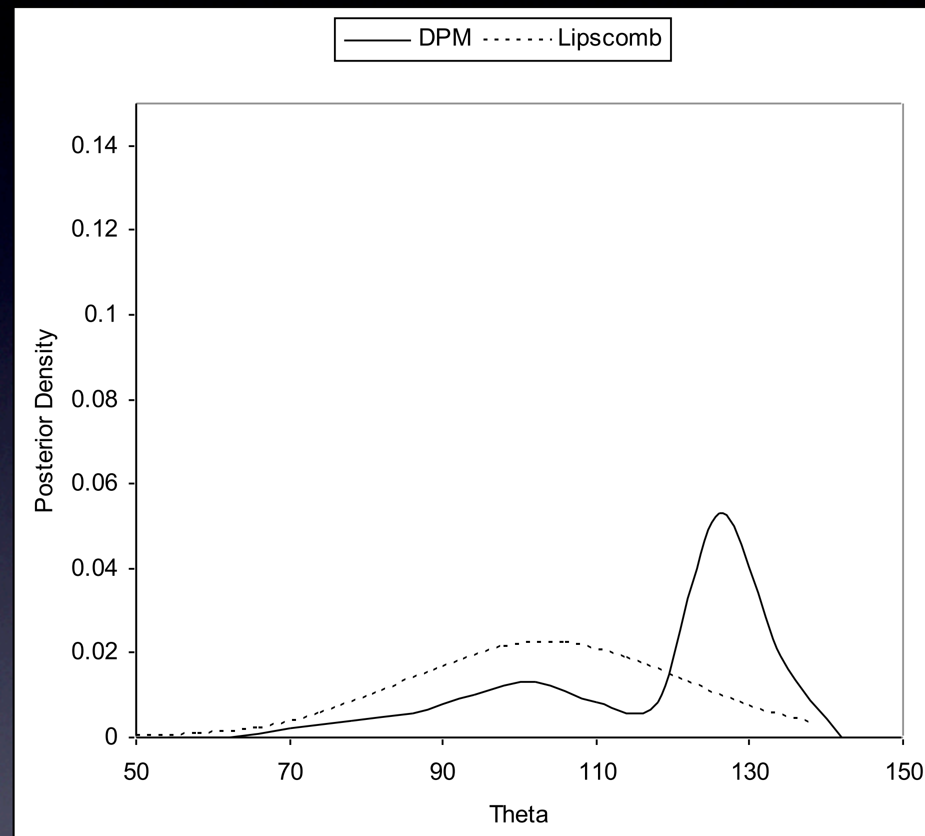

DPM
Lipscomb
Posterior Density
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100
Theta


Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
2
4
6
8
10
12
14
Theta


DPM
Lipscomb
Posterior Density
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
2
4
6
8
10
12
14
Theta



Expert 1
Expert 2
Expert 3
Expert 4
Assessed Density
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
50
70
90
110
130
150
Theta



DPM
Lipscomb
Posterior Density
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
50
70
90
110
130
150
Theta