Some problems around Mathematics of Planet Earth

Question

What would be the rise of the oceans if all glaciers of Antarctica and Greenland were to melt?

Greenland

Area of glaciers: 1,775,637 km^2

Volume of glaciers: 2,850,000 km^3

Antarctica

Area: 14,000,000 km^2

The major part of the continent is covered by ice with a mean thickness of at least 1,6 km

Total area of oceans

335,258,000 km^2

Rise of the sea level if all the melted water was uniformly spread over the oceans

It is given by

$$\frac{\text{Total volume of ice sheet}}{\text{Total area of oceans}} = \frac{25,250,000}{335,258,000} = 0.075 \text{ km}$$

hence approximately **75 meters**

Limits of the model

- Some of the water may evaporate. But then, will meteo rainfalls increase?
  - Will some of the water enrich ground water instead of staying at the surface of the Earth?
  - Part of the land will be covered by water, so water will be spread over a larger surface than that of the oceans.
  - Others, that scientists may or may not have planned…

Let us discuss 3: Part of the land will be covered by water, so water will be spread over a larger surface than that of the oceans.

- Percentage of the area of the Earth covered by the oceans: approximately 70%
  - The mean altitude of the Earth is 840m. Hence it is reasonable to assume that less than one fourth of the Earth will be covered by water, i.e. less than 1/4 x 30% = 7.5% of the surface of the Earth.
  - Height of the water if all the melted water is spread over the oceans plus ¼ of the land:

$$75 \times \frac{\text{Surface of the oceans}}{\text{Surface to cover}} = 75 \times \frac{70}{77.5} = 68 \text{ m}$$

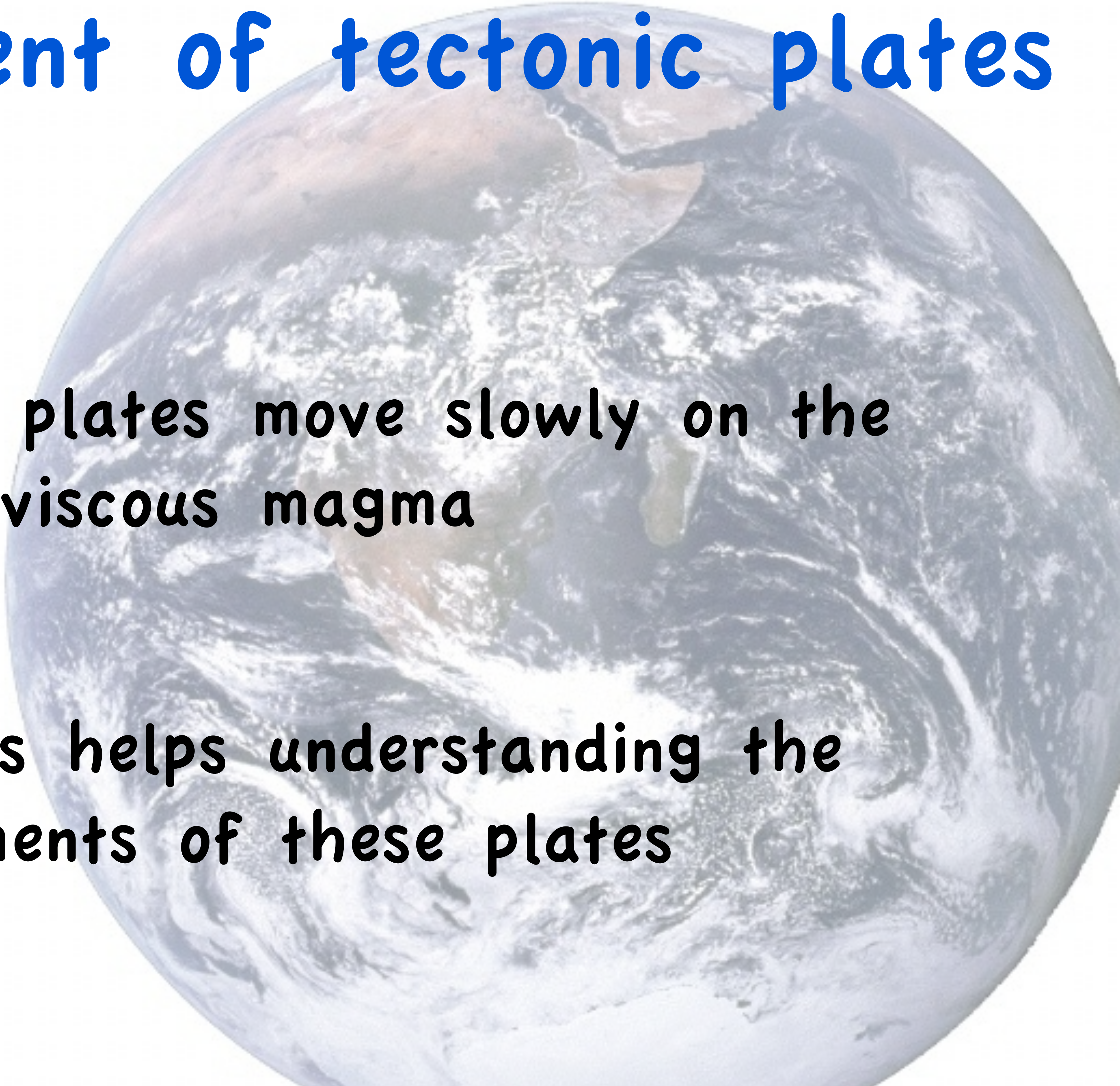
The rise will be at least 68 meters!

Hence why do the scientists only predict a rise of the order of one meter til 2100?

Because the ice cannot melt so fast

But we see that we must look beyond 2100...

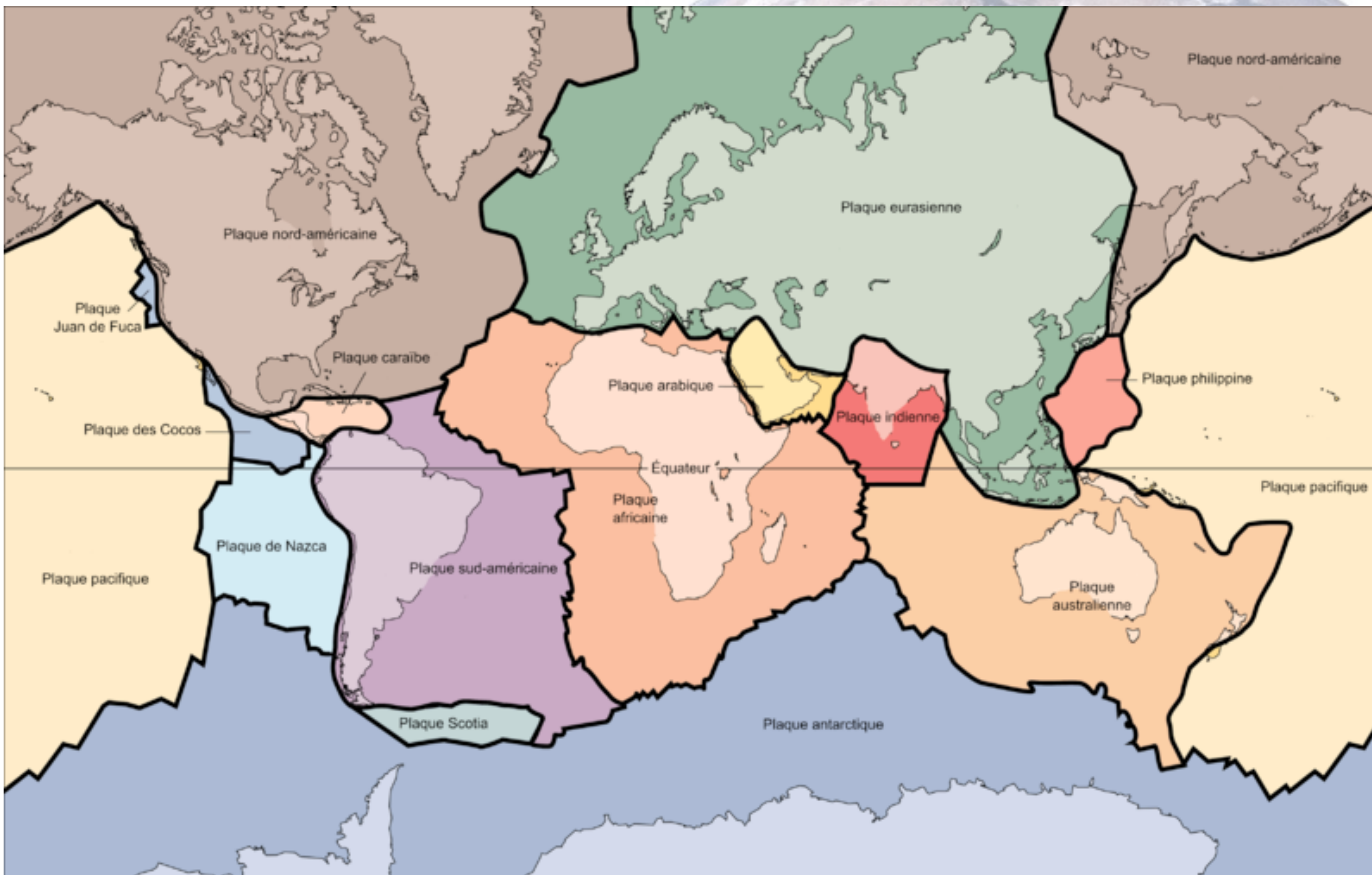
The movement of tectonic plates



The tectonic plates move slowly on the viscous magma

Mathematics helps understanding the movements of these plates

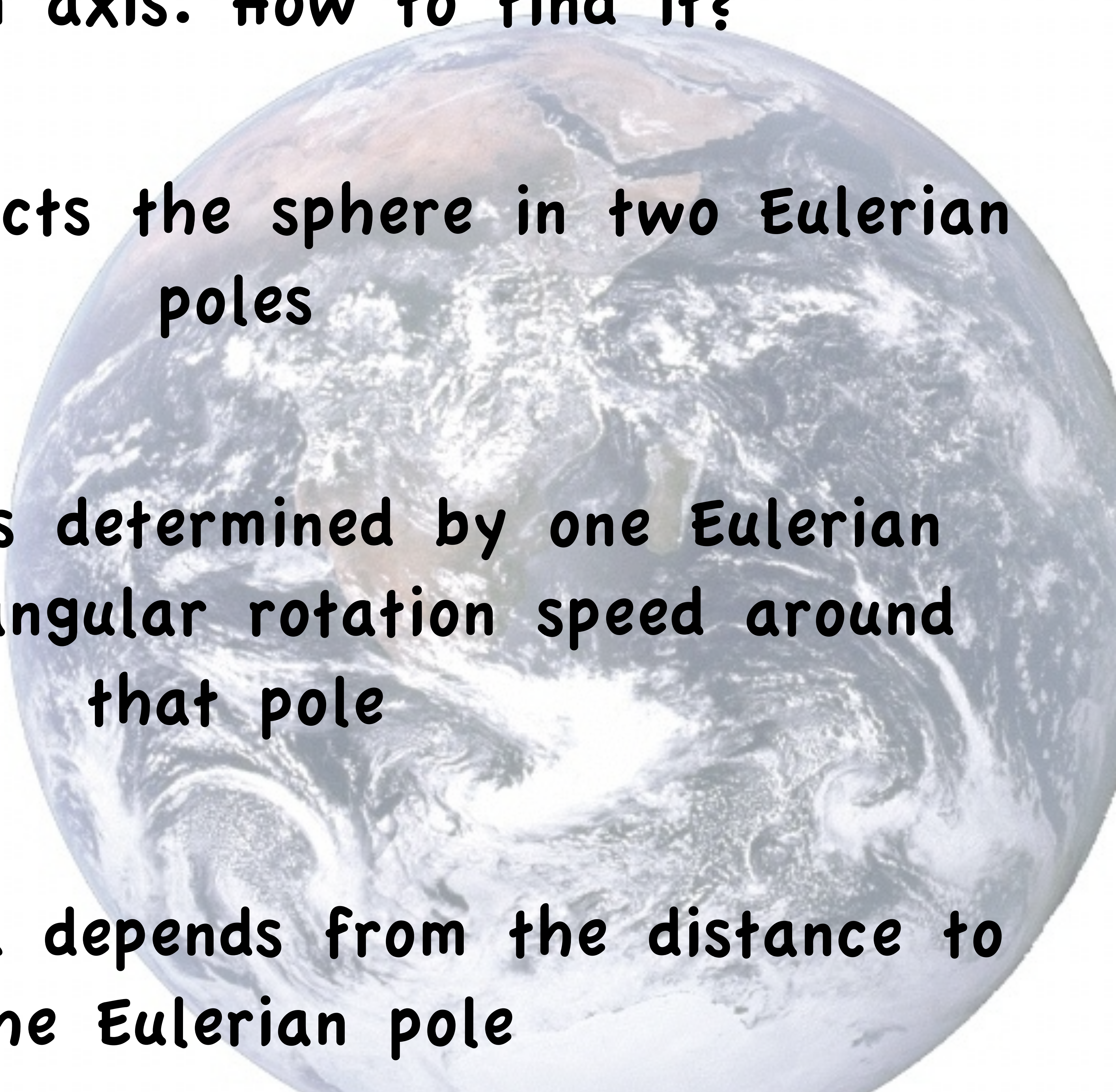
There are 12 large tectonic plates and many small ones (at least 40)



How to describe the movement of a tectonic plate

The movement of a tectonic plate is very slow. Moreover, the plate is rigid. Hence, each movement is well approximated by a linear transformation which preserves the distances and angles, hence an orthogonal transformation.

This transformation is close to the identity. Hence, it is a rotation around an axis!

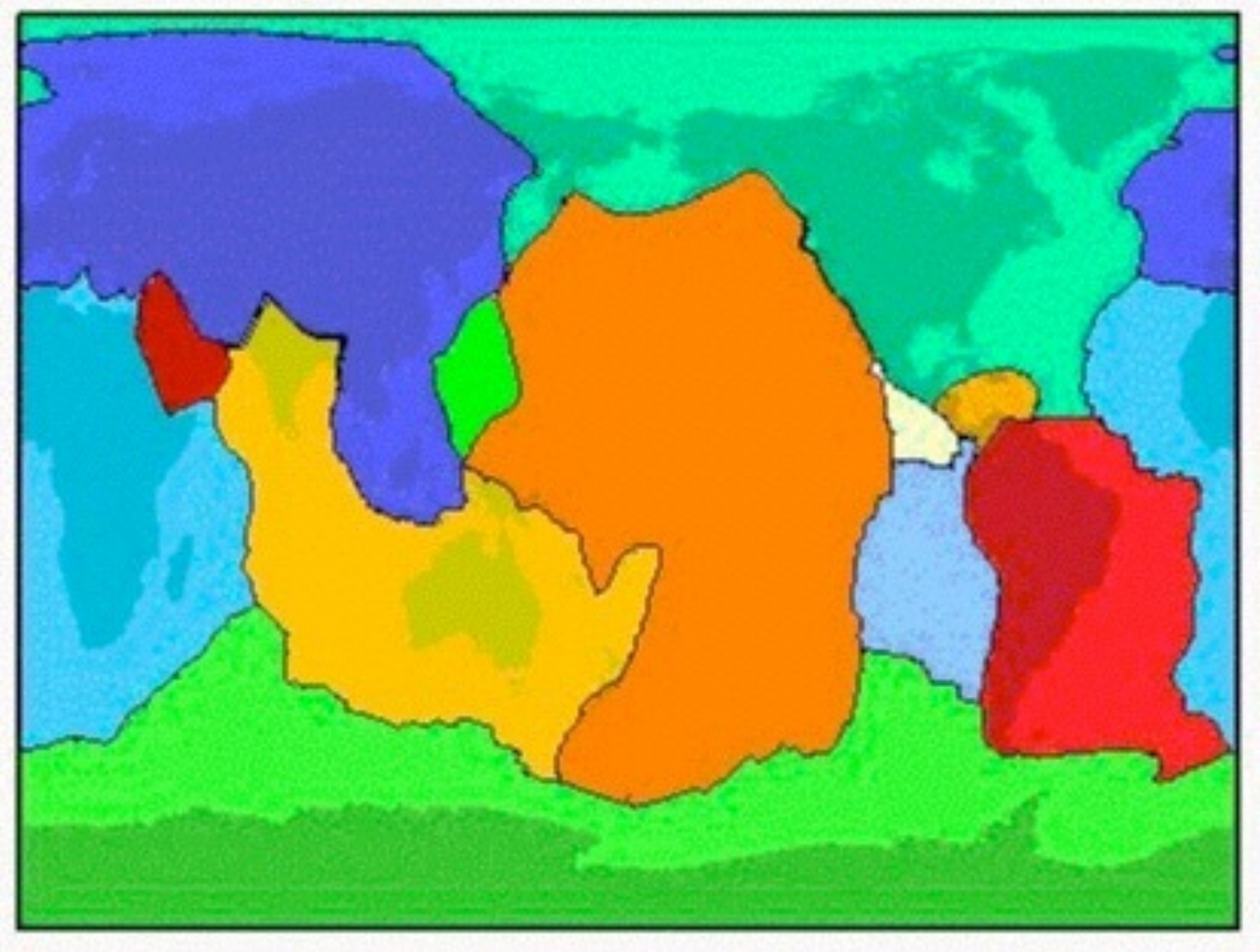


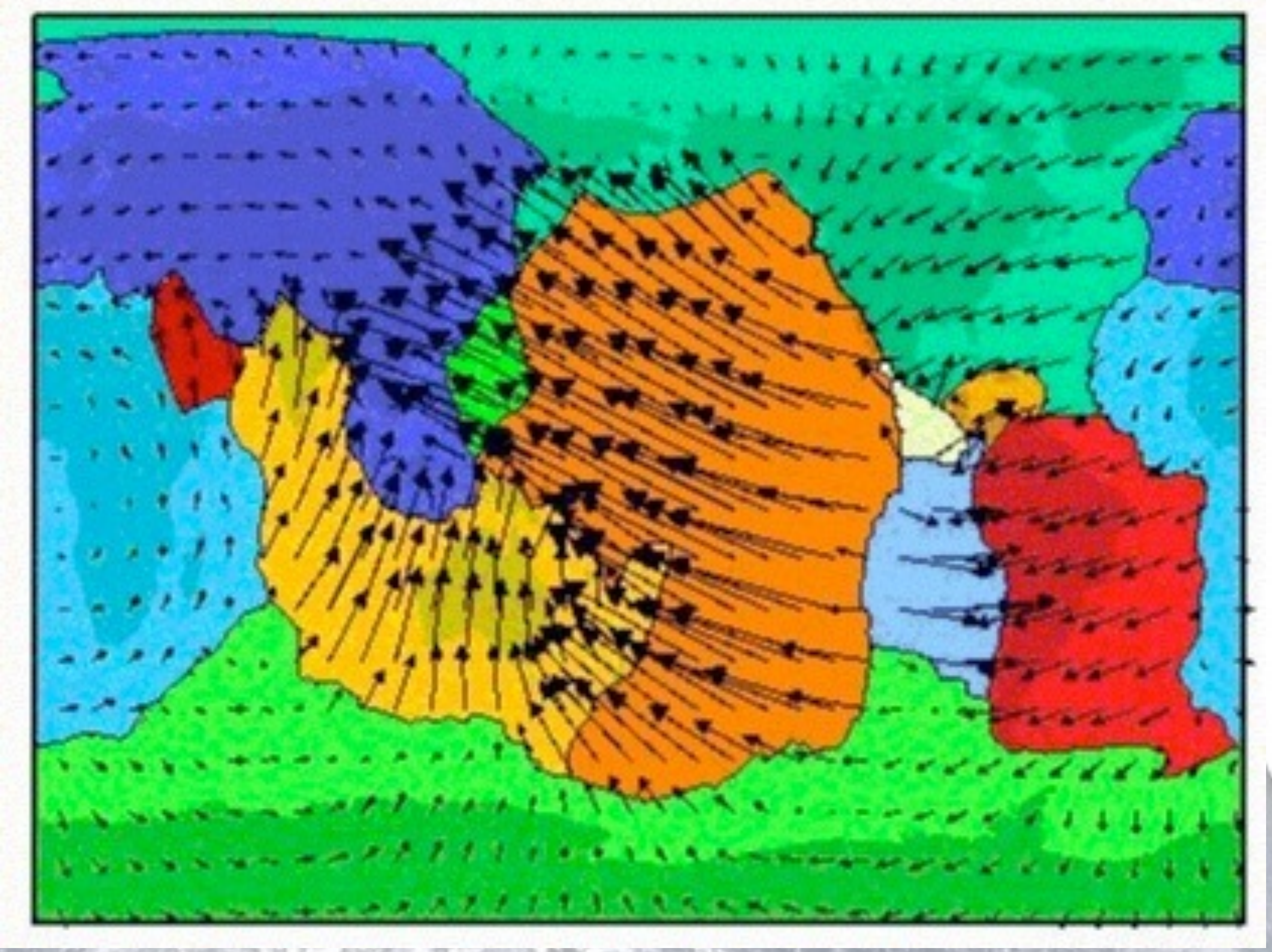
Each tectonic plate moves according to a rotation around an axis. How to find it?

This axis intersects the sphere in two Eulerian poles

The rotation is determined by one Eulerian pole and the angular rotation speed around that pole

The linear speed depends from the distance to the Eulerian pole

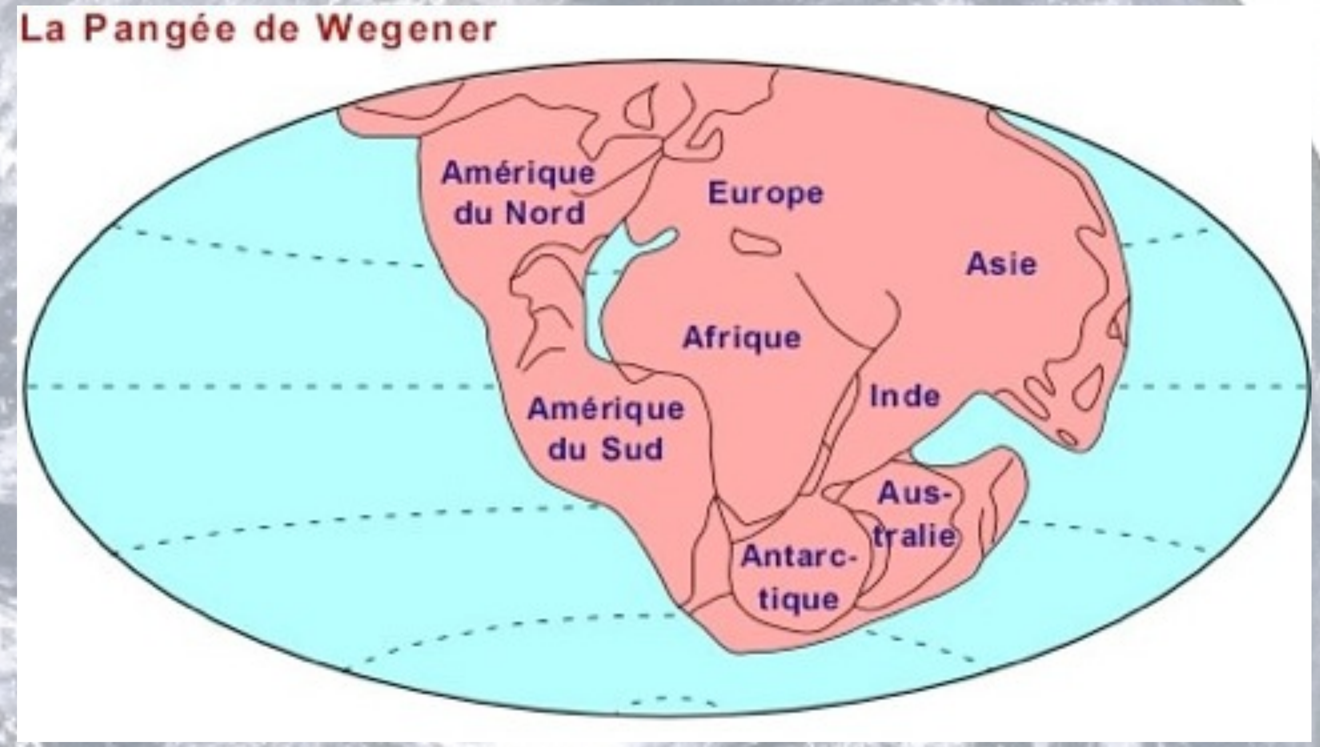




Hence the different types of faults ??

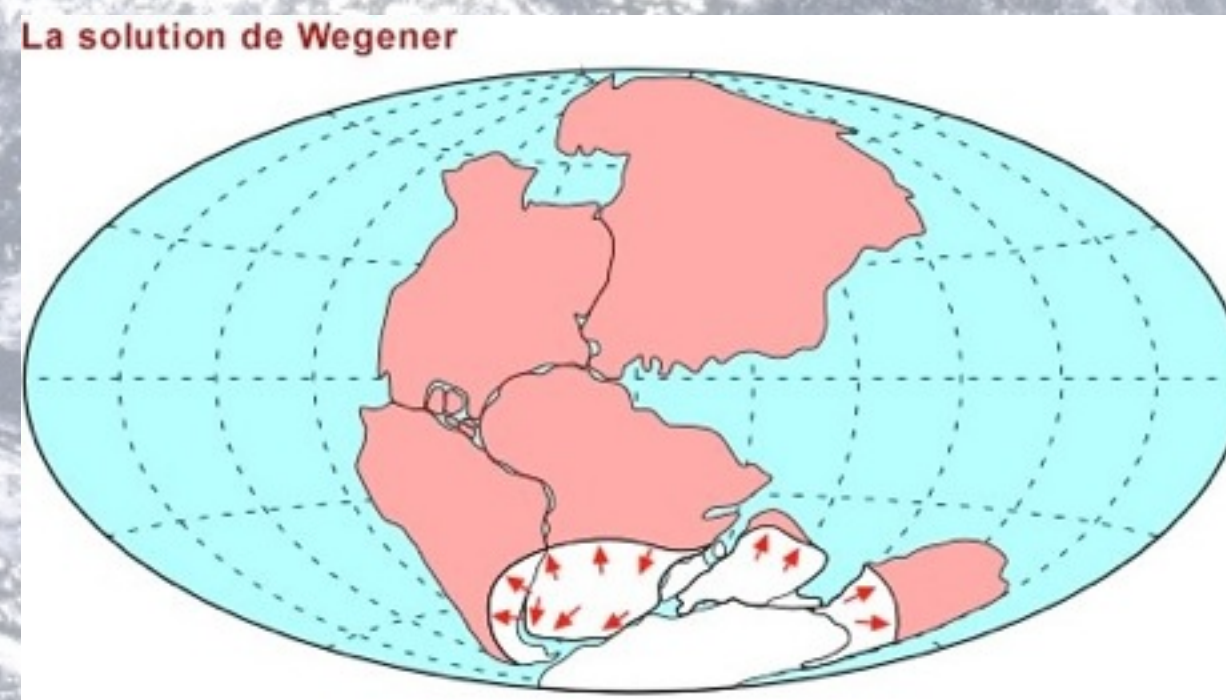
- divergence zones (African rift)
- convergence zones (rising of mountains
  subduction zone when one plate slides below another one)
- sliding zones

The drift of continents following Wegener (1915)



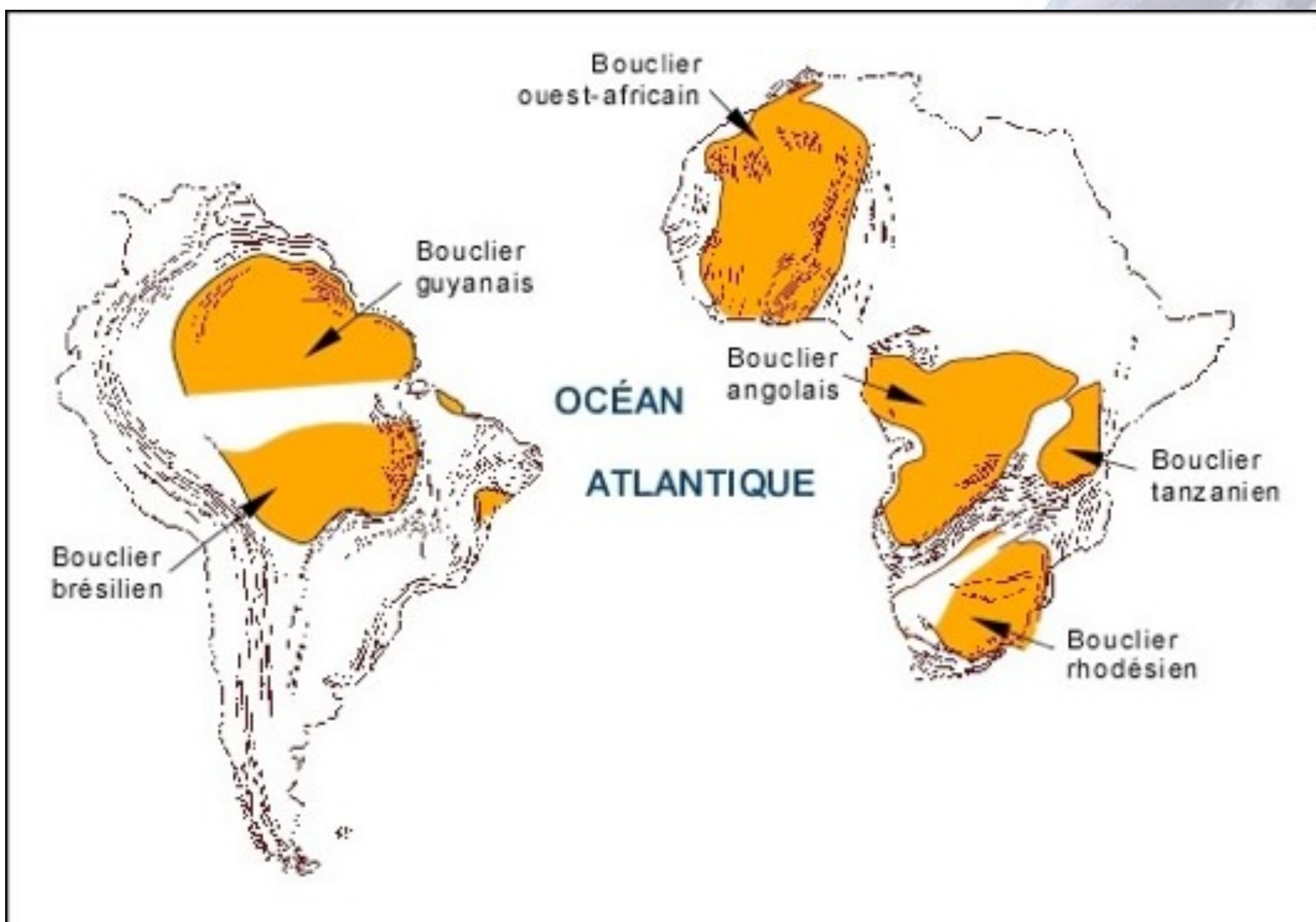
This theory relies on

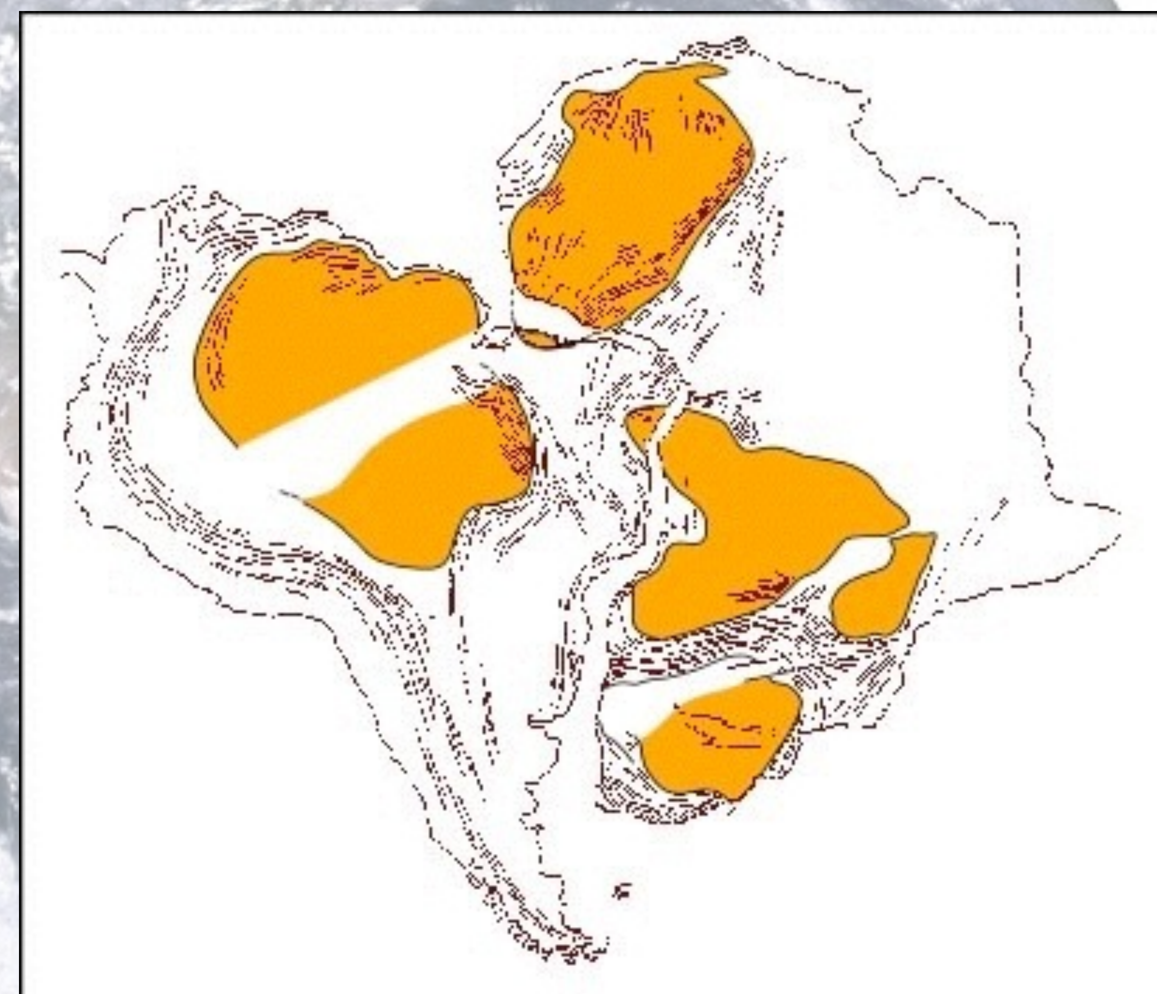
1. Marking from ancient glaciations,



2. Similarities in fossils in regions far one from the other

3. Correspondence between geological structures





Evaluating the mass of the Earth

We use Newton's gravitational law and deduce the mass of the Earth from the gravitational attraction of the Earth at the surface of the Earth. We get

$$M = 5.98 \times 10^{24}\ kg$$

The Earth is much too heavy to be homogeneous since the density of the crust is around 2.2-2.9 kg/dm^3 and the mean density of 5.52 kg/dm^3.

Hence, this tells us that the interior of the Earth is very heavy!

Discovering the Earth interior

Richard Dixon Oldham identified the different types of seismic waves recorded on seismographs:

- P-waves: the pressure waves travel through the viscous interior

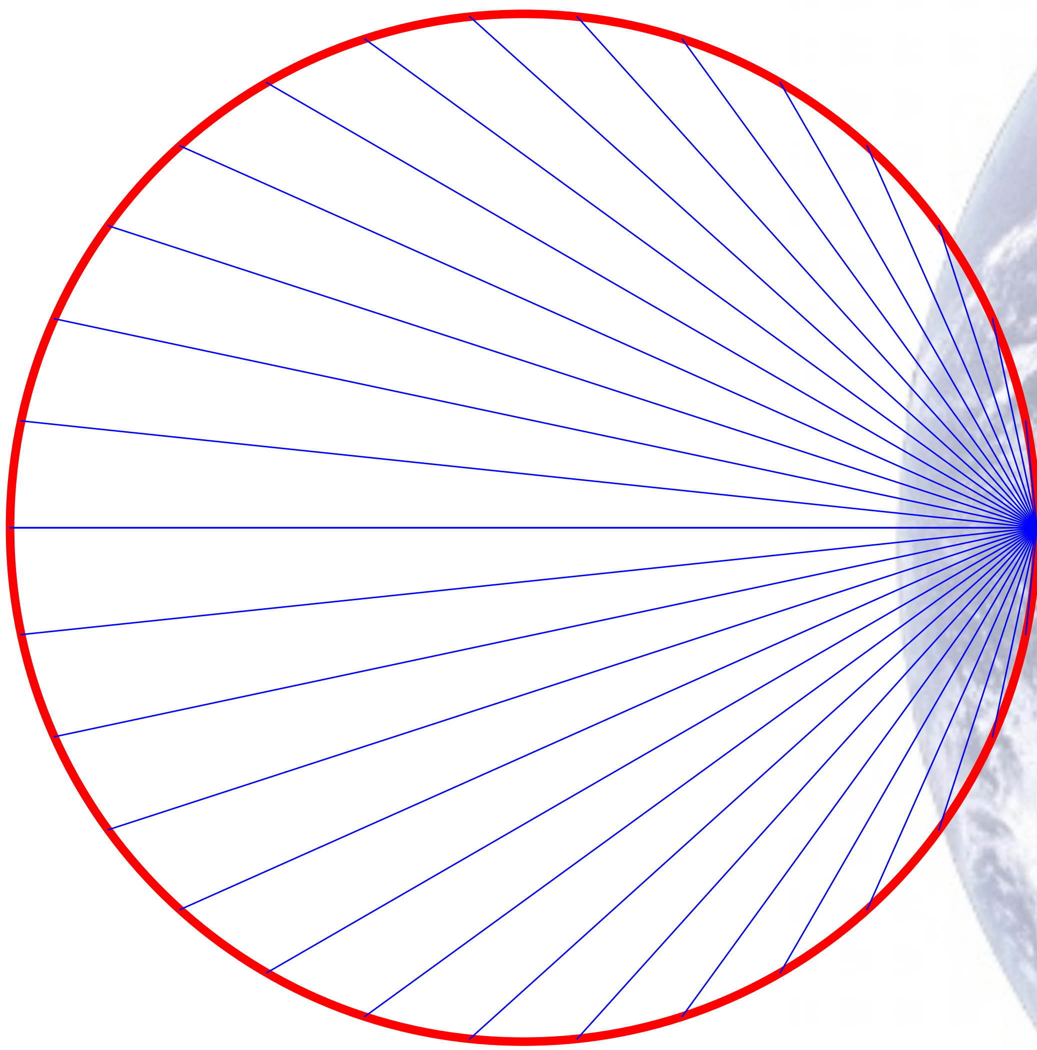
- S-waves: the shear waves are damped in the mantle, and hence not recorded far from the epicenter of an earthquake.

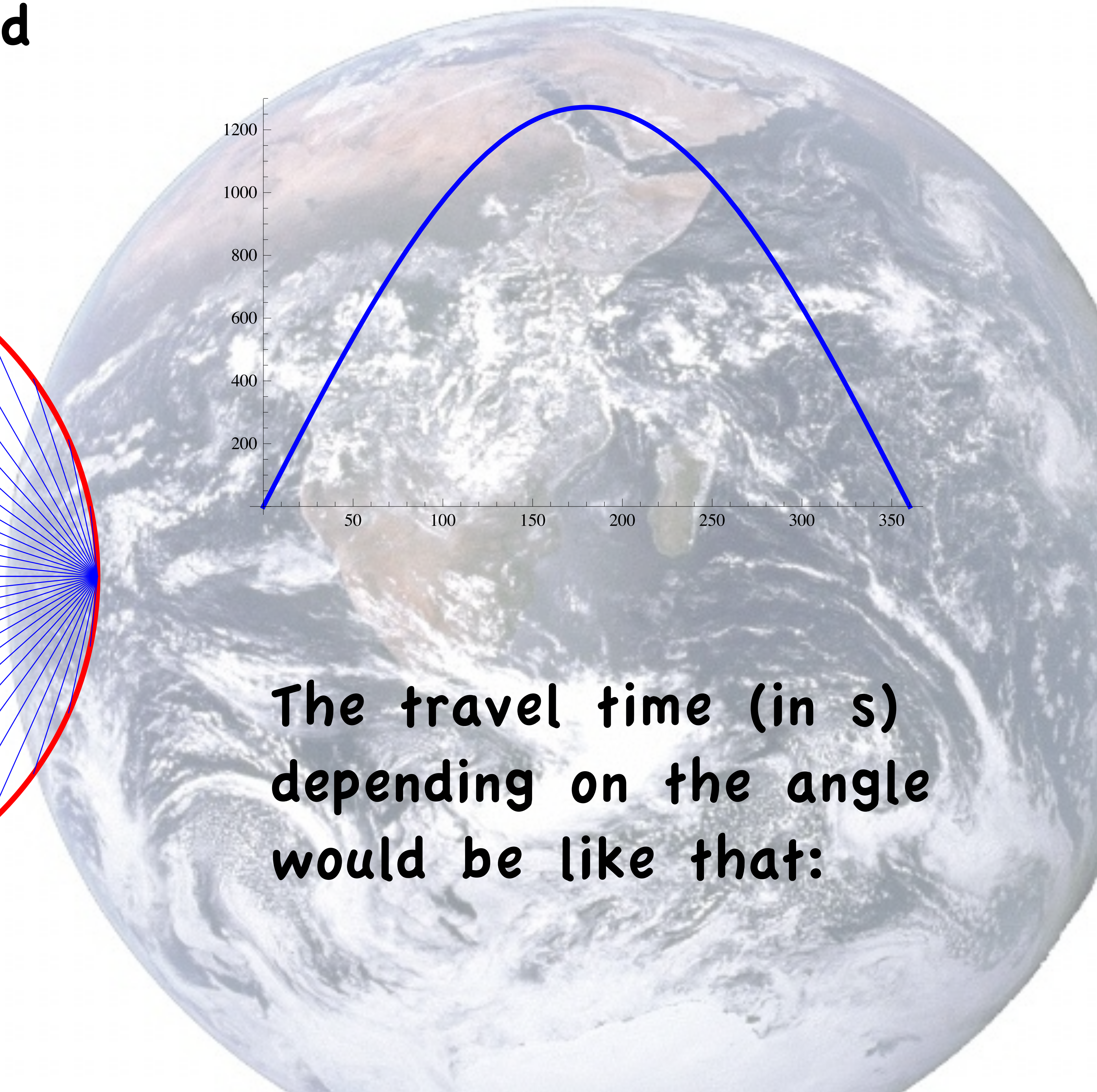
Inge Lehmann discovered the inner core of the Earth in 1936

Inge Lehmann was a mathematician. She worked at the Danish Geodetic Institute.

She used the measures of the different travel times of seismic waves generated by earthquakes to different stations over the Earth.

If the Earth were uniform then the signal would travel like that:

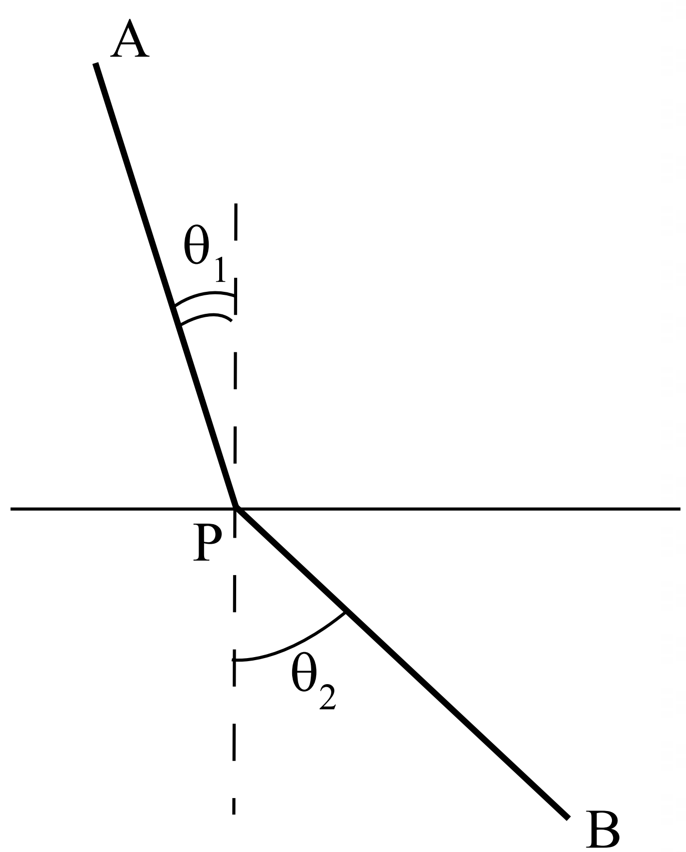




The travel time (in s) depending on the angle would be like that:

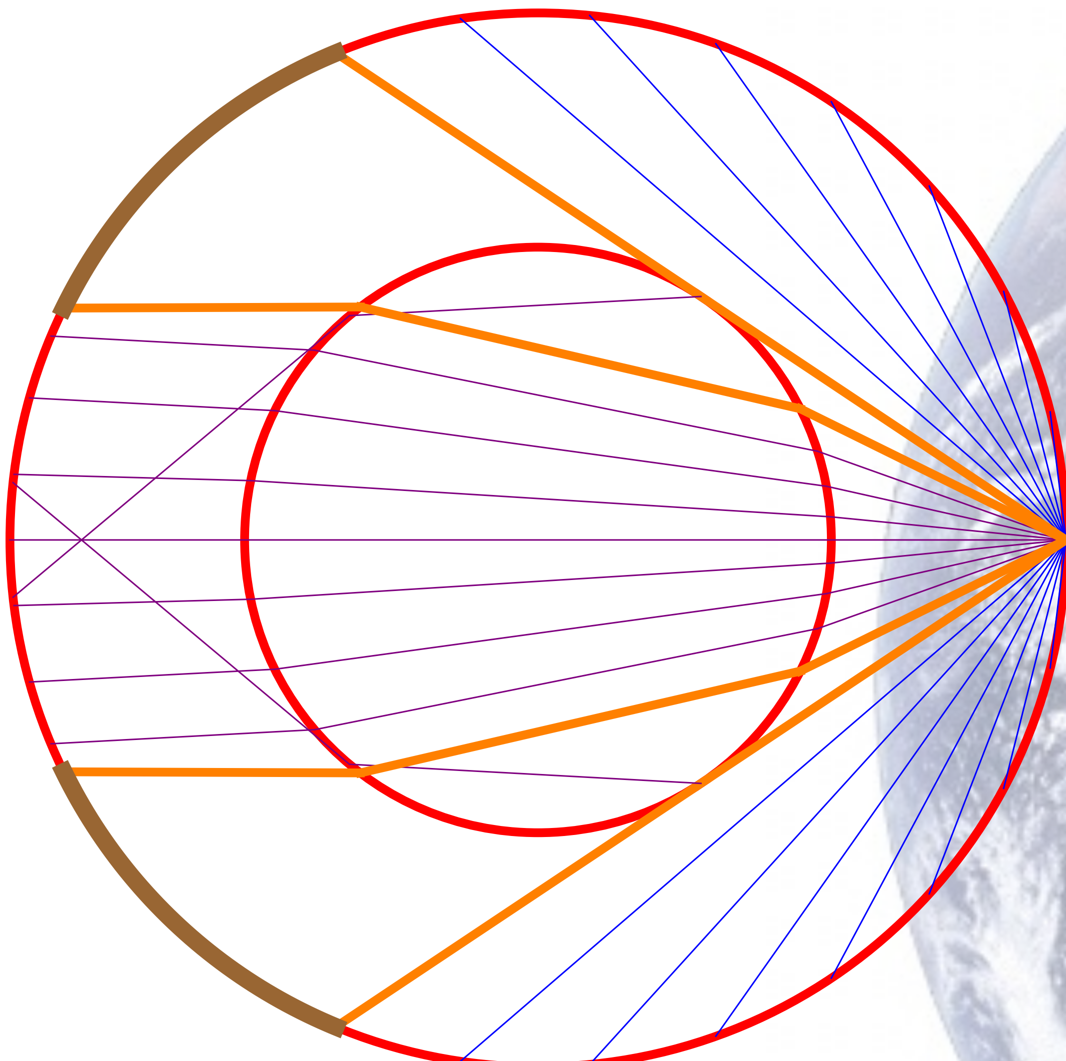
But the Earth has several layers in which the signal travels at different speeds.

When we change layer, the signal makes an angle according to the refraction law:



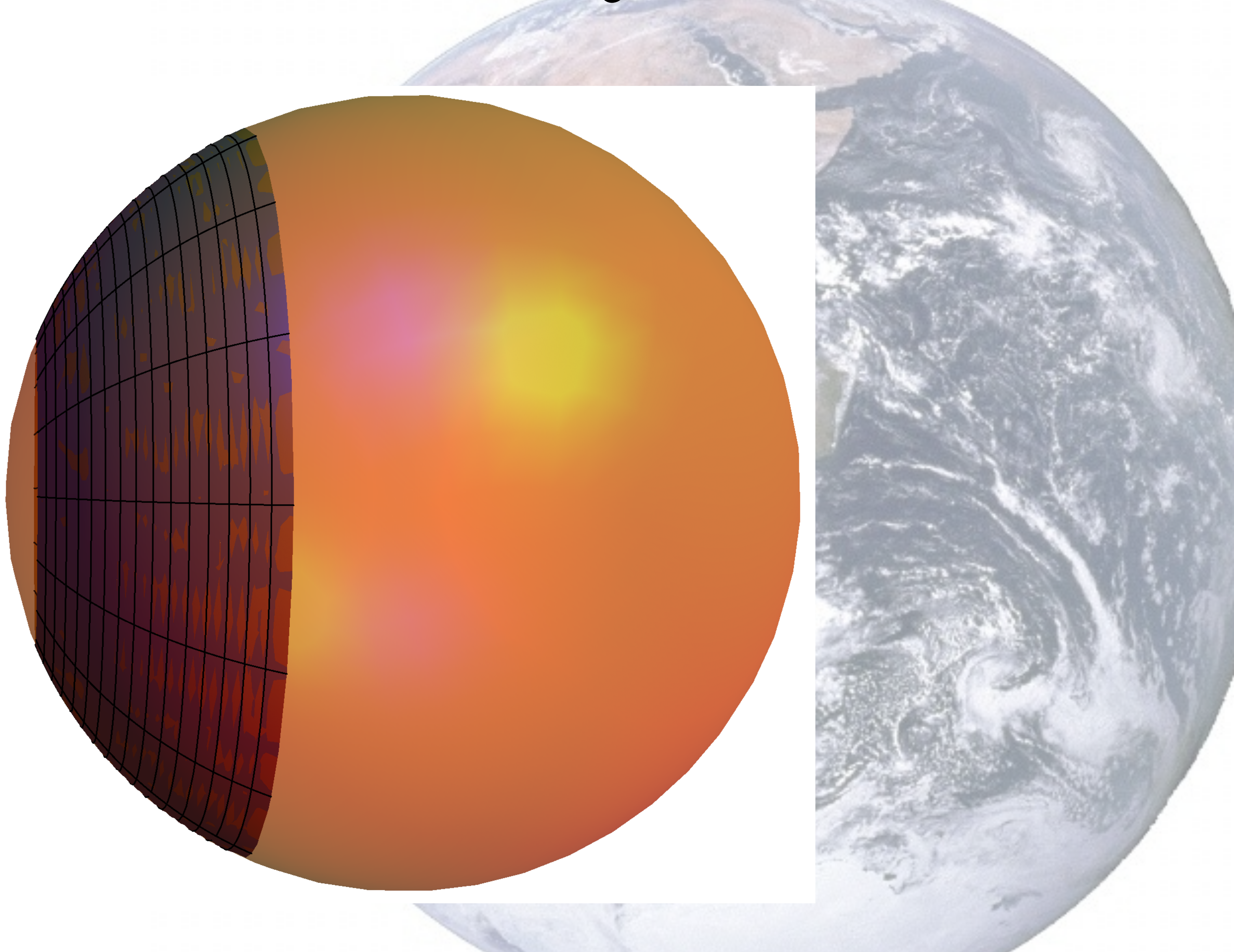
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

This is what occurs when leaving the mantle and entering the core in which the signal slows down.

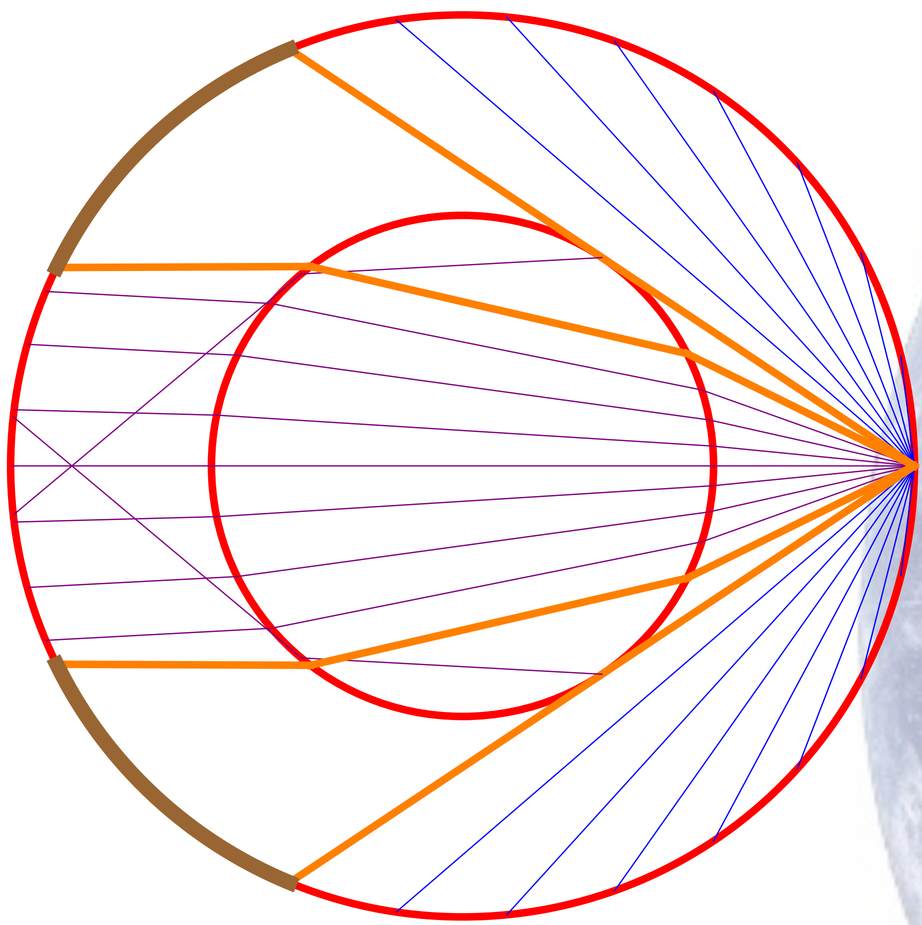


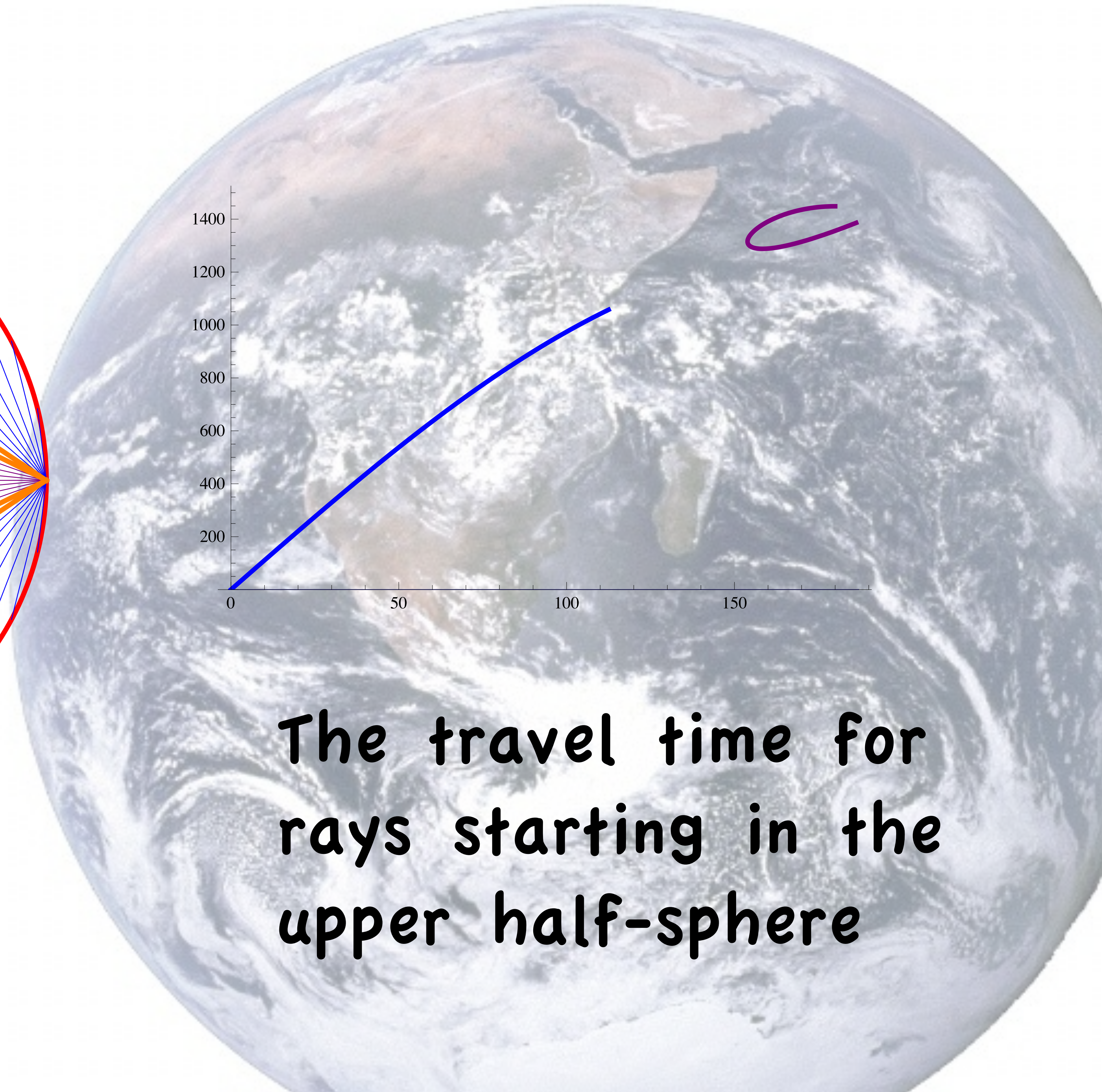
We see that **no signals can be detected along the two brown arcs** located between 112 degrees and 154 degrees.

This means that there should be no signal detected in an annular region like this



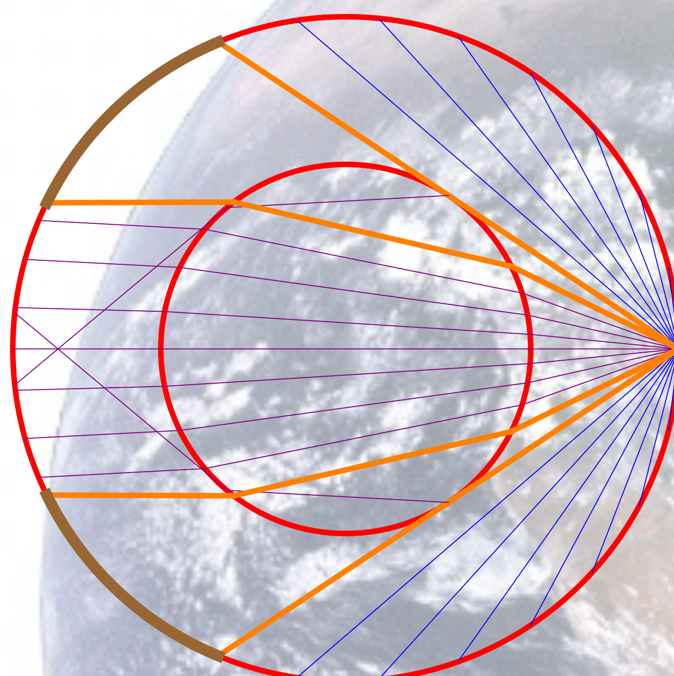
Also, the travel time is discontinuous





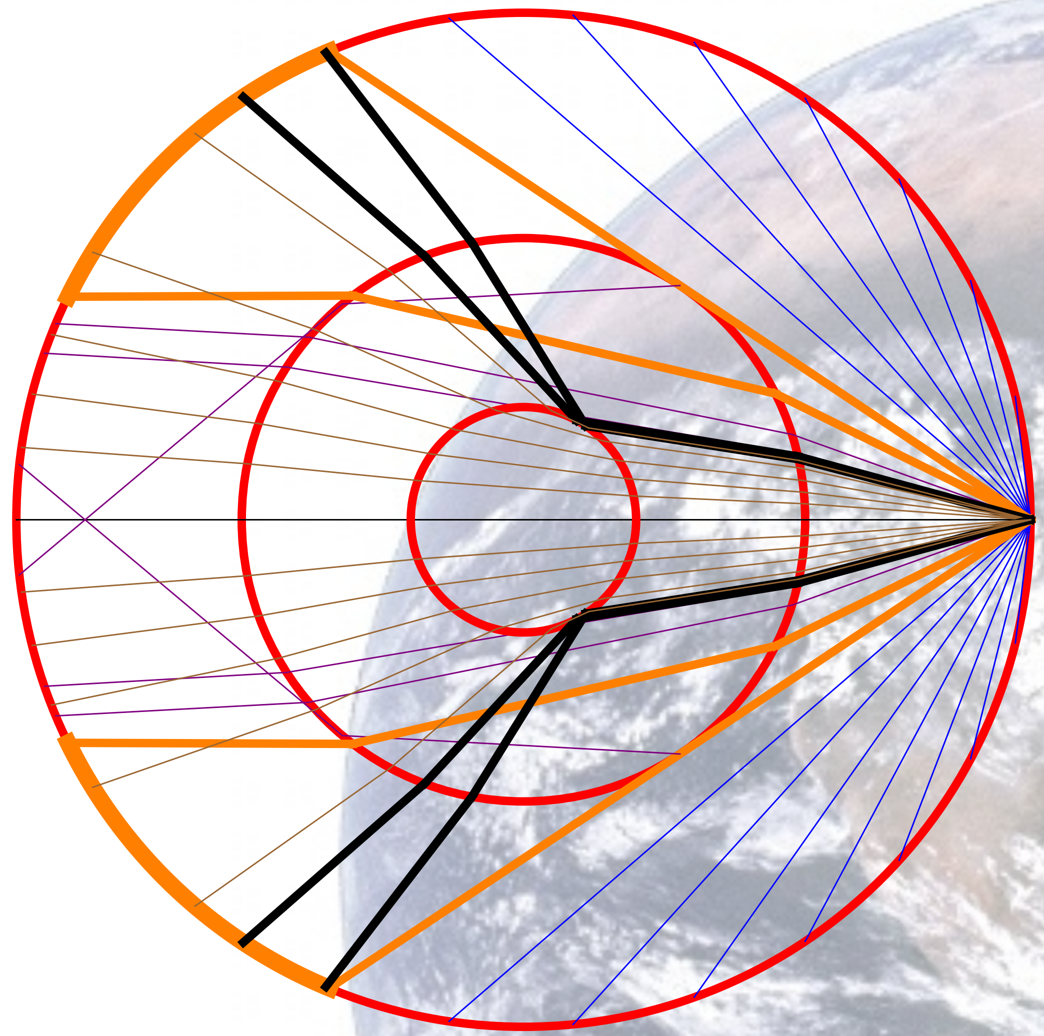
The travel time for rays starting in the upper half-sphere

But Inge Lehmann discovered that signals were registered in the forbidden region of the two brown arcs!



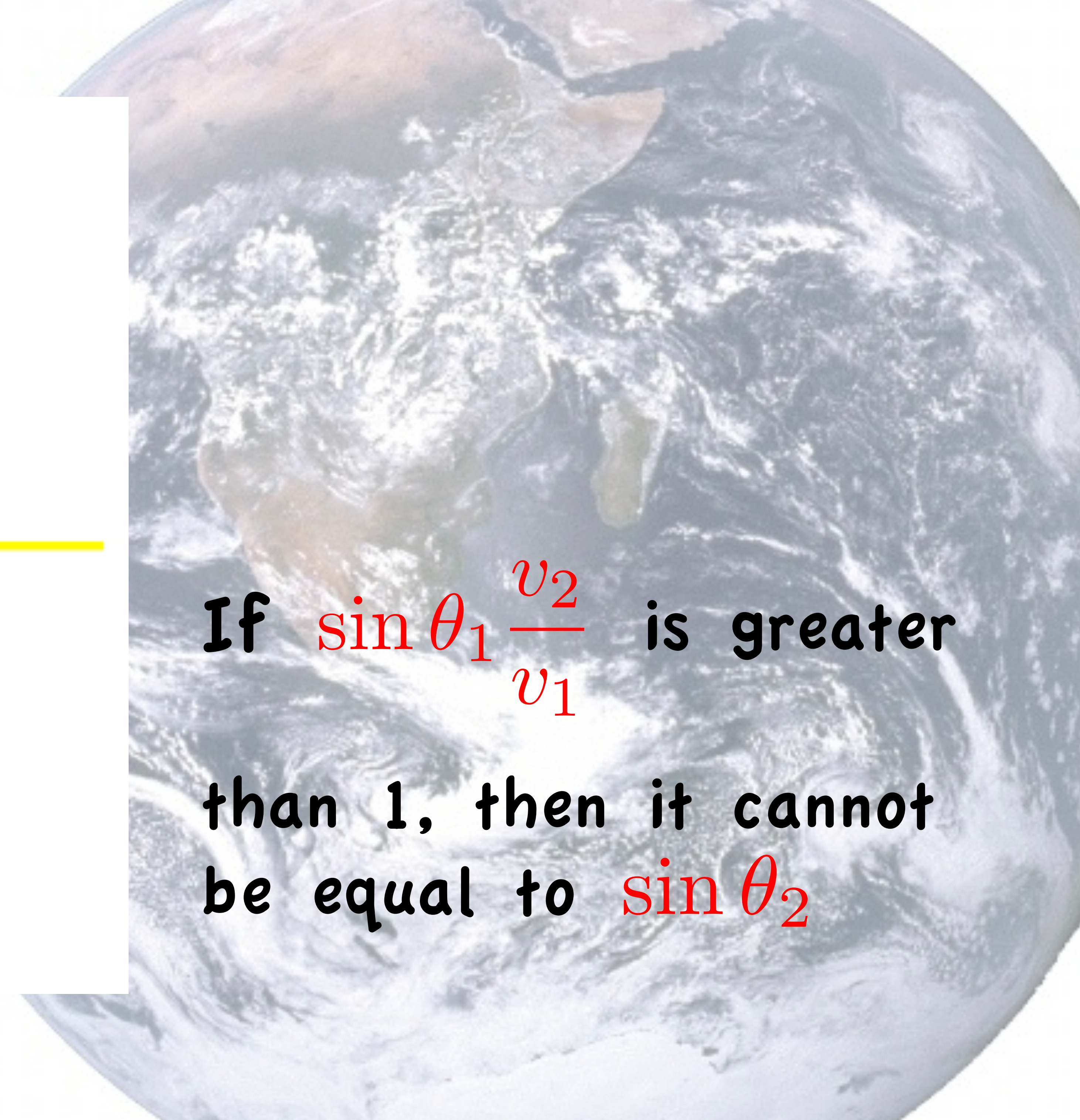
A model explaining the anomalies and the registered travel times for these signals is that the core is divided in two parts: the inner core and the outer core.

The outer core and the inner core



The signal travels faster in the inner core. So some rays cannot enter and are reflected. They are detected in the forbidden region of the orange arcs.

Indeed, in the refraction law, when θ_1 is too large and $v_1 < v_2$, the signal cannot enter the second layer and is reflected.



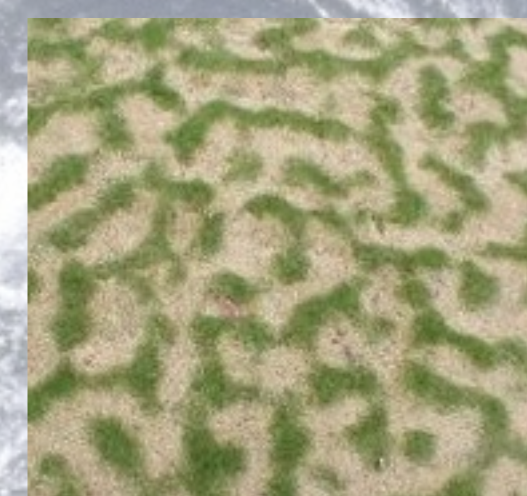
If $\sin\theta_1 \frac{v_2}{v_1}$ is greater than 1, then it cannot be equal to $\sin\theta_2$

The shapes of Earth

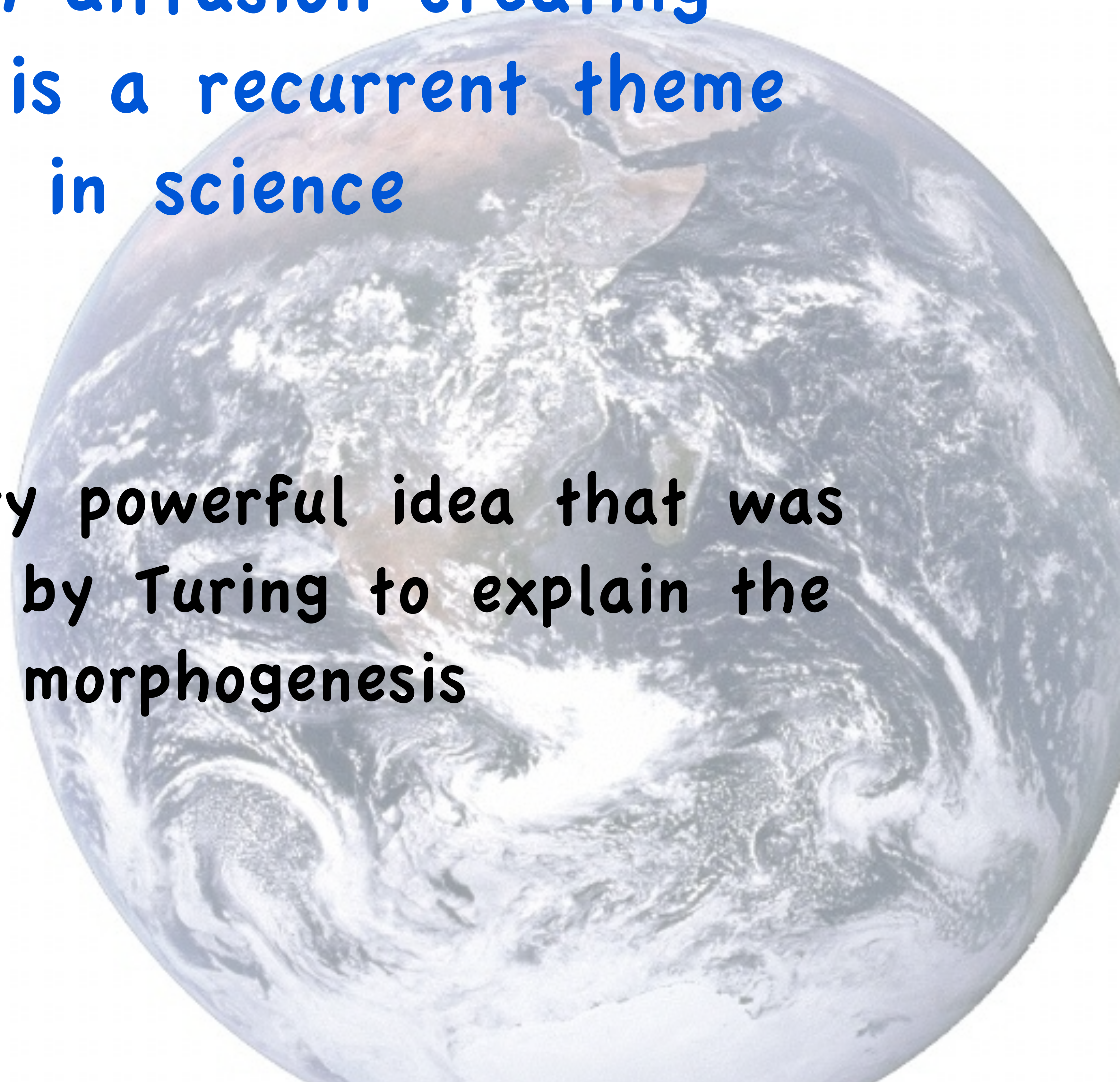
The loss of equilibrium through diffusion creates regular patterns:

- dunes
- waves
- vegetation patterns





The loss of equilibrium through diffusion creating patterns is a recurrent theme in science



It is a very powerful idea that was introduced by Turing to explain the morphogenesis

A model explaining sparse vegetation pattern

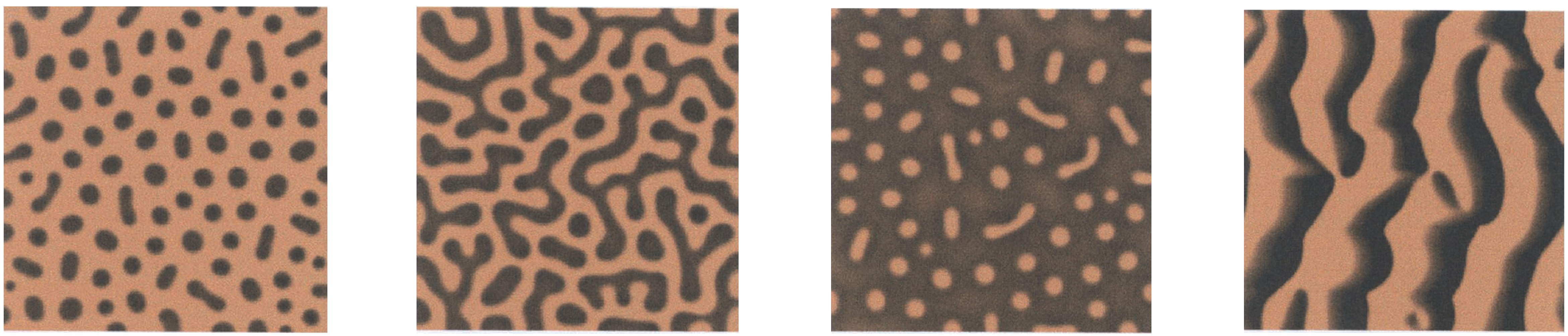
There is not enough water for a uniform covering with vegetation

There are feed-back mechanisms explaining why patches of vegetation can persist:

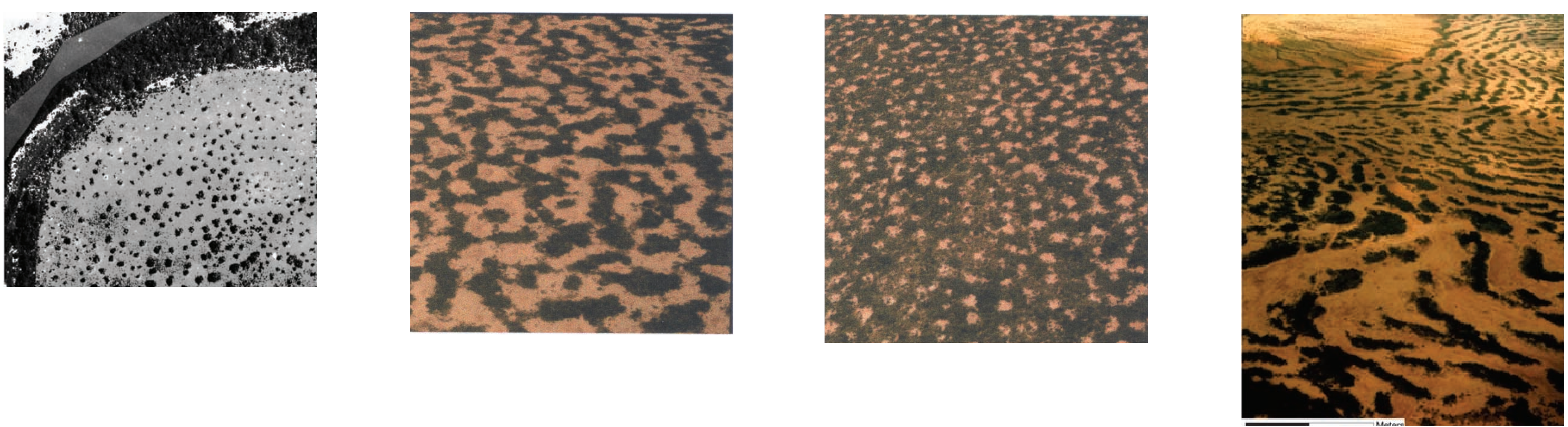
- the patches can drain the water from neighborhing empty spots
- the vegetation limits the evaporation

The four types of patterns depending on the quantity of water and the slope

generated by the model

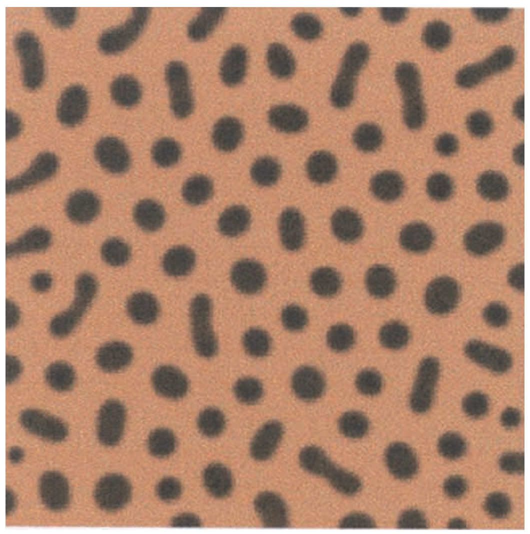
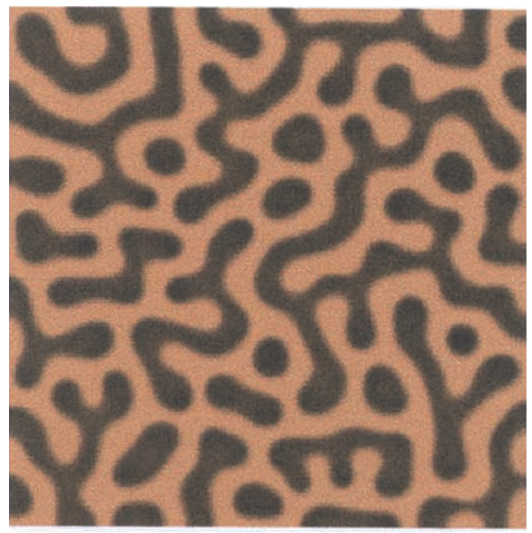
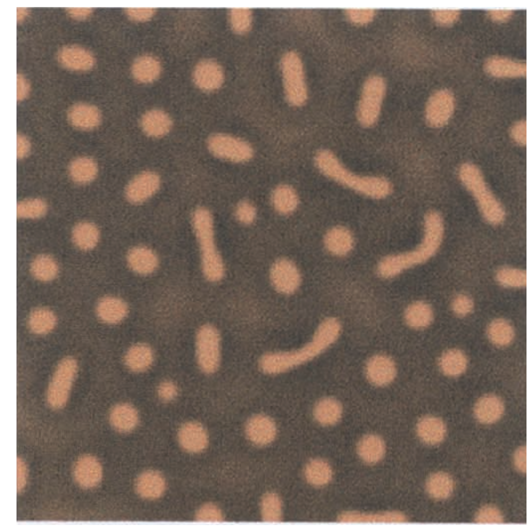
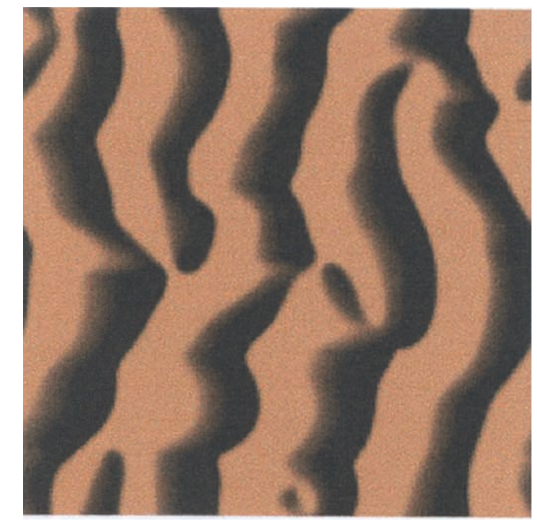


observed in nature



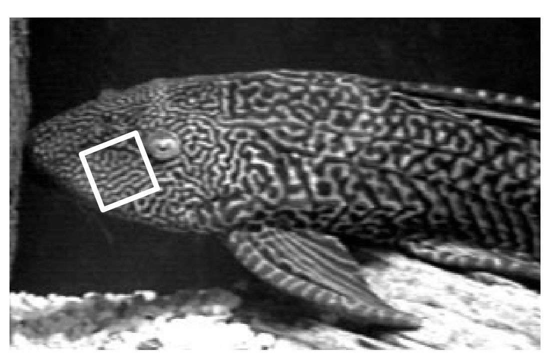
They are the same patterns observed for animal coatings!

generated by the model

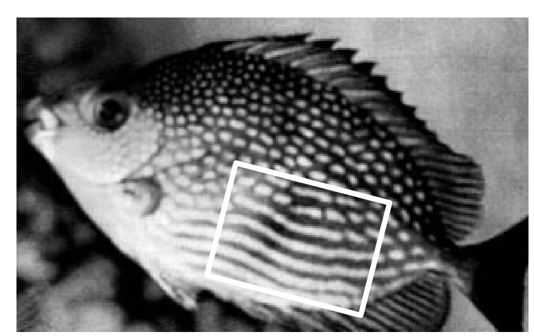





observed in nature

leopard



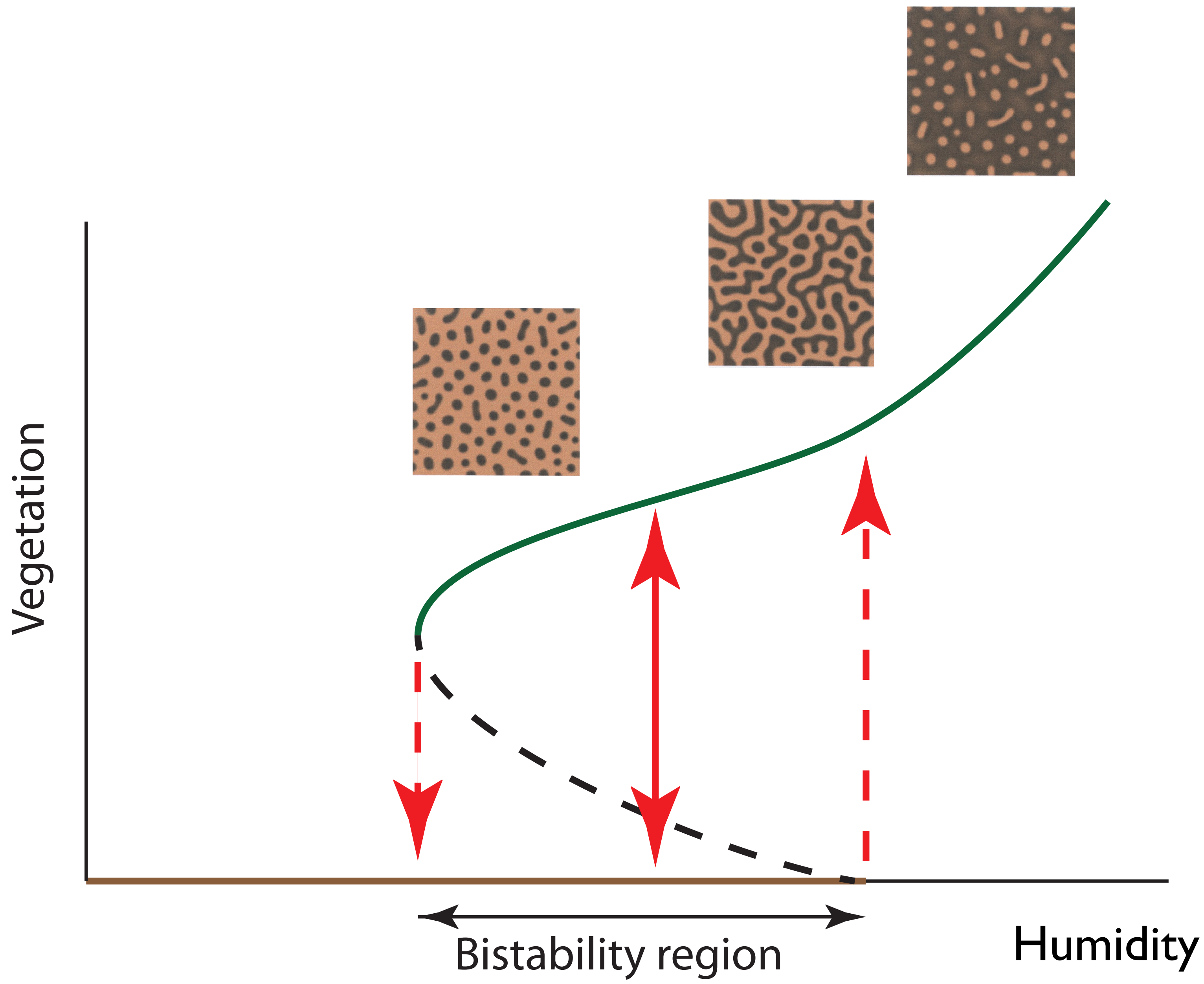
hypostomus plecostomus



siganus vermiculatus

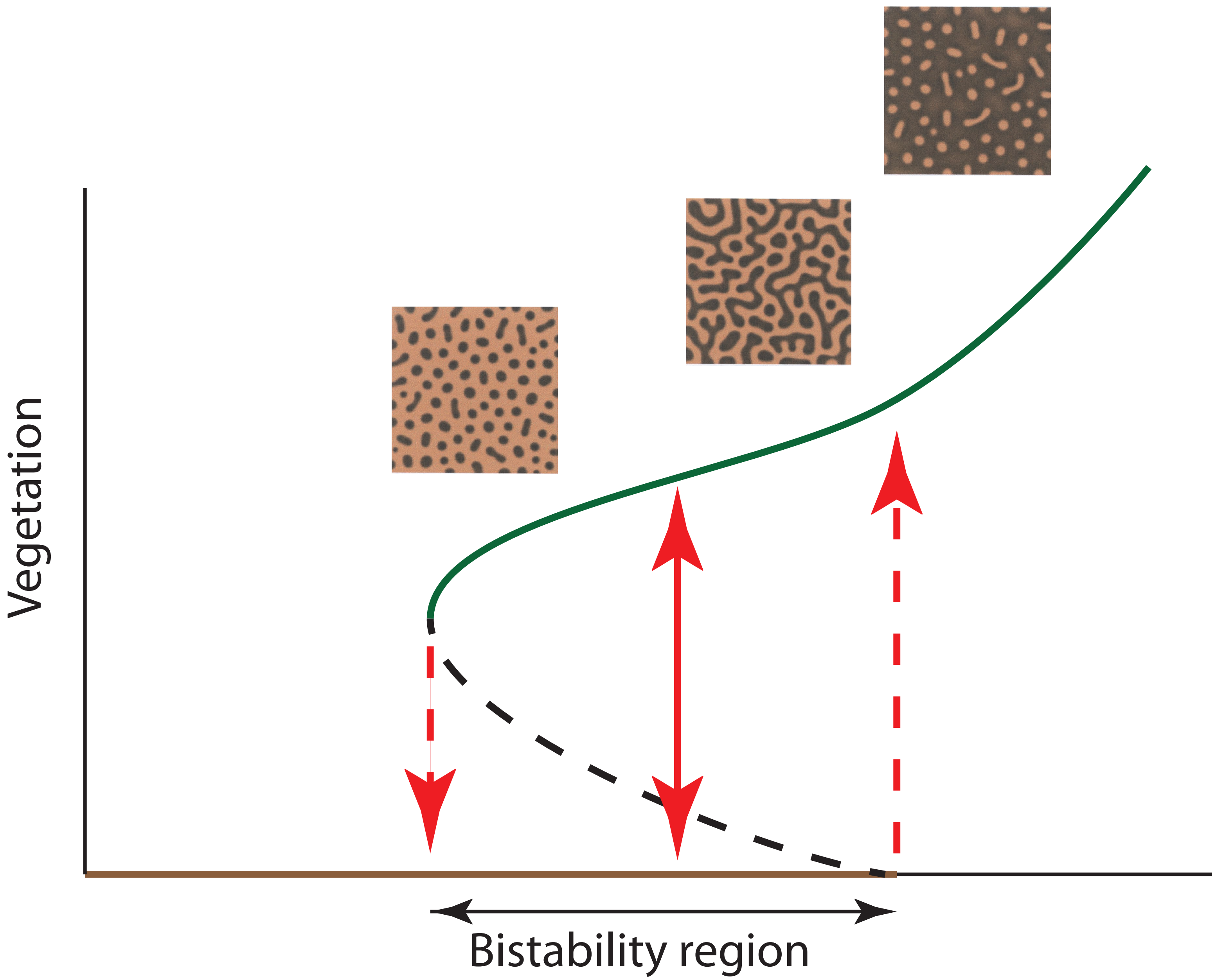
tiger

The hysteresis phenomenon



Attention! If the vegetation has disappeared in a region of low humidity following a drought period, then it will not spontaneously reappear because the feedback mechanisms cannot help.

The hysteresis phenomenon



We have passed a **tipping point:** no return is possible

Earth is inhabited by millions of living species

Where does all this biodiversity come from?

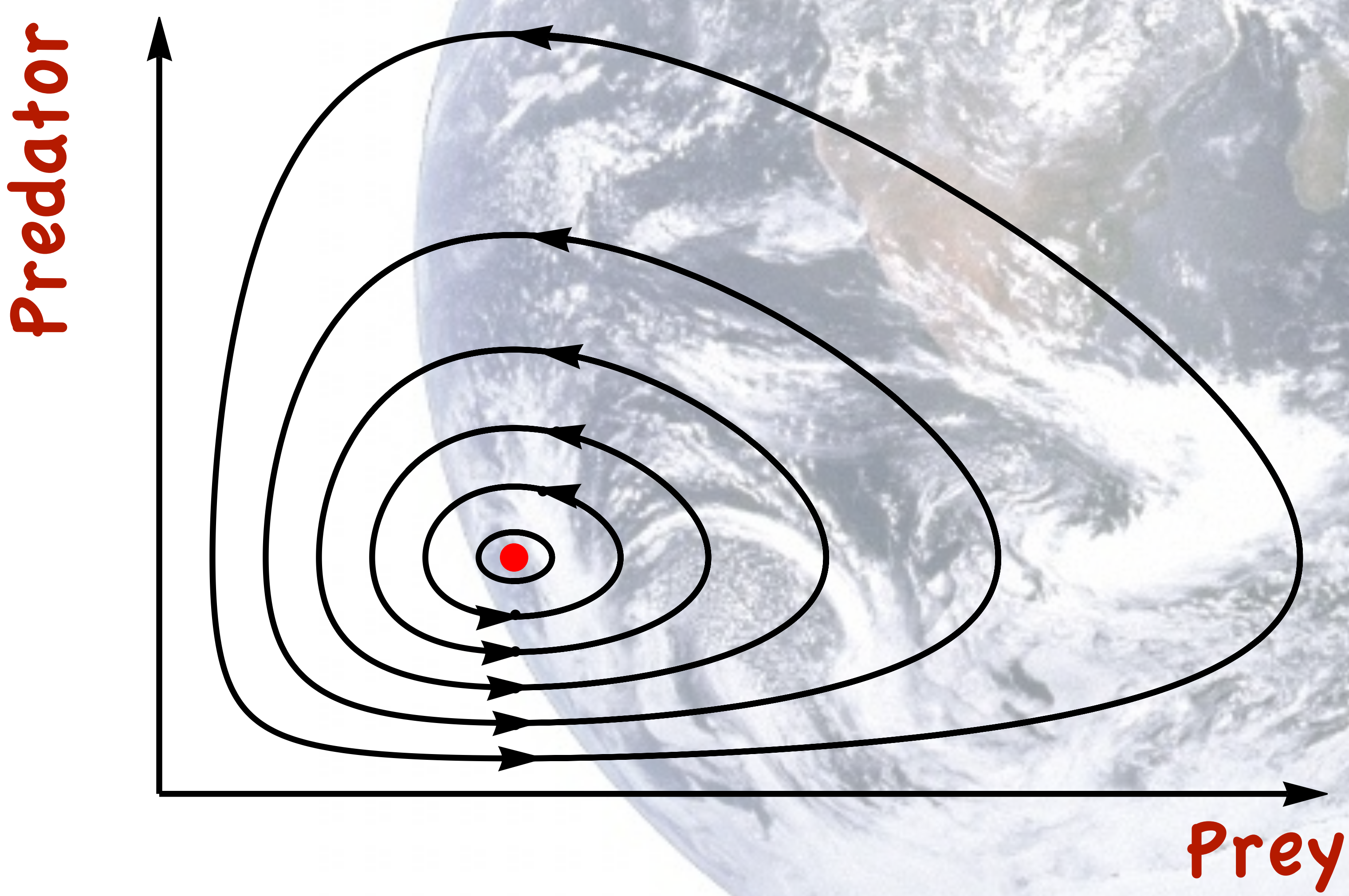
Mutations (randomness) create new species

These species interact to survive

How?

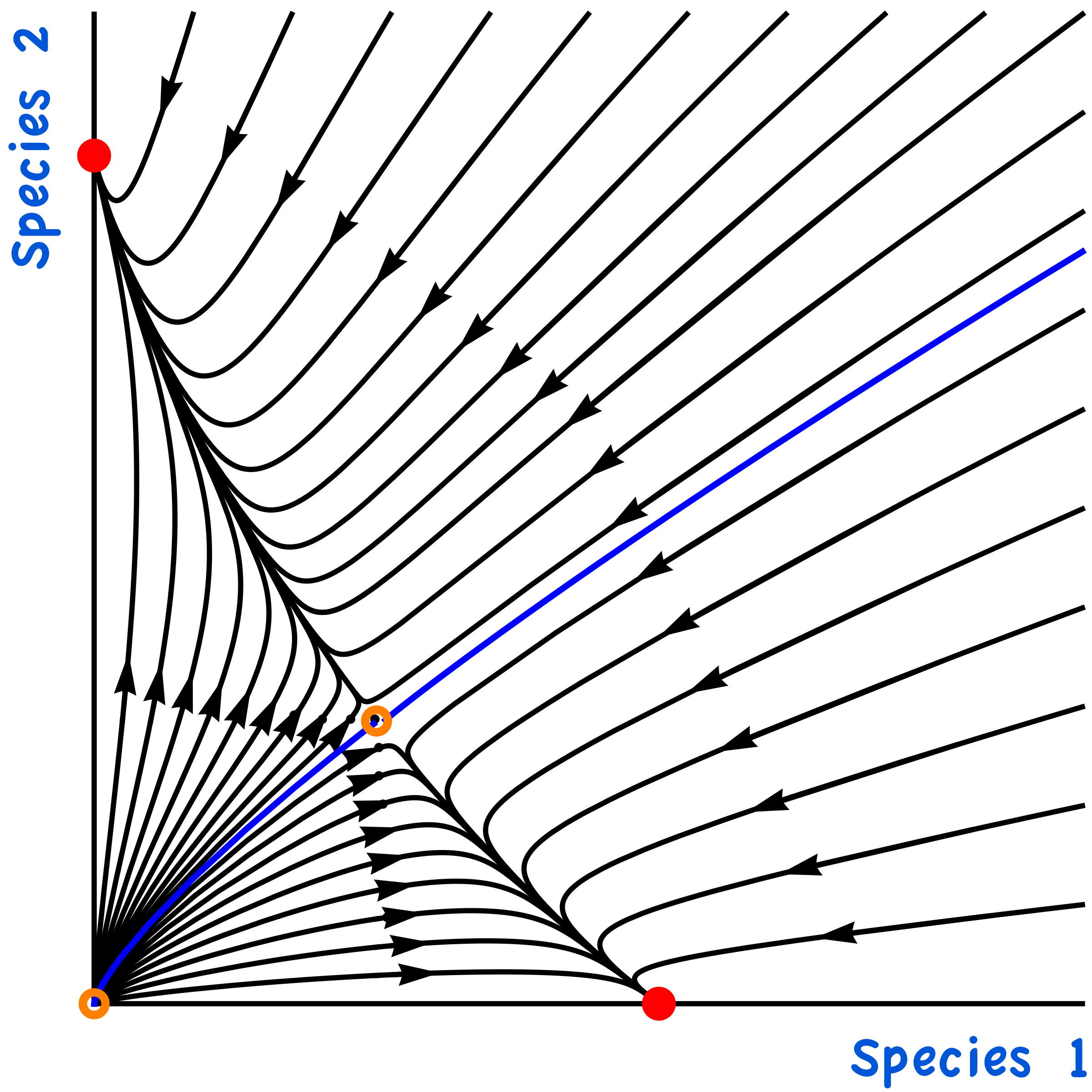
Predators and preys

Mathematicians represent their interaction by a geometric model

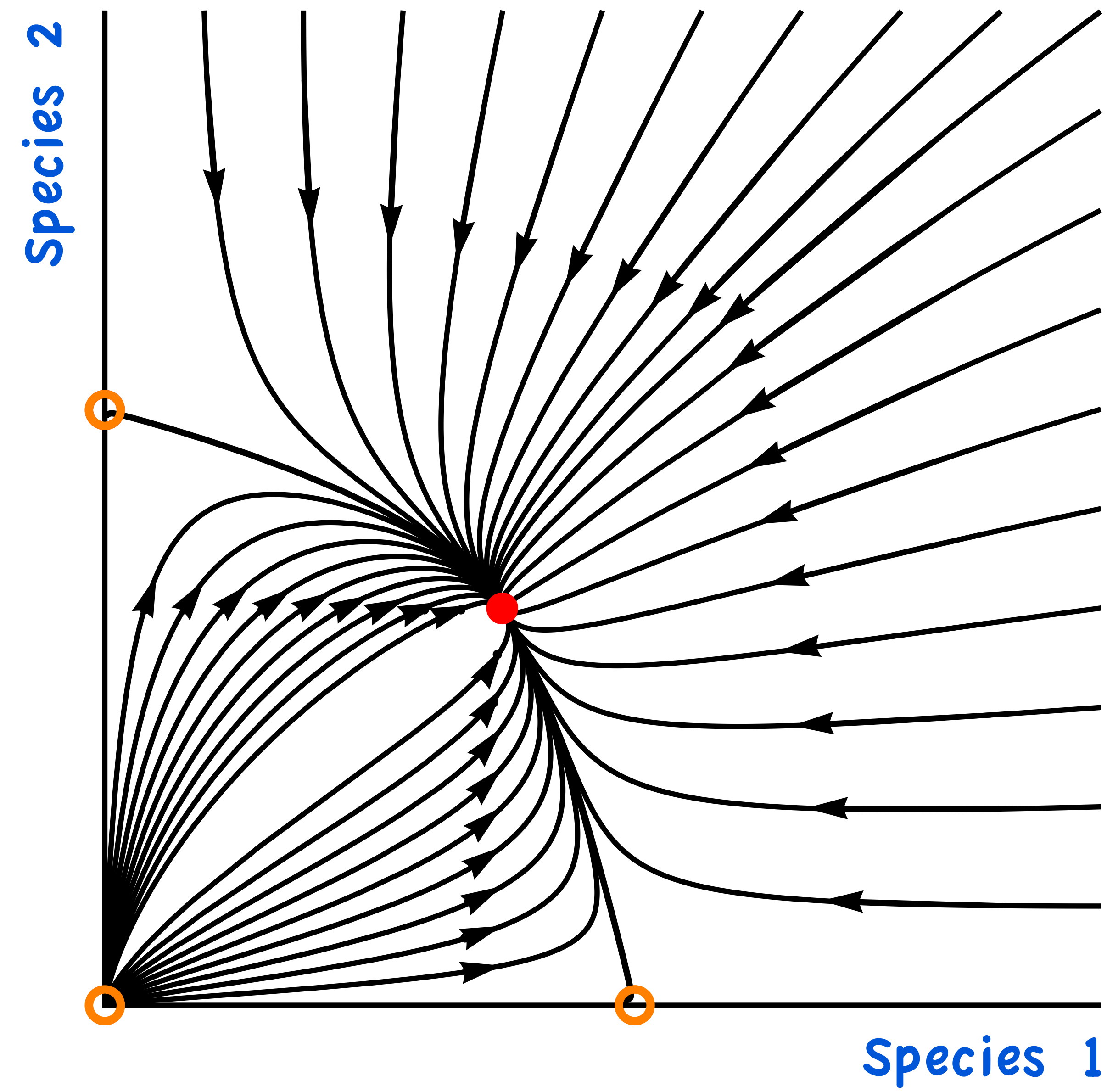


Competing species

We use the same type of models



Strong competition



Weak competition

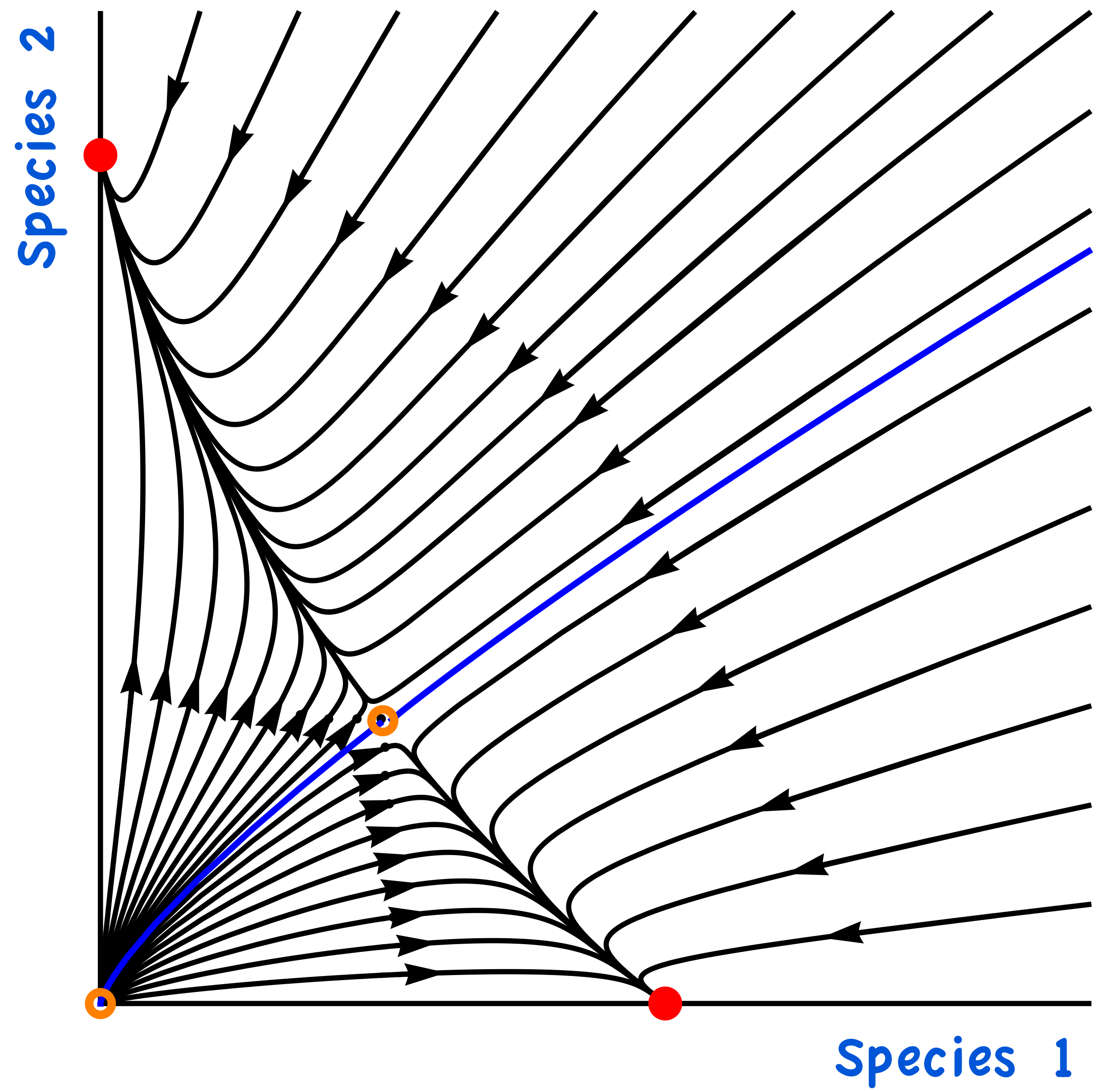
Strong competition for one resource leads to the extinction of one species

This has been generalized by Simon Levin:
in a model with n species

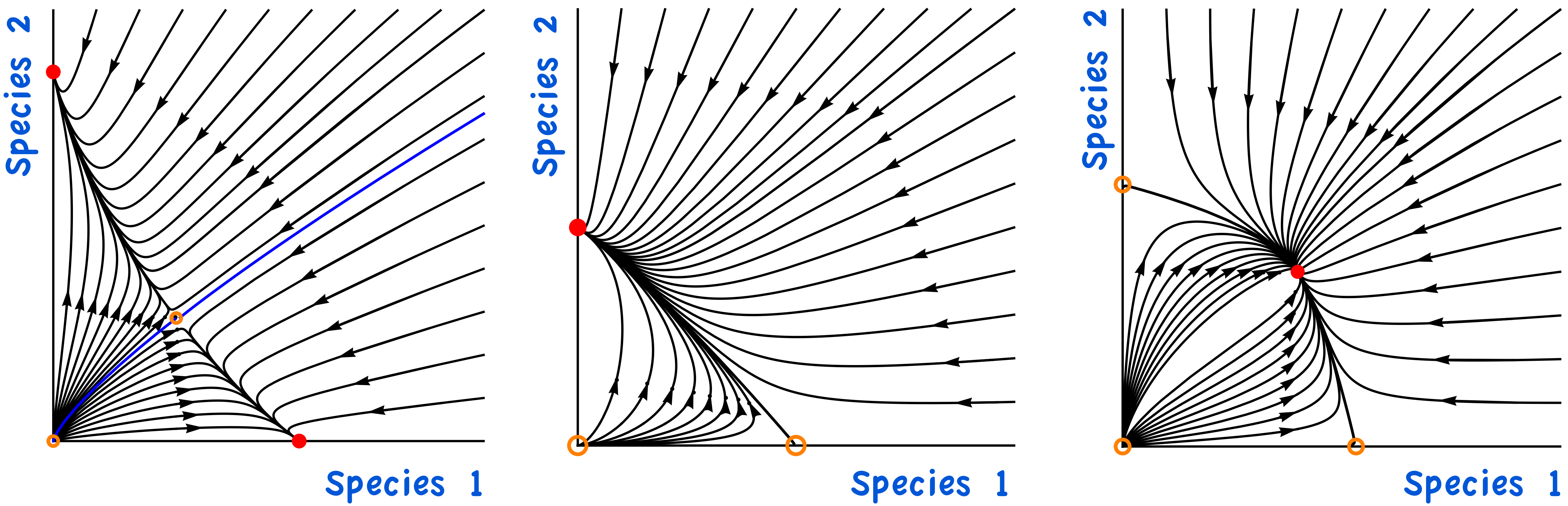
Hence, competition goes against biodiversity!

Other forces allow to maintain biodiversity

One of them is spatial heterogeneity



A second one is temporal heterogeneity



Founder control | One species win | Species coexist

Another force supports biodiversity

Cooperation!

(Martin Nowak)

The Prisoner's Dilemma

Individual 2

Individual 1	COOPERATE(remain silent)	DEFECT (confess)
COOPERATE(remain silent)	2 years in jail 2 years in jail	4 years in jail 1 year in jail
DEFECT (confess)	1 year in jail 4 years in jail	3 years in jail 3 years in jail

Martin Nowak identified five mechanisms leading to communities dominated by cooperators

1. Direct reciprocity: vampire bats share with the bat who found no blood. A winning strategy: win-stay, lose-shift

2. Spatial selection when cooperators and defectors are not uniformly distributed, leading to patches of cooperators and defectors: yeast cells

Five mechanisms

3. Kin selection: cooperation (including sacrifice) between genetically related individuals

4. Indirect reciprocity: help of another based on the needy's individual reputation: Japanese macaques

5. Group selection: employees competing among themselves, but cooperating for their company

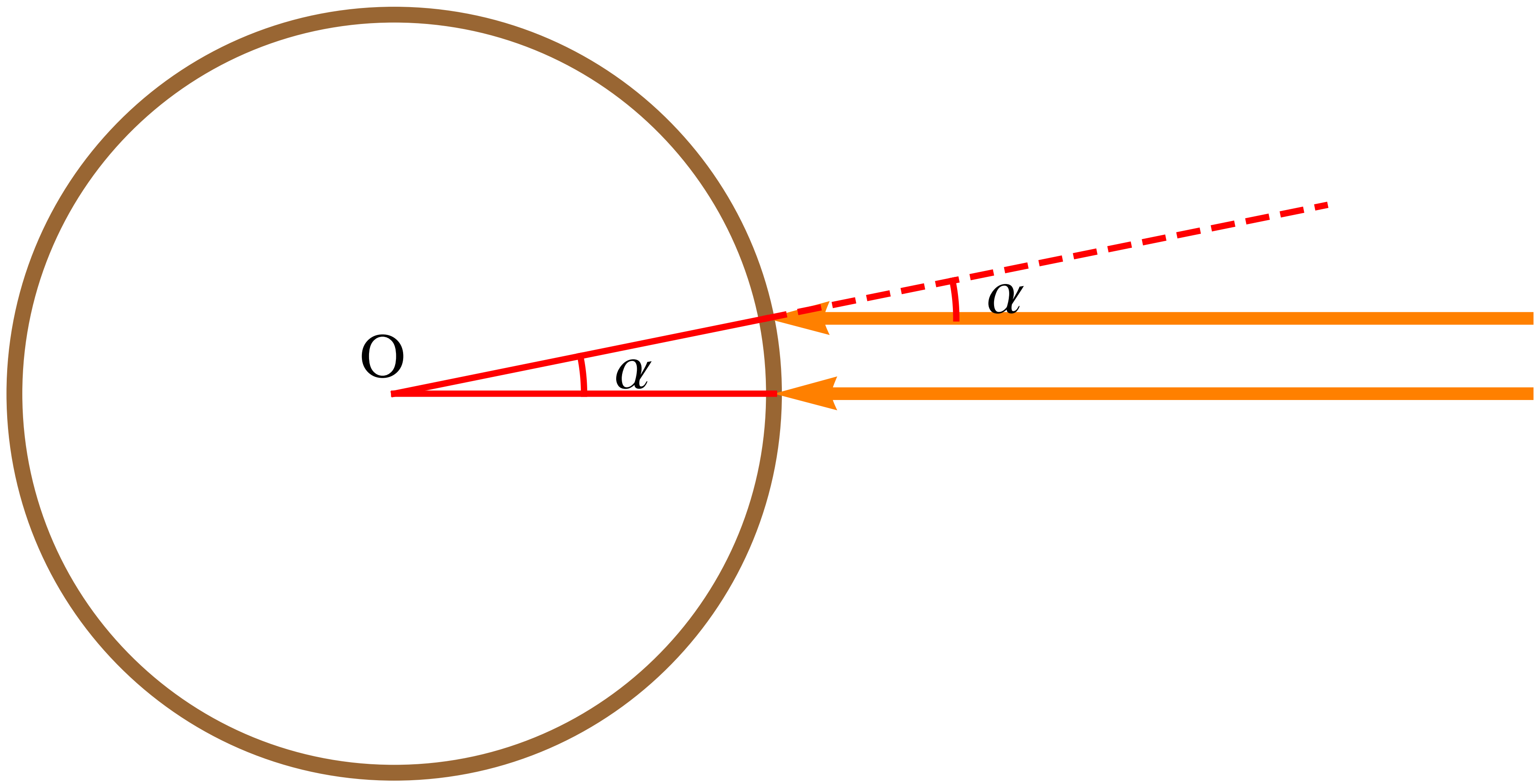
Cooperation has modeled the world as we know it

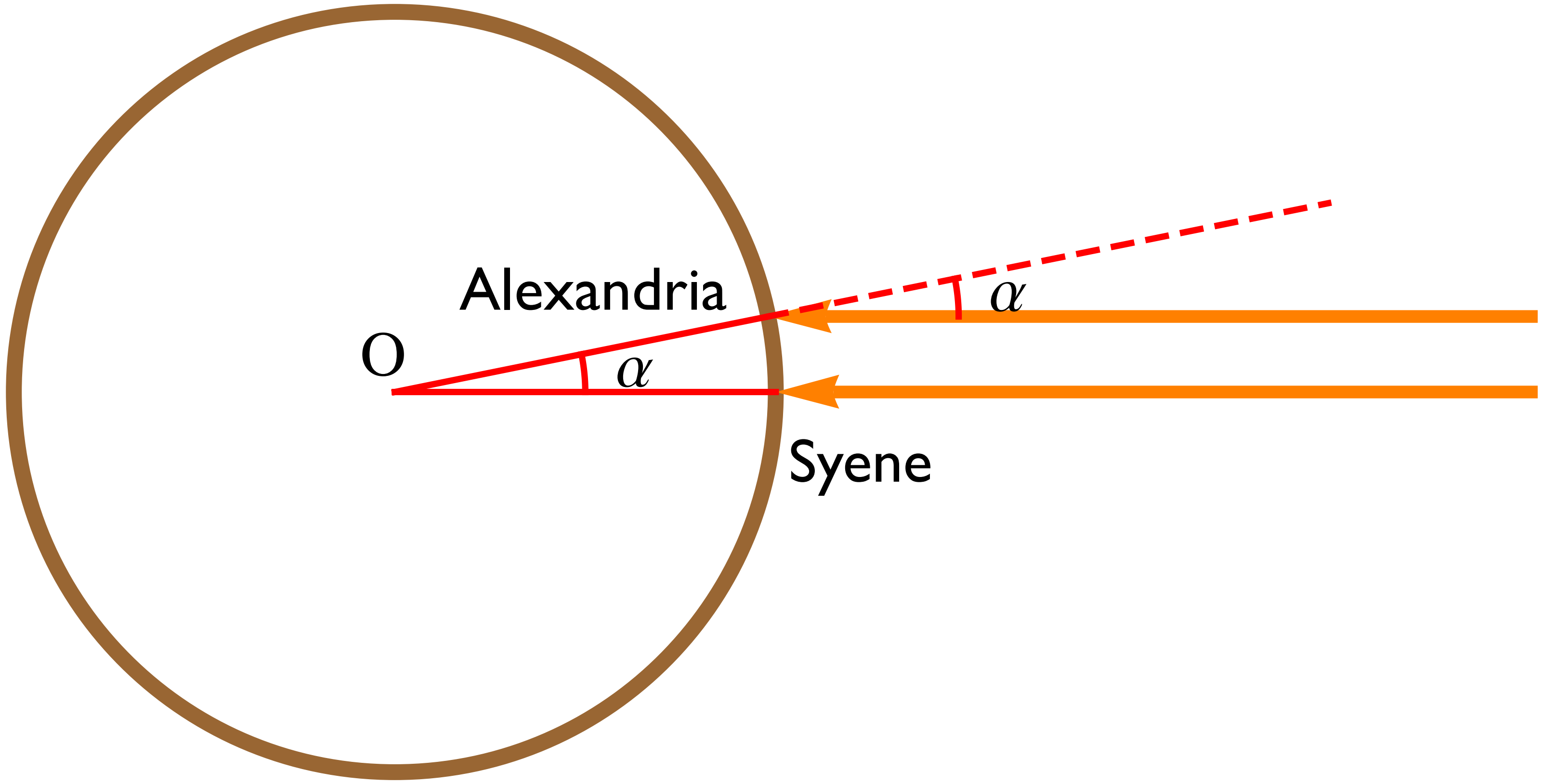
It explains the preservation of biodiversity

It is everywhere present in the human organization of the planet

Calculating the size of the Earth

The Greeks knew that the Earth is a sphere.
Eratosthenes had approximately calculated the circumference of the Earth



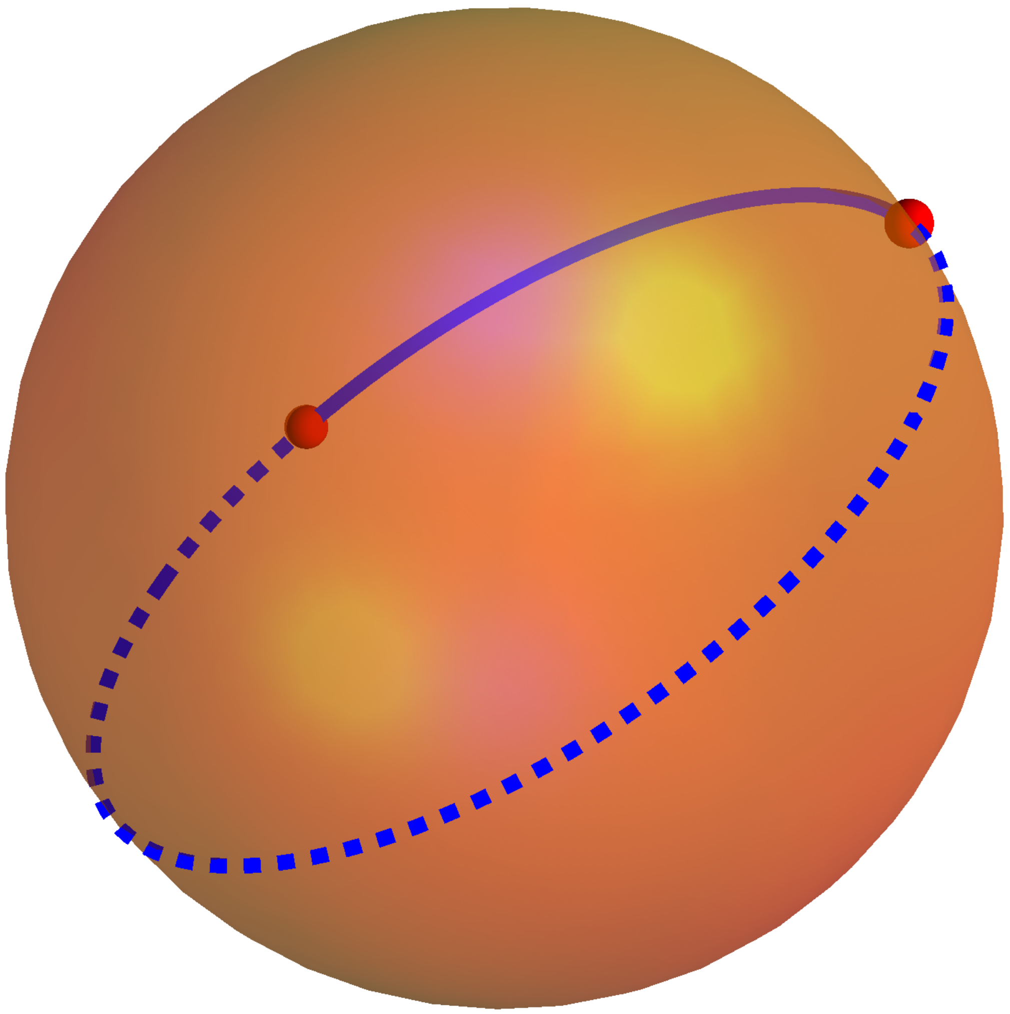


He knew that:

- the distance from Syene to Alexandria was 5000 stadiums (one stadium measured 157.5m),
- that the Sun was making an angle of 7.5 degrees with the vertical direction at Alexandria at the precise time when it was vertical at Syena,
- and that Alexandria and Syene are on the same meridian.

Problem

Question: what is the shortest path on Earth between two points?



Answer: It is the short arc of great circle joining these two points

Example

Calculate the shortest path on Earth between Dar es Salaam and Tokyo

Dar es Salaam:
Longitude: 39,27 E
Latitude: 6.82 S

Tokyo:
Longitude: 139,75 E
Latitude: 35,67 N

Considering that the Earth has an approximate radius $R = 6360$ km

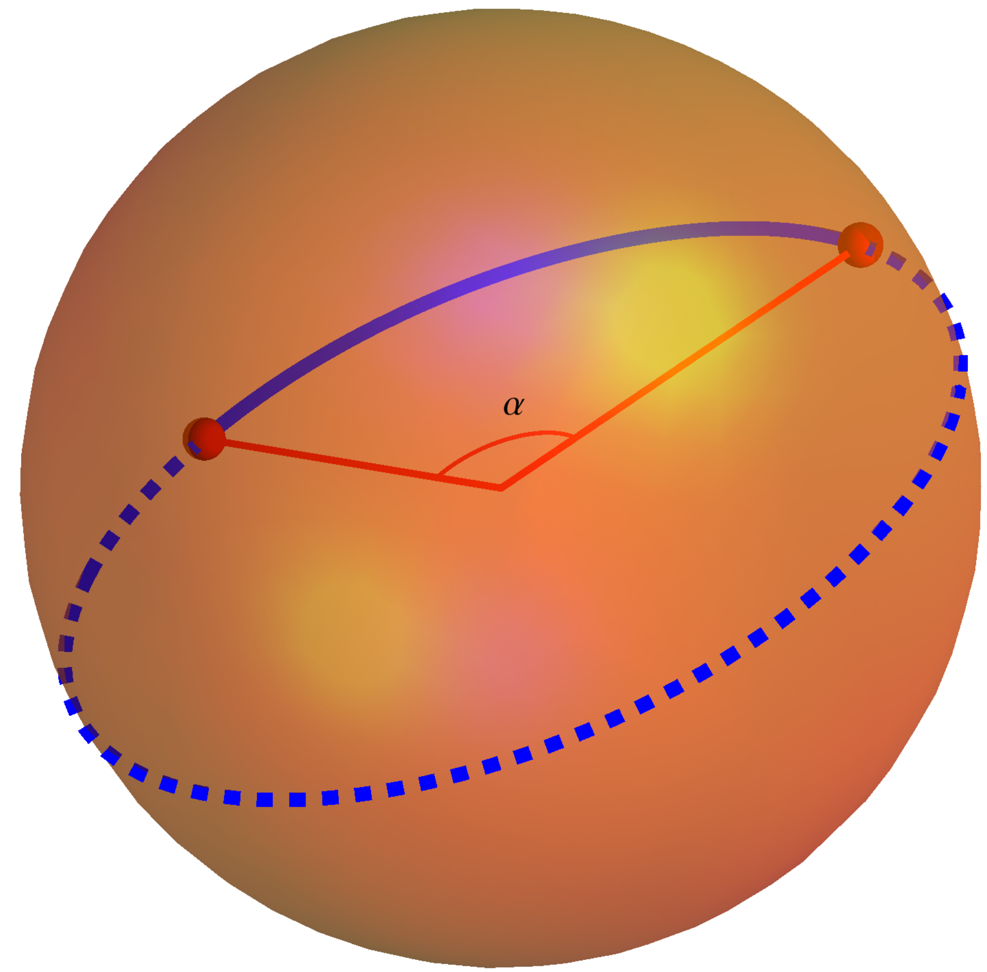
How to compute the distance between Dar es Salaam and Tokyo?

If a point has longitude L
and latitude ℓ, its coordinates are

$$(x, y, z) = R(\cos L \cos \ell, \sin L \cos \ell, \sin \ell)$$

Let us consider the two vectors joining the center of the Earth to Toronto and Tokyo

The cosine of the angle between the two vectors is given by their scalar product divided by the product of their lengths, $R \times R$



This yields α

The length of the arc of great circle is given by the angle **(in radians!)** multiplied by the radius $R = 6360$ km

Toronto

Longitude: 79.33 W

Latitude: 43.68 N

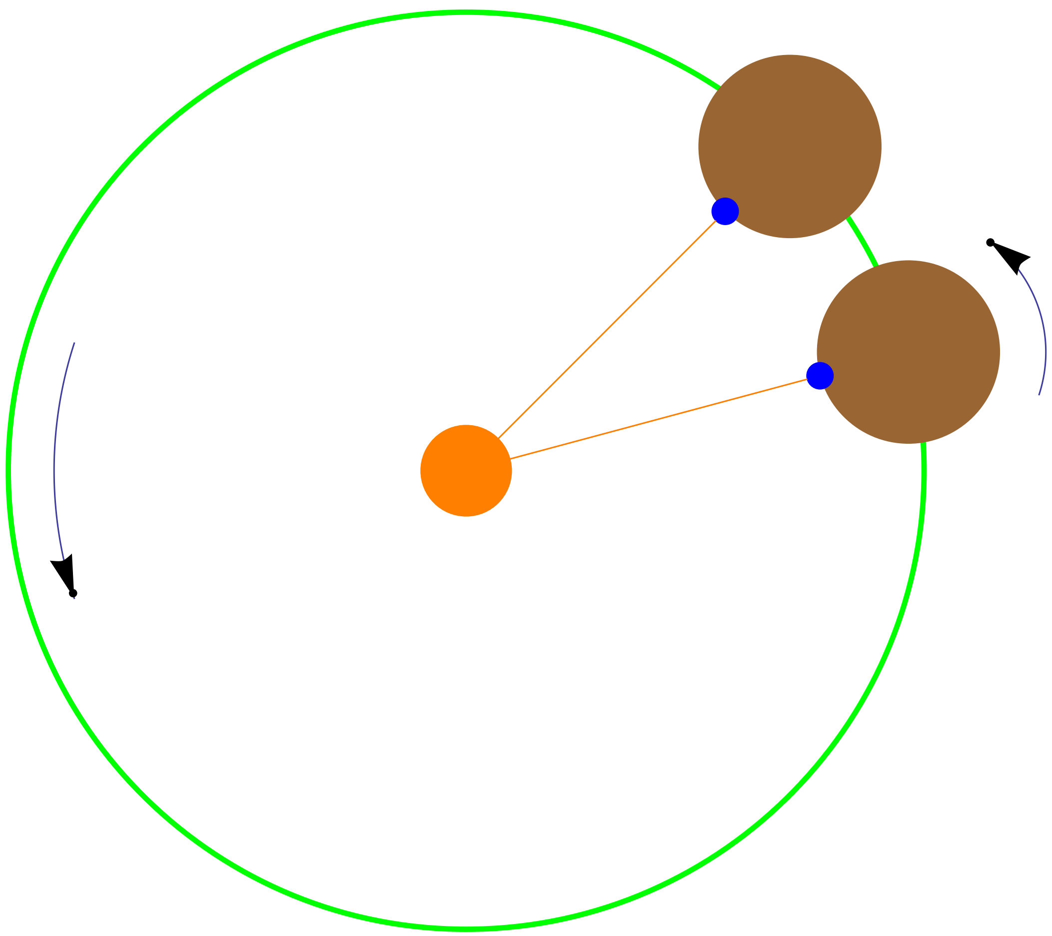
Tokyo

Longitude: 139.75 E

Latitude: 35.67 N

Question

Suppose that a year is 365 days. How many revolutions does the Earth make around its axis during one day? During one year?



Questions

The Tropic of Cancer has latitude 23.5 N and that of Capricorn, 23.5 S

The Arctic Polar Circle has latitude 66.5 N and the Antarctic Polar Circle has latitude 66.5 S

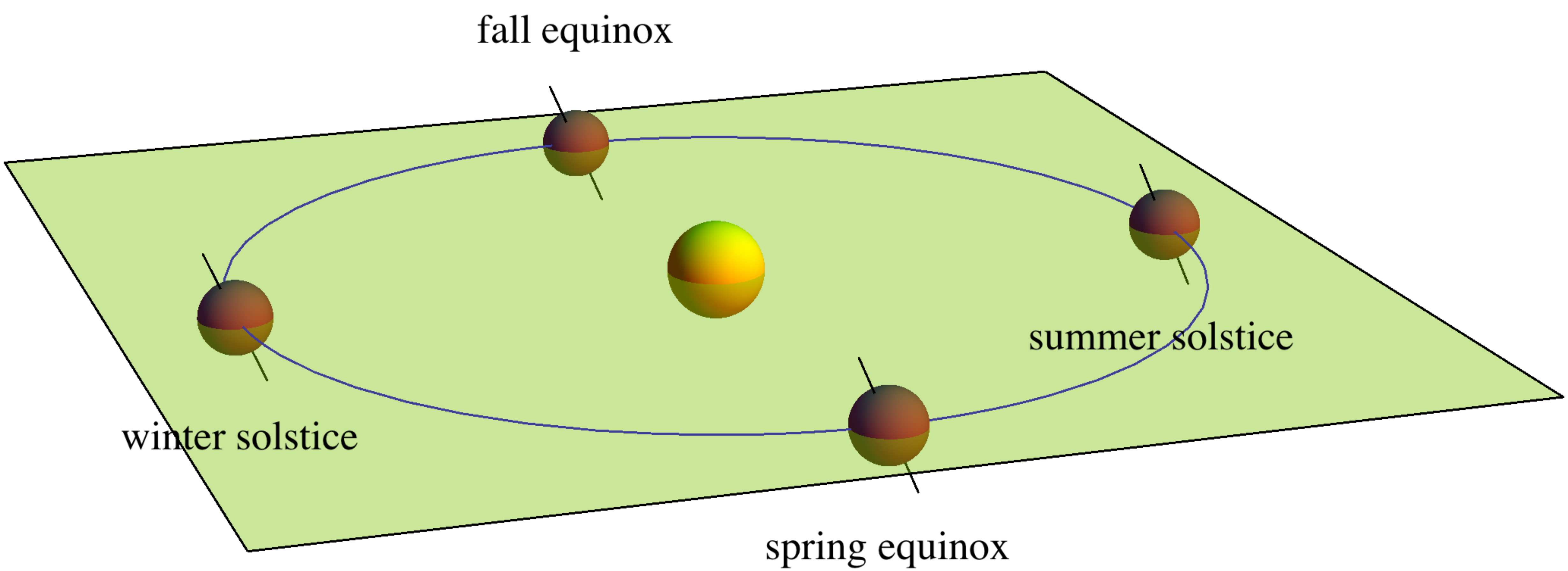
Explain what special phenomena occur at these latitudes, as well as at the Equator and at the poles.

The seasons

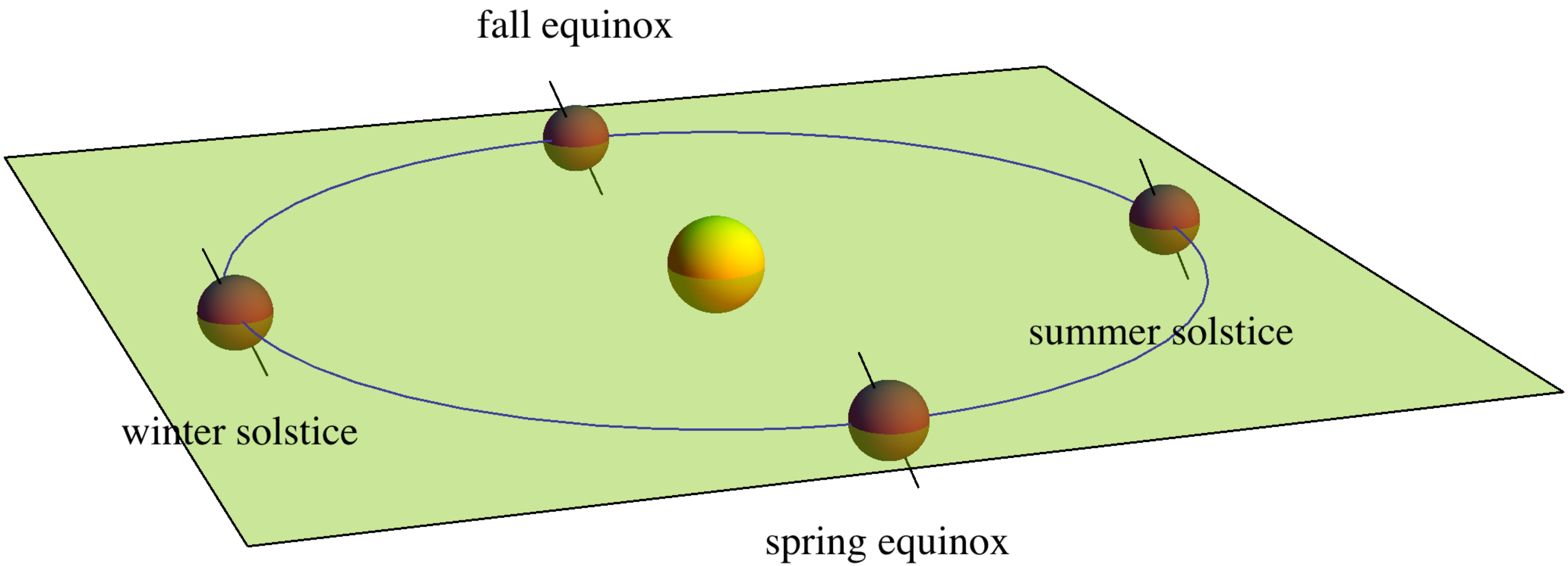
What is the height of the Sun at noon depending on the date and the latitude?

What is the length of the day depending on the date and the latitude?

The angle of the Earth's Axis with the equatorial plane is 23.5 degrees



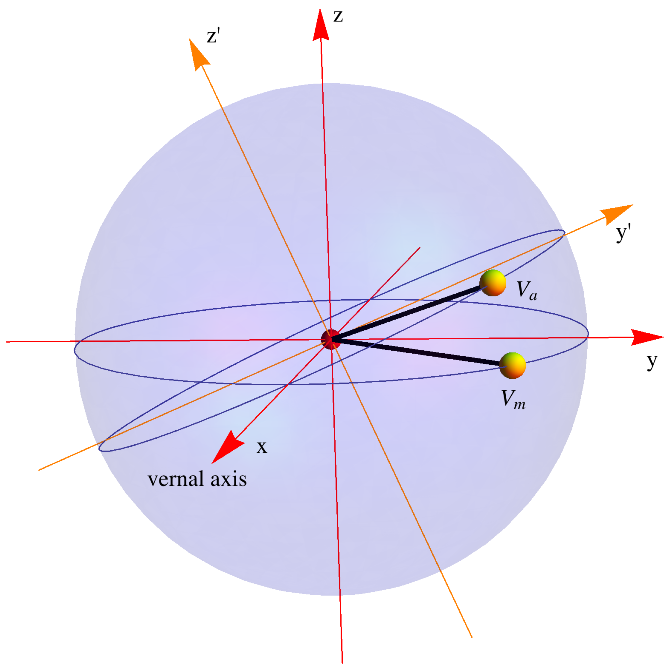
The season mechanism comes from the obliquity of the Earth axis

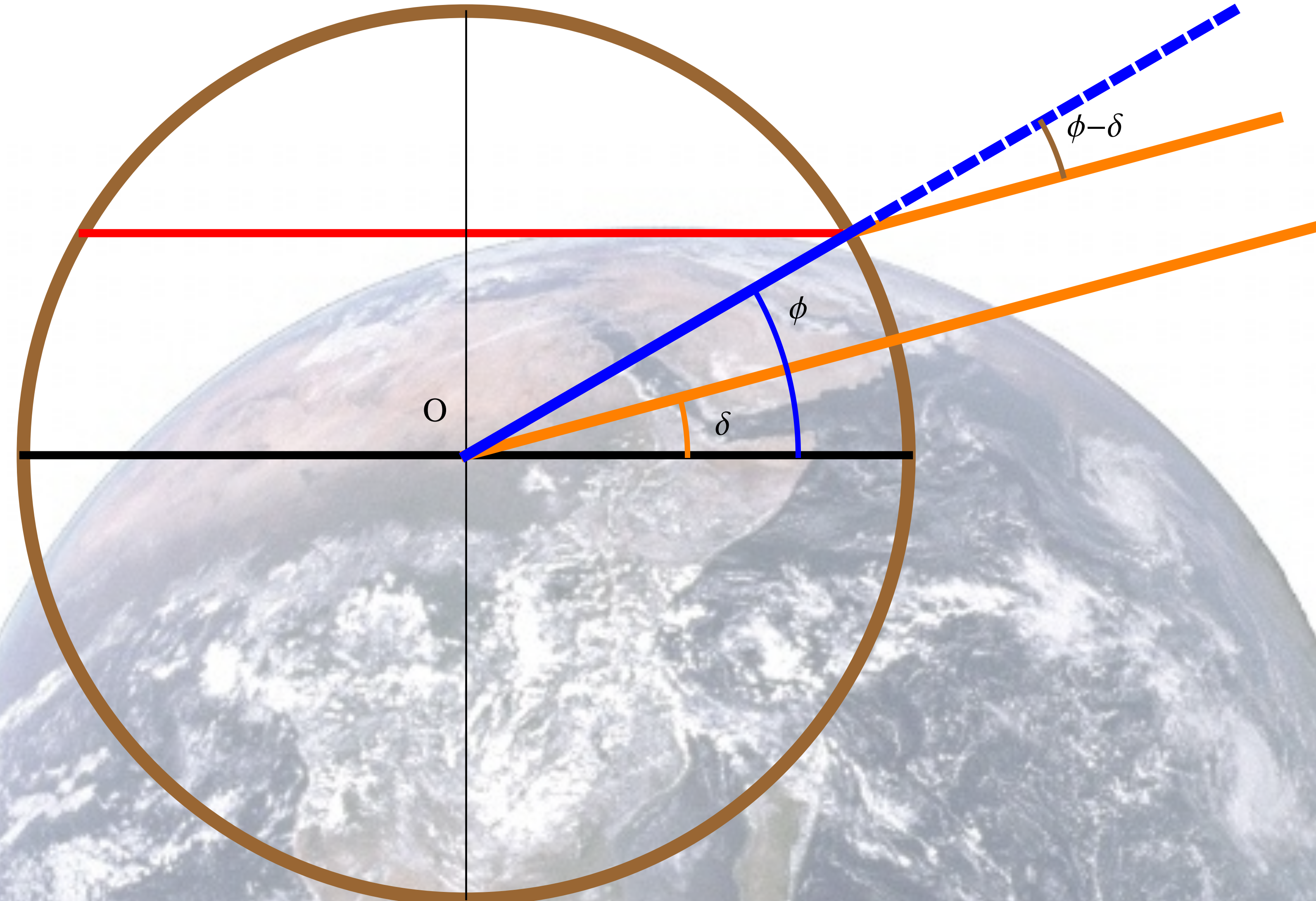


How to make the computation

The geocentric point of view:

- The earth is at the center
- The equatorial plane is horizontal and the polar axis vertical
- The Earth is at the center of the celestial sphere of infinite radius
- The Sun rotates around the Earth in a plane making an angle of 23.5 degrees with the equatorial plane



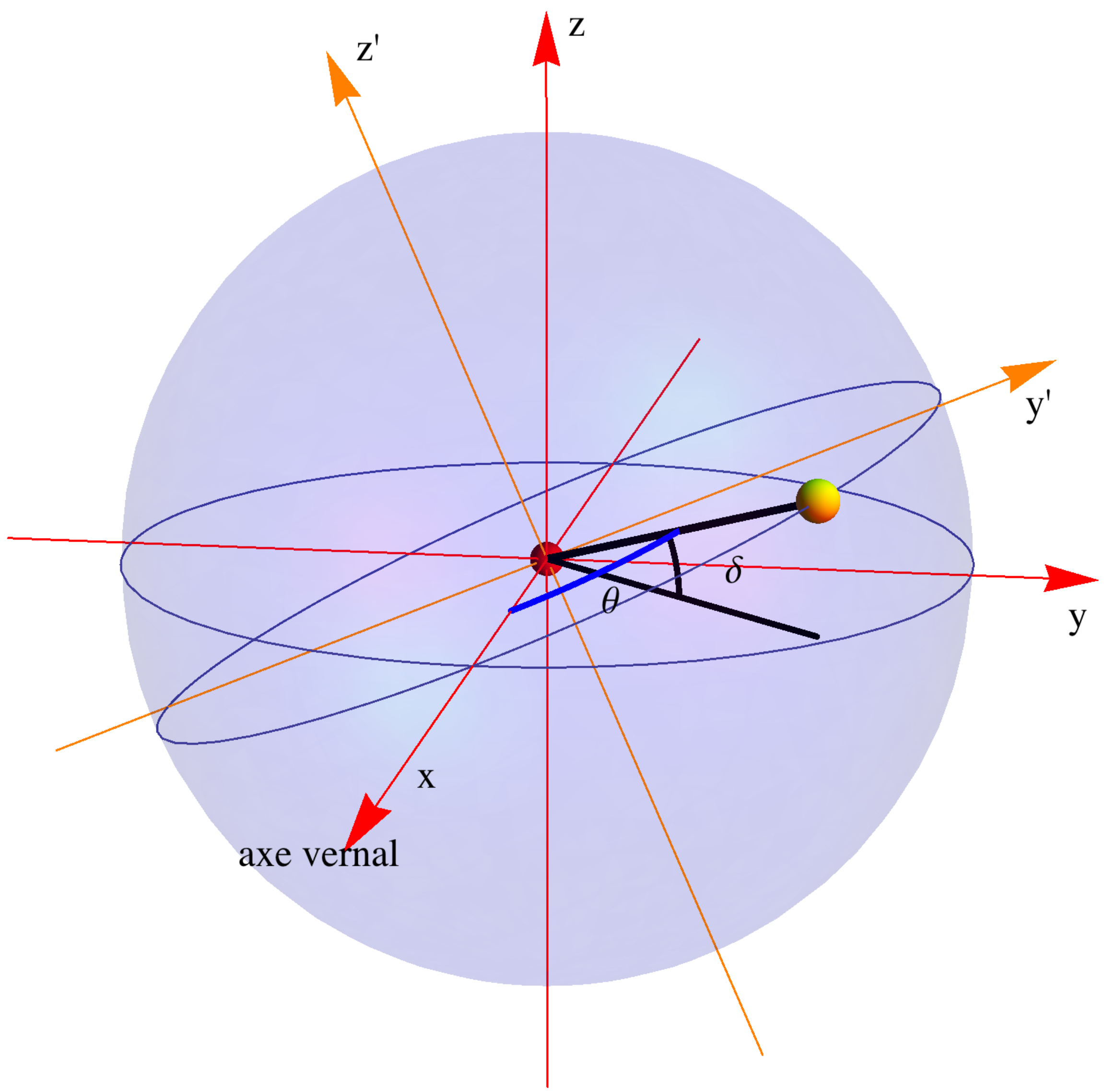


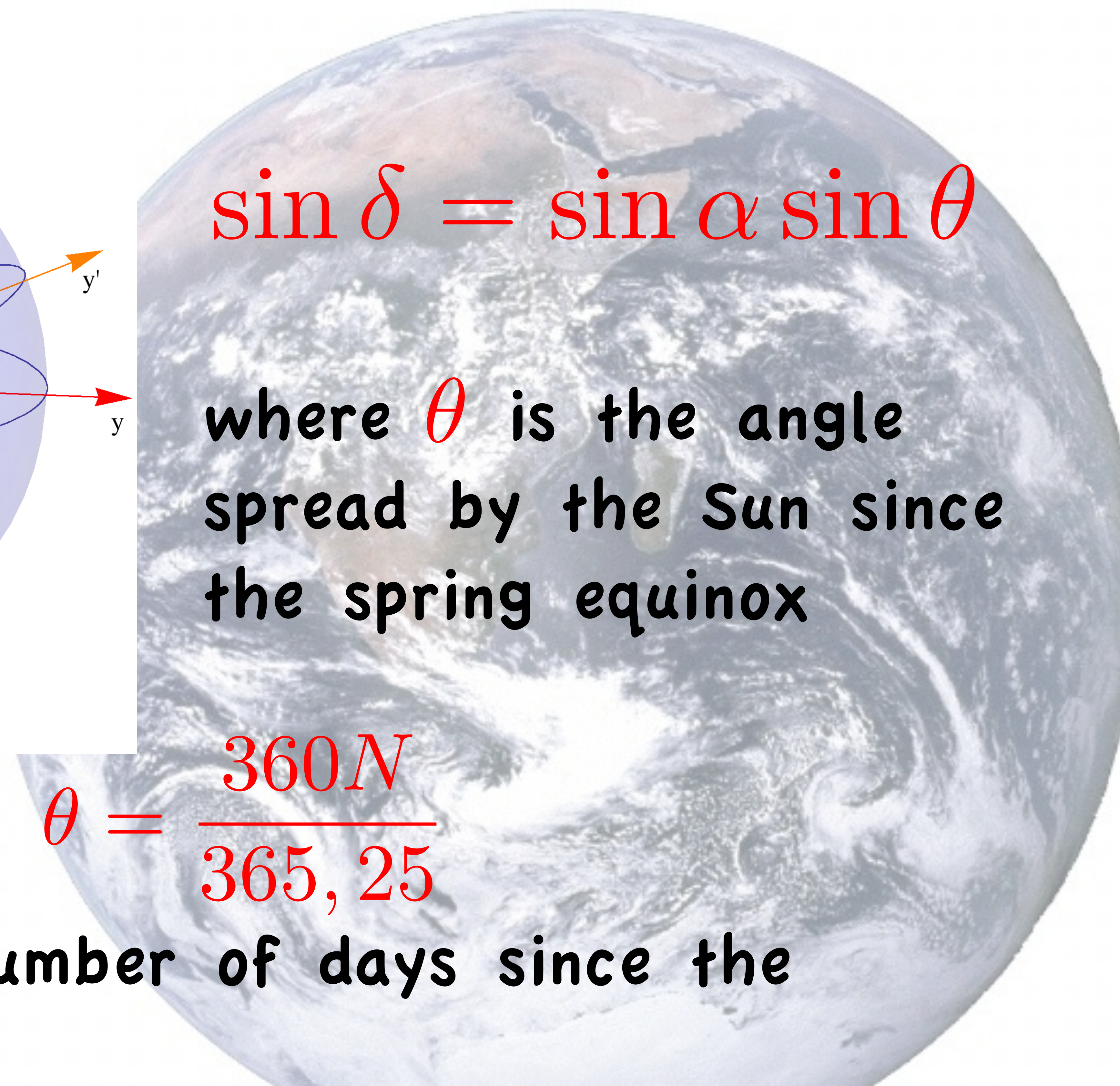
The height of the Sun at noon is given by

$$90^o - |\phi - \delta|$$

where ϕ is the latitude and δ is the declination (i.e the angle between the line Earth-Sun and the equatorial plane)

How to compute the declination δ ?



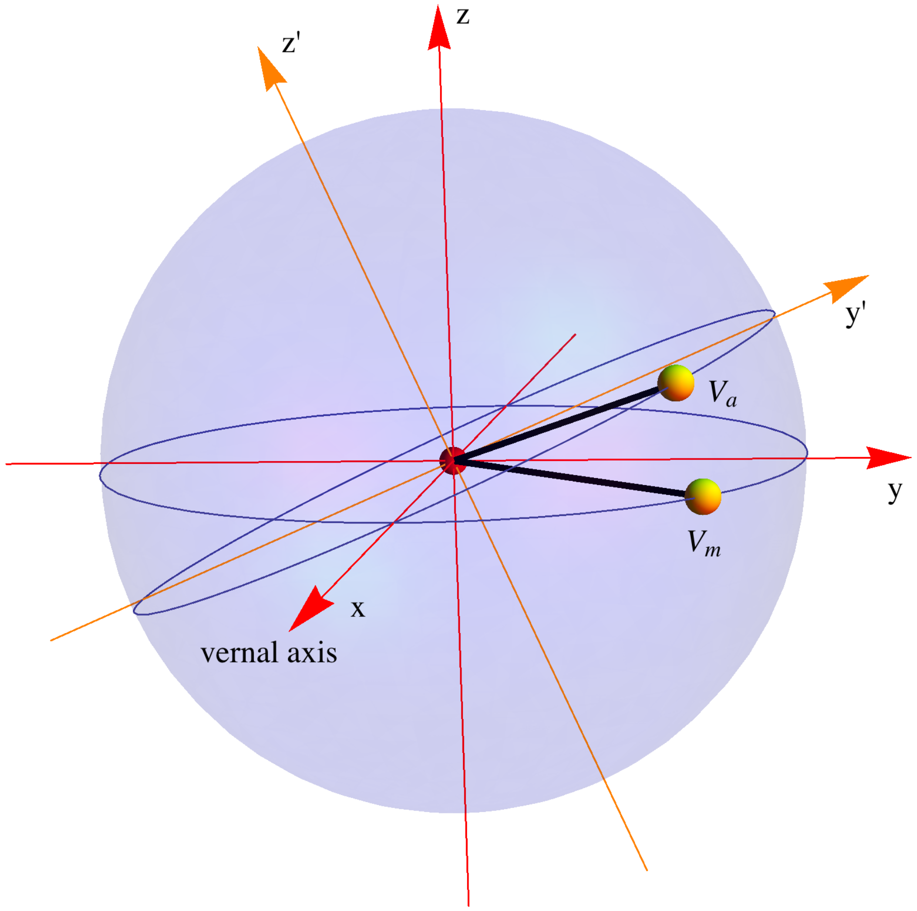


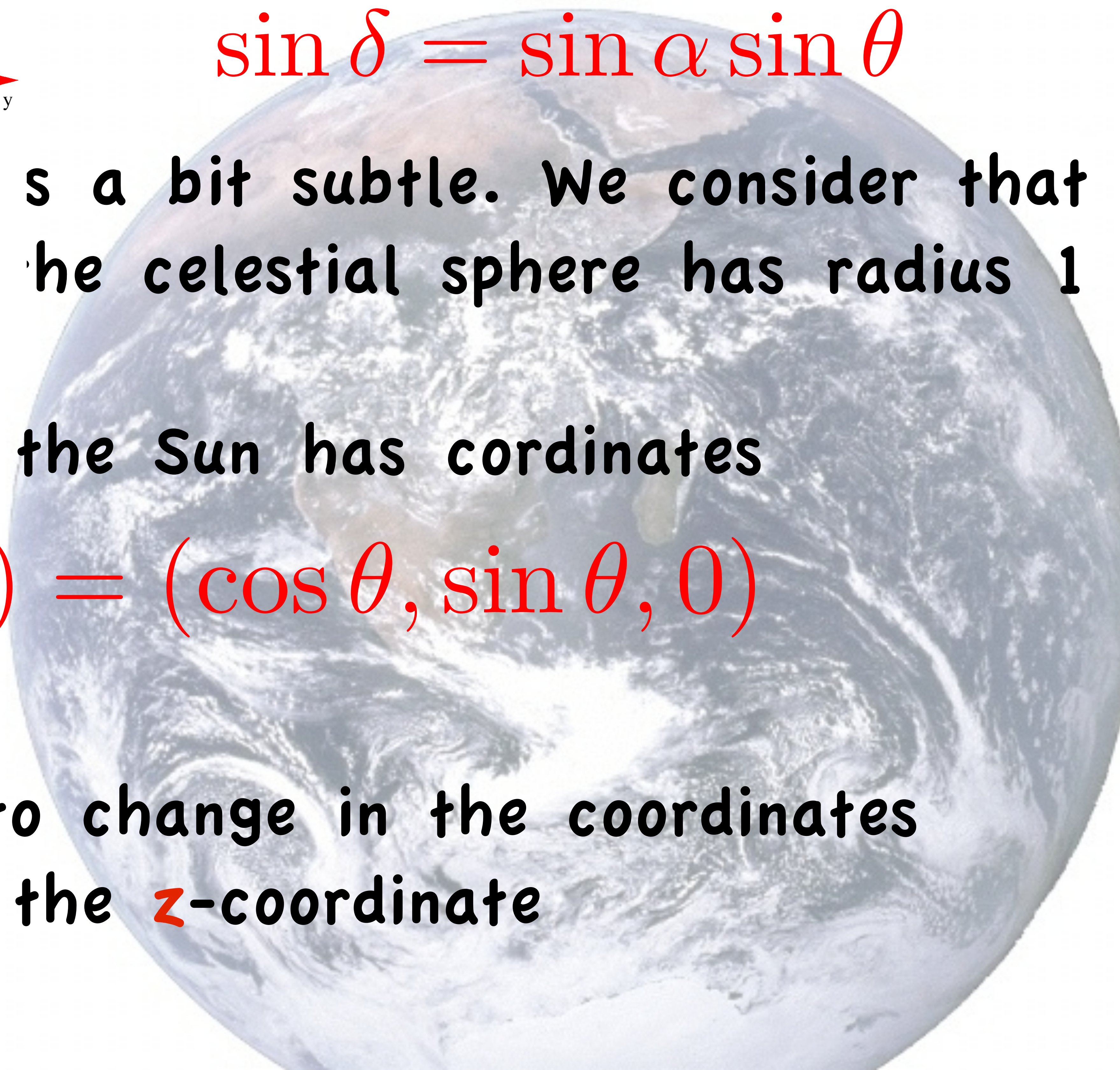
$$\sin\delta = \sin\alpha \sin\theta$$

where θ is the angle spread by the Sun since the spring equinox

i.e., in degrees $$\theta = \frac{360N}{365,25}$$

where N is the number of days since the spring equinox





Checking the formula

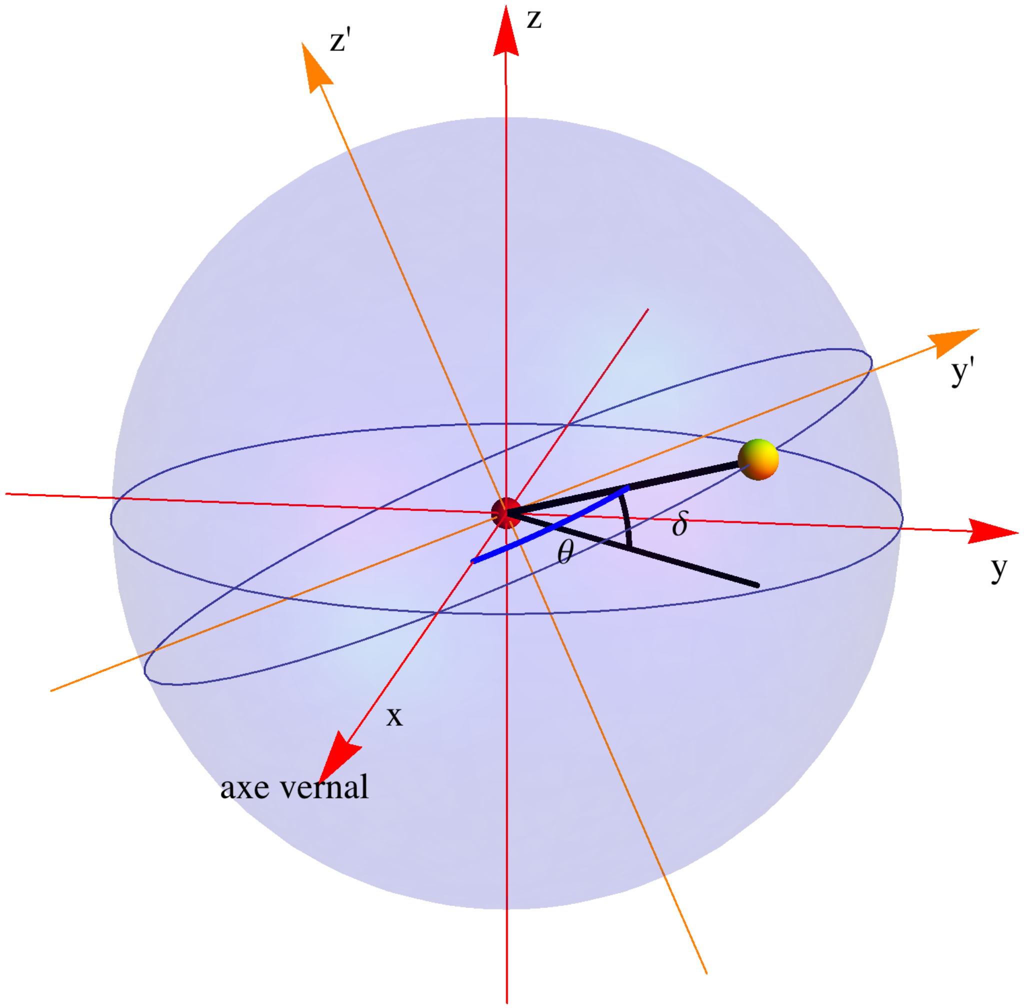
$$\sin\delta = \sin\alpha\sin\theta$$

s a bit subtle. We consider that
he celestial sphere has radius 1

First method: the Sun has cordinates

$$(x, y', z') = (\cos\theta, \sin\theta, 0)$$

One then needs to change in the coordinates **(x,y,z)** and take the **z**-coordinate

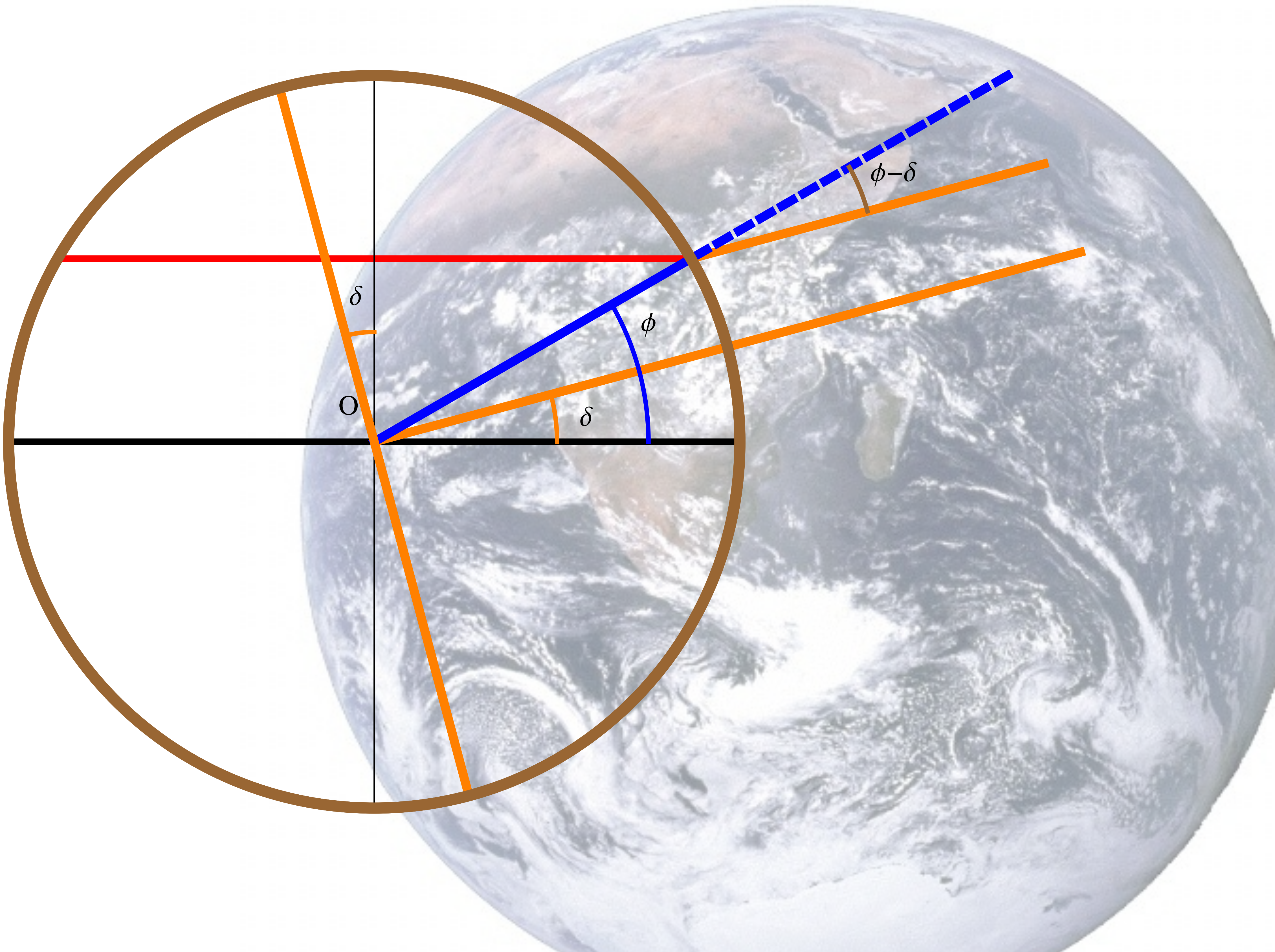


Checking the formula

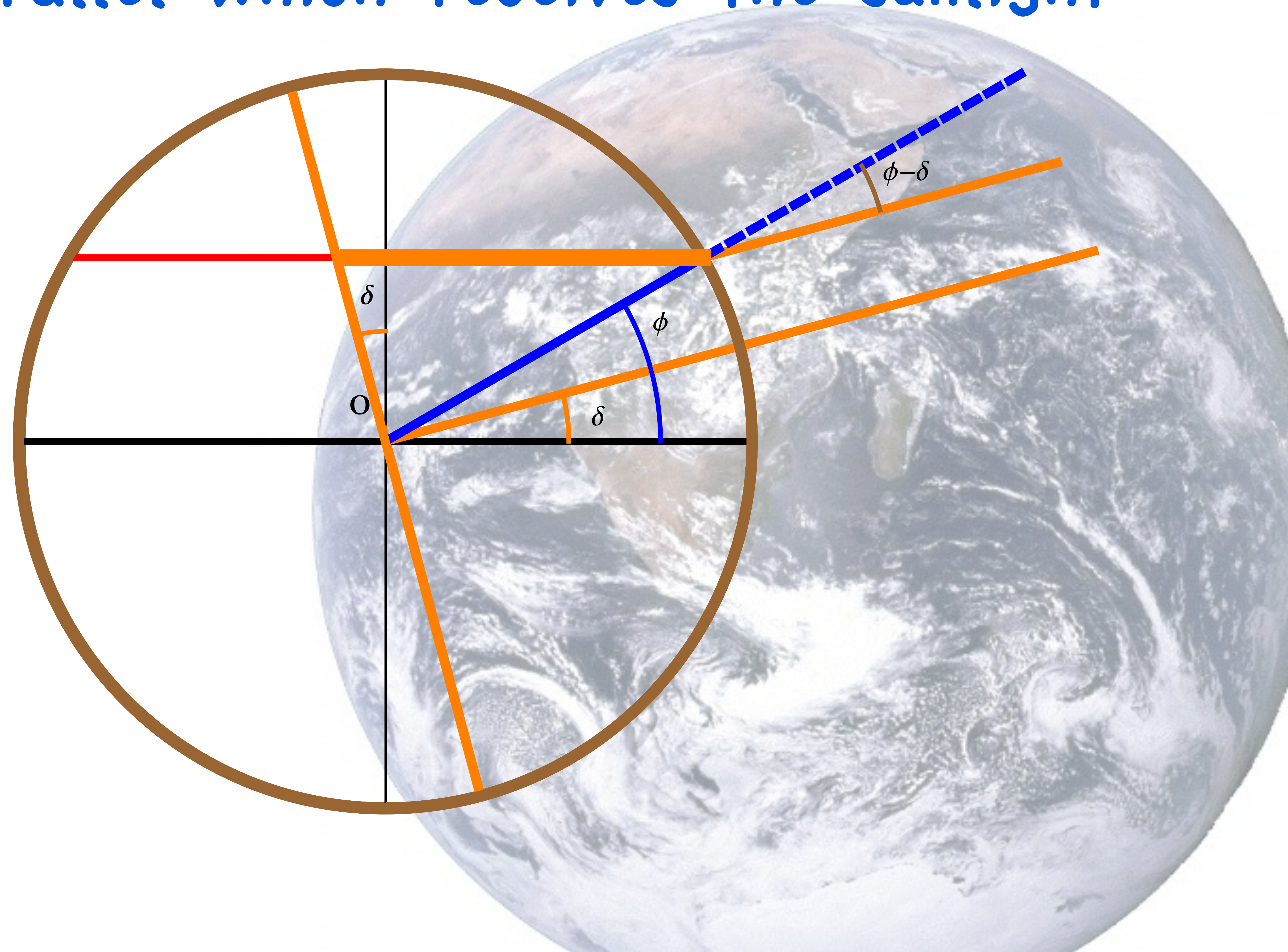
$$\sin\delta = \sin\alpha\sin\theta$$

Second method (more elementary but requiring mre computation): Consider similar traingles in the vertical plane through the Sun and the intersection point of the y'-axis with the celestial sphere.

What is the length of the day?



It is determined by the portion of the parallel which receives the sunlight



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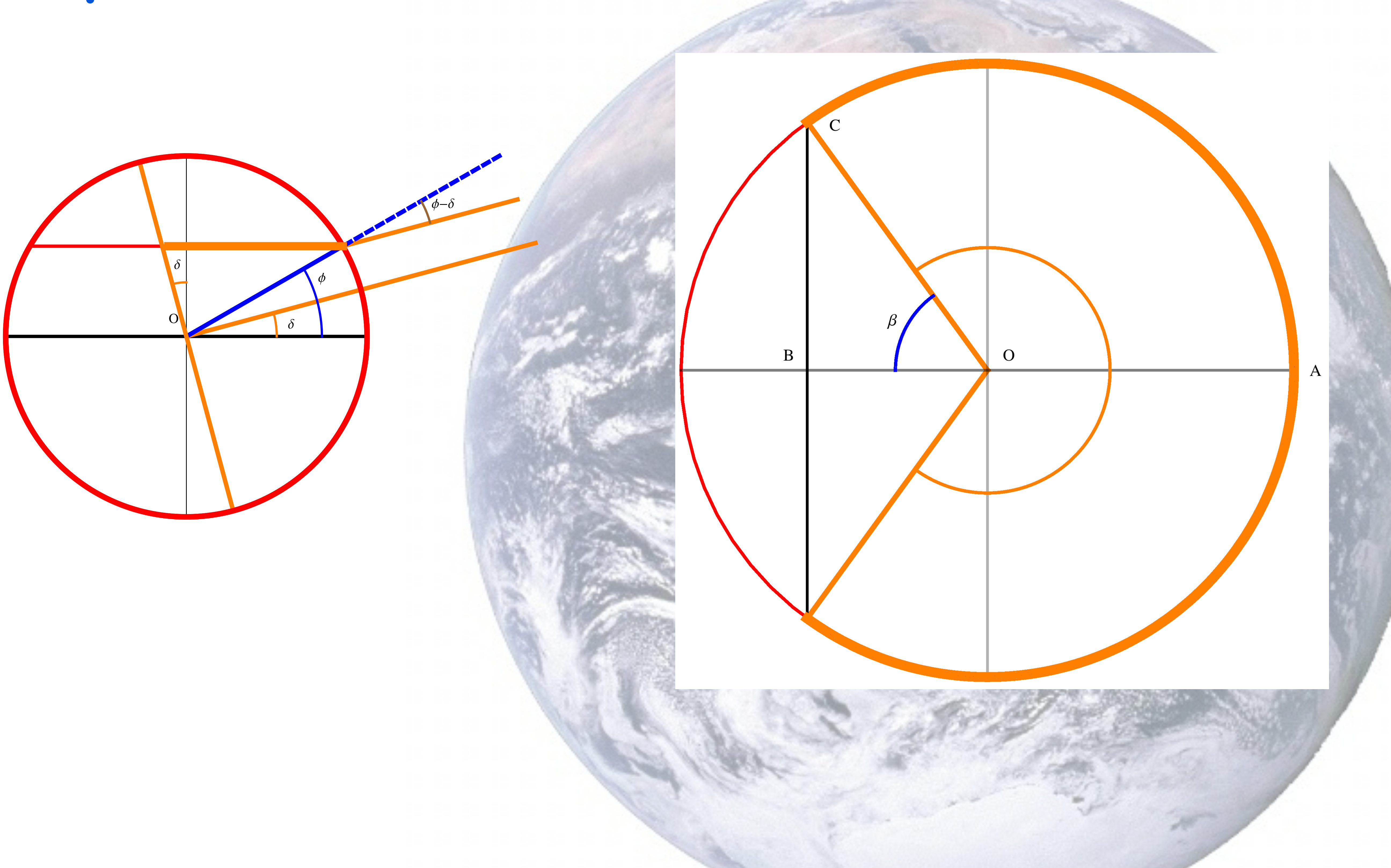
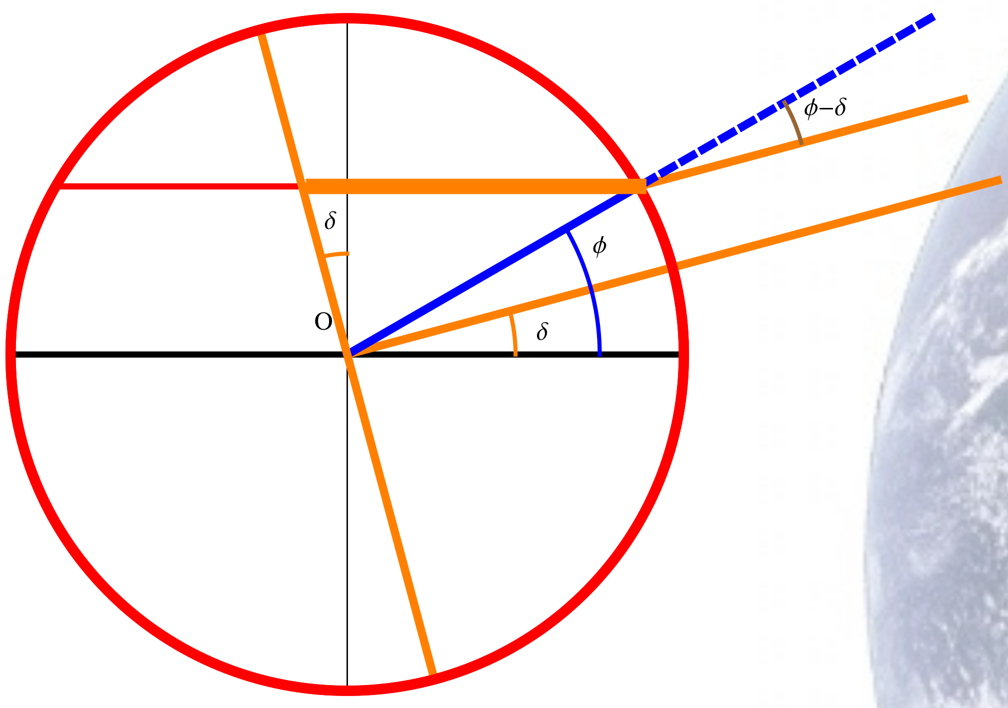
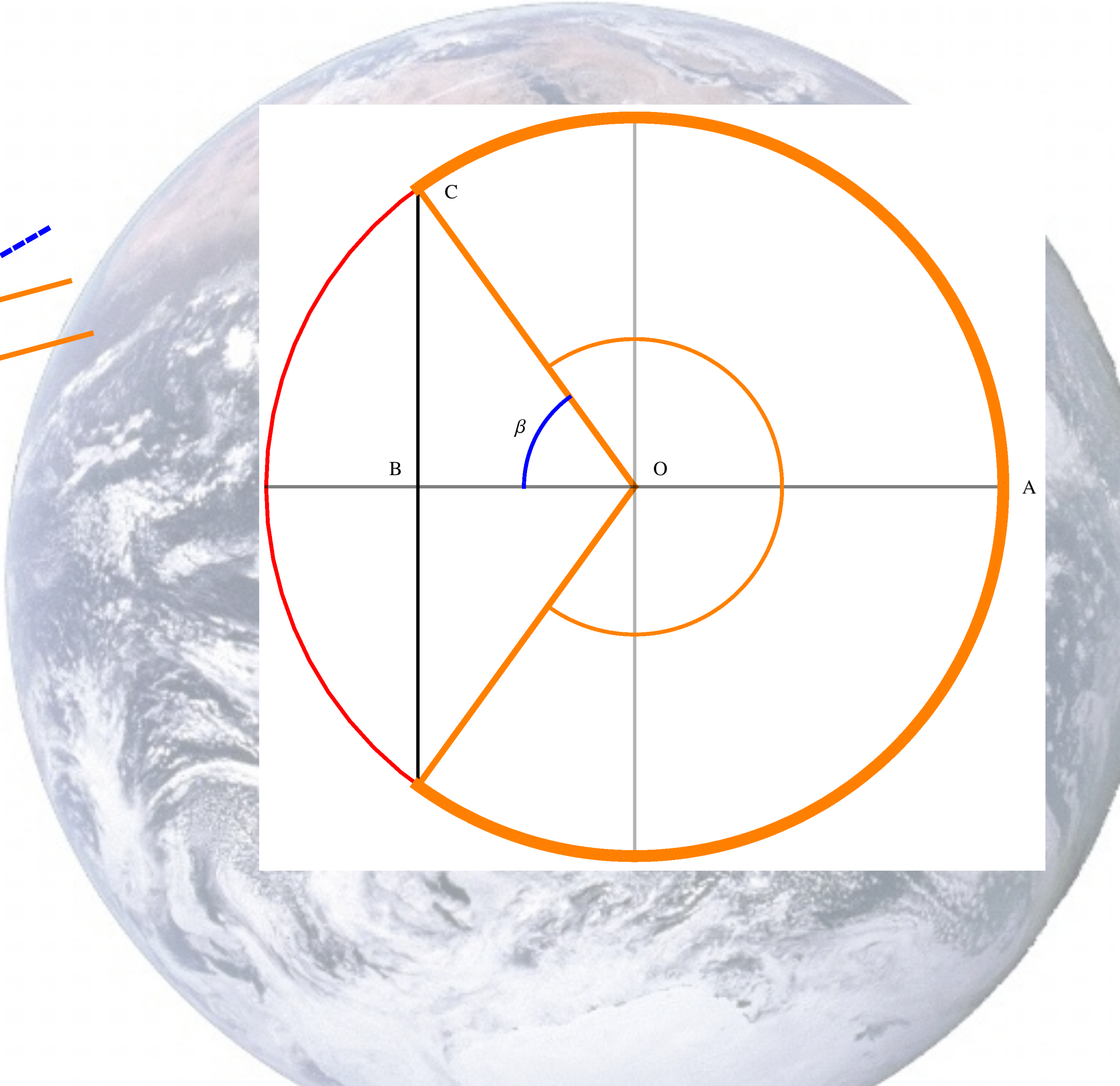
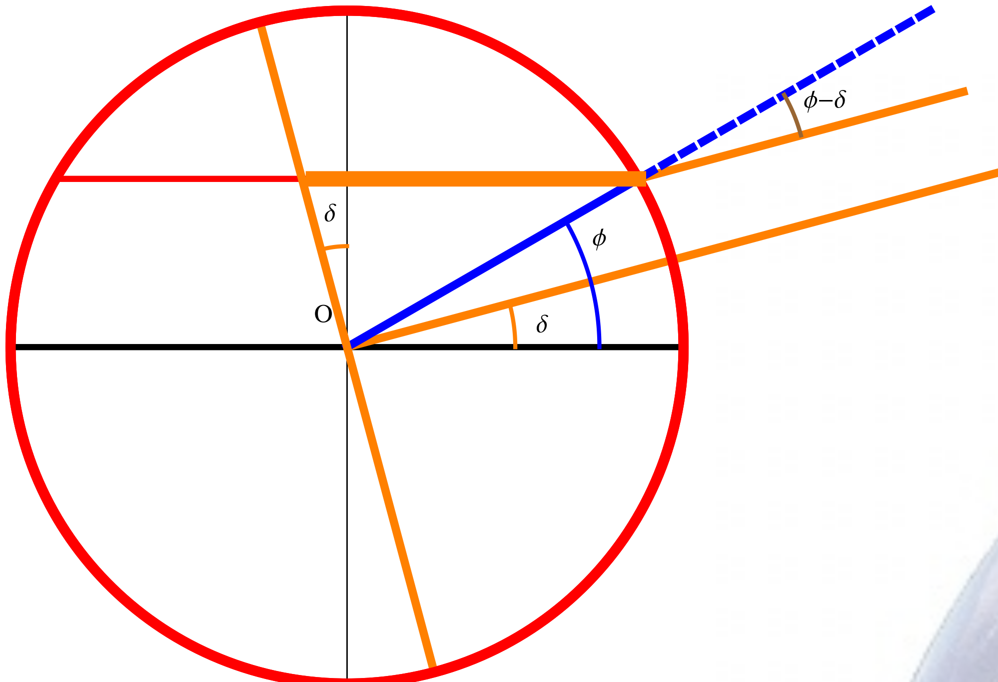


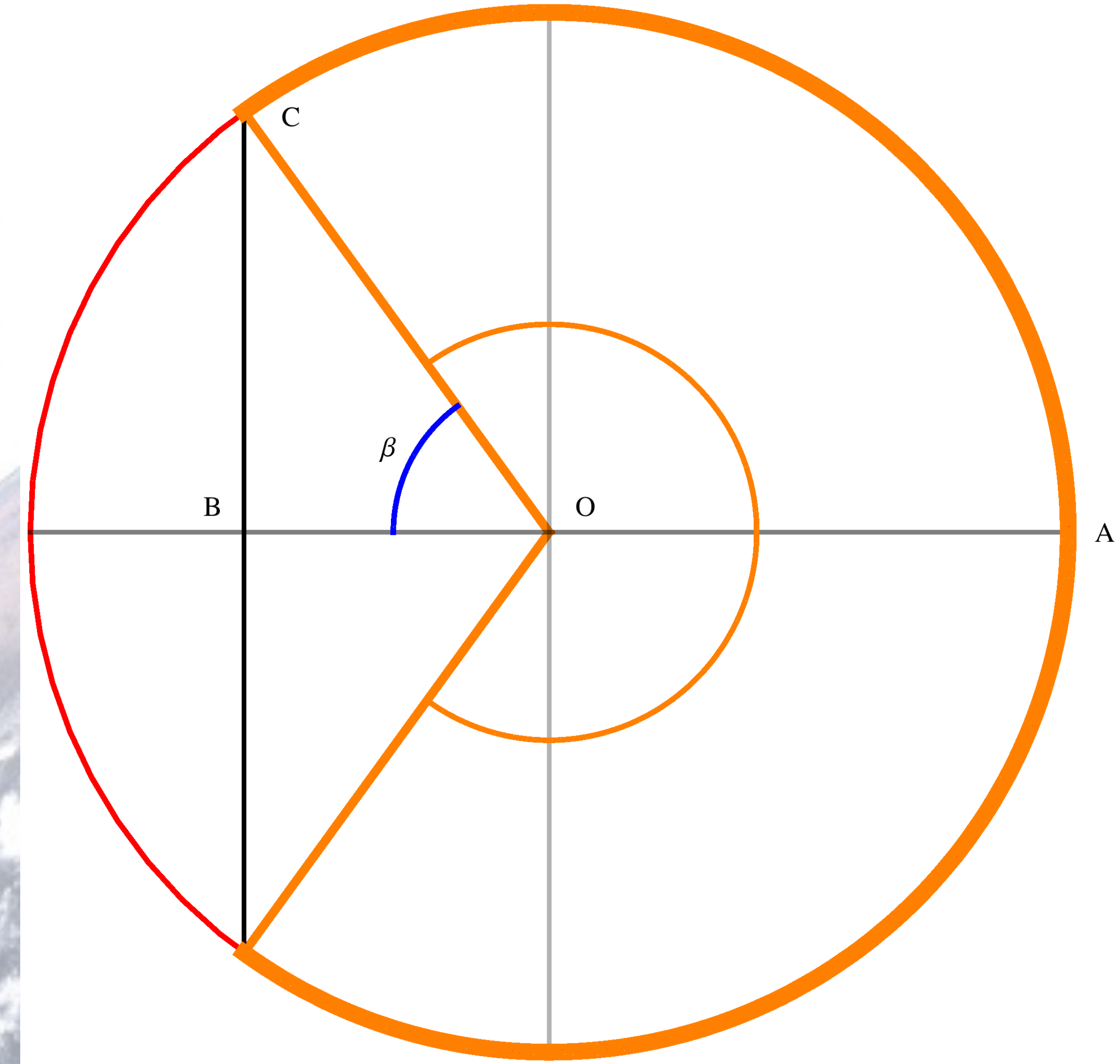
This corresponds to a central angle of

$$360^\circ - 2\beta$$









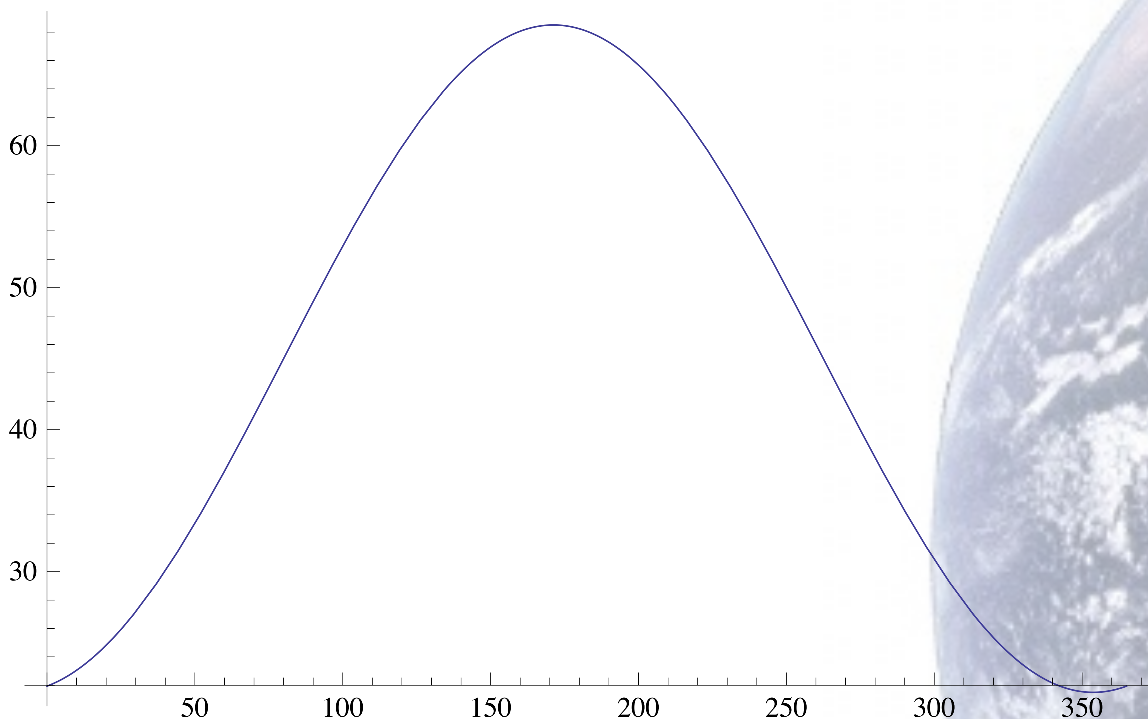
$$|OB| = R \sin\phi \tan\delta$$

$$\cos\beta = \frac{|OB|}{|OC|}$$

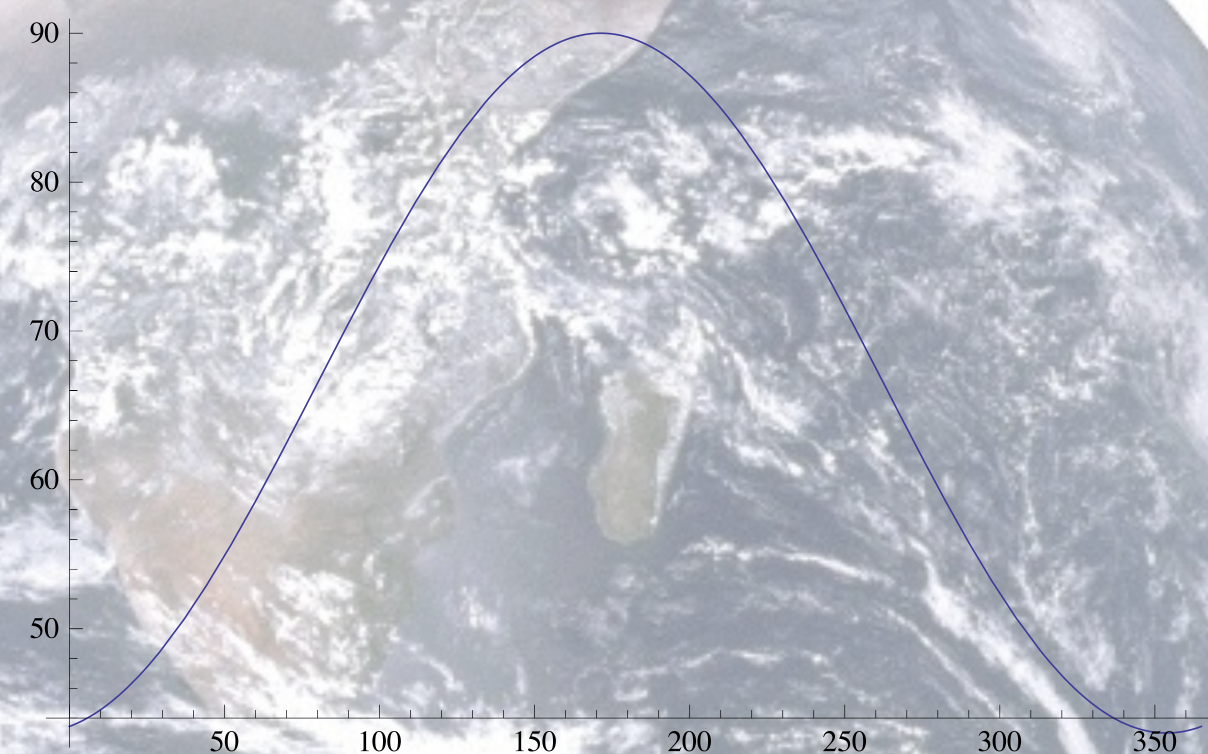
$$|OC| = R\cos\phi$$

$$\cos\beta = \tan\phi\tan\delta$$

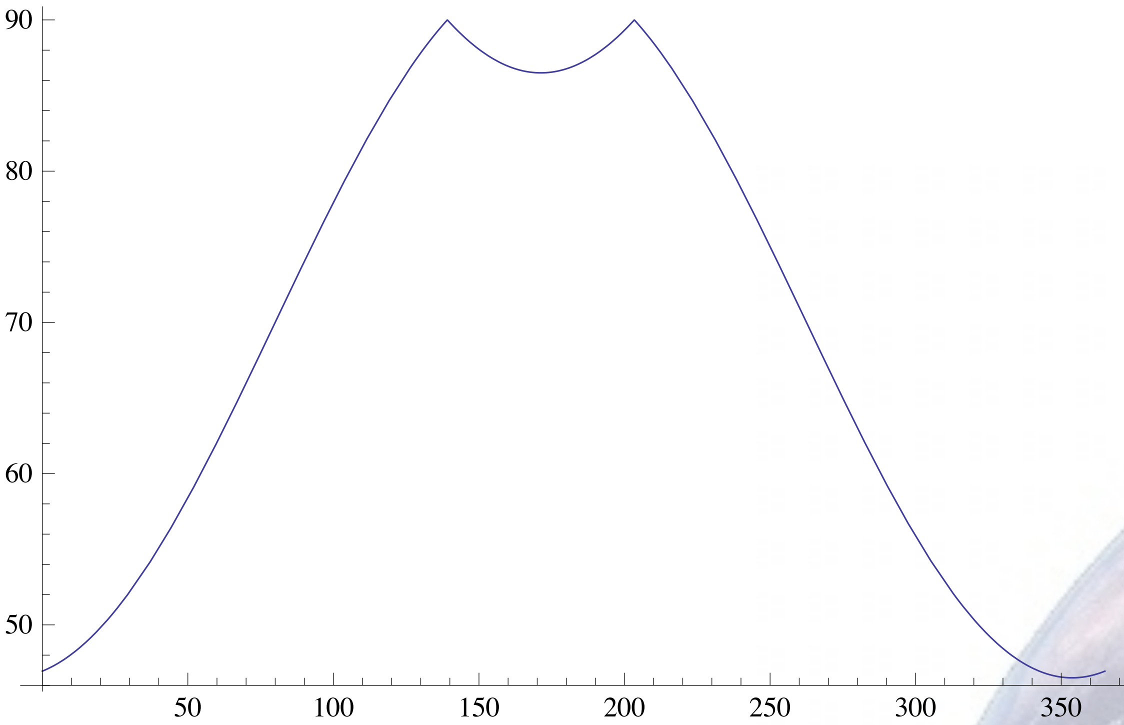
The height of the Sun at noon depending on the date and the latitude



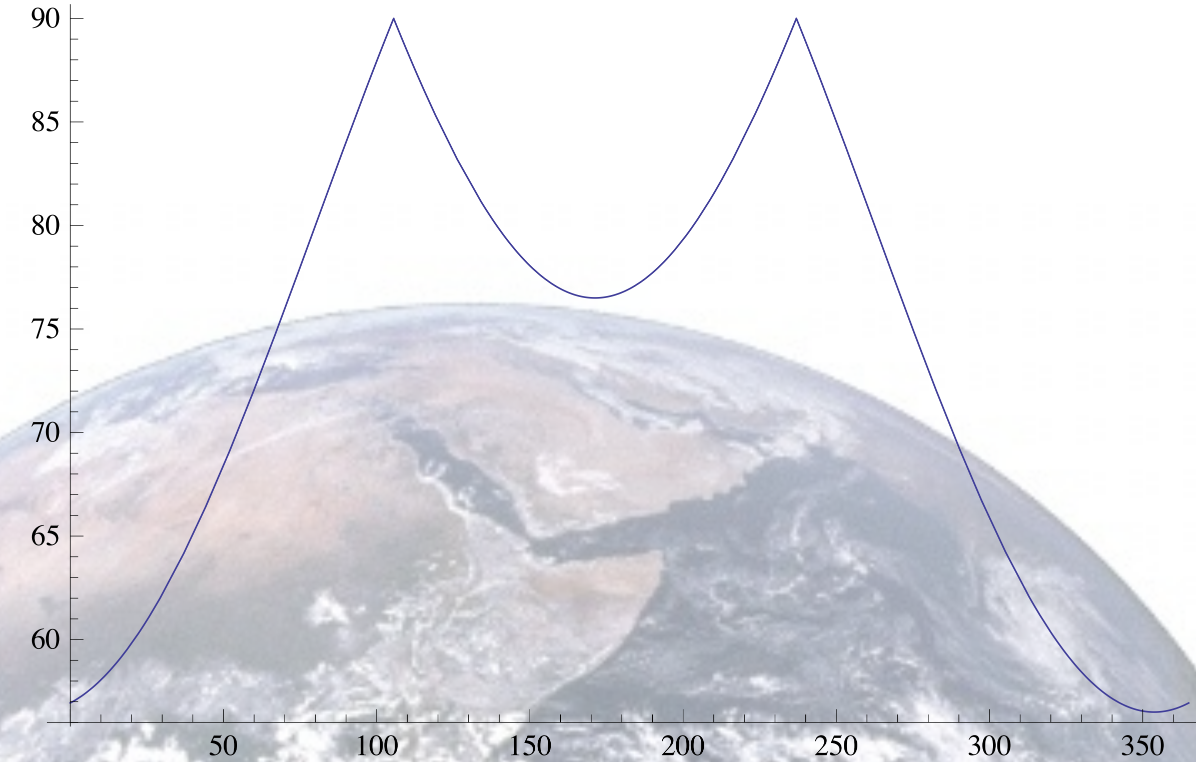
Latitude 45



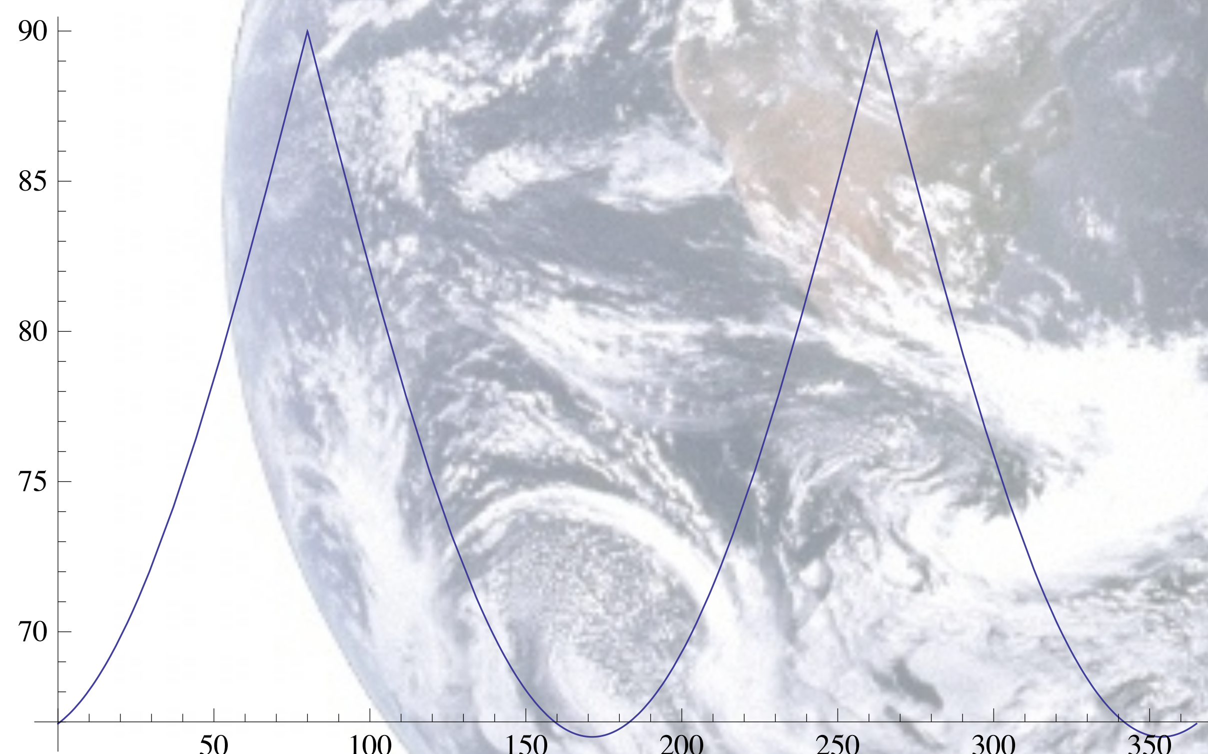
Latitude 23.5



Latitude 20



Latitude 10



At the Equator

The equation of time

Have you already noticed that in the Northern hemisphere the sunset is sooner on December 10 than at the winter solstice?

At a given place, the solar noon (when the Sun is at its highest position) is not always at the same time.

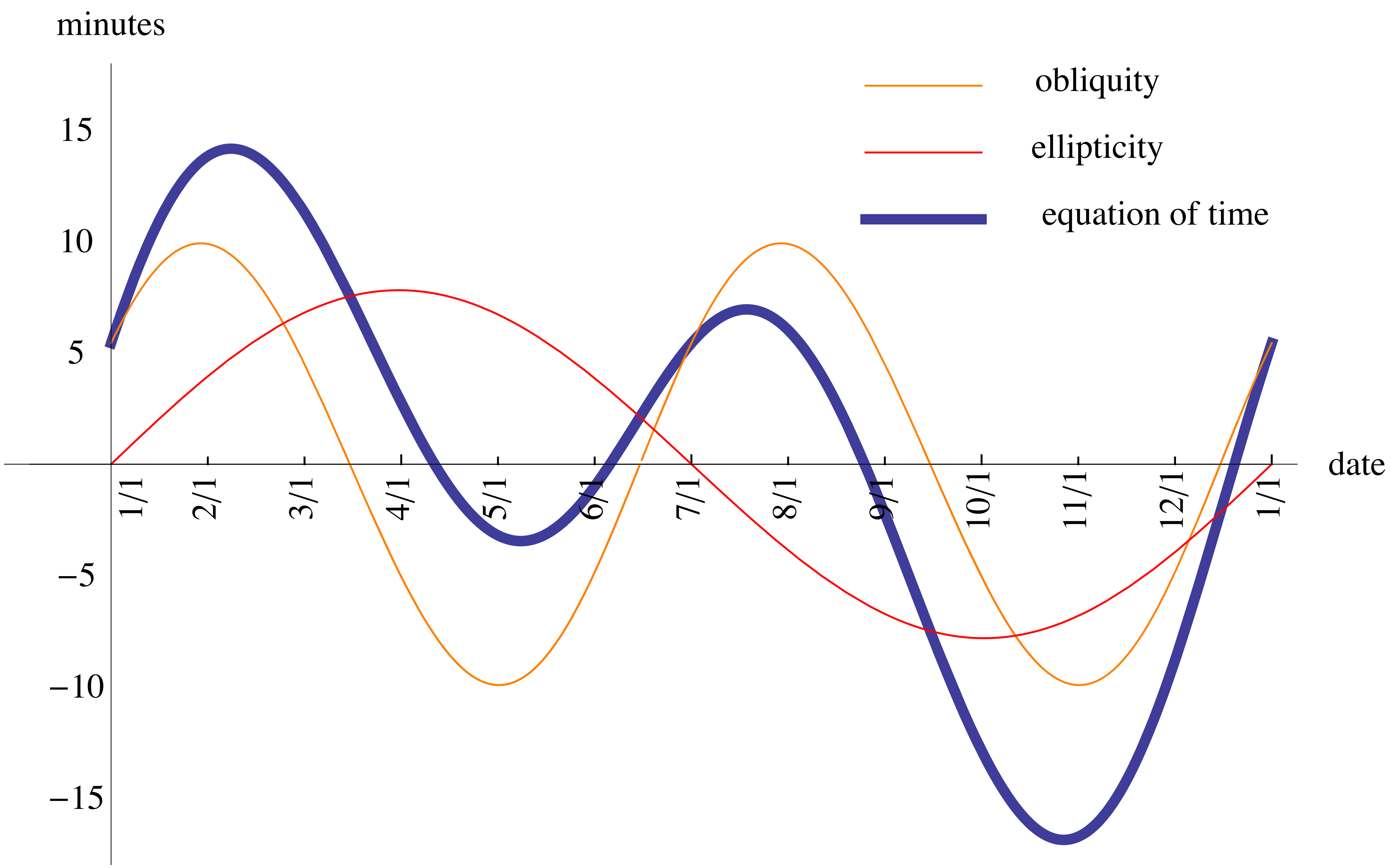
Solar noon at Dar es Salaam (at normal time)

date	sunrise	sunset	solar noon
January 1st	6:11	18:42	12:26
February 1st	6:24	18:48	12:36
March 1st	6:28	18:42	12:35
April 1st	6:25	18:27	12:26
May 1st	6:23	18:16	12:19
June 1st	6:28	18:13	12:20
July 1st	6:34	18:18	12:26
August 1st	6:34	18:23	12:29
September 1st	6:23	18:22	12:23
October 1st	6:07	18:17	12:12
November 1st	5:55	18:17	12:06
December 1st	5:57	18:26	12:12

The mean noon is at 12:23, taking into account that Dar es Salaam is East of the center of the time zone

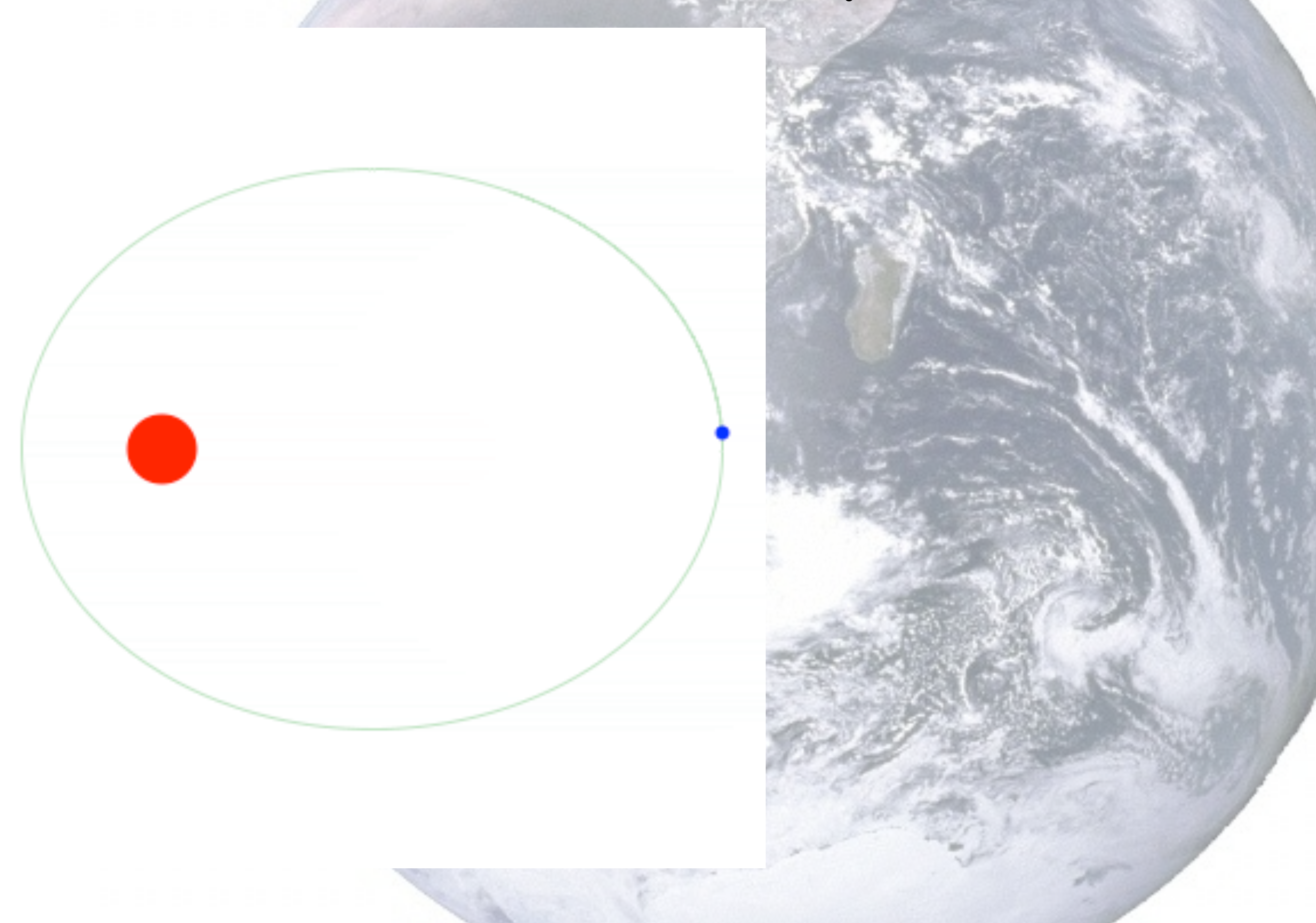
The equation of time is the difference

mean time - solar time



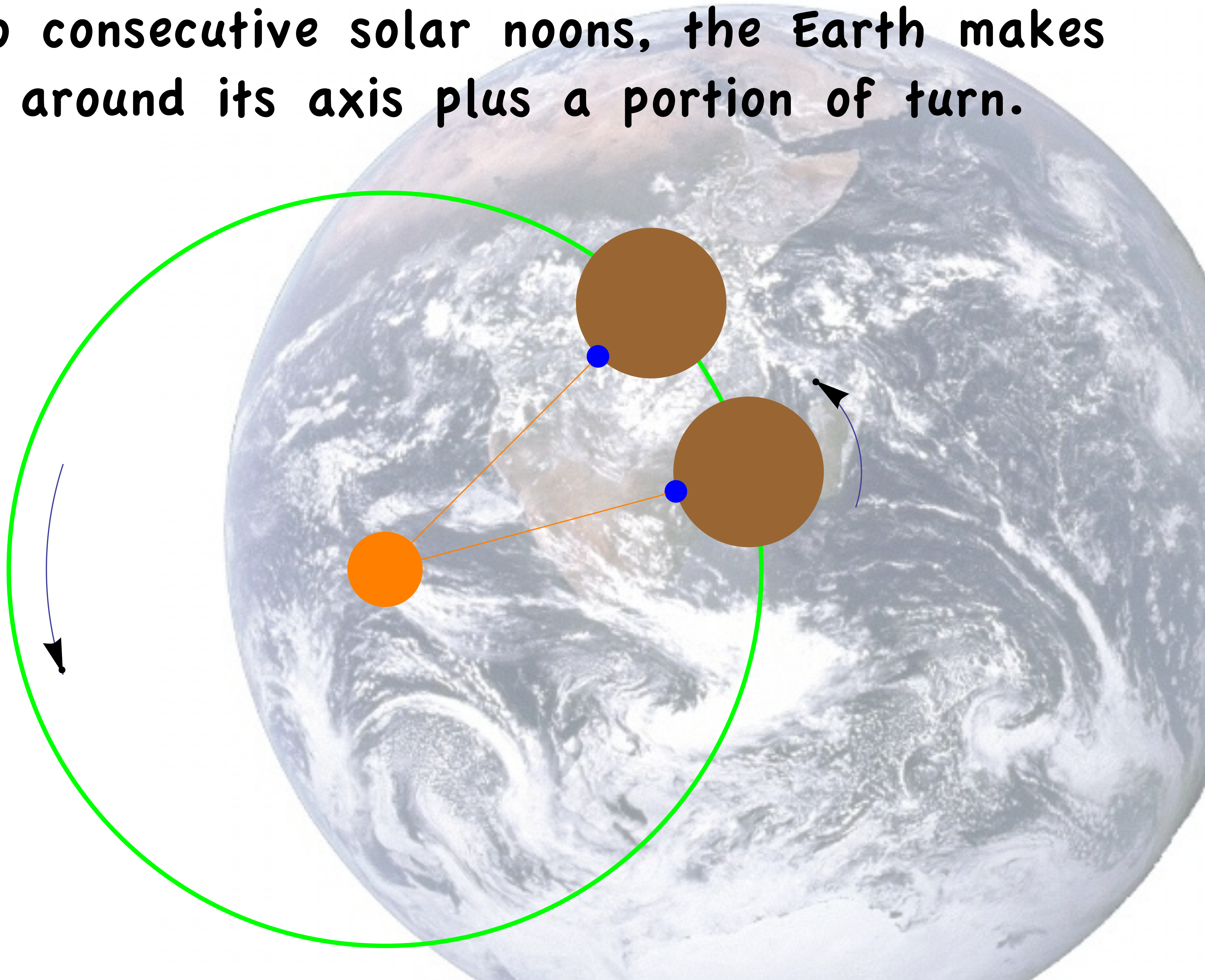
This difference comes from two reasons

The first is that the orbit of the Earth around the Sun is an ellipse

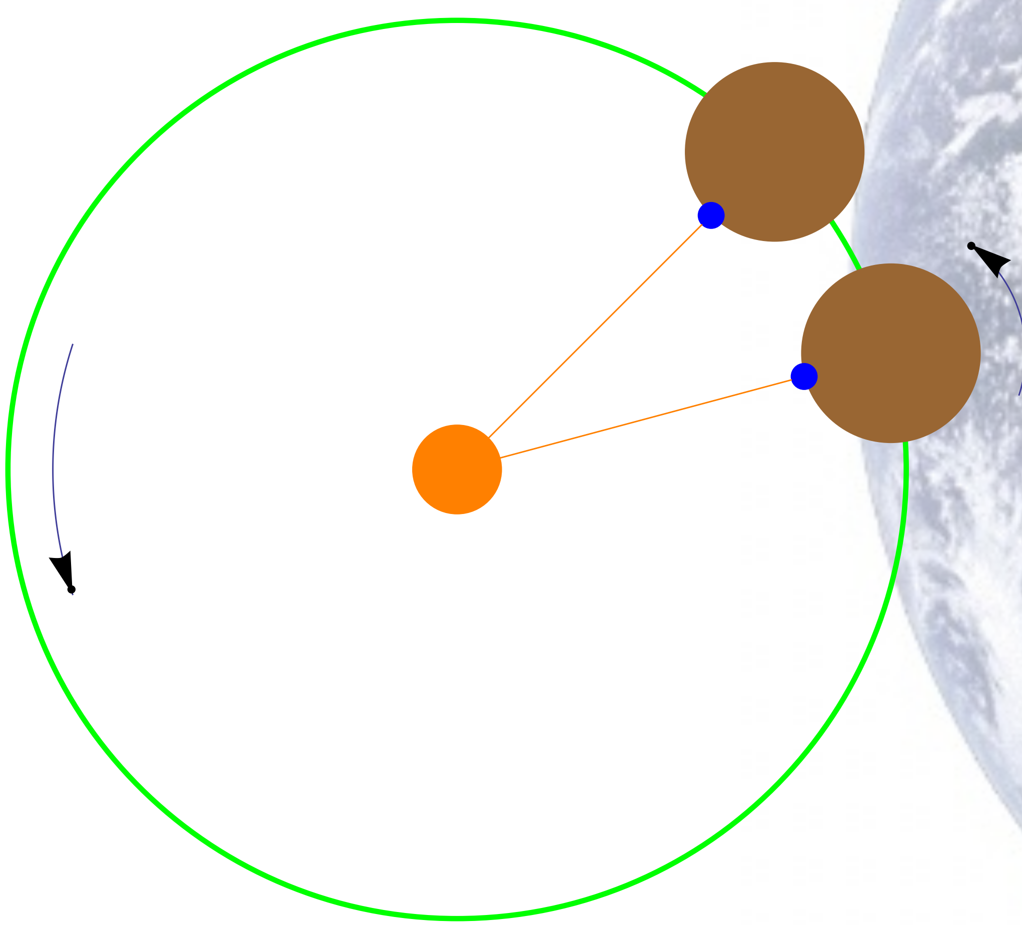


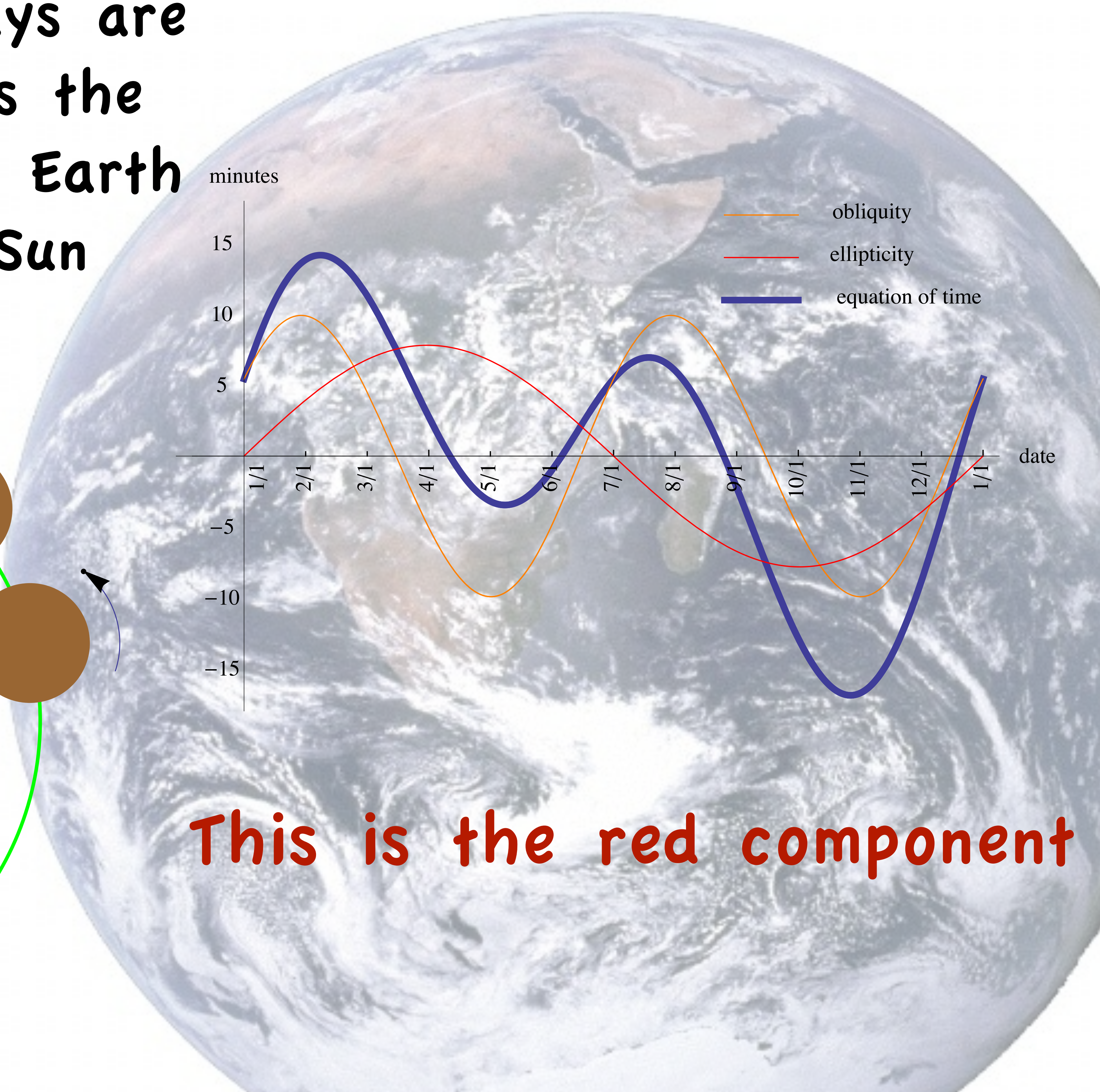
When the Earth is closer to the Sun (around January 4th) it has a higher speed.

Between two consecutive solar noons, the Earth makes one turn around its axis plus a portion of turn.



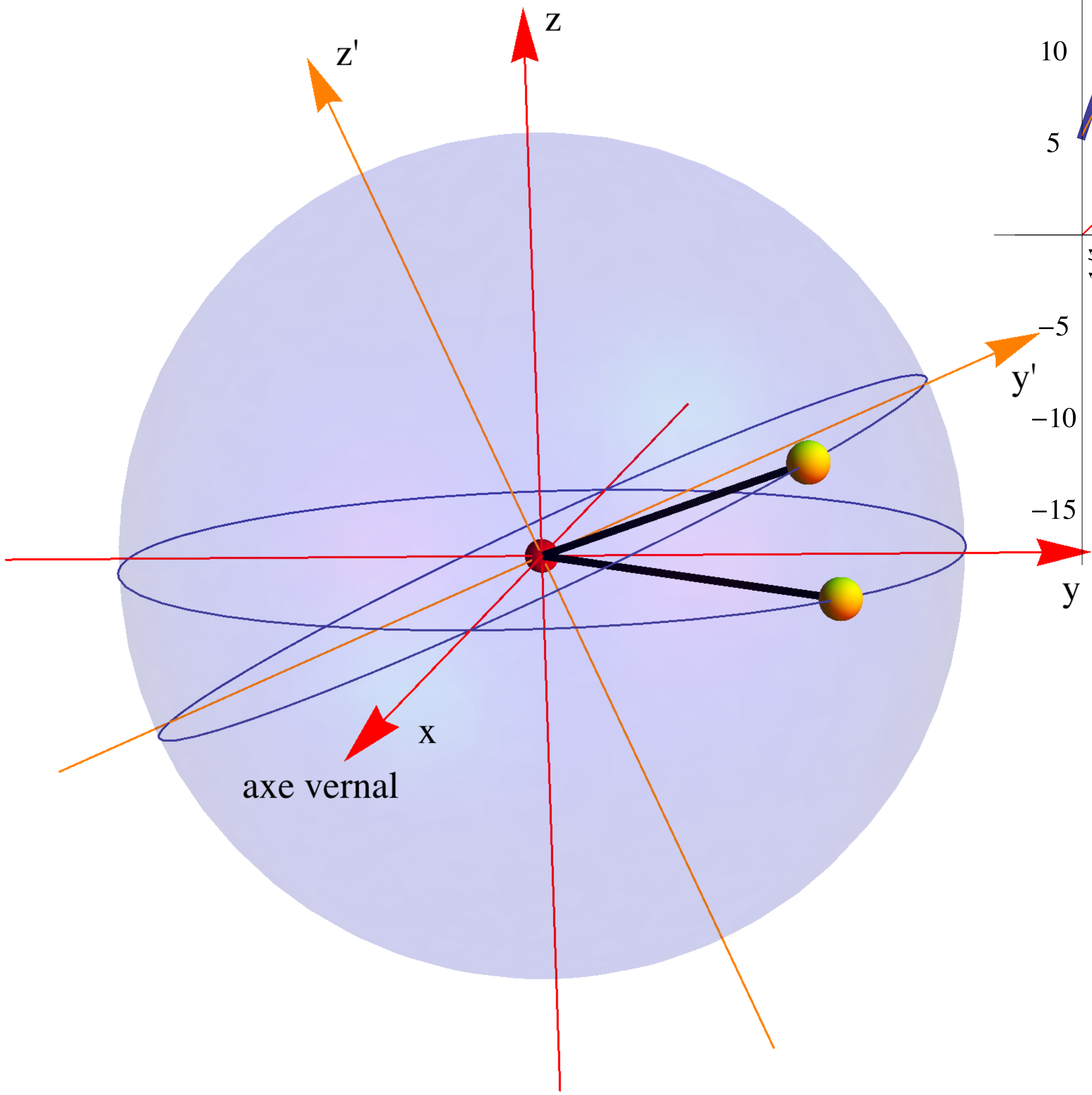
This portion of turn is larger when the speed is higher. The solar days are then longer. This is the case in winter when Earth is closer to the Sun

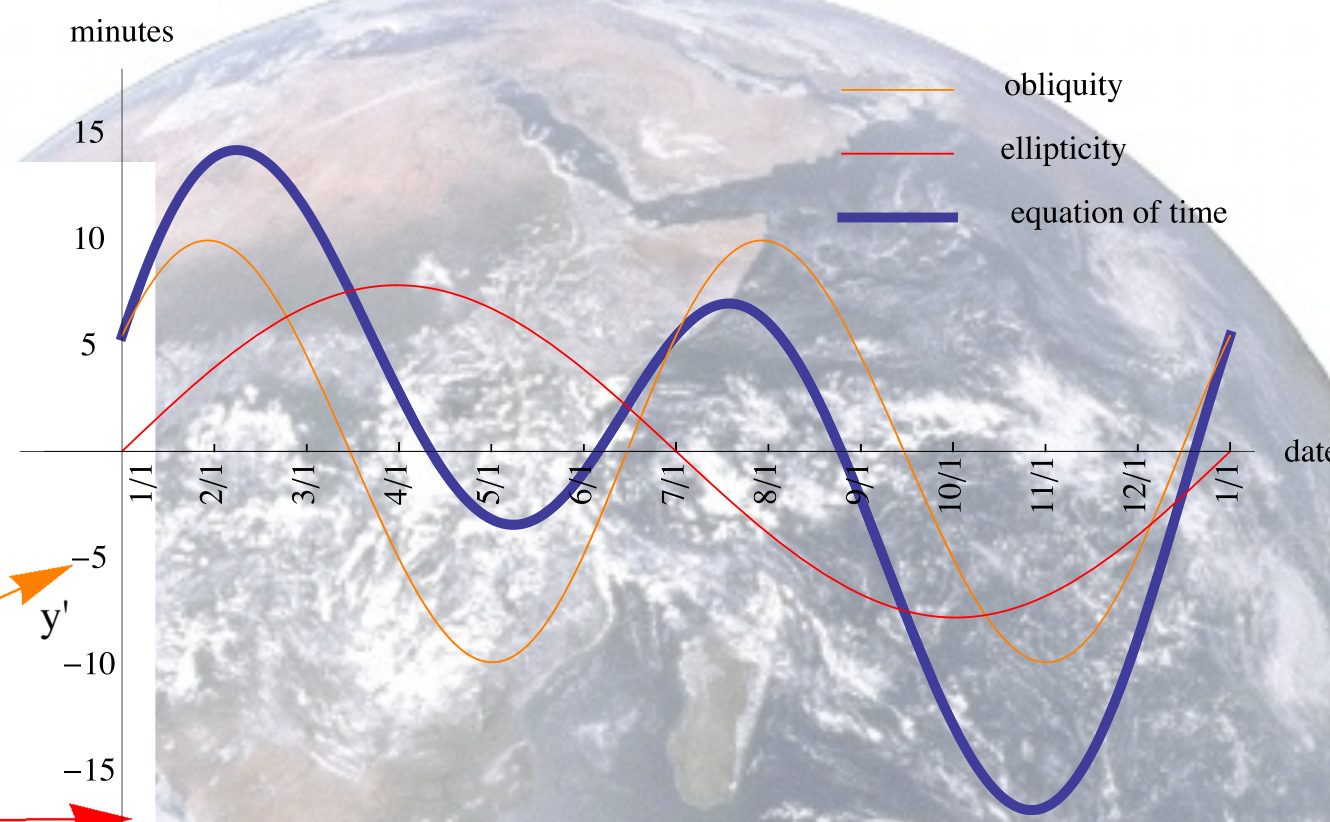




This is the red component

The second reason is the obliquity of the Earth axis

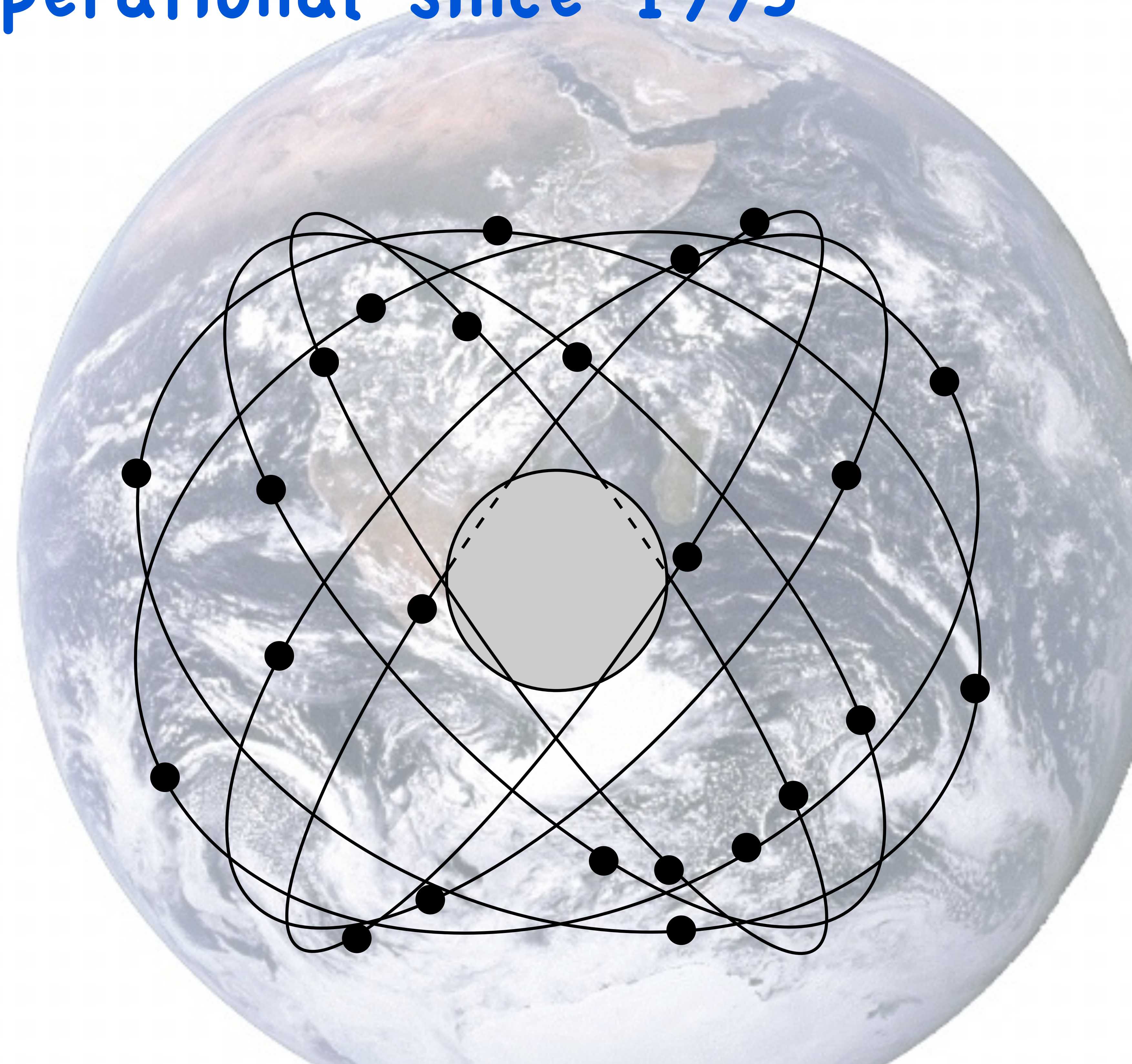




It is the orange component

The GPS (Global Positioning System)
Fully operational since 1995

Network of satellites whose position is known



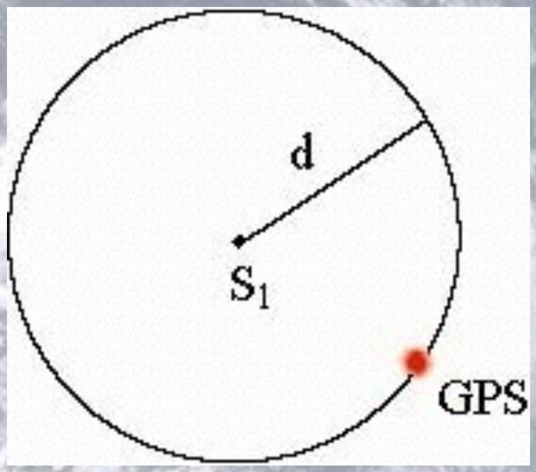
The receiver measures the travel time t of a signal emitted by a satellite to the receiver

The distance from the satellite to the receiver is

$$d = vt$$

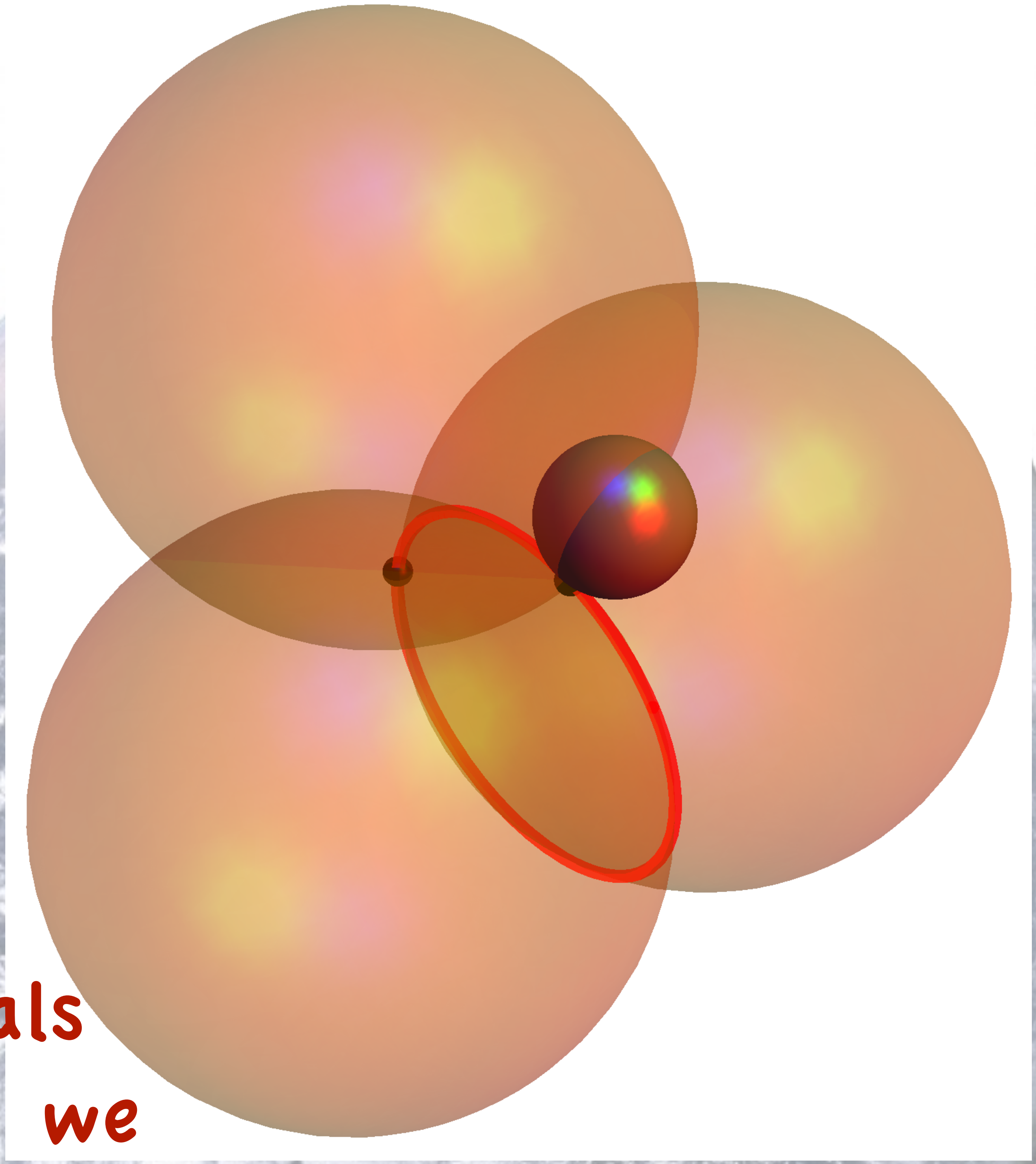
where v is the speed of light

The points at a distance d from the satellite are the points of a sphere centered at the satellite with radius d



The intersection of 2 spheres is a cercle

The intersection of three spheres is two points. One is excluded since unrealistic



Hence, if we know the travel times of 3 signals from 3 satellites, then we know the position of the receiver.

This is the theory...

In pratice ... the satellites have atomic clokcs perfectly synchronized

The receiver has a cheap clock

We have a fourth unknown: the shift between the clock of the satellites and that of the receiver

We then need to measure the travel time of the signal from a fourth satellite

4 measured times ⟺ 4 unknown

- The shift of the clocks
- The 3 coordinates of position

With this method we get a precision of 20 meters

Non standard applications of GPS

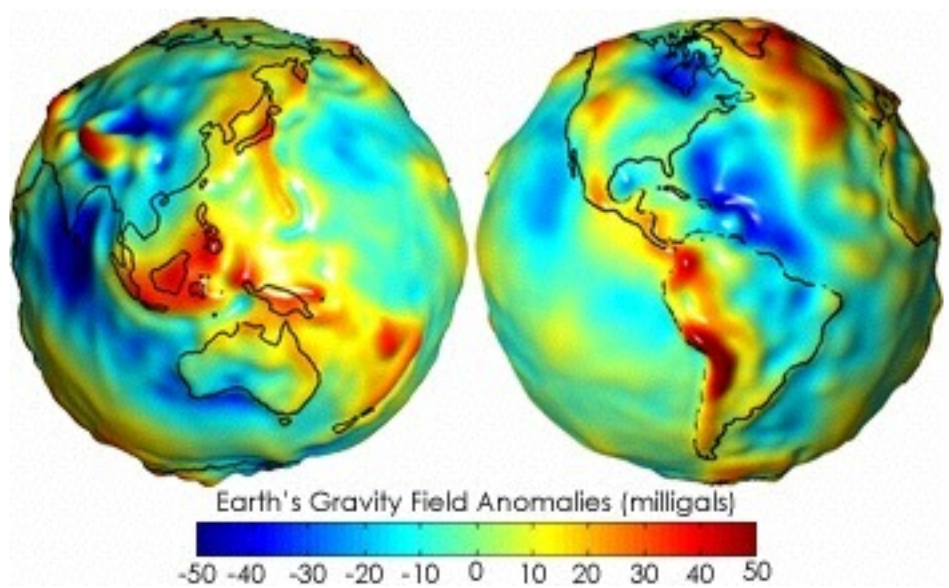
- Synchronizing electronic equipments

  The receiver calculates the shift of its clock with that of the satellites

- Measuring the displacement of tectonic plates

- Measuring Everest and observing its growth

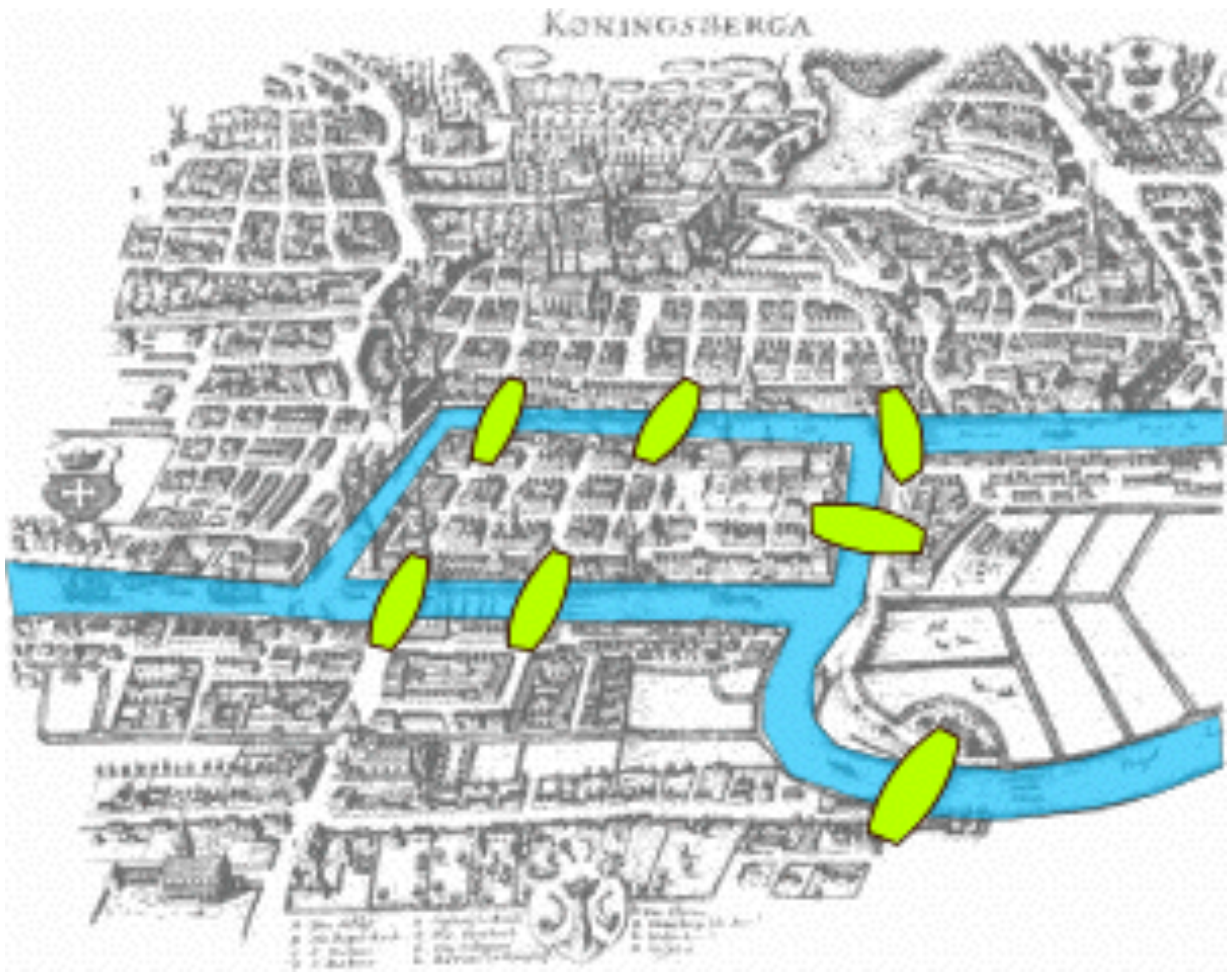
What means altitude if the Earth is not a sphere but a geoid?



Level surfaces of altitude are level surfaces of the gravitational field, with altitude zero corresponding to the level surface best approximating the surface of the oceans.

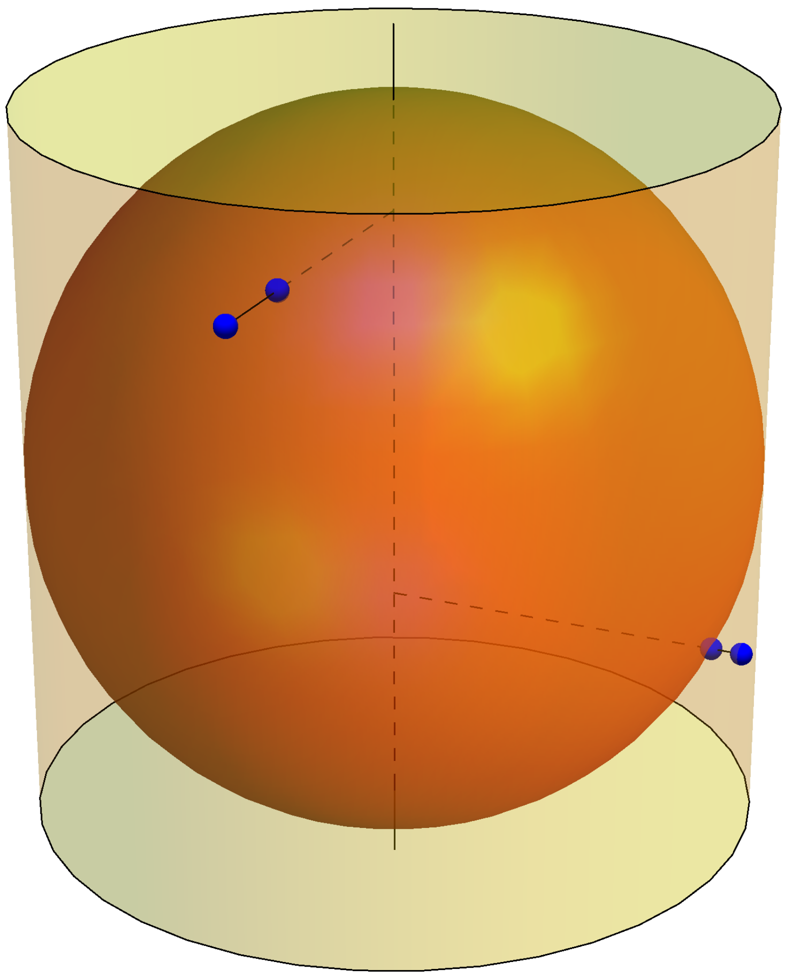
Königsberg's bridges

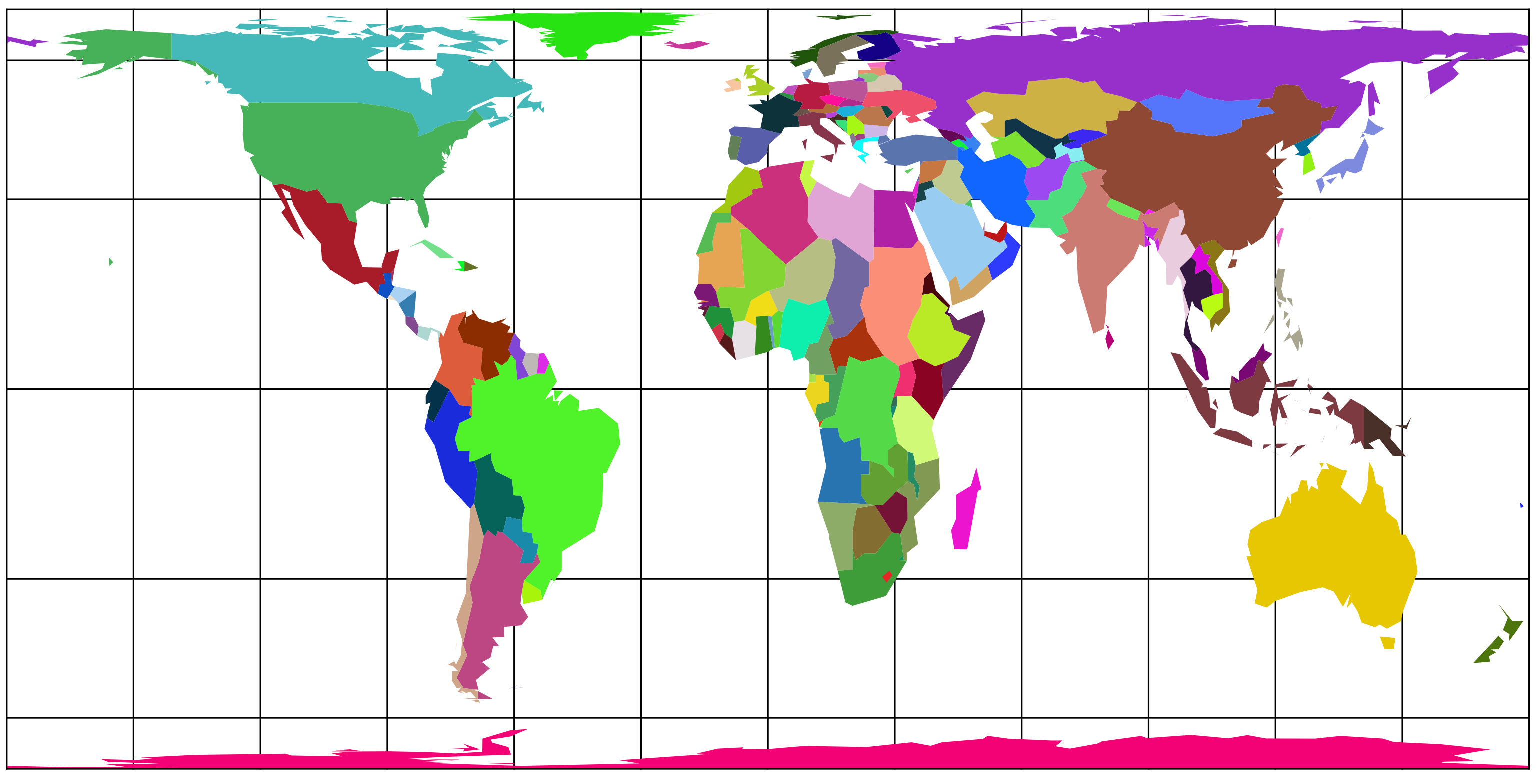
Can a person cross all bridges exactly once and come back to his(her) departure point?



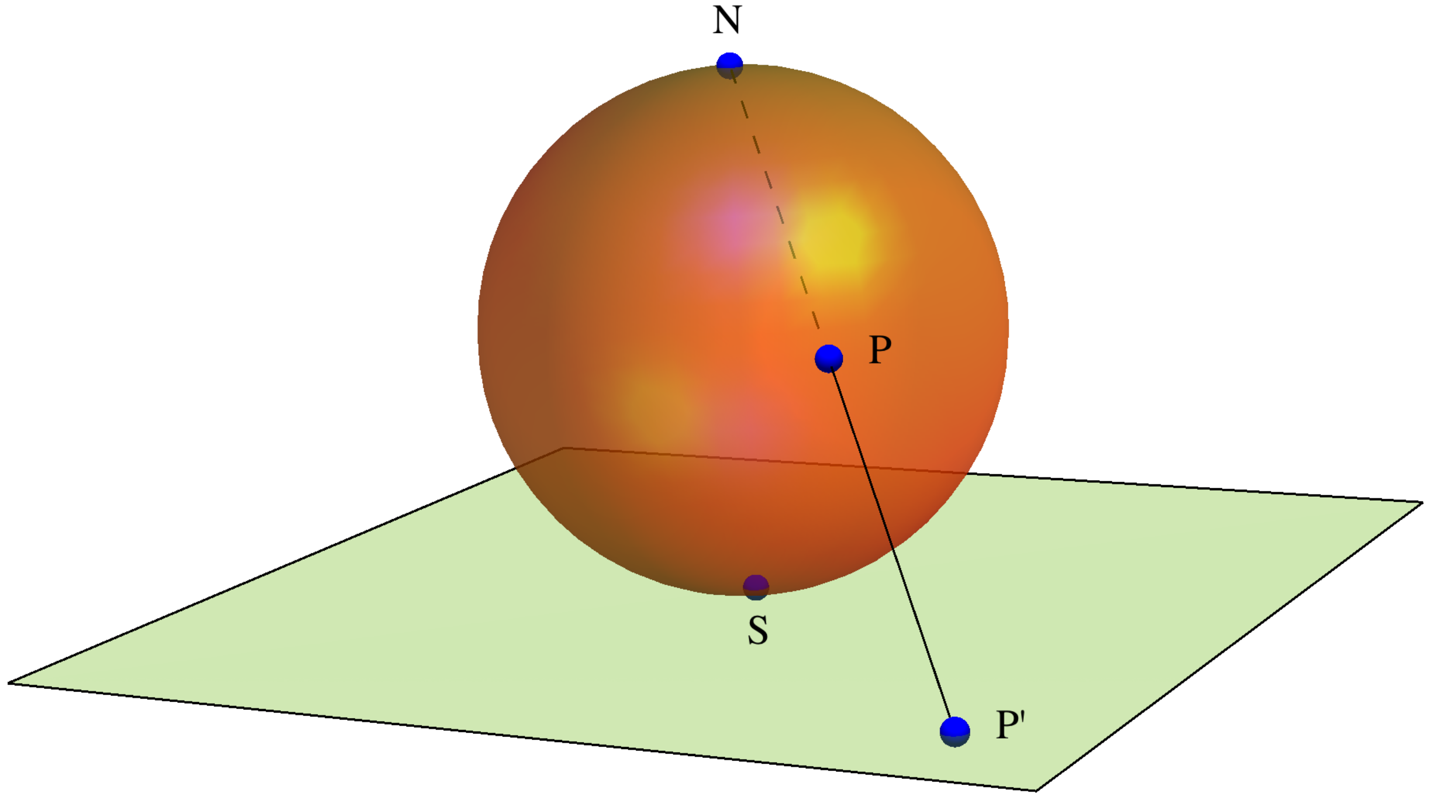
Cartography

Show that the orthogonal projection on the cylinder preserves ratios of areas

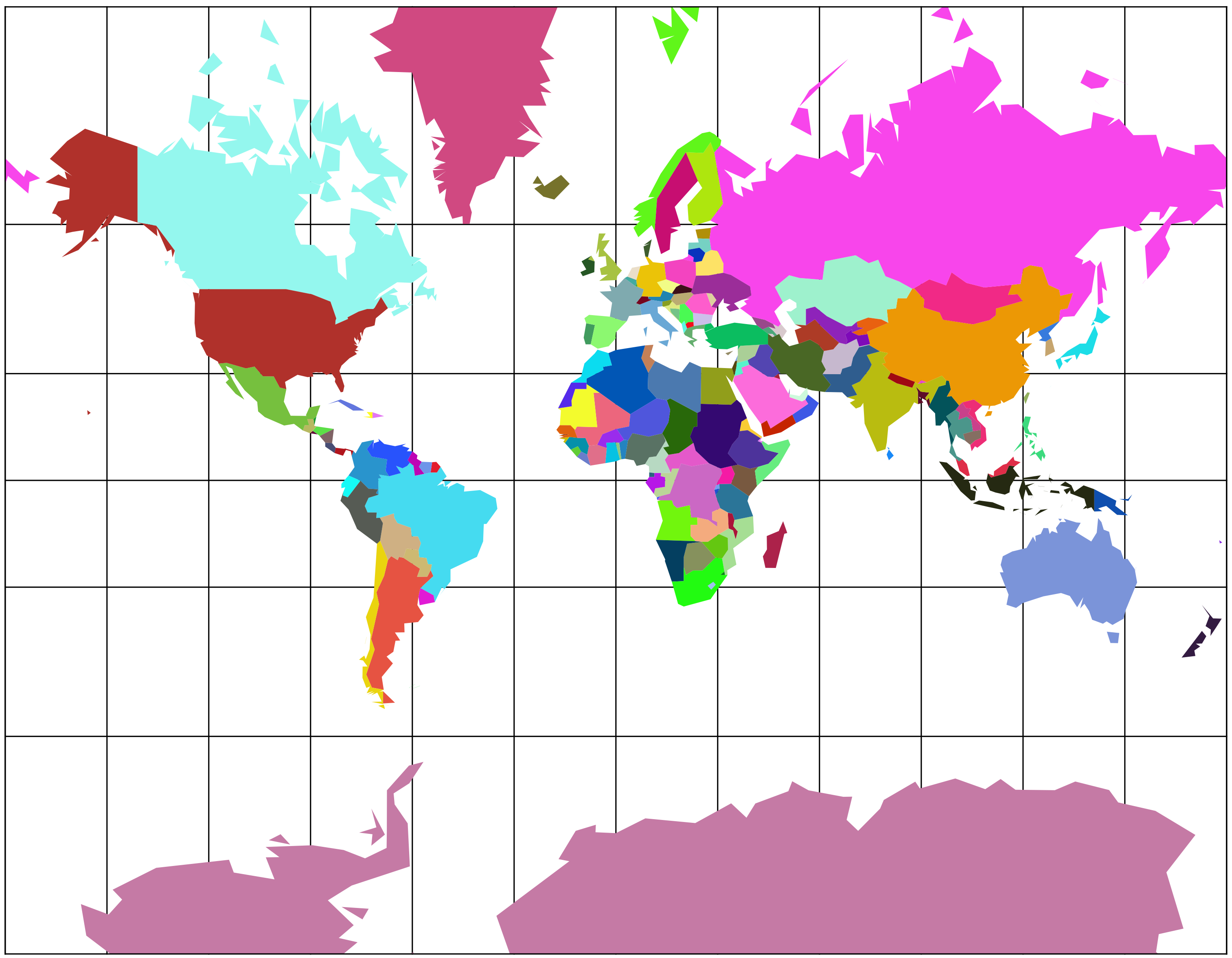


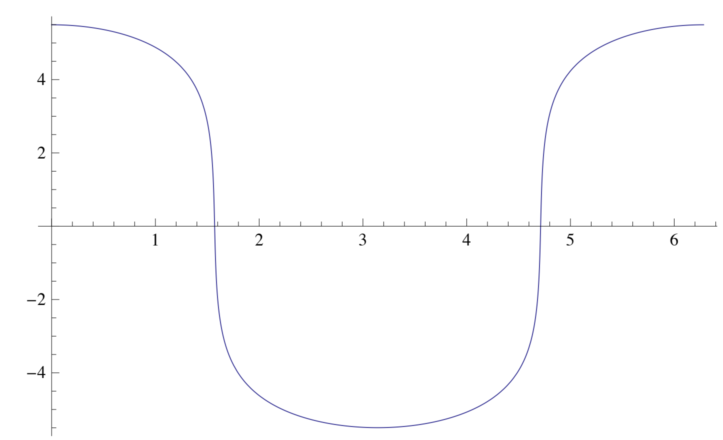


The stereographic map is conformal



Applying the function log(z) one gets the Mercator transformation





A sphere is conformally equivalent to a plane. But the Earth is not a sphere…

It is still possible to draw conformal maps!

Thank you!