Secretary Problem

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~~Secretary~~ Admin problem

- Observe a sequence of candidates
- Rejected candidates can’t be recalled
- Examples: jobs, air fare, spouse
- T potential candidates

Classic Solution

- Dynkin, 1963
- Goal: Maximize probability of finding the best
- Observe $k = T/e$ candidates and reject them
- Set an aspiration level max $\{v_1, \ldots, v_k\}$
- Search until meeting or exceeding the aspiration level
- Nothing beats this **on every distribution** in the limit as T gets large.

Dynkin Applied to Spouse Search

- 70 years to search
- Search for 25¾ years
- Best observed set as aspiration level
- Continue search until finding one or exhaust candidates
  - 51½ years of search
  - 37% (1/*e*) chance of failure

Problems

- Reject excellent candidates at *T*-1
  - Consequence of maximizing probability of identifying top candidate
- Time of acquisition doesn't matter
- Once and for all decision
- Extreme distribution

Hire Secretary

- Period of need *T*.
- Hire at t, employ secretary from *t* to *T*
- Discount factor δ ≤ 1
- If you hire in period *j* a secretary of value *v*, you obtain

$$v\sum_{t=j}^{T}\delta^{t} = v\frac{\delta^{j-1}-\delta^{T}}{1-\delta}$$

Assumptions

- Distribution *F* of values, *iid* draws
- *F*(0)=0
- Aspiration strategy is used
  - Observe distribution for *k* periods then set max as a min target

Expected Payoff

- The expected payoff of the aspiration strategy is

$$\pi = \int_0^\infty kF(y)^{k-1} f(y)\left(\sum_{j=k+1}^{T-1} \frac{\delta^{j-1} - \delta^T}{1-\delta} F(y)^{j-k-1} \int_y^\infty xf(x)dx \right.$$

$$\left. + \delta^{T-1} F(y)^{T-k-1} \int_0^\infty x f(x)dx \right) dy$$

Reduction

$$\pi_k = \frac{k\delta^{T-1}}{T-1}\int_0^1 F^{-1}(z)dz + k\sum_{j=k+1}^{T-1}\frac{1}{j-1}\frac{\delta^{j-1}-\delta^{T}}{1-\delta}\int_0^1 F^{-1}(z)z^{j-1}dz$$

$$= \int_0^1 F^{-1}(z)k\left(\frac{\delta^{T-1}}{T-1} + \sum_{j=k+1}^{T-1}\frac{z^{j-1}}{j-1}\frac{\delta^{j-1}-\delta^{T}}{1-\delta}\right)dz$$

Difference

$$\pi_k - \pi_m = \int_0^1 F^{-1}(z)\left[k \sum_{j=k+1}^{m} \frac{z^{j-1}}{j-1} \frac{\delta^{j-1} - \delta^T}{1-\delta} \right.$$

$$\left. - (m-k)\left(\frac{\delta^{T-1}}{T-1} + \sum_{j=k+1}^{m} \frac{z^{j-1}}{j-1} \frac{\delta^{j-1} - \delta^T}{1-\delta} \right) \right] dz$$

- For *m*>*k*, β=[•] satisfies
  - Negative at 0
  - If decreasing, stays decreasing

Maximize $\pi_k – \pi_m$, $m>k$

- Strategy: Minimize $\pi_k – \pi_m$ and conclude it is negative
- Note F^{-1} is non-decreasing
- Lemma: Suppose for all a, $\int_a^1 \beta(z)dz \leq 0$

Then $\pi_k – \pi_m \geq 0$.

Proof of Lemma

$$\pi_k - \pi_m = \int_0^1 F^{-1}(z)\beta(z)dz$$

$$\leq \int_x^y F^{-1}(z)\beta(z)dz + \int_y^1 F^{-1}(z)\beta(z)dz$$

$$\leq \int_x^y F^{-1}(y)\beta(z)dz + \int_y^1 F^{-1}(y)\beta(z)dz = F^{-1}(y)\int_x^1 \beta(z)dz$$

Standard Trick

$$\int_a^1 \beta(z)dz \le 0$$

- is equivalent to β(1)≤0, or

$$k\sum_{j=k}^{T-1}\frac{\delta^j-\delta^T}{j} \le m\sum_{j=m}^{T-1}\frac{\delta^j-\delta^T}{j}$$

Theorem

- Suppose k* maximizes $k\sum_{j=k}^{T-1}\frac{\delta^j-\delta^T}{j}$

then $\pi_{k*}-\pi_m \le 0$ for all m>k*.

Corollary: for δ=1, k* = arg max $k\sum_{j=k}^{T-1}\frac{T-j}{j}$

and k*/T ≈0.203

Conclusions: Spouse Search

- T = 70 years
- 10% annual discount
- Search for 4.1 years

Hazard Rate

- Standard approach permits very long tails
- Impose hazard rate: $\frac{1-F(\bullet)}{f(\bullet)}$ nondecreasing
- No discounting, one time acquistion

Same Starting Attack

- The expected payoff of the aspiration strategy is

$$\pi_k = \int_0^\infty kF(y)^{k-1} f(y) \left(\sum_{j=k+1}^{T-1} F(y)^{j-k-1} \int_y^\infty xf(x)dx + F(y)^{T-k-1} \int_0^\infty x f(x)dx \right) dy$$

$$= \int_0^1 F^{-1}(z)k \left(\frac{1}{T-1} + \sum_{j=k}^{T-2} \frac{z^j}{j} \right) dz$$

Integrate by Parts

$$\pi_{k+1} - \pi_k = \int_0^1 F^{-1}(z)\left(\frac{1}{T-1} + \sum_{j=k+1}^{T-2}\frac{z^j}{j} - z^k\right)dz$$

$$= -\int_0^1 F^{-1\prime}(z)\left(\frac{z^{k+1}}{k+1} - \frac{z}{T-1} - \sum_{j=k+1}^{T-2}\frac{z^{j+1}}{j(j+1)}\right)dz$$

Integrate by Parts

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$$= \int_0^1 F^{-1\prime}(z)(1-z)\left(\sum_{i=k}^{T-3} \frac{z^{i+1}}{i+1} - \sum_{i=0}^{T-3} \frac{z^{i+1}}{T-1}\right)dz$$

- Use the fact $\frac{1}{k+1} = \frac{1}{T-1} + \sum_{j=k+1}^{T-2} \frac{1}{j(j+1)}$

Hazard Rate

$$F^{-1\prime}(z)(1-z) = \frac{1-F(x)}{f(x)}$$

- Let $\beta(z) = \sum_{i=k}^{T-3} \frac{z^{i+1}}{i+1} - \sum_{i=0}^{T-3} \frac{z^{i+1}}{T-1}$
- Then $\pi_k - \pi_{k+1} = \int_0^1 F^{-1\prime}(z)(1-z)\beta(z)dz$

Hazard Rate

$$F^{-1'}(z)(1-z) = \frac{1-F(x)}{f(x)}$$

- Let $\beta(z) = \sum_{i=k}^{T-3} \frac{z^{i+1}}{i+1} - \sum_{i=0}^{T-3} \frac{z^{i+1}}{T-1}$
- Then $\pi_k - \pi_{k+1} = \int_0^1 \underbrace{F^{-1'}(z)(1-z)}_{\text{nondecreasing}} \beta(z)\,dz$

nondecreasing

Properties of β

- If k<(*T*-1)/*e*, β(1)>0
- If β(1)>0, β(z)>0 iff z>z*

- Lemma: Suppose β(z)>0 iff z>z*.

$$If \int_0^1 \beta(z)dz \le 0, then \max_{x' \ge 0} \int_0^1 x(z)\beta(z)dz \le 0$$

Applying the Lemma

$$If \int_0^1 \beta(z)dz \le 0, then\ \pi_{k+1} - \pi_k \le 0$$

- But $\int_0^1 \beta(z)dz = \frac{1}{k+1} - \frac{1}{T-1}\sum_{i=1}^{T-1}\frac{1}{i}$

$$If\ \ k \ge \frac{T-1}{\sum_{i=1}^{T-1}\frac{1}{i}} - 1,\ \ \pi_{k+1} - \pi_k \le 0$$

Observations

- Maximal discovery is $\frac{T-1}{Log(T)} - 1$
- At $T=10^6$, 6.9%
- Formula exact for exponential

Comparison

T	T/e Value	T/e Higher	$T/\text{Log}[T]$ Value	$T/\text{Log}[T]$ Higher	Optimal Search
50	64%	22%	69%	12%	88%
100	64%	27%	71%	15%	89%
500	64%	36%	76%	20%	92%
1000	63%	38%	78%	22%	92%
5,000	63%	43%	81%	25%	94%
10,000	63%	44%	82%	26%	94%

Optimal Search

- Given distribution, choose so that

$$v_t = \max_c F(c) v_{t+1} + \int_c^{\infty} x f(x) dx$$

- So that $c = v_{t+1}$, and

$$v_t = v_{t+1} + \int_{v_{t+1}}^{\infty} 1 - F(x) dx$$

Conclusions

- For problem of which ad to run
- Ads are durable
- Discounting fairly irrelevant
- Examine distribution, compute bound
- Uniform distribution

$$k^* = \tfrac{1}{2}\left(\sqrt{4T+1} - 3\right) \approx \sqrt{T-2}$$