Worst-Case Optimal Redistribution of VCG Payments in Multi-Unit Auctions

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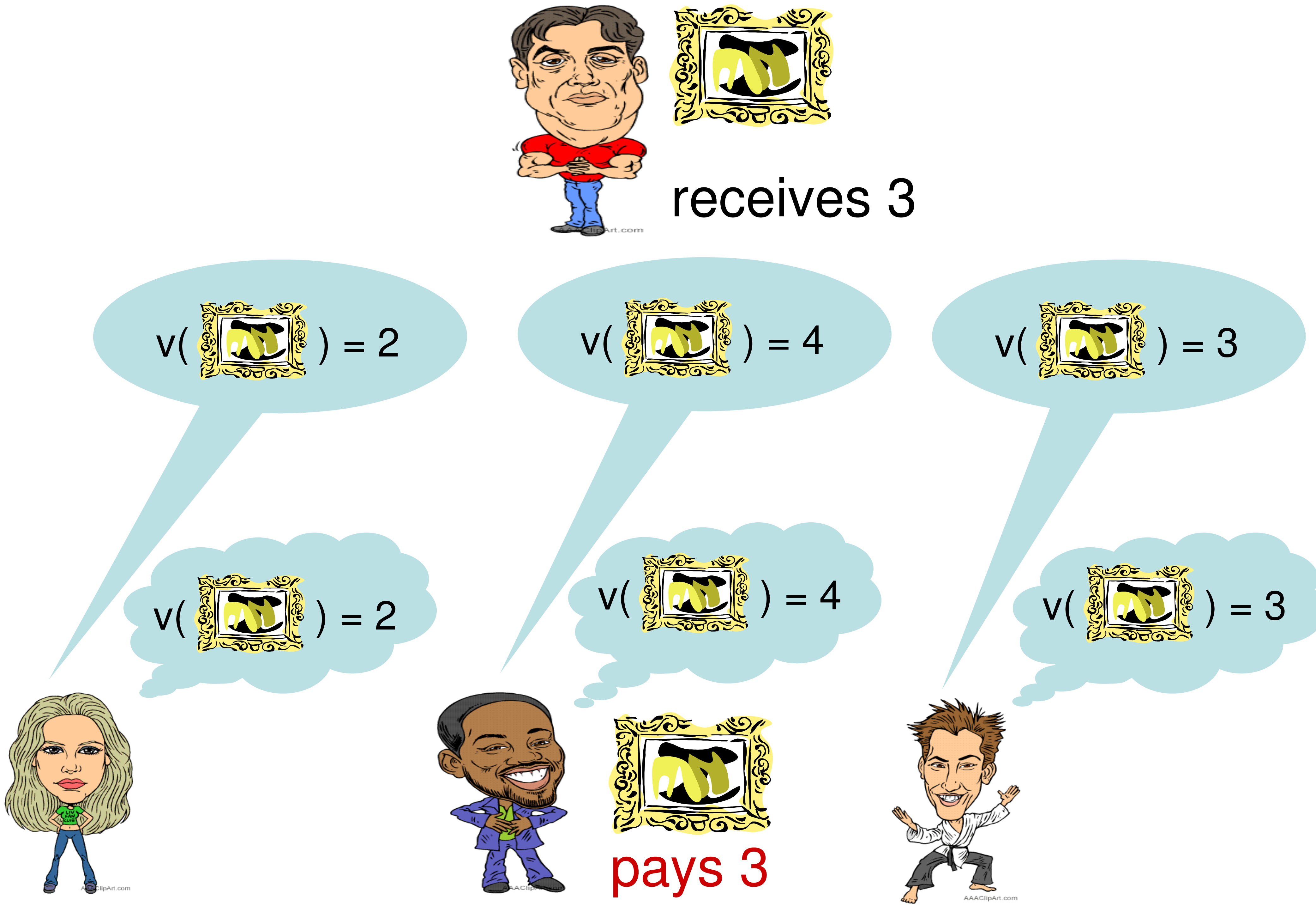
Joint work with Vincent Conitzer

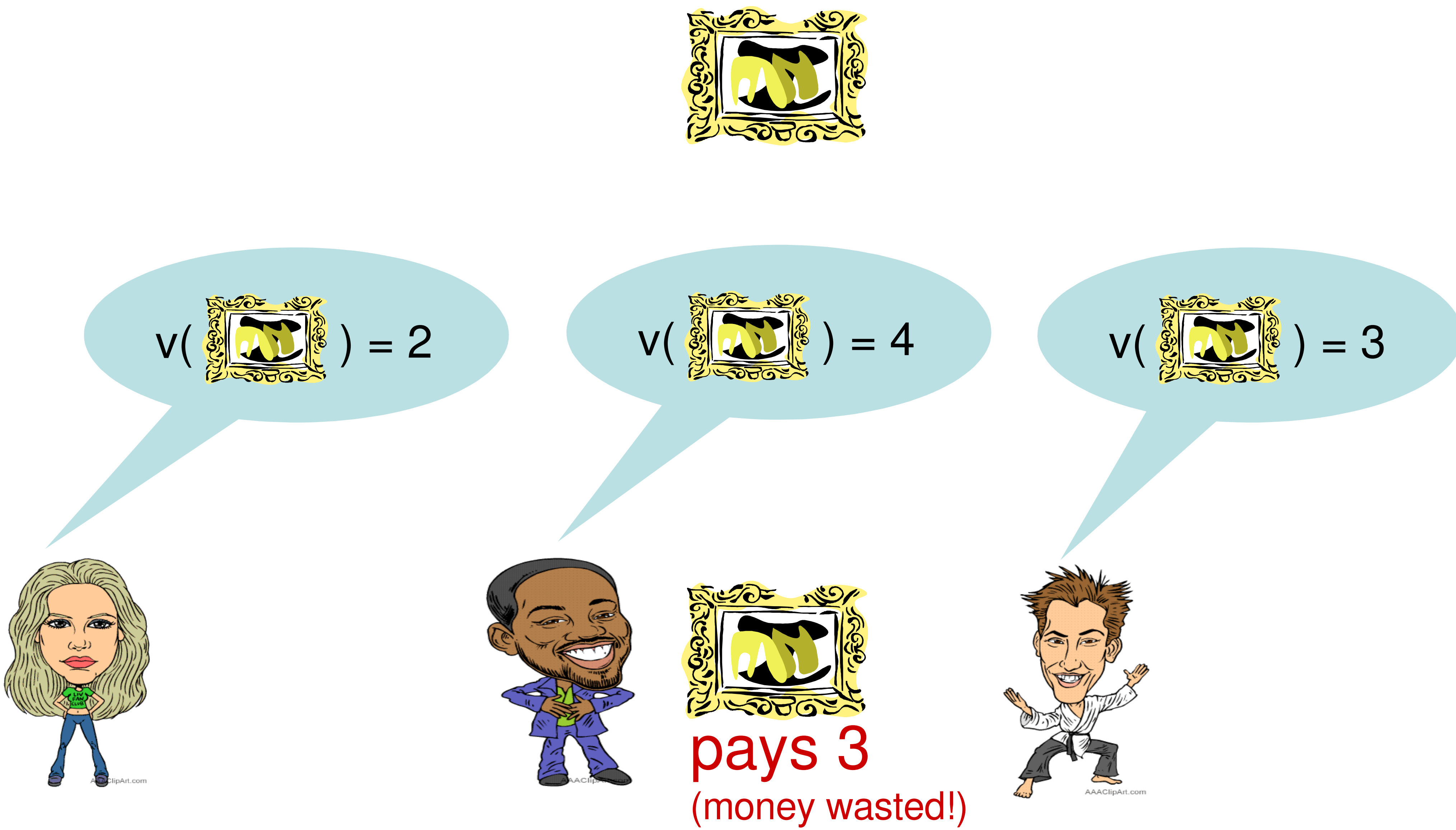
This talk covers material from:

Guo and Conitzer, “Worst-Case Optimal Redistribution of VCG Payments in Multi-Unit Auctions” (in submission, earlier version in EC 07)

Second-price (Vickrey) auction

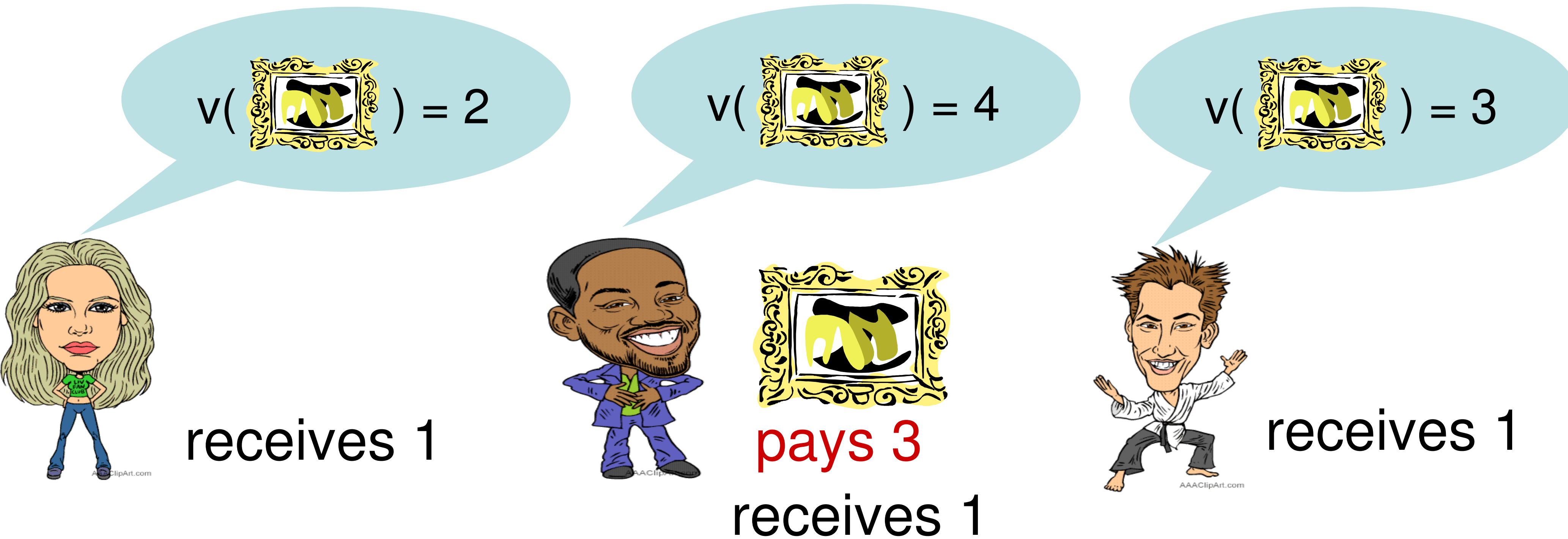


Vickrey auction without a seller



Can we redistribute the payment?

Idea: give everyone 1/n of the payment



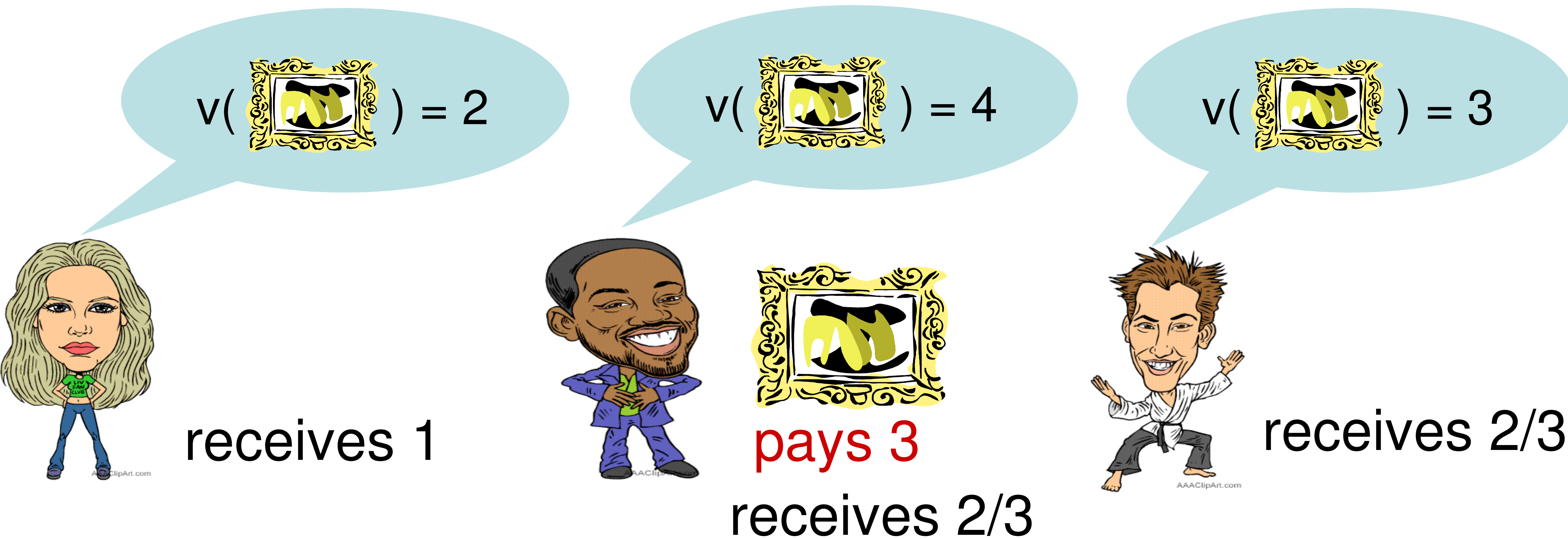
not incentive compatible

Bidding higher can increase your redistribution payment

Incentive compatible redistribution

[Bailey 97, Porter et al. 04, Cavallo 06]

Idea: give everyone 1/n of second-highest **other** bid



2/3 wasted (22%)

incentive compatible

Your redistribution does not depend on your bid;

incentives are the same as in Vickrey

Bailey-Cavallo mechanism…

- Bids: $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$
- First run Vickrey auction
- Payment is V_2
- First two bidders receive V_3/n
- Remaining bidders receive V_2/n
- Total redistributed: $2V_3/n+(n-2)V_2/n$

$R_1 = V_3/n$

$R_2 = V_3/n$

$R_3 = V_2/n$

$R_4 = V_2/n$

...

$R_{n-1} = V_2/n$

$R_n = V_2/n$

Can we do better?

Desirable properties

- Incentive compatibility
- Individual rationality: bidder's utility always nonnegative
- Efficiency: bidder with highest valuation gets item
- Non-deficit: sum of payments is nonnegative
  - i.e. total VCG payment ≥ total redistribution
- (Strong) budget balance: sum of payments is zero
  - i.e. total VCG payment = total redistribution
- **Impossible to get all**
- We sacrifice budget balance
  - Try to get approximate budget balance
- Other work sacrifices: incentive compatibility [Parkes 01], efficiency [Faltings 04], non-deficit [Bailey 97], budget balance [Cavallo 06]

Another redistribution mechanism

- Bids: $V_1 \geq V_2 \geq V_3 \geq V_4 \geq \ldots \geq V_n \geq 0$
- First run Vickrey
- Redistribution:

  Receive 1/(n-2) * second-highest **other** bid, - 2/[(n-2)(n-3)] third-highest **other** bid
- Total redistributed:

  $V_2 - 6V_4/[(n-2)(n-3)]$
- Efficient & incentive compatible
- Individually rational & non-deficit (for large enough n)

$R_1 = V_3/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_2 = V_3/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_3 = V_2/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_4 = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

...

$R_{n-1} = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

$R_n = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

Comparing redistributions

- Bailey-Cavallo: $\sum R_i = 2V_3/n+(n-2)V_2/n$
- Second mechanism: $\sum R_i = V_2 - 6V_4/[(n-2)(n-3)]$
- Sometimes the first mechanism redistributes more
- Sometimes the second redistributes more
- Both redistribute 100% in some cases
- What about the **worst** case?
- Bailey-Cavallo worst case: V3=0
  - percentage redistributed: 1-2/n
- Second mechanism worst case: V2=V4
  - percentage redistributed: 1-6/[(n-2)(n-3)]
- For large enough n, 1-6/[(n-2)(n-3)]≥1-2/n, so second is better (in the worst case)

Generalization: linear redistribution mechanisms

- Run Vickrey
- Amount redistributed to bidder:

$$C_0 + C_1 V_{-i,1} + C_2 V_{-i,2} + \ldots + C_{n-1} V_{-i,n-1}$$

  where $V_{-i,j}$ is the j-th highest **other** bid for bidder i
- Bailey-Cavallo: $C_2 = 1/n$
- Second mechanism: $C_2 = 1/(n-2)$, $C_3 = - 2/[(n-2)(n-3)]$

- Bidder's redistribution does not depend on own bid, so incentive compatible
- Efficient
- Other properties?

Redistribution to each bidder

Recall: $R = C_0 + C_1 V_{-i,1} + C_2 V_{-i,2} + \ldots + C_{n-1} V_{-i,n-1}$

$R_1 = C_0+C_1V_2+C_2V_3+C_3V_4+\ldots+C_iV_{i+1}+\ldots+C_{n-1}V_n$

$R_2 = C_0+C_1V_1+C_2V_3+C_3V_4+\ldots+C_iV_{i+1}+\ldots+C_{n-1}V_n$

$R_3 = C_0+C_1V_1+C_2V_2+C_3V_4+\ldots+C_iV_{i+1}+\ldots+C_{n-1}V_n$

$R_4 = C_0+C_1V_1+C_2V_2+C_3V_3+\ldots+C_iV_{i+1}+\ldots+C_{n-1}V_n$

...

$R_{n-1}= C_0+C_1V_1+C_2V_2+C_3V_3+\ldots+C_iV_i \quad +\ldots+C_{n-1}V_n$

$R_n = C_0+C_1V_1+C_2V_2+C_3V_3+\ldots+C_iV_i \quad +\ldots+C_{n-1}V_{n-1}$

Individual rationality & non-deficit

- Individual rationality:

  equivalent to

$$R_n = C_0 + C_1V_1 + C_2V_2 + C_3V_3 + \ldots + C_iV_i + \ldots + C_{n-1}V_{n-1} \geq 0$$

$$\text{for all } V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0$$

- Non-deficit:

  $\sum R_i \leq V_2$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq V_n \geq 0$

Worst-case optimal (linear) redistribution

Try to maximize worst-case redistribution %

Variables: C_i ,K

Maximize K

subject to:

$R_n \geq 0$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0$

$\sum R_i \leq V_2$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$

$\sum R_i \geq K\, V_2$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$

R_i as defined in previous slides

Transformation into linear program

- **Claim**: $C_0=0$
- **Lemma**: $Q_1X_1+Q_2X_2+Q_3X_3+\ldots+Q_kX_k\geq 0$ for all $X_1\geq X_2\geq\ldots\geq X_k\geq 0$

  is equivalent to

  $Q_1+Q_2+\ldots+Q_i\geq 0$ for $i=1$ to k
- Using this lemma, can write all constraints as linear inequalities over the C_i

Worst-case optimal **remaining** %

n=5: 27% (40%)

n=6: 16% (33%)

n=7: 9.5% (29%)

n=8: 5.5% (25%)

n=9: 3.1% (22%)

n=10: 1.8% (20%)

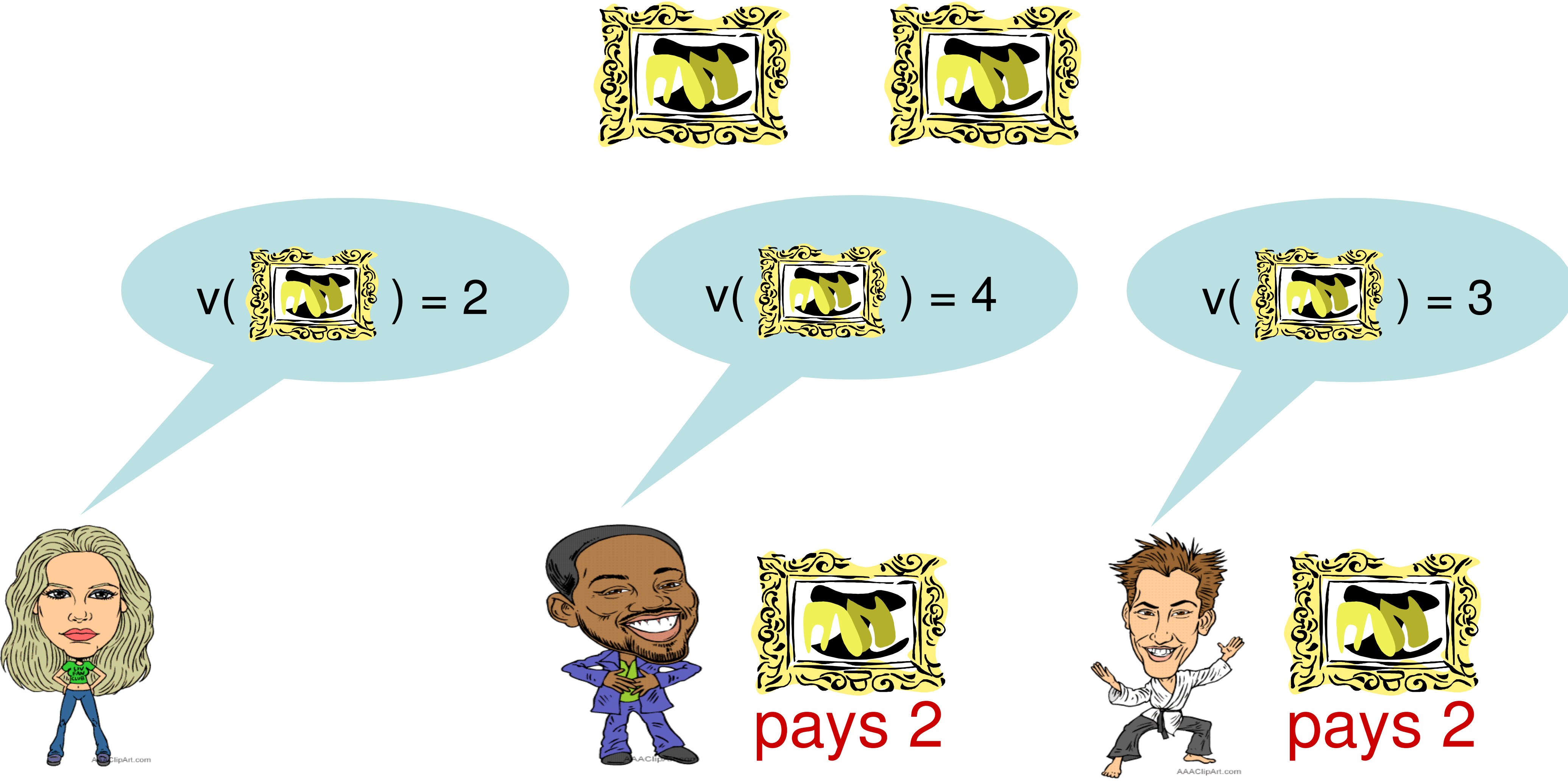
n=15: 0.085% (13%)

n=20: 3.6 e-5 (10%)

n=30: 5.4 e-8 (7%)

the data in the parenthesis are for Bailey-Cavallo mechanism

m-unit auction with unit demand: VCG (m+1th price) mechanism



Incentive compatible

Our techniques can be generalized to this setting

m+1th price mechanism

Variables: C_i ,K

Maximize K

subject to:

$R_n \geq 0$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0$

$\sum R_i \leq V_2$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$

$\sum R_i \geq K\, V_2$ for all $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$

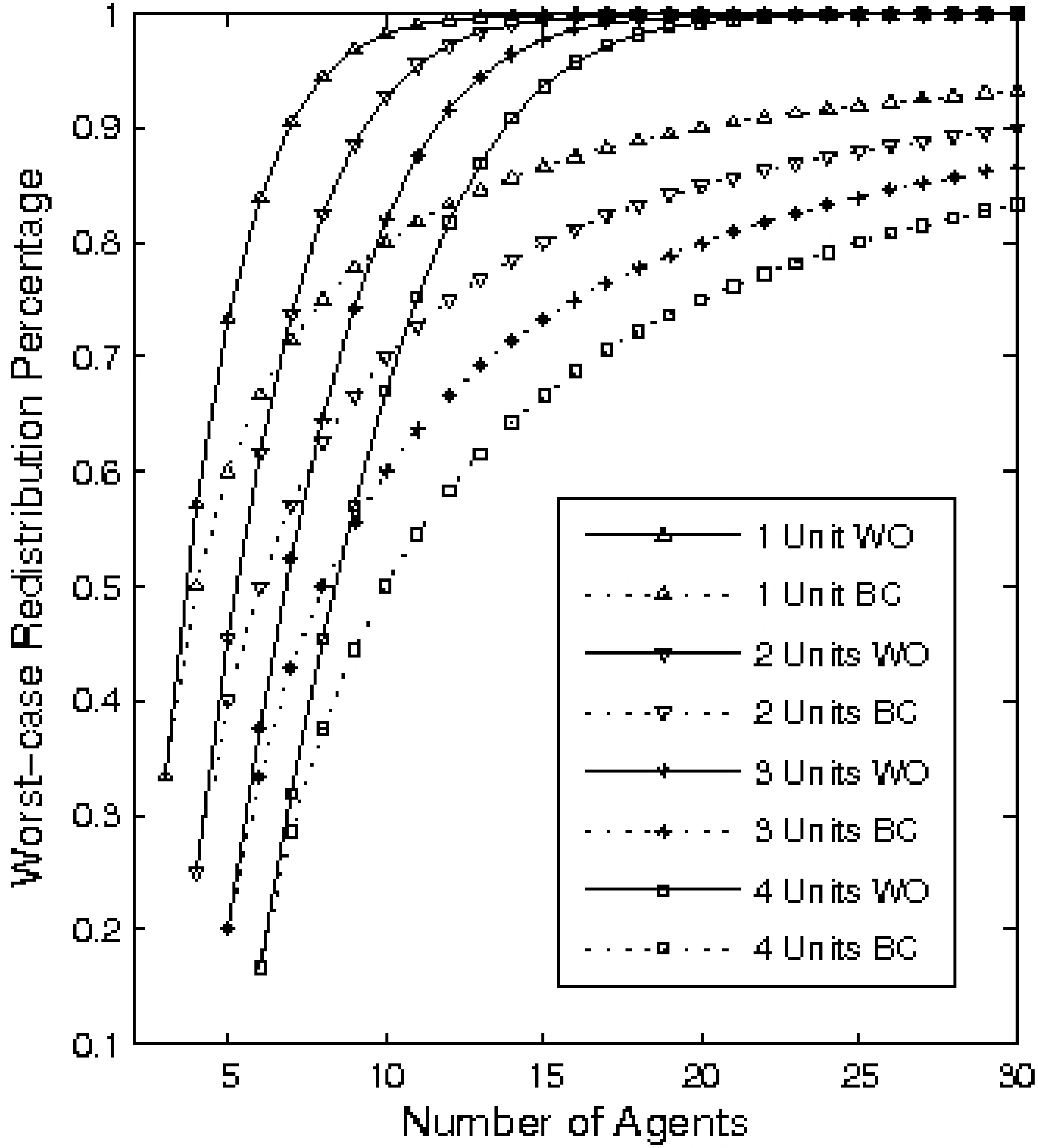
R_i as defined in previous slides

Only need to change V_2 into mV_{m+1}

Results for m+1th price auction

BC = Bailey-Cavallo

WO = Worst-case Optimal



Analytical characterization of WO mechanism

$$k^* = 1 - \frac{\binom{n-1}{m}}{\Sigma_{j=m}^{n-1}\binom{n-1}{j}}$$

$$c_i^* = \frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{i\Sigma_{j=m}^{n-1}\binom{n-1}{j}} \frac{1}{\binom{n-1}{i}} \Sigma_{j=i}^{n-1}\binom{n-1}{j}$$

$$for\ i = m+1, \ldots, n-1$$

- Unique optimum
- Can show: for fixed m, as n goes to infinity, worst-case redistribution percentage approaches 100% linearly
- Rate of convergence 1/2

Worst-case optimality outside the linear family

- **Theorem:** The worst-case optimal **linear** redistribution mechanism is also worst-case optimal among **all** VCG redistribution mechanisms that are
  - deterministic,
  - anonymous,
  - incentive compatible,
  - efficient,
  - non-deficit

- Individual rationality is not mentioned
  - Sacrificing individual rationality does not help
- Not **uniquely** worst-case optimal

Remarks

- Moulin's working paper "Efficient, strategy-proof and almost budget-balanced assignment"

  pursues different worst-case objective (minimize waste/efficiency)

  - Results in same mechanism in the unit-demand setting (!)
  - Different mechanism results after removing individual rationality
  - Also mentions the idea of removing non-deficit property, without solving for the actual mechanism

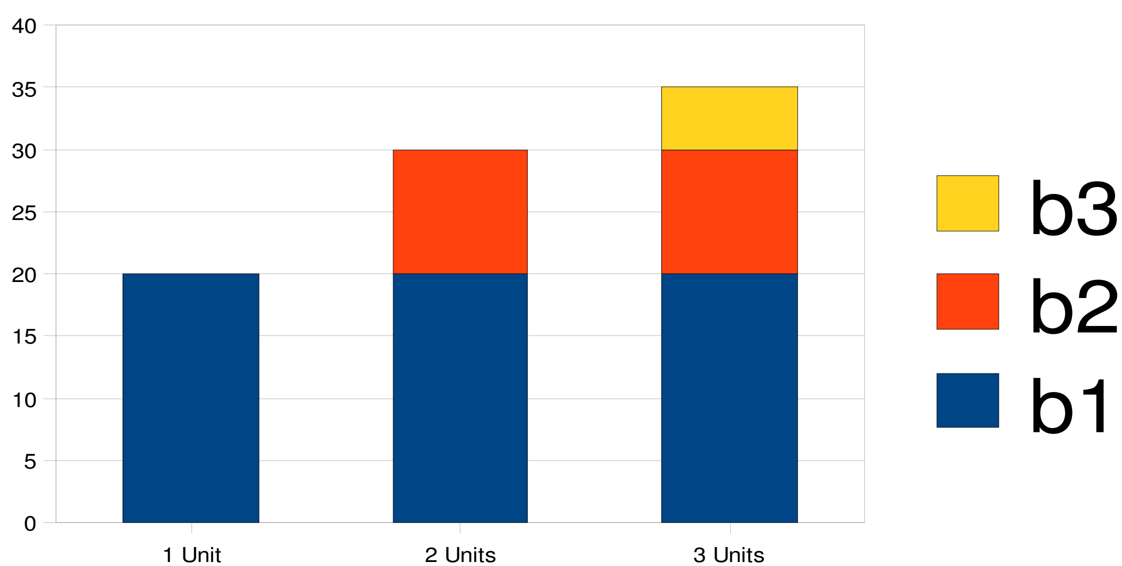
Multi-unit auction with nonincreasing marginal values

- A bid consists of m elements: $b_1, b_2, \ldots, b_m$

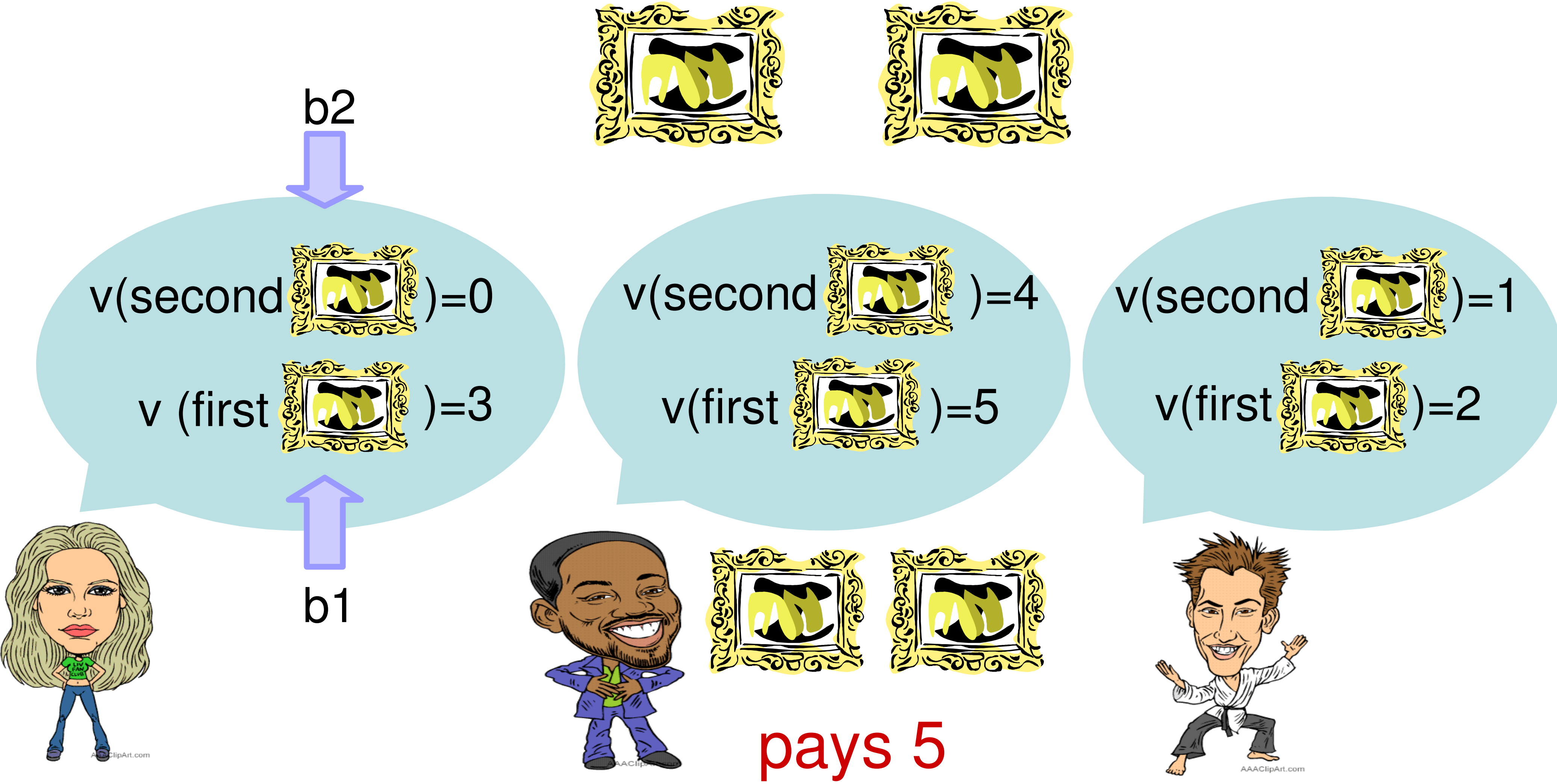
  b_i = utility(i units) – utility(i-1 units)

  $b_1 \geq b_2 \geq \ldots \geq b_m \geq 0$

  b_1 is called the initial marginal value



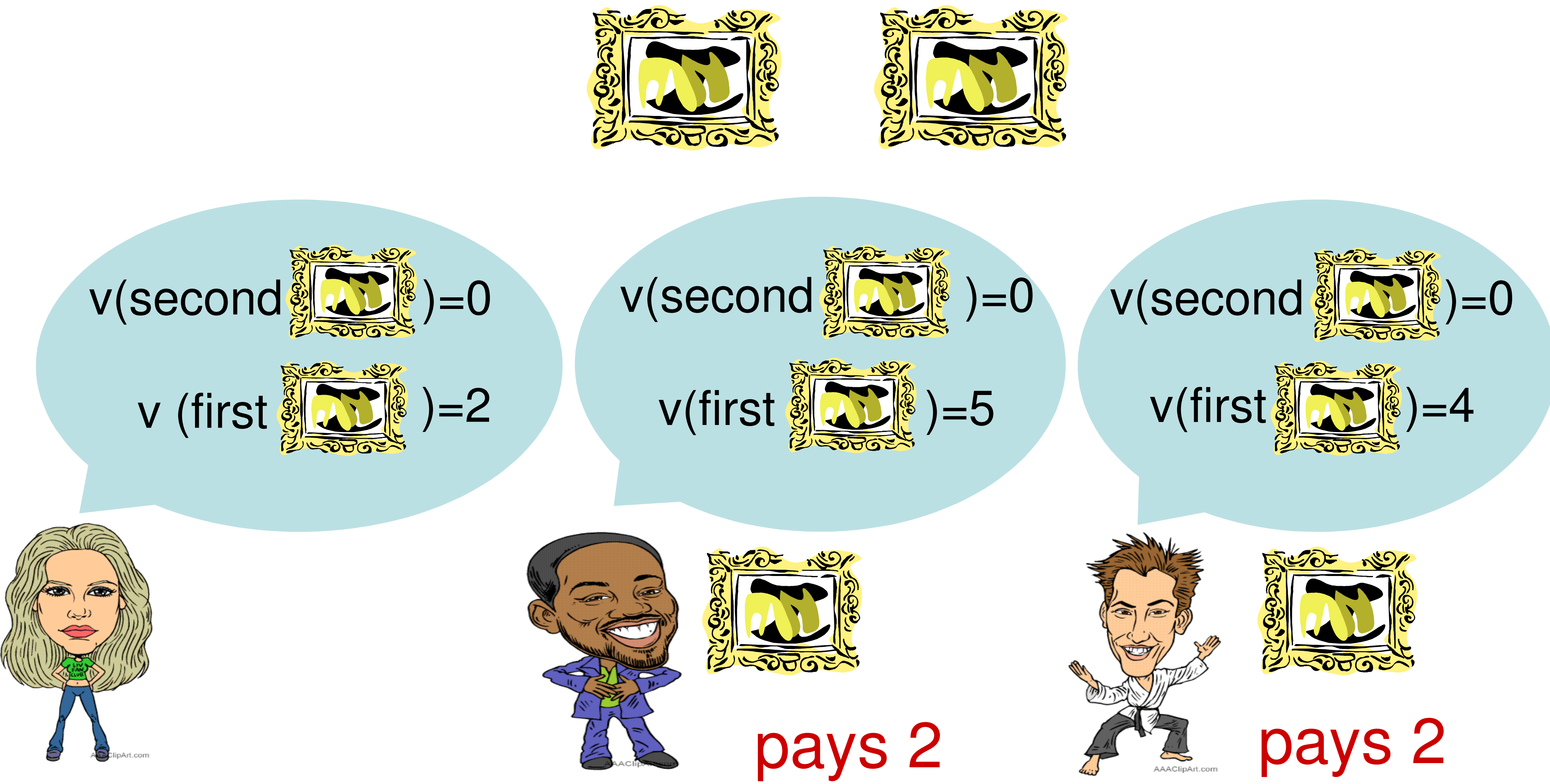
Multi-unit auction with nonincreasing marginal values



payment of i = others' total utility when i is not present

– others' total utility when i is present

Another example



Approach

- We construct a mechanism that has the same worst-case performance as the earlier WO mechanism.
- Multi-unit auction with unit demand is a special case of multi-unit auction with nonincreasing marginal value.

- The new mechanism is optimal in the worst case.

Gadgets

- Let S be a set of bidders. Define function R recursively:
- R(S,0)=VCG(S)
  - total VCG payment from selling all units (using VCG mechanism) to the set of bidders S
- R(S,i) is defined as
  - remove 1 bidder from the first m+i bidders of S (order by initial marginal value)
  - denote the new set by S'
  - average over all R(S', i-1) (m+i choices)
  - Domain: $i \leq |S|-m$

Example

m=2, S={s1, s2, s3, s4, s5, s6}
R(S,2) is computed as the average of m+2 = 4 choices

R({s1, s2, s3, s4, s5, s6},1)

R({s1, s2, s3, s4, s5, s6},0)
R({s1, s2, s3, s4, s5, s6},0)
R({s1, s2, s3, s4, s5, s6},0)

R({s1, s2, s3, s4, s5, s6},1)

R({s1, s2, s3, s4, s5, s6},0)
R({s1, s2, s3, s4, s5, s6},0)
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R({s1, s2, s3, s4, s5, s6},0)
R({s1, s2, s3, s4, s5, s6},0)
R({s1, s2, s3, s4, s5, s6},0)

Mechanism construction

- The set of all bidders: $A=\{a_1,a_2,\ldots,a_n\}$
  - a_i is the bidder with the ith highest initial marginal value
  - the set of other bidders for a_i: $A_{-i} = A - \{a_i\}$
- We redistribute to bidder i

$$1/m \sum_{j=m+1..n-1} C_j \, R(A_{-i}, j-m-1)$$

  - the Ci are the same as in unit demand setting
  - The mechanism is incentive compatible: redistribution is independent of your own bid
- This mechanism is worst-case optimal

Proof of optimality

- Let $U_i = R(A,i)$
- Claim: U_i is nonincreasing in i
- Claim: The total redistribution is

$$1/m \sum_{j=m+1..n-1} C_j ((n-j)\, U_{j-m-1} + jU_{j-m})$$

- In unit demand setting, the total redistribution is

$$\sum R_i = \sum_{j=m+1..n-1} C_j ((n-j)V_j + jV_{j+1})$$

- For **all** $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$,

$$K m V_{m+1} \leq \sum R_i \leq m V_{m+1}$$

K is the optimal guaranteed percentage of redistribution

- By shifting subscripts, we have

$$KU_0 \leq 1/m \sum_{j=m+1..n-1} C_j ((n-j)U_{j-m-1} + jU_{j-m}) \leq U_0$$

End of proof

- $KU_0 \leq$ Total Redistribution $\leq U_0$
  - U_0=VCG(A)=total VCG payment
  - The new mechanism never incurs a deficit and performs as well as the WO mechanism
  - Also individually rational and anonymous

Next, we will use two properties of the setting to prove the claims

Revenue Monotonicity

- VCG(S) = total VCG payment from selling all units (using VCG mechanism) to the set of bidders S
- When marginal values are nonincreasing, we have
  - VCG(U) ≥ VCG(V) given $U \supseteq V$ [Lehmann et al. 02], [Gul and Stacchetti 99]
- Not true in general [Rastegari et al. 07], [Ausubel and Milgrom 06], [Conitzer and Sandholm 06], [Yokoo 03], [Yokoo et al. 01], [Yokoo et al. 04]

Observation

- Let S = {$s_1,s_2,...,s_n$} where s_i is the bidder with the ith highest initial marginal value in S.
- The total utility is determined by $s_1,s_2,...,s_m$
  - only $s_1,s_2,...,s_m$ can possibly win units
  - proof sketch: if s_j (j>m) wins some units, at least one s_i (i in 1..m) does not win any units. Taking one unit from s_j and give it to s_i will only increase the overall utility (ignoring ties)

Observation

- VCG(S) depends only on $s_1, s_2, \ldots, s_m, s_{m+1}$
  - Payment of s_i = others' total utility when s_i is not present – others' total utility when si is present
- others' total utility when s_i is present

  $s_1, s_2, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_m, s_{m+1}, s_{m+2}, \ldots, s_n$

- others' total utility when s_i is not present

  $s_1, s_2, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_m, s_{m+1}, s_{m+2}, \ldots, s_n$

Previous Example

m=2, S={s1, s2, s3, s4, s5, s6}
R(S,2) is computed as the average of m+2 = 4 choices

R({s1, s2, s3, s4, s5, s6},1)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)

R({s1, s2, s3, s4, s5, s6},1)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)

R({s1, s2, s3, s4, s5, s6},1)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)

R({s1, s2, s3, s4, s5, s6},1)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)
- R({s1, s2, s3, s4, s5, s6},0)

- R(S,i) is nonincreasing in i by revenue monotonicity (proves the first claim)
- R(S,i) depends only on the first m+1+i bidders

Second Claim

- $\sum_{i=1..n} 1/m \sum_{j=m+1..n-1} C_j R(A_{-i}, j\text{-}m\text{-}1)$

  $=1/m \sum_{j=m+1..n-1} C_j \sum_{i=1..n} R(A_{-i}, j\text{-}m\text{-}1)$

  $=1/m \sum_{j=m+1..n-1} C_j ((n\text{-}j) U_{j-m-1} + jU_{j-m})$

$\sum_{j=1..n} R(A_{-j}, i)=\sum_{j=1..m+1+i} R(A_{-j}, i)+\sum_{j=m+1+i+1...n} R(A_{-j}, i)$

$=(m+1+i)R(A, i+1) + \sum_{j=m+1+i+1...n} R(A_{-j}, I)$

(definition of R)

$=(m+1+i)U_{i+1} + \sum_{j=m+1+i+1...n} R(A_{-j}, i)$

(R(A,i) depends only on the first m+1+i bidders in A. When j>m+1+i, whether bidder j is present or not does not change anything)

$=(m+1+i)U_{i+1} + (n\text{-}m\text{-}1\text{-}i)U_i$

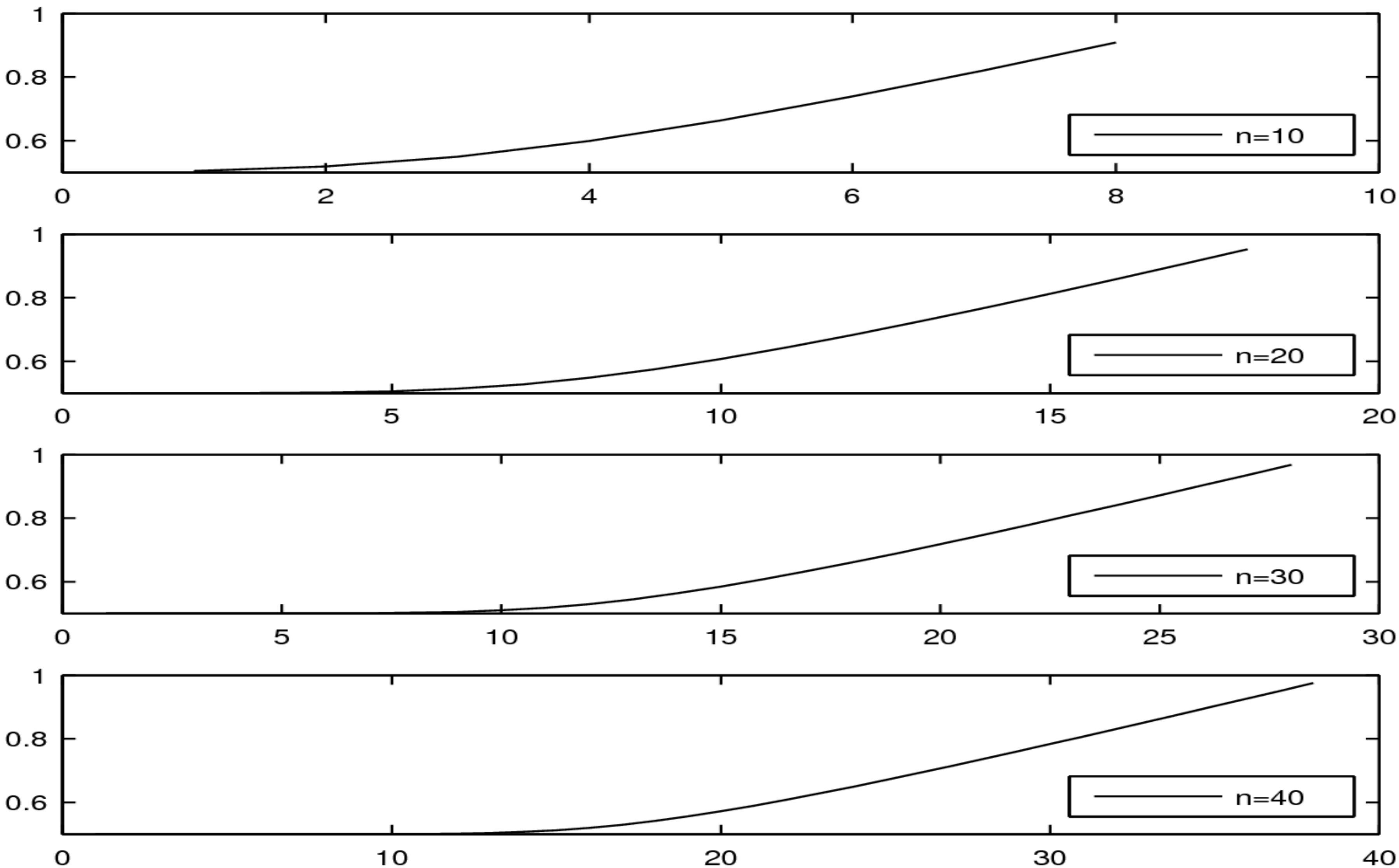
Removing non-deficit property

- We can get even closer to budget balance if we remove the non-deficit property.
- To solve for the mechanism that deviates the least from budget balance, we change the constraint $KmV_{m+1} \leq \sum R_i \leq mV_{m+1}$

  into $(1-K)mV_{m+1} \leq \sum R_i \leq (1+K)mV_{m+1}$
- Then instead of maximizing K, we minimize K.
- Can also be generalized to multi-unit auction with nonincreasing marginal values.

Up to 50% closer to BB for small m

x axis=m, y axis=waste w. deficit/ waste w.o. deficit



More general setting?

- If marginal values are not required to be nonincreasing, the worst-case redistribution percentage is at most 0
  - example (nothing can be redistributed, details omitted)
  - one bidder bids 1 on m units
  - one bidder bids 1 on 1 unit
  - the other bidders bid 0
  - The original VCG mechanism is already worst-case optimal
- Similar example for general combinatorial auction with single-minded bidders.

Thank you for your attention!