Complexity of some aspects of control and manipulation in weighted voting games

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Abstract

An important aspect of mechanism design in social choice protocols and multiagent systems is to discourage insincere behaviour. Manipulative behaviour has received increased attention since the famous Gibbard-Satterthwaite theorem. We examine the computational complexity of manipulation in weighted voting games which are ubiquitous mathematical models used in economics, political science, neuroscience, threshold logic, reliability theory and distributed systems. It is a natural question to check how changes in weighted voting game may affect the overall game. Tolerance and amplitude of a weighted voting game signify the possible variations in a weighted voting game which still keep the game unchanged. We characterize the complexity of computing the tolerance and amplitude of weighted voting games. Tighter bounds and results for the tolerance and amplitude of key weighted voting games are also provided. Moreover, we examine the complexity of manipulation and show limits to how much the Banzhaf index of a player increases or decreases if it splits up into sub-players. It is shown that the limits are similar to the previously examined limits for the Shapley-Shubik index. A pseudo-polynomial algorithm to find the optimal split is also provided.

Key words : weighted voting games, voting power, cooperative game theory, algorithms and complexity

1 Introduction

1.1 Motivation

Weighted voting games (WVGs) are mathematical models which are used to analyze voting bodies in which the voters have different number of votes. In WVGs, each voter is

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assigned a non-negative weight and makes a vote in favour of or against a bill. The bill is passed if and only if the total weight of those voting in favour of the bill is greater than or equal to some fixed quota. Power indices such as the Banzhaf index measure the ability of a player in a WVG to determine the outcome of the vote. WVGs have been applied in various political and economic organizations [21, 20, 1]. Voting power is used in joint stock companies where each shareholder gets votes in proportion to the ownership of a stock [13].

WVGs have received increased interest in the artificial intelligence and agents community due to their ability to model various coalition formation scenarios [10, 11]. Such games have also been examined from the point of view of susceptibility to manipulations [3, 31]. WVGs are also encountered in threshold logic, reliability theory, neuroscience and logical computing devices [28, 29]. There are many parallels between reliability theory and voting theory [27]. Parhami [26] points out that voting has a long history in reliability systems dating back to von Neumann [30]. Nordmann et al. [24] deal with reliability and cost evaluation of weighted dynamic-threshold voting-systems. Systems of this type are used in various areas such as target and pattern recognition, safety monitoring and human organization systems.

Elkind et al. [9] note that since WVGs have only two possible outcomes, they do not fall prey to manipulation of the type characterized by Gibbard-Satterthwaite [15]. However, there are various ways WVGs can be manipulated and controlled. We examine some of the aspects. Tolerance and amplitude of WVGs signify the possible variances in a WVG which still keep the game unchanged. They are significant in mathematical models of reliability systems and shareholdings. For reliability systems, the weights of a WVG can represent the significance of the components, whereas the quota can represent the threshold for the overall system to fail. It is then a natural requirement to provide a framework which can help identify similar reliability systems. In shareholding scenarios [2], there is a need to check the maximum changes in shares which still maintain the status quo. In political settings, the amplitude of a WVG signifies the maximum percentage change in various votes without changing the voting powers of the voters. In this paper, the computational aspects of amplitude and tolerance of WVGs are examined.

Moreover, splitting of a player into sub-players can be seen as a false-name manipulation by an agent where it splits itself into more agents so that the sum of the utilities of the split-up players is more than the utility of the original player. Elkind et al. [4] examined this manipulation from the point of view of Shapley-Shubik indices and asked the question of how the analysis of false-name manipulation will look in the case of Banzhaf indices. We examine situations when a player splitting up into smaller players may be advantageous or disadvantageous in the context of WVGs and Banzhaf indices. This gives a better idea of how to devise WVGs in which manipulation can be deterred.

1.2 Outline

In Section 2, some basic definitions of simple games, weighted voting games and computational complexity are provided.

Section 3 provides a background of tolerance and amplitude. In Section 4, computational aspects of tolerance and amplitude are examined. It is seen that computing the amplitude and tolerance of a WVG is NP-hard. We give tighter bounds and results for the tolerance and amplitude of key WVGs such as uniform (symmetric) WVGs and unanimity WVGs.

In Section 5, the case of players splitting up into sub-players in a WVG to increase their Banzhaf index is analysed. We check the limits to how much the Banzhaf index of a player can increase or decrease if it splits up into sub-players.

From a computational perspective, it is #P-hard for a manipulator to find the ideal splitting to maximize his payoff. A prospective manipulator could still be interested in enabling a beneficial split even if the improvement in payoff is not high. In Section 6, we prove that it is NP-hard even to decide whether a split is beneficial or not. In the end a pseudo-polynomial algorithm is proposed which returns 'no' if no beneficial split is available and returns the optimal split otherwise.

The final section presents conclusions and ideas for future work.

2 Preliminaries

In this section we give definitions and notations of key terms. The set of voters is $N = \{1, ..., n\}$.

Definitions 1. *A* simple voting game *is a pair* (N, v) *where the valuation function* $v : 2^N \to \{0, 1\}$ *has the properties that* $v(\emptyset) = 0$*,* $v(N) = 1$ *and* $v(S) \leq v(T)$ *whenever* $S \subseteq T$*. A* coalition $S \subseteq N$ *is* winning *if* $v(S) = 1$ *and* losing *if* $v(S) = 0$*. A simple voting game can alternatively be defined as* (N, W) *where* W *is the set of winning coalitions.*

Definitions 2. *The simple voting game* (N, W) *where*
$W = \{X \subseteq N, \sum_{x \in X} w_x \geq q\}$ *is called a* weighted voting game *(WVG). A WVG is denoted by* $[q; w_1, w_2, ..., w_n]$ *where* $w_i \geq 0$ *is the voting weight of player* i*. By convention, we take* $w_i \geq w_j$ *if* $i < j$*.*

Usually, $q > \frac{1}{2} \sum_{1 \leq i \leq n} w_i$ so that there are no two mutually exclusive winning coalitions at the same time. WVGs with this property are termed *proper*. Proper WVGs are also desirable because they satisfy the criterion of the majority getting preference. If the valuation function of a WVG v is same as another WVG v', then v' is called a *representation* of v. If the quota q' of v' is such that for all $S \subseteq N$, $\sum_{i \in S} w_i{}' \neq q'$, then v' is called a *strict representation* of v.

Definitions 3. *A player i is* critical in *a winning coalition S when $S \in W$ and $S \setminus \{i\} \notin W$. For each $i \in N$, we denote the number of coalitions in which i is critical in game v by $\eta_i(v)$. The* Banzhaf index *of player i in WVG v is $\beta_i = \frac{\eta_i(v)}{\sum_{i \in N} \eta_i(v)}$. The* probabilistic Banzhaf index, *β_i' of player i in game v is equal to $\eta_i(v)/2^{n-1}$.*

The following are non-technical definitions of some basic complexity classes.

Definition 4. *A problem is in complexity class* P *if it can be solved in time which is polynomial in the size of the input. A problem is in the complexity class* NP *if its solution can be verified in time which is polynomial in the size of the input of the problem. A problem is in complexity class* co-NP *if and only if its complement is in NP. A problem is in the complexity class* NP-hard *if any problem in* NP *is polynomial time reducible to that problem.* NP-hard *problems are as hard as the hardest problems in* NP. *A* $\#P$-hard *problem is a counting problem which is as hard as the counting version of any NP-hard problem.*

3 Tolerance & Amplitude: background

3.1 Background

The question we are interested in is to find the maximum possible variations in the weights and quotas of a WVG which still do not change the game. The two key references which address this question are [16] and [12]. Hu [16] worked within the theory of switching functions. He set forth the idea of *linearly separable switching functions* which are equivalent to each other. Freixas and Puente [12] extended the theory by framing it in the context of strict representations of WVGs, which are equivalent to linearly separable switching functions.

3.2 Tolerance

The setting of the problem is that we look at a transformation, $f_{(\lambda_1,\ldots,\lambda_n),\Lambda}$ which maps a WVG, $v = [q; w_1, \ldots, w_n]$ to $v' = [q'; w_1{}', \ldots, w_n{}']$ such that $w_i{}' = (1 + \lambda_i)w_i$ and $q' = (1+\Lambda)q$. Let A be the maximum of $w(S)$ for all $\{S|v(S) = 0\}$. and let B be the minimum of $w(S)$ for all $\{S|v(S) = 1\}$. Then $A < q \leq B$ (and $q < B$ if the representation is strict). Moreover, let $m = \mathsf{Min}(q - A, B - q)$ and $M = q + w(N)$. Hu [16] and then Freixas and

Puente [12] showed that if for all $1 \leq i \leq n$, $|\lambda_i| < m/M$ and $|\Lambda| < m/M$, then v' is just another representation of v. They defined $\tau[q; w_1, \ldots, w_n] = m/M$ as the *tolerance* of the system. Freixas and Puente [12] also showed that the tolerance is less than or equal to $1/3$ for strict representations of a WVG and less than or equal to $1/5$ for a not necessarily monotonic[0] WVG.

3.3 Amplitude

Freixas and Puente defined the *amplitude* as the maximum μ such that $f_{(\lambda_1,\ldots,\lambda_n),\Lambda}$ is a representation of v whenever $\mathsf{Max}(|\lambda_1|, \ldots, |\lambda_n|, |\Lambda|) < \mu(v)$. For a strict representation of a WVG $[q; w_1, \ldots, w_n]$, for each coalition $S \subseteq N$, let $a(S) = |w(S) - q|$ and $b(S) = q + w(S)$.

Freixas and Puente [12] showed that the amplitude of a WVG is $\mu(v) = \mathsf{Inf}_{S \subseteq N} \frac{a(S)}{b(S)}$. Although both tolerance and amplitude have been used in the WVG literature to signify the maximum possible variation in the weights and the quota without changing the game, the amplitude is a more precise and accurate indicator of the maximum possible variation than tolerance.

4 Tolerance & Amplitude: some results

4.1 Complexity

In all the complexity proofs in this section, we assume that the weights in a WVG are positive integers. We let WVG-STRICT be the problem of checking whether a WVG $v = [q; w_1, \ldots, w_n]$ is strict or not, i.e., WVG-STRICT = $\{v$: v is strict$\}$. Then we have the following proposition:

Proposition 5. *WVG-STRICT is co-NP-complete.*

Proof. Let WVG-NOT-STRICT = $\{v$: v is not strict$\}$. WVG-NOT-STRICT is in NP since a certificate of weights can be added in linear time to confirm that they sum up to q. Moreover v is not strict if and only if there is a subset of weights which sum up to q. Therefore the NP-complete problem SUBSET-SUM (see Garey and Johnson [14]) reduces to WVG-NOT-STRICT. Hence WVG-NOT-STRICT is NP-complete and WVG-STRICT is co-NP-complete. ☐

Corollary 6. *The problem of checking whether the amplitude of a strict WVG is zero is NP-hard.*

[0] Freixas and Puente also consider WVGs where players' weights can be negative.

Proposition 7. *The problem of computing the amplitude of a WVG v is NP-hard.*

Proof. Let us assume that weights in v are even integers whereas the quota q is an odd integer $2k-1$. Then the minimum possible difference between q and A, the weight of the maximal losing coalition, or q and B, the weight of minimal winning coalition is 1. So $A \leq 2k-2$ and $B \geq 2k$. We see that $\mu(v) \leq 1/(4k-1)$ if and only if there exists a coalition C such that $w(C) = 2k$. Thus the problem of computing $\mu(v)$ for a WVG is NP-hard by a reduction from the SUBSET-SUM problem. □

A similar proof can be used to prove the following proposition:

Proposition 8. *The problem of computing the tolerance of a strict WVG is NP-hard.*

4.2 Uniform and unanimity WVGs

We show that the bound for the maximum possible tolerance can be improved when we restrict to strict representations of special cases of WVGs. We first look at uniform WVGs which are an important subclass of WVGs which model many multi-agent scenarios where each agent has the same voting power.

Proposition 9. *For a strict representation of a proper uniform WVG $v = [q; \underbrace{w, \ldots, w}_{n}]$, $\tau(v) \leq \frac{1}{3n}$.*

Proof. Since $\frac{q-A}{q+w(N)} = 1 - \frac{w(N)+A}{q+w(N)}$ is an increasing function of q and $\frac{B-q}{q+w(N)}$ is a decreasing function of q, the tolerance reaches its maximum when $q - A = B - q$, i.e. when q is the arithmetic mean $\frac{A+B}{2}$. We let the size of the maximal losing coalition be r and the size of the minimal winning coalition be $r+1$. Then the weight of a maximal losing coalition is rw and the weight of the minimal winning coalition is $(r+1)w$ and $m = w/2$. Since v is proper, $q \geq \frac{1}{2}(nw)$, and $M = q + w(N) \geq \frac{3nw}{2}$. Then,

$$\tau(v) = m/M \leq \frac{1}{3n}.$$

□

Proposition 10. *For a uniform WVG $v = [q; \underbrace{w, \ldots, w}_{n}]$, we have $B = w\lceil \frac{q}{w} \rceil$ and $A = B - w$. Then,*

$$\mu(v) = \begin{cases} \frac{q-A}{A+q}, & \textit{if } q \leq \sqrt{AB} \\ \frac{B-q}{B+q}, & \textit{otherwise.} \end{cases}$$

Proof. It is clear that B, the weight of the minimal winning coalition is $w\lceil\frac{q}{w}\rceil$ and A, the weight of the maximal losing coalition is $B - w$. Note that, $\frac{q-A}{q+A} \leq \frac{B-q}{q+B}$ if and only if $q \leq \sqrt{AB}$. For losing coalitions with weight w, $\frac{q-w}{q+w}$ is a decreasing function for w. For winning coalitions with weight w, $\frac{w-q}{q+w} = 1 - \frac{2q}{q+w}$ is an increasing function for w. Thus if $q \leq \sqrt{AB}$, $\mu(v) = \frac{q-A}{A+q}$. Otherwise, $\mu(v) = \frac{B-q}{B+q}$. □

Corollary 11. *The amplitude $\mu(v)$ of a uniform WVG v can be found in $O(1)$, i.e., constant, time.*

Proof. The corollary immediately follows from the previous theorem. □

We now look at unanimity WVGs, which are another important subclass of WVGs in which a coalition is winning if and only if it is the grand coalition N.

Proposition 12. *For a unanimity WVG $v = [q; w_1, \ldots, w_n]$, $\tau(v) \leq \frac{w_n}{4w(N)-w_n} \leq \frac{1}{4n-1}$.*

Proof. We know that $B = w(N)$ and $A = w(N) - w_n$ which means that $w(N) - w_n < q \leq w(N)$. For maximum tolerance, $q = \frac{A+B}{2} = w(N) - \frac{w_n}{2}$. Therefore $m = w_n/2$ and $M = w(N) - \frac{w_n}{2} + w(N)$. Then the tolerance of v satisfies:

$$\tau(v) \leq \frac{m}{M} = \frac{w_n}{4w(N) - w_n} \leq \frac{1}{4n-1},$$

since $w_n \leq w(N)/n$. □

A multiple weighted voting game is composed of more than one weighted voting game and a coalition wins if and only if it is winning in each of the weighted voting games:

Definition 13. *An m-*multiple weighted voting game, $(N, v_1 \wedge ... \wedge v_m)$ *is the simple game (N, v), where the games (N, v_t) are the weighted voting games $[q^t; w_1^t, ...w_n^t]$ for $1 \leq t \leq m$. Then $v = v_1 \wedge ... \wedge v_m$ is defined as: $v(S) = 1 \iff v_t(S) = 1$ for $1 \leq t \leq m$. The game v is called the* meet *of the v_is.*

Note that we do not insist that $w_i^t \geq w)j^t$ for all $i < j$ and $1 \leq t \leq m$. Let $(N, v) = (N, v_1 \wedge ... \wedge v_m)$ be a multiple weighted voting game. Then we can see that $\mu(v) \geq \mathsf{Inf}(\mu(v_1), \ldots, \mu(v_m))$. The reason is that for v to change, at least one constituent game has to change. However it is not necessary that a change in any one game v_i changes v. As a simple example, suppose $v_1 = [2; 2, 1]$ and $v_2 = [2; 1, 2]$. Then $\mu(v_1 \wedge v)2) = \sqrt{3/2} - 1$, as witnessed by the coalition $\{1, 2\}$. However, $\mu(v_i) = 0$, as witnessed by $\{i\}$, for $i = 1, 2$.

5 Manipulation via splitting

5.1 Background

In the real world, WVGs may be dynamic. Players might have incentive to split up into smaller players or merge into voting blocks. *Imputations* of the players are acceptable distributions of the payoff of the grand coalition. Imputations of players in a coalitional games setting can be based on fairness, i.e., power indices, or they can be based on the notion of stability which includes many cooperative game theoretic solutions such as core, nucleolus etc. We examine the situation when the Banzhaf indices of agents can be used as imputations in a cooperative game theoretic situation. Falsenthal and Machover [22] refer to this notion of voting power as P-power since the motivation of agents is prize-seeking as opposed to influence-seeking. Banzhaf indices have been considered as possible payments in cooperative settings [7, 4] since they satisfy useful axioms [8]. Splitting of a player can be seen as a false-name manipulation by an agent in which it splits itself into more agents so that the sum of the utilities of the split-up players is more than the utility of the original player.

Splitting is not always beneficial. We give examples where, if we use Banzhaf indices as payoffs of players in a WVG, splitting can be advantageous, neutral or disadvantageous.

Example 14. *Splitting can be advantageous, neutral or disadvantageous:*

- Disadvantageous splitting. *We take the WVG* $[5; 2, 2, 2]$ *in which each player has a Banzhaf index of* $1/3$*. If the last player splits up into two players, the new game is* $[5; 2, 2, 1, 1]$*. In that case, the split-up players have a Banzhaf index of* $1/8$ *each.*

- Neutral splitting. *We take the WVG* $[4; 2, 2, 2]$ *in which each player has a Banzhaf index of* $1/3$*. If the last player splits up into two players, the new game is* $[4; 2, 2, 1, 1]$*. In that case, the split-up players have a Banzhaf index of* $1/6$ *each.*

- Advantageous splitting. *We take the WVG* $[6; 2, 2, 2]$ *in which each player has a Banzhaf index of* $1/3$*. If the last player splits up into two players, the new game is* $[6; 2, 2, 1, 1]$*. In that case, the split-up players have a Banzhaf index of* $1/4$ *each.*

We analyse the splitting of players in the unaninimity WVG.

Proposition 15. *In a unanimity WVG with* $q = w(N)$*, if Banzhaf or Shapley-Shubik indices are used as imputations of agents in a WVG, then it is beneficial for an agent to split up into several agents.*

Proof. In a WVG with $q = w(N)$, the Banzhaf index of each player is $1/n$. Let player i split up into $m + 1$ players. In that case there is a total of $n + m$ players and the Banzhaf index of each player is $1/(n + m)$. In that case the total Banzhaf index of the split up players is $\frac{m+1}{n+m}$, and for $n > 1$, $\frac{m+1}{n+m} > \frac{1}{n}$. An exactly similar analysis holds for the Shapley-Shubik index. $\square$

However there is the same motivation for all agents to split up into smaller players, which returns us to the status quo.

5.2 General case

We recall that a player is critical in a winning coalition if the player's exclusion makes the coalition losing. We will also say that a player is *critical for* a losing coalition C if the player's inclusion results in the coalition winning.

Proposition 16. *Let WVG v be $[q; w_1, \ldots, w_n]$. If v transforms to v' by the splitting of player i into i' and i'', then $\beta_{i'}(v') + \beta_{i''}(v') \leq 2\beta_i(v)$.*

Proof. We assume that a player i splits up into i' and i'' and that $w_{i'} \leq w_{i''}$. We consider a losing coalition C for which i is critical in v. Then $w(C) < q \leq w(C) + w_i = w(C) + w_{i'} + w_{i''}$.

- If $q - w(C) \leq w_{i'}$, then i' and i'' are critical for C in v'.
- If $w_{i'} < q - w(C) \leq w_{i''}$, then i' is critical for $C \cup \{i''\}$ and i'' is critical for C in v'.
- If $q - w(C) > w_{i''}$, then i' is critical for $C \cup \{i''\}$ and i'' is critical for $C \cup \{i'\}$ in v'.

Therefore we have $\eta_{i'}(v') + \eta_{i''}(v') = 2\eta_i(v)$ in each case.

Now we consider a player x in v which is other than player i. If x is critical for a coalition C in v then x is also critical for the corresponding coalition C' in v' where we replace $\{i\}$ by $\{i', i''\}$. Hence $\eta_x(v) \leq \eta_x(v')$. Of course x may also be critical for some coalitions in v' which contain just one of i' and i'', so the above inequality will not in general be an equality.

Moreover,

$$\begin{aligned}
\beta_{i'}(v') + \beta_{i''}(v') &= \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x\in N(v')\setminus\{i',i''\}} \eta_x(v')} \\
&\le \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x\in N(v)\setminus\{i\}} \eta_x(v)} \\
&\le \frac{2\eta_i(v)}{\eta_i(v) + \sum_{x\in N(v)\setminus\{i\}} \eta_x(v)} = 2\beta_i(v)
\end{aligned}$$

□

We can give an example which shows that the upper bound of the improvement in payoff by splitting into two players is tight:

Example 17. Advantageous splitting. *We take a WVG $[n; 2, 1, \dots, 1]$ with $n+1$ players. We find that $\eta_1 = n + \binom{n}{2}$ and for all other x, $\eta_x = 1 + \binom{n-1}{2}$. Therefore*

$$\beta_1 = \frac{n + \binom{n}{2}}{n + \binom{n}{2} + n(1 + \binom{n-1}{2})} = \frac{n+1}{(n-2)^2} \sim 1/n.$$

In case player 1 splits up into $1'$ and $1''$ with weights 1 each, then for all players j, $\beta_j = \frac{1}{n+2}$. Thus for large n, $\beta_{1'} + \beta_{1''} = \frac{2}{n+2} \sim 2\beta_1$.

Moreover, we show that splitting into two players can decrease the Banzhaf index payoff by as much as a factor of almost $\sqrt{\frac{\pi}{2n}}$:

Example 18. Disadvantageous splitting. *We take a WVG v on n players where $v = [3n/2; 2n, 1, \dots, 1]$. For the sake of simplicity, we assume that n is even. It is easy to see that player 1 is a dictator. Now we consider the case where v changes into v' with player 1, splitting up into $1'$ and $1''$ with weight n each. For player $1'$ to be critical for a losing coalition in v', the coalition much exclude $1''$ and have from $n/2$ to $n-1$ players with weight 1 or it must include $1''$ and have from 0 to $(n/2)-1$ players with weight 1. So $\eta_{1'}(v') = \eta_{1''}(v') = \sum_{i=0}^{n} \binom{n-1}{i} = 2^{n-1}$. Moreover, for a smaller player x with weight 1 to be critical for a coalition in v', the coalition must include only one of $1'$ or $1''$ and $(n-2)/2$ of the $n-2$ other smaller players. So, $\eta_x(v') = 2\binom{n-2}{(n-2)/2}$. By using Stirling's formula, we can approximate $\eta_x(v')$ by $\sqrt{\frac{2}{\pi(n-2)}}2^{n-1}$. We see that:*

$$\begin{aligned}
\beta_{i'}(v') &= \beta_{i''}(v') \\
&\approx \frac{2^{n-1}}{2^{n-1}+2^{n-1}+(n-1)\sqrt{\frac{2}{\pi(n-2)}}2^{n-1}} \\
&= \frac{1}{2+\frac{(n-1)}{\sqrt{n-2}}\sqrt{\frac{2}{\pi}}} \\
&\sim \sqrt{\frac{\pi}{2n}}.
\end{aligned}$$

Remark 19. *We notice that the bounds on the effect of splitting on the Banzhaf index are quite similar to those in the Shapley-Shubik case.*

6 Complexity of finding a beneficial split

From a computational perspective, it is #P-hard for a manipulator to find the ideal splitting to maximize his payoff. An easier question is to check whether a beneficial splitting exists or not. We define a Banzhaf version of the BENEFICIAL SPLIT problem defined in [4].

Name: BENEFICIAL-BANZHAF-SPLIT
Instance: (v, i) where v is the WVG $v = [q; w_1, \ldots, w_n]$ and player $i \in \{1, \ldots, n\}$.
Question: Is there a way for player i to split his weight w_i between sub-players $i_1, \ldots, i_m$ so that, in the new game v', $\sum_{j=1}^{k} \beta_{i_k}(v') > \beta_i(v)$?

Proposition 20. *BENEFICIAL-BANZHAF-SPLIT is NP-hard, and remains NP-hard even if the player can only split into two players with equal weights.*

Proof. We prove this by a reduction from an instance of the classical NP-hard PARTITION problem to BENEFICIAL-BANZHAF-SPLIT.

Name: PARTITION
Instance: A set of k integer weights $A = \{a_1, \ldots, a_k\}$.
Question: Is it possible to partition A, into two subsets $P_1 \subseteq A$, $P_2 \subseteq A$ so that $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = A$ and $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A_2} a_i$?

Given an instance of PARTITION $\{a_1, \ldots, a_k\}$, we can transform it to a WVG $v = [q; w_1, \ldots, w_n]$ with $n = k + 2$ where $w_i = 8a_i$ for $i = 1$ to $n - 2$, $w_{n-1} = 2$, $w_n = 1$ and

$q = 4\sum_{i=1}^{k} a_i + 3$. After that we want to see whether it can be beneficial for player $(n-1)$ with weight 2 to split into two sub-players $(n-1)_1$ and $(n-1)_2$ each with weight 1 to form a new WVG $v' = [q; w_1, \ldots, w_{n-2}, w_{(n-1)_1}, w_{(n-1)_2}, 1]$. Note that, since the weights are integral, it is certainly not beneficial to split up a weight of 2 other than into 1 and 1.

If A is a 'no' instance of PARTITION, then we see that a subset of the weights $\{w_1, \ldots, w_{n-2}\}$ cannot sum to $4\sum_i a_i$. This implies that player $(n-1)$ is a dummy. We see that even if player $(n-1)$ splits into sub-players, the sub-players are also dummies. Therefore $(v, n-1)$ is a 'no' instance of BENEFICIAL-BANZHAF-SPLIT.

Now let us assume that A is a 'yes' instance of PARTITION. In that case, let the number of subsets of weights $\{w_1, \ldots w_{n-2}\}$ summing to $4\sum_i a_i$ be x. Then $\eta_{n-1}(v) = \eta_n(v) = x$. If player $(n-1)$ splits into two players $(n-1)_1, (n-1)_2$ with weights 1 and 1, then $\eta_{(n-1)_1}(v) = \eta_{(n-1)_2}(v) = \eta_n(v) = x$. We see that $\eta_i(v) = \eta_i(v')$ for $i = 1$ to $n-2$. Suppose that $\sum_{i=1}^{n-2} \eta_i(v) = S$. Then for the split to be beneficial $\beta_{n-1_1}(v') + \beta_{n-1_2}(v') > \beta_{n-1}(v)$, i.e., $\frac{x}{S+3x} + \frac{x}{S+3x} > \frac{x}{S+2x}$. Since $x > 0$, this is equivalent to $S + x > 0$. Since $S + x \geq x > 0$, a 'yes' instance of PARTITION implies a 'yes' instance of BENEFICIAL-BANZHAF-SPLIT. $\square$

In terms of minimizing chances of manipulation, we see that computational complexity comes to our rescue. This idea of using computational complexity to model bounded rationality is well explained by Papadimitriou and Yannakakis [25]. In the context of complexity of voting, it was a series of groundbreaking papers by Bartholdi, Orlin, Tovey, and Trick [5, 17, 18, 19] that showed how important computational complexity consideration is in terms of ease of computing winners and difficulty of manipulating elections.

6.1 Pseudopolynomial algorithm

It is well known that although, computing Banzhaf indices of players in a WVG is NP-hard, there are polynomial time algorithm using dynamic programming [23] or generating functions [6] to compute Banzhaf indices if the weights of players are polynomial in n. Let this pseudo-polynomial algorithm be $\mathsf{BanzhafIndex}(v, i)$ which takes a WVG v and an index i as input and returns $\beta_i(v)$, the Banzhaf index of player i in v. We use a similar argument to that in [4] to show that a polynomial algorithm exists to find a beneficial split if the weights of players are polynomial in n and the player i in question can split into up to a constant k number of sub-players. Whenever game player i in WVG v splits up according to a split s, we denote the new game by $v_{i,s}$.

Algorithm 1 BeneficialSplitInWVG

Input: (v, i) where $v = [q; w_1, \ldots, w_n]$ and i is the player which wants to split into a maximum of k sub-players.

Output: Returns NO if there is no beneficial split. Otherwise returns the optimal split $(w_{i_1}, \ldots, w_{i_{k'}})$ where $k' \leq k$, and $\sum_{j=1}^{k'} w_{i_j} = w_i$.

$\mathsf{BeneficialSplitExists} = $ **false**
$\mathsf{BestSplit} = \emptyset$
$\mathsf{BestSplitValue} = -\infty$
$\beta_i = \mathsf{BanzhafIndex}(v, i)$
for $j = 2$ to k **do**
  for Each possible split s where $w_i = w_{i_1} + \ldots + w_{i_j}$ **do**
    $\mathsf{SplitValue} = \sum_{a=1}^{j} \mathsf{BanzhafIndex}(v_{i,s}, i_a)$
    if $\mathsf{SplitValue} > \beta_i$ **then**
      $\mathsf{BeneficialSplitExists} = $ **true**
      if $\mathsf{SplitValue} > \mathsf{BestSplitValue}$ **then**
        $\mathsf{BestSplit} = s$
        $\mathsf{BestSplitValue} = \mathsf{SplitValue}$
      end if
    end if
  end for
end for
if $\mathsf{BeneficialSplitExists} = $ **false then**
  return false
else
  return $\mathsf{BestSplit}$
end if

We see that the total number of disjoint splits for player i is equal to $q(w_i, k)$ where $q(n, k)$ is the partition function which gives the number of partitions of n with k or fewer addends. It is clear that for a constant k, the number of splits of player i is less than $(w_i)^k$ which is a polynomial in n. Since the computational complexity for each split is also a polynomial in n, Algorithm 1 is polynomial in n if weights are polynomial in n.

7 Conclusion and future work

We have examined the computational complexity of computing the tolerance and amplitude of WVGs. The tolerance and amplitude of uniform and unanimity games is also analysed. There is a need to devise approximation algorithms for computing the amplitude of a WVG. The analysis of amplitude and tolerance motivates the formulation of an overall framework to check the 'sensitivity' of voting games under fluctuations and susceptibility to control. It will be interesting to explore the limit of changes in WVGs in alternative representations of simple games.

We have also investigated the impact on the Banzhaf index distribution due to a player splitting into smaller players in a weighted voting game. It is seen that manipulation by splitting into sub-players may be discouraged by keeping to weights which are large or reals. There is more scope to analyse such situations with respect to other cooperative game theoretic solutions.

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